

Hybrid seesaw neutrino model in SUSY $SU(5) \times \mathbb{A}_4$

R. Ahl Laamara, M. A. Loualidi, M. Miskaoui, and E. H. Saidi*

*LPHE-Modeling and Simulations, Faculty of Sciences, Mohammed V University, Rabat 10090, Morocco
and Center of Physics and Mathematics, CPM, Faculty of Science, Rabat 10090, Morocco*

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Motivated by recent results from neutrino experiments, we study the neutrino masses and mixing in the framework of a SUSY $SU(5) \times \mathbb{A}_4$ model. The hybrid of Type I and Type II seesaw mechanisms leads to the nonzero value of the reactor angle $\theta_{13} \neq 0$ and to the recently disfavored maximal atmospheric angle $\theta_{23} \neq 45^\circ$ by the NOvA experiment. The phenomenological consequences of the model are studied for both normal and inverted mass hierarchies. The obtained ranges for the effective Majorana neutrino mass $m_{\beta\beta}$, the electron neutrino mass m_{ν_e} , and the CP violating phase δ_{CP} lie within the current experimental allowed ranges where we find that the normal mass hierarchy is favored over the inverted one.

DOI: [10.1103/PhysRevD.98.015004](https://doi.org/10.1103/PhysRevD.98.015004)**I. INTRODUCTION**

The neutrino oscillation experiments performed in the past two decades provided many decisive evidences of nonzero neutrino masses and large neutrino mixing [1–6]. The atmospheric, solar, and reactor neutrino experiments have provided the measurements of the mass-squared differences Δm_{ij}^2 , as well as the mixing angles θ_{ij} ; the current neutrino oscillation data can be found in the latest global fit analysis [7–9]. To understand the origin of these masses—which are very tiny—and mixing, we must go beyond the standard model (SM) that predicts massless neutrinos. Theoretically, the most prominent way to generate such tiny masses for neutrinos is through the famous seesaw mechanism, which requires the introduction of extra heavy fermions (Type I and Type III seesaws) or scalars (Type II seesaw) into the SM [10,11], giving rise to neutrino masses of Majorana type. For the neutrino mixing angles, it was not until 2012 that the reactor angle θ_{13} was discovered to be different from zero [3], but unlike the other two mixing angles θ_{12} and θ_{23} , its value is relatively small. Furthermore, the NOvA experiment has disfavored recently the maximal atmospheric neutrino mixing $\sin^2 \theta_{23} = 0.5$ [12]; however, whether its value is less or greater than $\pi/4$ is yet to be discovered. In the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix that describes these angles, θ_{13} always appears in combination with the Dirac phase, and thus, the discovery of its nonzero value has a crucial

influence on the Dirac CP violating (CPV) phase δ_{CP} where its measurement is the ultimate objective of the long baseline neutrino oscillation experiments [13]. The recent progress in neutrino physics motivated theoretical as well as experimental physicists to search for new physics beyond the SM. This concerns the preexisting theories and models such as supersymmetric grand unified theories (SUSY GUTs) which unlike the non-SUSY GUTs solve the hierarchy problem and unification of gauge couplings just by introducing supersymmetry; thus, they are adopted as one of the most appealing extensions of the SM [14]. Moreover, an attractive way to outline the observed neutrino mass hierarchies and mixing within SUSY-GUT models is through discrete flavor symmetries. Indeed, several models beyond SM have used different non-Abelian groups and described successfully all the neutrino mixing angles; see Table 3 of Refs. [15] and [16]. In fact, these non-Abelian discrete groups are widely adopted to describe the large mixing angles in the lepton sector. In particular, these groups lead to a specific form of the neutrino mass matrix which is consistent with tribimaximal mixing (TBM). This special mixing induces $\theta_{13} = 0$ and $\theta_{23} = \pi/4$; however, it is now ruled out by the discovery of the nonzero reactor angle as mentioned above. Thus a small deviation from TBM is required to reconcile with the small value of θ_{13} as well as a small deviation from the maximal value of the atmospheric angle θ_{23} . In this regard, several ways have been proposed to generate a small deviation of these mixing angles. For example, the deviation from TBM in flavor symmetry-based models can arise from (i) the diagonalization of the charged lepton mass matrix [17], (ii) perturbing the vacuum expectation value (VEV) alignment [18], (iii) the Yukawa sector [19], or (iv) the Majorana sector [16,20]. These deviations are generally realized by introducing next-to-leading-order effective operators while

*h-saidi@fsr.ac.ma

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the leading contribution is produced by one of the seesaw mechanisms. On the other hand, it was claimed in Ref. [21] that the required deviations from the TBM matrix can be interpreted as the interplay of two different seesaw mechanisms making what is known as hybrid neutrino masses. This hybrid has been used by many authors to account for the nonzero reactor angle $\theta_{13} \neq 0$ in the framework of the SM and GUTs; see, for example, Ref. [22].

