Supersymmetry breakdown for an extended version of the Nicolai supersymmetric fermion lattice model

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Sannomiya *et al.* have recently studied an extension of the Nicolai supersymmetric fermion lattice model, which is named "the extended Nicolai model." The extended Nicolai model is parametrized by an adjustable constant $g \in \mathbb{R}$ in its defining supercharge and satisfies $\mathcal{N} = 2$ supersymmetry. We show that for any nonzero g the extended Nicolai model breaks supersymmetry dynamically, and the energy density of any homogeneous ground state for the model is strictly positive.

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I. INTRODUCTION

In Ref. [1] Sannomiya *et al.* investigated supersymmetry breakdown for an extended version of the Nicolai supersymmetric fermion lattice model [2]. This model is called *the extended Nicolai model*. It satisfies the algebraic relation of $\mathcal{N} = 2$ supersymmetry, and it is same as the original Nicolai model when its adjustable parameter g is equal to 0.

Supersymmetry breakdown for the extended Nicolai model has been shown for any nonzero g on finite systems [1]. In the infinite-volume limit, however, Sannomiya *et al.* verified supersymmetry breakdown of the model only when $g > g_0 := 4/\pi$. This restriction upon the parameter g seems to be technical, and its physics meaning is unclear. The purpose of this paper is to remove this restriction upon g in the case of the infinite-volume limit. We show that for any $g \neq 0$ the extended Nicolai model defined on $\mathbb{Z}$ breaks supersymmetry dynamically. Furthermore, we prove that for any $g \neq 0$ the energy density of any (homogeneous) ground state for the extended Nicolai model is strictly positive.

It is noted in Ref. [1] that, even if supersymmetry (SUSY) is broken for any finite subsystem, supersymmetry may be restored in the infinite-volume limit as exemplified by some SUSY quantum mechanical model [3]. We, however, show that such restoration does not happen for the extended Nicolai model. We formulate the extended Nicolai model as supersymmetric C^*-dynamics [4] and verify its breaking of supersymmetry in a rather model-independent manner. In more detail, our proof makes essential use of a crucial finding by Sannomiya *et al.* (Eq. (15) of Ref. [1]) that will be reformulated in terms of superderivations.

II. EXTENDED NICOLAI SUPERSYMMETRIC FERMION LATTICE MODEL

We consider spinless fermions over an infinitely extended lattice $\mathbb{Z}$. Let c_i and c_i^* denote the annihilation operator and the creation operator of a spinless fermion at $i \in \mathbb{Z}$, respectively. Those obey the canonical anticommutation relations (CARs). For $i, j \in \mathbb{Z}$,

$$\{c_i^*, c_j\} = \delta_{i,j}1, \qquad \{c_i^*, c_j^*\} = \{c_i, c_j\} = 0. \tag{1}$$

For any $g \in \mathbb{R}$, we take the following infinite sum of local fermion operators:

$$Q(g) := \sum_{k\in\mathbb{Z}} (gc_{2k-1} + c_{2k-1}c_{2k}^* c_{2+1}). \tag{2}$$

It is perturbation of the supercharge of the original Nicolai model $Q(0)$ by another supercharge $\sum_{k\in\mathbb{Z}} c_{2k-1}$ multiplied by g. Note that the perturbed term $\sum_{k\in\mathbb{Z}} c_{2k-1}$ itself generates a trivial model. By some formal computation using the canonical anticommutation relations (1), we see that $Q(g)$ is nilpotent:

$$0 = Q(g)^2 = Q(g)^{*2}. \tag{3}$$

Let a supersymmetric Hamiltonian be given as

$$H(g) := \{Q(g), Q(g)^*\}. \tag{4}$$

For any $g \in \mathbb{R}$, the model has $\mathcal{N} = 2$ supersymmetry by definition. As noted above, if $g = 0$, it corresponds to the supersymmetric fermion lattice model defined by Nicolai in Ref. [2].

We note that in the infinite-volume system either $Q(g)$ or $Q(g)^*$, or both, cannot exist as a well-defined linear operator if the supersymmetry associated with them breaks dynamically. In fact, we will show that this is the case unless $g = 0$. Nevertheless, its supersymmetric dynamics always makes sense in the infinitely extended system, as we will see later.