In this paper we propose a neutrino model in the framework of a supersymmetric $SU(5)$ GUT extended by three right-handed neutrinos N_i and a 15-dimensional Higgs H_{15} transforming respectively as a triplet and a nontrivial singlet under \mathbb{A}_4 flavor symmetry. The theoretical predictions of our proposal concerning the mixing angles and masses are compatible with the latest neutrino experimental data. The main line of our proposal is as follows: First, we consider SUSY $SU(5) \times \mathbb{A}_4$ theory and generate the neutrino mass matrix by the hybrid seesaw mechanism. In this hybrid, the dominant mass contribution comes from Type I seesaw, leading to the TBM [23]. A small perturbation responsible for nonzero reactor angle θ_{13} and nonmaximal atmospheric angle θ_{23} is realized by the 15-dimensional $SU(5)$ Higgs that contains an $SU(2)_L$ Higgs triplet Δ_d via Type II seesaw mechanism. Then, we perform a numerical study, where we use the experimental allowed ranges of the mixing angles and the mass-squared differences, to examine the octant degeneracy of θ_{23} for both normal and inverted mass hierarchies. Next, we use the current neutrino oscillation data as well as the cosmological limit on the sum of neutrino masses to study the phenomenological consequences of our proposal for both normal and inverted mass hierarchies. We find that the allowed ranges of the effective Majorana neutrino mass m_{ee} , the sum of neutrino masses $\sum_{i=1}^3 |m_i|$, the effective electron neutrino mass m_β , and the Dirac CPV phase δ_{CP} are within the current experimental data.

To perform this study, we use known results on SUSY $SU(5)$ as well as properties of the alternating group \mathbb{A}_4 . This flavor symmetry is generally admitted as the most natural and economical discrete group that captures the family symmetry as motivated in the literature [24]. The discrete \mathbb{A}_4 possesses two generators S, T and four irreducible representations that can be labeled by their characters as $\mathbf{1}_{(1,1)}$, $\mathbf{1}_{(1,\omega)}$, $\mathbf{1}_{(1,\omega^2)}$, and $\mathbf{3}_{(-1,0)}$. These four representations, which are related to the \mathbb{A}_4 group order by the formula $\mathbf{1}_{(1,1)}^2 + \mathbf{1}_{(1,\omega)}^2 + \mathbf{1}_{(1,\omega^2)}^2 + \mathbf{3}_{(-1,0)}^2 = 12$, are also used to host the matter and Higgs content of the SUSY $SU(5) \times \mathbb{A}_4$ proposal. For general properties on \mathbb{A}_4 group representations and their characters, see [25,26].

This paper is organized as follows. In Sec. II, we present the superfield content for the neutrino sector in SUSY $SU(5) \times \mathbb{A}_4$. Then, we study the Dirac and Majorana neutrino mass matrices as well as the deviations of θ_{13} and θ_{23} from their TBM values. In Sec. III, we study the phenomenological implications of the proposal and provide the predictions regarding the effective Majorana mass m_{ee} , the effective mass m_β , the sum $\sum_{i=1}^3 |m_i|$, and the CPV

phase δ_{CP} . In Sec. IV, we give our conclusion. In order to make the paper more self-contained we add Appendix on the charged sector where we show that a $U(1)$ flavor symmetry is needed to control the couplings of the model. We also add in the same appendix a brief discussion on the well-known dangerous four- and five-dimensional operators leading to the rapid proton decay and show how they are suppressed in our model due to the flavor symmetry.

II. $SU(5)$ GUT WITH \mathbb{A}_4 FLAVOR SYMMETRY

In this section, we first describe the superfield content of our $SU(5) \times \mathbb{A}_4$ GUT proposal. Then, we use a hybrid seesaw mechanism to study the deviation of the θ_{13} and θ_{23} angles in this proposal. After that, we study the mass-squared differences as functions of the space parameters of the model and the θ_{23} and θ_{13} mixing angles.