We shall consider the model under periodic-boundary conditions as in Ref. [1]. Let M, $N \in 2\mathbb{N}$. Define

$$\tilde{Q}(g)_{[-M+1,N]} := \sum_{k=-M/2+1}^{N/2} (gc_{2k-1} + c_{2k-1}c_{2k}^{*}c_{2k+1}), \quad (5)$$

where $N+1$ is identified with $-M+1$. We see that

$$0 = \tilde{Q}(g)^{2}_{[-M+1,N]} = \tilde{Q}(g)^{*2}_{[-M+1,N]}. \quad (6)$$

Then, we define the corresponding local supersymmetric Hamiltonian on the same region $[-M+1, N]$ as

$$\tilde{H}(g)_{[-M+1,N]} := \{\tilde{Q}(g)_{[-M+1,N]}, \tilde{Q}(g)^{*}_{[-M+1,N]}\}. \quad (7)$$

Also, we may consider free-boundary conditions upon supercharges. Let

$$\hat{Q}(g)_{[-M+1,N+1]} := \sum_{k=-M/2+1}^{N/2} (gc_{2k-1} + c_{2k-1}c_{2k}^{*}c_{2k+1}) + gc_{N+1}. \quad (8)$$

We see that

$$0 = \hat{Q}(g)^{2}_{[-M+1,N+1]} = \hat{Q}(g)^{*2}_{[-M+1,N+1]}. \quad (9)$$

We give a local supersymmetric Hamiltonian upon the same region $[-M+1, N+1]$ by the supersymmetric form:

$$\hat{H}(g)_{[-M+1,N+1]} := \{\hat{Q}(g)_{[-M+1,N+1]}, \hat{Q}(g)^{*}_{[-M+1,N+1]}\}. \quad (10)$$

We have introduced two different local Hamiltonians for finite subsystems. Note that those give rise to the same time evolution on the total system by taking the infinite-volume limit as we will see in the next section.

III. MATHEMATICALLY RIGOROUS FORMULATION

In Ref. [4], a general framework of supersymmetric fermion lattice models is given. By using this framework, we shall reformulate the extended Nicolai model introduced in the preceding section as supersymmetric C^*-dynamics [5].

For each finite subset $\mathrm{I} \Subset \mathbb{Z}$, let $\mathcal{A}(\mathrm{I})$ denote the finite-dimensional algebra generated by $\{c_i, c_i^*; i \in \mathrm{I}\}$. For $\mathrm{I} \subset \mathrm{J} \Subset \mathbb{Z}$, $\mathcal{A}(\mathrm{I})$ is imbedded into $\mathcal{A}(\mathrm{J})$. We define

$$\mathcal{A}_\circ := \bigcup_{\mathrm{I} \Subset \mathbb{Z}} \mathcal{A}(\mathrm{I}), \quad (11)$$

where all finite subsets I of $\mathbb{Z}$ are taken. The norm completion of the *-algebra $\mathcal{A}_\circ$ (with the operator norm) yields a C^*-algebra $\mathcal{A}$, which is known as the CAR algebra.

The dense subalgebra $\mathcal{A}_\circ$ is called the local algebra. A linear functional ω of $\mathcal{A}$ is called a state if it is positive, i.e., $\omega(A^*A) \geq 0$ for any $A \in \mathcal{A}$, and also normalized, i.e., $\omega(1) = 1$.