A. Implementing \mathbb{A}_4 in neutrino sector

In supersymmetric $SU(5)$ GUT, matter superfields are unified into two irreducible representations of $SU(5)$, namely $\mathbf{10}_m^i$ and $\bar{\mathbf{5}}_m^i$ where $i = 1, 2, 3$ refers to the three possible generations of matter. On the other hand, the Higgs doublets H_u and H_d of the minimal supersymmetric standard model (MSSM) sit in representations $\mathbf{5}_{H_u} = H_5$ and $\bar{\mathbf{5}}_{H_d} = H_{\bar{5}}$.

Here we focus our attention on the neutrino sector in SUSY $SU(5)$ GUT promoted by an \mathbb{A}_4 flavor symmetry. Thus, we give only the superfield content needed to generate the mass terms for the neutrinos. In our construction of SUSY $SU(5) \times \mathbb{A}_4$ GUT, we proceed as follows:

- (i) First, we extend the fermion sector of $SU(5)$ GUT by adding three right-handed neutrinos N_i which are $SU(5)$ gauge singlets and sit together in the \mathbb{A}_4 triplet $\mathbf{3}_{(-1,0)}$. These N_i 's allow us to use Type I seesaw formula $m_\nu^I = -m_D M_R^{-1} m_D^T$ to generate light neutrino masses. One \mathbb{A}_4 flavon triplet superfield Φ is added to get a neutrino mass matrix m_ν^I consistent with the leading order TBM pattern. The addition of one flavon in the neutrino sector is actually the minimal setup if we consider only the four-dimensional $SU(5) \times \mathbb{A}_4$ models that describe successfully all the mixing angles. Some of these models that used at least three flavon superfields in the neutrino sector are given in Ref. [27].
- (ii) Second, we extend the Higgs sector of SUSY $SU(5)$ GUT by adding a 15-dimensional Higgs $\mathbf{15}_{\Delta_d} \equiv H_{15}$ which contains a $Y = 2$ $SU(2)_L$ Higgs triplet Δ_d . This leads to a Majorana mass matrix M_ν^{II} via Type II seesaw mechanism as exhibited by the Yukawa coupling $\bar{\mathbf{5}}_m \otimes \mathbf{15}_{\Delta_d} \otimes \bar{\mathbf{5}}_m$. When added to m_ν^I , the matrix M_ν^{II} will play the role of a perturbation inducing a deviation from the TBM values. Notice that H_{15} has been first used in non-SUSY $SU(5)$ without flavor symmetry to achieve the gauge coupling unification and the generation of tiny neutrino masses [28]. Notice also that the deviation

from TBM by Type II seesaw mechanism with discrete flavor \mathbb{A}_4 has also been considered in SM to reconcile with the experimental value of θ_{13} [29]. In our SUSY $SU(5) \times \mathbb{A}_4$ proposal which extends this approach to supersymmetric GUT models building, we took into account the latest experimental results on neutrino masses and mixing, and we successfully produced the nonzero value of θ_{13} as well as the nonmaximal value of θ_{23} .

So the superfield content of our proposal is as follows: (a) matter containing three generations of $\bar{\mathbf{5}}_m^i$ denoted as F_i , $\mathbf{10}_m^i$ denoted as T_i , and the three right-handed neutrinos N_i . Below, we will mainly focus on F_i and N_i couplings relevant for the neutrino sector, while the contribution of the $\mathbf{10}_m^i$'s in the charged lepton and quark sectors will be discussed in the Appendix. (b) The Higgs sector containing: (i) the two usual Higgses H_5 and $H_{\bar{5}}$ as well as the added H_{15} and $H_{\bar{15}}$; the $H_{\bar{5}}$ and $H_{\bar{15}}$ are required by supersymmetry. (ii) The usual 24-dimensional adjoint Higgs H_{24} needed to break the $SU(5)$ group to the standard model gauge group. (iii) An extra flavon chiral superfield Φ to generate the TBM matrix.

These superfields are the minimal set we need to generate neutrino masses and mixing compatible with experimental data. The quantum numbers of these superfields under $SU(5) \times \mathbb{A}_4$ are as listed in Table I.

Besides N_i and Φ , which are gauge singlets, T_i , F_i , H_5 , and H_{15} are given in standard model representations as follows:

$$\begin{array}{lcl}
 SU(5) \rightarrow & SU(3)_c \times SU(2)_L \times U(1)_Y & \\
 \hline
 \bar{\mathbf{5}}_m^i & : & (\bar{\mathbf{3}}, 1)_{2/3} + (1, 2)_{-1} = (D_i^c, L_i) \\
 \mathbf{5}_{H_u} & : & (3, 1)_{-2/3} + (1, 2)_1 = (T_u, H_u) \\
 \mathbf{15}_{\Delta_d} & : & (1, 3)_2 + (3, 2)_{1/3} + (6, 1)_{-4/3} = (\Delta_d, \Delta'_d, \Delta''_d) \\
 \mathbf{10}_m^i & : & (3, 2)_{1/3} + (\bar{\mathbf{3}}, 1)_{-4/3} + (1, 1)_2 = (Q_i, U_i^c, E_i^c) \\
 \hline
 \end{array} \quad (2.1)$$

where the decompositions of $\mathbf{5}_{H_d}$ and $\bar{\mathbf{15}}_{\Delta_u}$ are understood.

B. Deviation of θ_{13} and θ_{23} in $SU(5) \times \mathbb{A}_4$ hybrid seesaw

We start with the leading approximation where the neutrino mass matrix is generated through Type I seesaw mechanism and is consistent with TBM predicting the mixing angles: $\sin^2 \theta_{12} = \frac{1}{3}$, $\sin^2 \theta_{23} = \frac{1}{2}$, and $\sin^2 \theta_{13} = 0$. Then, we make use of the 15-dimensional $SU(5)$ Higgs $\mathbf{15}_{\Delta_d}$ that contains an $SU(2)_L$ Higgs triplet Δ_d leading to Majorana mass term via Type II seesaw mechanism. Hence, the total neutrino mass matrix combines both Type I and

TABLE I. Superfield content and their quantum numbers under $SU(5) \times \mathbb{A}_4$.

Fields	F_i	T_1	T_2	T_3	N_i	H_5	$H_{\bar{5}}$	Φ	H_{15}
$SU(5)$	$\bar{\mathbf{5}}_m^i$	$\mathbf{10}_m^1$	$\mathbf{10}_m^2$	$\mathbf{10}_m^3$	1_v^i	$\mathbf{5}_{H_u}$	$\mathbf{5}_{H_d}$	1	$\mathbf{15}_{\Delta_d}$
\mathbb{A}_4	$3_{-1,0}$	$1_{(1,\omega)}$	$1_{(1,\omega^2)}$	$1_{(1,1)}$	$3_{-1,0}$	$1_{(1,1)}$	$1_{(1,\omega)}$	$3_{-1,0}$	$1_{(1,\omega)}$

Type II seesaws, allowing a reconciliation with the experimental values of the mixing angles θ_{13} and θ_{23} .

1. TBM from Type I seesaw mechanism

The Type I seesaw formula incorporates both Dirac and Majorana mass matrices where the Dirac mass matrix m_D is obtained from the superpotential term involving the couplings among the superfields N_i , F_i , and H_5 while the Majorana mass matrix M_R is obtained from the superpotential involving the coupling of right-handed neutrinos N_i with themselves. As we mentioned before, both F_i and N_i live in the \mathbb{A}_4 triplet $3_{-1,0}$ while the Higgs H_5 is assigned to the trivial singlet. The leading order superpotential for neutrino Yukawa couplings respecting gauge and \mathbb{A}_4 symmetries is given by

$$W_D = \lambda_1 N F H_5, \quad (2.2)$$

where λ_1 is a Yukawa coupling constant. Using the tensor product of \mathbb{A}_4 irreducible representations in the Altarelli-Feruglio basis [25,30], the superpotential (2.2) reads

$$W_D = \lambda_1 (N_1 F_1 H_5 + N_2 F_3 H_5 + N_3 F_2 H_5). \quad (2.3)$$

When the Higgs doublet develops its VEV as the usual $\langle H_u \rangle = v_u$, we get the Dirac mass matrix of neutrinos as

$$m_D = v_u \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & \lambda_1 & 0 \end{pmatrix}. \quad (2.4)$$