Let σ denote the shift-translation automorphism group on $\mathcal{A}$. For each $k \in \mathbb{Z}$,

$$\sigma_k(c_i) = c_{i+k}, \qquad \sigma_k(c_i^*) = c_{i+k}^*, \quad \forall\, i \in \mathbb{Z}. \quad (12)$$

Let γ denote the grading automorphism on the C^*-algebra $\mathcal{A}$ determined by

$$\gamma(c_i) = -c_i, \qquad \gamma(c_i^*) = -c_i^*, \quad \forall\, i \in \mathbb{Z}. \quad (13)$$

The total system $\mathcal{A}$ is decomposed into the even part and the odd part:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \text{ where } \mathcal{A}_+ := \{A \in \mathcal{A} | \gamma(A) = A\},$$
$$\mathcal{A}_- := \{A \in \mathcal{A} | \gamma(A) = -A\}. \quad (14)$$

For each $\mathrm{I} \Subset \mathbb{Z}$,

$$\mathcal{A}(\mathrm{I}) = \mathcal{A}(\mathrm{I})_+ \oplus \mathcal{A}(\mathrm{I})_-, \quad \text{where}$$
$$\mathcal{A}(\mathrm{I})_+ := \mathcal{A}(\mathrm{I}) \cap \mathcal{A}_+, \quad \mathcal{A}(\mathrm{I})_- := \mathcal{A}(\mathrm{I}) \cap \mathcal{A}_-. \quad (15)$$

The graded commutator is defined as

$$[F_+, G]_\gamma = [F_+, G] \quad \text{for } F_+ \in \mathcal{A}_+, G \in \mathcal{A},$$
$$[F_-, G_+]_\gamma = [F_-, G_+] \quad \text{for } F_- \in \mathcal{A}_-, G_+ \in \mathcal{A}_+,$$
$$[F_-, G_-]_\gamma = \{F_-, G_-\} \quad \text{for } F_- \in \mathcal{A}_-, G_- \in \mathcal{A}_-. \quad (16)$$

From the canonical anticommutation relations (1), the graded locality follows:

$$[A, B]_\gamma = 0 \quad \text{for all } A \in \mathcal{A}(\mathrm{I}) \quad \text{and}$$
$$B \in \mathcal{A}(\mathrm{J}) \quad \text{if } \mathrm{I} \cap \mathrm{J} = \emptyset, \mathrm{I}, \mathrm{J} \Subset \mathbb{Z}. \quad (17)$$

The formal expression of the supercharge $Q(g)$ of Eq. (2) will give a well-defined infinitesimal fermionic transformation. Define a superderivation (a linear map that satisfies the graded Leibniz rule) from $\mathcal{A}_\circ$ into $\mathcal{A}_\circ$ by

$$\delta_g(A) := [Q(g), A]_\gamma \quad \text{for every } A \in \mathcal{A}_\circ. \quad (18)$$

Similarly, the conjugate superderivation is given by

$$\delta_g^*(A) := [Q(g)^*, A]_\gamma \quad \text{for every } A \in \mathcal{A}_\circ. \quad (19)$$

Let us explain the formula (18) in some depth. For each *fixed* local element $A \in \mathcal{A}_\circ$, only finite terms in the summation formula of $Q(g)$ are involved in $[Q(g), A]_\gamma$, because there is a least $\mathrm{I} \Subset \mathbb{Z}$ (with respect to the inclusion) such that $A \in \mathcal{A}(\mathrm{I})$, and by the graded locality (17), only local fermion terms of (2) that have nontrivial intersection with I may contribute to $[Q(g), A]_\gamma$. The other infinite number of terms vanish. Therefore, there exist M_0, $N_0 \in 2\mathbb{N}$ such that $[-M_0+1, N_0] \supset \mathrm{I}$ and the identity

$$\delta_g(A) = [\tilde{Q}(g)_{[-M+1,N]}, A]_\gamma \tag{20}$$

holds for all $M(\in 2\mathbb{N}) \geq M_0$ and $N(\in 2\mathbb{N}) \geq N_0$, where the periodic-boundary condition as in (5) is used. For example, if I is a finite interval of the type $[-S+1, -S+2, \ldots, T-1, T]$ with $(S, T \in 2\mathbb{N})$, then it is enough to take $M_0 = S+2$, $N_0 = T+2$. An analogous identity for free-boundary supercharges given in (8) is possible. For each $A \in \mathcal{A}_\circ$, we have the asymptotic formula

$$\delta_g(A) = \lim_{N\to\infty} [\tilde{Q}(g)_{[-N+1,N]}, A]_\gamma, \tag{21}$$

and similarly

$$\delta_g(A) = \lim_{N\to\infty} [\hat{Q}(g)_{[-N+1,N+1]}, A]_\gamma. \tag{22}$$

The nilpotent condition (3) is expressed by the superderivation δ_g as

$$\delta_g \circ \delta_g = \delta_g^* \circ \delta_g^* = \mathbf{0}. \tag{23}$$

Define the derivation generated by the Hamiltonian $H(g)$:

$$d_g(A) \coloneqq [H(g), A] \quad \text{for every } A \in \mathcal{A}_\circ. \tag{24}$$

This is the infinitesimal time generator of the model. We can immediately verify the following supersymmetric relation:

$$d_g(A) = \delta_g^* \circ \delta_g(A) + \delta_g \circ \delta_g^*(A) \quad \text{for every } A \in \mathcal{A}_\circ. \tag{25}$$

It has been known that short-range interactions of fermion lattice systems give Hamiltonian dynamics in the infinite-volume limit [6]. Somewhat heuristically, we have for any $A \in \mathcal{A}$ and $t \in \mathbb{R}$

$$\alpha_g(t)(A) \coloneqq \lim_{N\to\infty} \exp(it\tilde{H}(g)_{[-N+1,N]}) \times A \exp(-it\tilde{H}(g)_{[-N+1,N]}) \in \mathcal{A}.$$

Here, special local Hamiltonians given in (7) are used for concreteness. However, we may take any boundary condition upon local Hamiltonians.

With the above mathematical preliminary, we can construct supersymmetric dynamics in the infinitely extended system corresponding to the extended Nicolai model as in the following theorem. Note that it holds irrespective of broken-unbroken supersymmetry.

Theorem III.I: For each $g \in \mathbb{R}$, the superderivation δ_g generates a supersymmetric dynamics in $\mathcal{A}$. Precisely, there exists a strongly continuous one-parameter group of *-automorphisms $\alpha_g(t)$ $(t \in \mathbb{R})$ on $\mathcal{A}$ of which the pregenerator is given by the derivation $d_g \equiv \delta_g^* \circ \delta_g + \delta_g \circ \delta_g^*$ on the local algebra $\mathcal{A}_\circ$.

Proof: From our work [4], the statement follows immediately. ■

We need to fix the crucial terminology "supersymmetry breakdown." In physics literature, SUSY breakdown is usually identified with strict positivity of the SUSY Hamiltonian; see, e.g., Ref. [7]. However, one should be cautious when dealing with models on noncompact space. (We see a somewhat relevant remark in Ref. [3].) As shown in Theorem III.1, superderivations provide building blocks of supersymmetric dynamics. So, let us propose the following definition based on superderivations. We consider that it is a straightforward formulation of the physics concept of symmetry and symmetry breakdown.

Definition III.2: Suppose that a superderivation generates a supersymmetric dynamics as in Theorem III.1. If a state of $\mathcal{A}$ is invariant under the superderivation defined on the local system $\mathcal{A}_\circ$, then it is called a supersymmetric state. If no supersymmetric state exists, then it is said that SUSY is spontaneously broken.

Sannomiya *et al.* employed a different definition [1]: SUSY is spontaneously broken if the energy density of ground states is strictly positive.

This alternative definition based on the energy density seems not satisfactory in some respects. First, it is only limited to homogeneous ground states. There may be nonperiodic ground states that do not have a well-defined energy density; we have given such states for the original Nicolai model [8,9]. It is not obvious how the status of SUSY for homogeneous states implies that for nonhomogeneous states in the infinite-volume limit. Second, its full justification has not yet been done even for the particular model (i.e., the extended Nicolai model). We investigate the above second issue in the next section.

IV. SUPERSYMMETRY BREAKDOWN

A. Supersymmetry breakdown in the infinite-volume system

The first theorem is a direct consequence of a crucial property of the extended Nicolai model found by Sannomiya *et al.* for finite systems. It is stated below in (26) and (27). We only need to show that it remains valid in the infinite-volume limit.

Theorem IV.1: For any $g \neq 0$, the extended Nicolai supersymmetric fermion lattice model breaks SUSY spontaneously.