As for the Majorana mass matrix, the superpotential respecting gauge and flavor symmetries of our model are given by

$$W_R = m_R N N + \lambda_2 N N \Phi, \quad (2.5)$$

where we have added the second term involving the flavon Φ to satisfy the TBM texture and to generate appropriate masses for the neutrinos. This term—which is at the renormalizable level—will contribute to all the entries in the Majorana mass matrix. By using the multiplication rules of \mathbb{A}_4 , the superpotential W_R develops into

$$\begin{aligned}
 W_R = & m_R (N_1 N_1 + N_2 N_3 + N_3 N_2) \\
 & + \frac{\lambda_2}{3} (2N_1 N_1 - N_2 N_3 - N_3 N_2) \Phi_1 \\
 & + \frac{\lambda_2}{3} (2N_3 N_3 - N_1 N_2 - N_2 N_1) \Phi_3 \\
 & + \frac{\lambda_2}{3} (2N_2 N_2 - N_1 N_3 - N_3 N_1) \Phi_2, \quad (2.6)
 \end{aligned}$$

and by taking the VEV of the flavons Φ as $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = \langle \Phi_3 \rangle = v_\Phi$, we find the Majorana neutrino mass matrix M_R given by

$$M_R = m_R \begin{pmatrix} 1+2\alpha & -\alpha & -\alpha \\ -\alpha & 2\alpha & 1-\alpha \\ -\alpha & 1-\alpha & 2\alpha \end{pmatrix} \text{ with } \alpha = \frac{\lambda_2 v_\Phi}{3m_R}. \quad (2.7)$$

The light neutrino mass matrix is obtained using Type I seesaw mechanism formula $m_\nu^I = -m_D M_R^{-1} m_D^T$ with the Dirac mass matrix as in Eq. (2.4), and we find

$$m_\nu^I = -m_0 \begin{pmatrix} a & b & b \\ b & c & a+b-c \\ b & a+b-c & c \end{pmatrix}, \quad (2.8)$$

where we have adopted the following parametrization:

$$\begin{aligned} a &= \frac{\alpha + 1}{3\alpha + 1}, & b &= \frac{\alpha}{3\alpha + 1}, \\ c &= \frac{3\alpha^2 + 2\alpha}{9\alpha^2 - 1}, & m_0 &= \frac{\lambda_1^2 v_u^2}{m_R}. \end{aligned} \quad (2.9)$$

Moreover, the values of the parameters a and b are related as $a = 1 - 2b$; this property will be used in our numerical study. The matrix m_ν^I respects the well-known $\mu - \tau$ reflection symmetry [31], and the condition among the elements $(m_\nu^I)_{11} + (m_\nu^I)_{12} = (m_\nu^I)_{22} + (m_\nu^I)_{23}$ required to diagonalize m_ν^I by the TBM matrix as $m_\nu^I = U_{\text{TBM}}^T m_\nu U_{\text{TBM}}$ = $\text{diag}(m_1, m_2, m_3)$ where the U_{TBM} is given by

$$U_{\text{TBM}} = \begin{pmatrix} -\sqrt{2/3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (2.10)$$

2. Deviation using Type II seesaw mechanism

Now we turn to study the deviation from TBM, which consists of inducing a small perturbation in the neutrino mass matrix. This deviation is motivated by the fact that the current experimental data on solar and atmospheric mixing angles are inadequate with the TBM values. The current 3σ ranges of the three mixing angles obtained from the global analysis in Ref. [9] are given by

$$\begin{aligned} 0.271 &\leq \sin^2 \theta_{12} \leq 0.345, \\ 0.385(0.393) &\leq \sin^2 \theta_{23} \leq 0.635(0.640), \\ 0.01934(0.01953) &\leq \sin^2 \theta_{13} \leq 0.02393(0.02408) \end{aligned} \quad (2.11)$$

for a normal (inverted) mass hierarchy. As mentioned above, the perturbation is carried out through Type II seesaw, which implies the introduction of a scalar $SU(2)_L$ triplet Δ_d belonging to the 15-dimensional representation H_{15} of the $SU(5)$ gauge group. The $SU(5) \times \mathbb{A}_4$ -invariant

superpotential induces the Yukawa coupling involving Δ_d as

$$W^{\text{II}} = \lambda_3 \bar{5}_m \bar{5}_m 15_{\Delta_d} = \lambda_3 F F H_{15}. \quad (2.12)$$