Proof: As given in Eq. (15) of Ref. [1], for each $k \in \mathbb{Z}$, let [10]

$$X_k \coloneqq c^*_{2k-1}\left(1 - \frac{1}{g}(c^*_{2k}c_{2k+1} + c_{2k-3}c^*_{2k-2}) + \frac{2}{g^2}c_{2k-3}c^*_{2k-2}c^*_{2k}c_{2k+1}\right). \tag{26}$$

We shall show that for all $k \in \mathbb{Z}$

$$\delta_g(X_k) = g. \tag{27}$$

As the model is σ_2 invariant, it is enough to show the statement for a specific $k \in \mathbb{Z}$. So, let us consider

$$X_2 = c^*_3\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right) \in \mathcal{A}([1,2,3,4,5,6])_-.$$

Then, for the identity (20) to be valid for X_2, it is enough to take $M_0 = 0 - 2 = -2$ and $N_0 = 6 + 2 = 8$. We compute

$$\begin{aligned}
\delta_g(X_2) &= [\tilde{Q}(g)_{[-1,8]}, X_2]_\gamma \\
&= \left[\sum_{k=0}^{4}(gc_{2k-1} + c_{2k-1}c^*_{2k}c_{2k+1}), X_2\right]_\gamma (9 = -1) \\
&= [g(c_{-1} + c_1 + c_3 + c_5 + c_7) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7 + c_7c^*_8c_{-1}), X_2]_\gamma \\
&= [g(c_1 + c_3 + c_5) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7), X_2]_\gamma,
\end{aligned}$$

where the identification $9 = -1$ is made and the graded locality (17) is noted. Similarly, we can verify that

$$\begin{aligned}
\delta_g(X_2) &= [\hat{Q}(g)_{[-1,7]}, X_2]_\gamma \\
&= [g\,(c_{-1} + c_1 + c_3 + c_5 + c_7) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7), X_2]_\gamma \\
&= [g\,(c_1 + c_3 + c_5) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7), X_2]_\gamma.
\end{aligned}$$

By direct computation using the canonical anticommutation relations (1), we have

$$\begin{aligned}
\delta_g(X_2) &= \left[g(c_1 + c_3 + c_5) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7), c^*_3\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right)\right]_\gamma \\
&= \{gc_3 + c_1c^*_2c_3 + c_3c^*_4c_5, c^*_3\}\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right) \\
&\quad - c^*_3\delta_g\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right) \\
&= (g + c_1c^*_2 + c^*_4c_5)\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right) - c^*_3 \cdot 0 \\
&= g - (c^*_4c_5 + c_1c^*_2) + \frac{2}{g}c_1c^*_2c^*_4c_5 + (c_1c^*_2 + c^*_4c_5) - 2 \times \frac{1}{g}c_1c^*_2c^*_4c_5 \\
&= g,
\end{aligned}$$

where we have noted

$$\begin{aligned}
\delta_g\left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right) &= \Bigg[g(c_1 + c_3 + c_5) + (c_{-1}c^*_0c_1 + c_1c^*_2c_3 + c_3c^*_4c_5 + c_5c^*_6c_7), \\
&\qquad \left(1 - \frac{1}{g}(c^*_4c_5 + c_1c^*_2) + \frac{2}{g^2}c_1c^*_2c^*_4c_5\right)\Bigg]_\gamma \\
&= 0.
\end{aligned}$$

Analogously, we obtain (27) for any $k \in \mathbb{Z}$.

Now, take any (not necessarily homogeneous) state ω of $\mathcal{A}$. Then, for any $k \in \mathbb{Z}$,

$$\omega(\delta_g(X_k)) = \omega(g1) = g. \tag{28}$$

Thus, if $g \neq 0$, then ω is not invariant under δ_g. As ω is arbitrary, there exists no invariant state under δ_g, and hence SUSY is spontaneously broken for any $g \neq 0$. ■

B. Strict positivity of energy density due to supersymmetry breakdown

We shall discuss the energy density for homogeneous ground states. Let us fix relevant notation. A state ω on $\mathcal{A}$ is called translation invariant if $\omega(A) = \omega(\sigma_1(A))$ for all $A \in \mathcal{A}$. A state ω on $\mathcal{A}$ is called homogeneous (with periodicity 2) if $\omega(A) = \omega(\sigma_2(A))$ for all $A \in \mathcal{A}$.

For any homogeneous state ω, we can define the energy density for the extended Nicolai model by the expectation value of the local Hamiltonians per site in the infinite-volume limit. As we can choose any boundary condition upon local Hamiltonians as noted in Ref. [11], we have

$$\begin{aligned} e(g)(\omega) &:= \lim_{N\to\infty} \frac{1}{2N} \omega(\tilde{H}(g)_{[-N+1,N]}) \\ &= \lim_{N\to\infty} \frac{1}{2N} \omega(\hat{H}(g)_{[-N+1,N+1]}). \end{aligned} \tag{29}$$

Since all $\tilde{H}(g)_{[-N+1,N]}$ [as well as $\hat{H}(g)_{[-N+1,N+1]}$] are positive operators by definition, we have

$$e(g)(\omega) \geq 0. \tag{30}$$

A general definition of ground states for C^*-systems is given in terms of the infinitesimal time evolution; see Ref. [6]. Now, it is d_g given in (25). For homogeneous states, this general characterization of ground states is known to be equivalent to the minimum energy-density condition [12]. The extended Nicolai model on $\mathbb{Z}$ is a homogeneous model of 2-periodicity. It has been known that for any translation invariant model there is at least one translation invariant ground state [13]. Hence, there exists at least one (not necessarily pure) homogeneous ground state φ of the periodicity 2 for the extended Nicolai model. The second theorem is as follows.

Theorem IV.2: Let φ be any homogeneous ground state for the extended Nicolai model. Then, its energy density $e(g)(\varphi)$ is strictly positive if the parameter g of the model is not 0.

Proof: The basic idea of our proof is owing to Refs. [14,15]. We will make use of the formulation of the Gelfand-Naimark-Segal (GNS) construction; see Ref. [16]. By $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ we denote the GNS representation associated to the state φ of $\mathcal{A}$. Precisely, π_φ is a homomorphism from $\mathcal{A}$ into $\mathfrak{B}(\mathcal{H}_\varphi)$ (the set of all bounded linear operators on the Hilbert space $\mathcal{H}_\varphi$), and $\Omega_\varphi \in \mathcal{H}_\varphi$ is a cyclic vector such that $\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle$ for all $A \in \mathcal{A}$.

We consider finite averages of local operators $\{X_k\}$ defined in (26) under shift translations: for $n \in \mathbb{N}$, let

$$\chi(n) := \frac{1}{n}\sum_{k=1}^{n} X_k \in \mathcal{A}([-1, 2n+1])_-. \tag{31}$$

We shall study the asymptotic behavior of $\varphi(\delta_g(\chi(n)))$ as $n \to \infty$. By (27), we have

$$\varphi(\delta_g(\chi(n))) = \frac{1}{n}\sum_{k=1}^{n} \varphi(\delta_g(X_k)) = \frac{1}{n}\sum_{k=1}^{n} g = g. \tag{32}$$

We can rewrite $\delta_g(\chi(n)) \in \mathcal{A}_\circ$ in terms of finite supercharges, which are located in a slightly larger region including the support region of $\chi(n)$. As in (20), by using local supercharges (5) under periodic-boundary conditions, we have

$$\delta_g(\chi(n)) = [\tilde{Q}(g)_{[-3,2(n+2)]}, \chi(n)]_\gamma. \tag{33}$$

Similarly, we may use free-boundary supercharges (8). By using the GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$, we have

$$\begin{aligned} \varphi(\delta_g(\chi(n))) &= \langle \Omega_\varphi, \pi_\varphi([\tilde{Q}(g)_{[-3,2(n+2)]}, \chi(n)]_\gamma)\Omega_\varphi \rangle \\ &= \langle \Omega_\varphi, (\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\pi_\varphi(\chi(n)) \\ &\quad + \pi_\varphi(\chi(n))\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]}))\Omega_\varphi \rangle \\ &= \langle \pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})^*\Omega_\varphi, \pi_\varphi(\chi(n))\Omega_\varphi \rangle \\ &\quad + \langle \pi_\varphi(\chi(n))^*\Omega_\varphi, \pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi \rangle. \end{aligned} \tag{34}$$

As Ω_φ is a normalized vector, by using the triangle inequality and the Cauchy-Schwarz inequality, Eq. (34) above yields the following estimate:

$$\begin{aligned} |\varphi(\delta_g(\chi(n)))| &\leq \|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})^*\Omega_\varphi\| \cdot \|\pi_\varphi(\chi(n))\Omega_\varphi\| \\ &\quad + \|\pi_\varphi(\chi(n))^*\Omega_\varphi\| \cdot \|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\| \\ &\leq (\|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})^*\Omega_\varphi\| \\ &\quad + \|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\|) \cdot \|\chi(n)\|. \end{aligned} \tag{35}$$

By applying Lemma IV.3 and Lemma IV.4, which will be shown later, to the above estimate (35), we obtain

$$\lim_{n\to\infty} |\varphi(\delta_g(\chi(n)))| = 0. \tag{36}$$

This contradicts Eq. (32) when $g \neq 0$. Thus, when $g \neq 0$, the assumption of Lemma IV.4 does not hold, and accordingly $e(g)(\varphi)$ should be nonzero. ■

Now, we will show the lemmas used in the proof of Theorem IV.2 above. We recall the Landau notation: O is called "big-O," and o is called "little-o." Let $f(n)$ and $g(n)$ be real-valued functions on $\mathbb{N}$. We write $f(n) = \mathrm{O}(g(n))$ as $n \to \infty$ if there exists a positive $M < \infty$ and $n_0 \in \mathbb{N}$ such that $|f(n)| \leq M|g(n)|$ for all $n > n_0$. We write $f(n) = \mathrm{o}(g(n))$ as $n \to \infty$ if for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|f(n)| \leq \varepsilon |g(n)|$ for all $n > n_0$.

The following lemma states the nonexistence of averaged fermion operators (fermion observables at infinity [6]) in the infinite-volume limit. As the estimate is also essential, we shall recapture its derivation from the original work [17].

Lemma IV.3:

$$\|\chi(n)\| = \mathrm{O}\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \to \infty. \tag{37}$$

In particular, $\lim_{n\to\infty}\chi(n) = 0$ in norm of the C^*-algebra.

Proof: For any $F \in \mathcal{A}$, the inequality $\|F\|^2 = \|F^*F\| \leq \|F^*F + FF^*\|$ holds. By using this and the triangle inequality, we obtain

$$\|\chi(n)\|^2 \leq \frac{1}{n^2}\sum_{k=1}^{n}\sum_{k'=1}^{n}\|\{X_k^*, X_{k'}\}\|. \tag{38}$$

Each term is estimated from the above by some constant:

$$\begin{aligned}\|\{X_k^*, X_{k'}\}\| &\leq \|X_k^* X_{k'}\| + \|X_{k'} X_k^*\| \\ &\leq 2\|X_k^*\| \cdot \|X_{k'}\| \\ &= 2\|X_1\|^2 \equiv C^2/5 \quad (C > 0).\end{aligned} \tag{39}$$

By the graded locality (17) and the definition of X_k given in (26), we have

$$\{X_k^*, X_{k'}\} = 0 \quad \text{if } |k - k'| > 2. \tag{40}$$

Thus, for each fixed $k \in \{1, 2, \ldots, n\}$, there are at most five $k' \in \{1, 2, \ldots, n\}$ such that $\{X_k^*, X_{k'}\}$ does not vanish. By applying (39) and (40) to (38), we obtain

$$\|\chi(n)\|^2 \leq \frac{1}{n^2}\sum_{k=1}^{n} 5 \times C^2/5 = \frac{C^2}{n}. \tag{41}$$

It is equivalent to $\|\chi(n)\| \leq \frac{C}{\sqrt{n}}$, and so (37) is obtained.■

Lemma IV.4: If the energy density $e(g)(\varphi)$ of a homogeneous ground state φ is equal to 0, then

$$\|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})^*\Omega_\varphi\| = \mathrm{o}(\sqrt{n}) \quad \text{as } n \to \infty,$$

and

$$\|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\| = \mathrm{o}(\sqrt{n}) \quad \text{as } n \to \infty. \tag{42}$$

Proof: By the assumption $e(g)(\varphi) = 0$, we have

$$\begin{aligned}0 = e(g)(\varphi) &= \lim_{n\to\infty}\frac{1}{2n+8}\varphi(\tilde{H}(g)_{[-3,2(n+2)]}) \\ &= \lim_{n\to\infty}\frac{1}{2n}\varphi(\tilde{H}(g)_{[-3,2(n+2)]}).\end{aligned} \tag{43}$$

By (7),

$$\begin{aligned}&\varphi(\tilde{H}(g)_{[-3,2(n+2)]}) \\ &= \varphi(\{\tilde{Q}(g)_{[-3,2(n+2)]}, \tilde{Q}(g)^*_{[-3,2(n+2)]}\}) \\ &= \langle \pi_\varphi(\tilde{Q}(g)^*_{[-3,2(n+2)]})\Omega_\varphi, \pi_\varphi(\tilde{Q}(g)^*_{[-3,2(n+2)]})\Omega_\varphi\rangle \\ &\quad + \langle \pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi, \pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\rangle \\ &= \|\pi_\varphi(\tilde{Q}(g)^*_{[-3,2(n+2)]})\Omega_\varphi\|^2 + \|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\|^2.\end{aligned} \tag{44}$$

By this together with (43), we have

$$\begin{aligned}&\|\pi_\varphi(\tilde{Q}(g)^*_{[-3,2(n+2)]})\Omega_\varphi\|^2 = \mathrm{o}(2n) \quad \text{as } n \to \infty, \\ &\|\pi_\varphi(\tilde{Q}(g)_{[-3,2(n+2)]})\Omega_\varphi\|^2 = \mathrm{o}(2n) \quad \text{as } n \to \infty.\end{aligned} \tag{45}$$

By taking square roots, we obtain (42). ■

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[1] N. Sannomiya, H. Katsura, and Y. Nakayama, Phys. Rev. D **94**, 045014 (2016).
[2] H. Nicolai, J. Phys. A **9**, 1497 (1976).
[3] E. Witten, Nucl. Phys. **B202**, 253 (1982).
[4] H. Moriya, Ann. Inst. Henri. Poincaré **17**, 2199 (2016).
[5] Usually, one first considers finite systems and then takes their infinite-volume limit. We have a different viewpoint here. First, we are given an infinitely extended system

upon $\mathbb{Z}$ that represents the total system. The total system includes subsystems as its subalgebras. See Ref. [6] for C^*-algebraic treatment of quantum statistical mechanics.
[6] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, 2nd ed., (Springer-Verlag, 1997).
[7] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 2000), Vol. III.
[8] H. Moriya, J. Stat. Phys., DOI: 10.1007/s10955-018-2100-3 (2018).
[9] H. Katsura, H. Moriya, and Y. Nakayama, arXiv:1710.04385.
[10] In Ref. [1], this is denoted by O_k.
[11] B. Simon, *The Statistical Mechanics of Lattice Gases* (Princeton University Press, Princeton, NJ, 1993).
[12] O. Bratteli, A. Kishimoto, and D. W. Robinson, Comm. Math. Phys. **64**, 41 (1978).
[13] We do not exclude the possibility of symmetry breakdown of translation symmetry. So, such a translation-invariant ground state is not necessarily a pure state.
[14] D. Buchholz, Lect. Notes Phys. **539**, 211 (2000).
[15] D. Buchholz and I. Ojima, Nucl. Phys. **B498**, 228 (1997).
[16] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*, 2nd ed., (Springer-Verlag, 1987).
[17] O. E. Lanford, III and D. W. Robinson, J. Math. Phys. **9**, 1120 (1968).