Using the VEV $\langle \Delta_d \rangle = v_{\Delta_d}$ of the $SU(2)_L$ triplet component of $H_{15} \equiv 15_{\Delta_d}$, the Majorana neutrino mass matrix reads as follows:

$$M_\nu^{\text{II}} = m_0 \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \end{pmatrix} \text{ with } \varepsilon = \lambda_3 \frac{v_{\Delta_d}}{m_0}, \quad (2.13)$$

where we factored this matrix by m_0 to form a dimensionless deviation parameter ε as well as to ease the hybridization between the seesaw mechanisms. Even though the tiny mass of neutrinos is encoded in the VEV of the Higgs triplet—which is expressed as the ratio of the Higgs doublets VEVs and the Higgs triplet mass [11]—in ordinary seesaw Type II models, in the present paper we will discuss its contribution only through the deviation parameter ε as we will see later when we perform a numerical study concerning the oscillation parameters. In addition, it is well known that the phenomenological constraint from the ρ parameter that measures the ratio between the neutral and charged currents [32] restricts the VEVs of the Higgs multiplets higher than dimension two [33]. As in our model the calculation of the ρ parameter requires taking into consideration at least three kinds of Higgs superfields—namely an $SU(2)$ triplet that belongs to 15_{Δ_d} with hypercharge $Y = 2$, an $SU(2)$ triplet that belongs to $\bar{15}_{\Delta_u}$ with $Y = -2$, and an $SU(2)$ triplet that belongs to 24_H with $Y = 0$ —we leave detailed investigations to future work.

Now, we turn to the total neutrino mass matrix generated by the hybrid seesaw mechanism that consists of combining the contribution of Type II seesaw in Eq. (2.13) and the one arisen from Type I seesaw in Eq. (2.8) as $m_\nu = m_\nu^I + M_\nu^{\text{II}}$ with

$$m_\nu = m_0 \begin{pmatrix} -a & -b & \varepsilon - b \\ -b & \varepsilon - c & c - b - a \\ \varepsilon - b & c - b - a & -c \end{pmatrix}, \quad (2.14)$$

where a , b , and c are as given in Eq. (2.9). The neutrino mass matrix is diagonalized by a transformation such as $m_\nu^{\text{diag}} = \tilde{U}^T m_\nu \tilde{U}$ where the system of eigenvectors and eigenvalues can be developed as power series of ε ; we find up to order $\mathcal{O}(\varepsilon^2)$, the matrix \tilde{U} given in terms of its eigenvectors as

$$\tilde{U} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{\varepsilon}{2\sqrt{2}(a-c)} \\ \frac{1}{\sqrt{6}} - \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(a-c)} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} - \frac{\varepsilon}{4\sqrt{2}(a-c)} \\ \frac{1}{\sqrt{6}} + \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(a-c)} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\varepsilon}{4\sqrt{2}(a-c)} \end{pmatrix} + \mathcal{O}(\varepsilon^2) \quad (2.15)$$

and eigenvalues

$$\begin{aligned} m_1 &= m_0 \left(b - a - \frac{\varepsilon}{2} \right), & m_2 &= -m_0(a + 2b - \varepsilon), \\ m_3 &= m_0 \left(b + a - 2c + \frac{\varepsilon}{2} \right). \end{aligned} \quad (2.16)$$

Consequently, the mixing angles θ_{13} and θ_{23} become

$$\sin \theta_{13} = \left| \frac{\varepsilon}{2\sqrt{2}(a-c)} \right|, \quad \sin \theta_{23} = \left| \frac{\varepsilon}{4\sqrt{2}(a-c)} + \frac{1}{\sqrt{2}} \right| \quad (2.17)$$

while the solar angle maintains its TBM (maximal) value $\sin \theta_{12} = 1/\sqrt{3}$. We have now a nonvanishing reactor angle θ_{13} and a small shift from the TBM value for the atmospheric angle θ_{23} .

C. Mass-squared differences and mixing angles

Concerning neutrino masses, the current neutrino oscillation experiments are only sensitive to mass-squared differences where we distinguish between two mass hierarchies: normal mass hierarchy (NH) where $m_1 < m_2 < m_3$ and inverted mass hierarchy (IH) where $m_3 < m_1 < m_2$. Their 3σ experimental ranges are given by [9]

$$\begin{aligned} 0.0000703 \leq \Delta m_{21}^2 \leq 0.0000809, \\ (0.002399)0.002407 \leq |\Delta m_{3l}^2| \leq 0.002643(0.002635) \end{aligned} \quad (2.18)$$

with $l = 1$ ($l = 2$) for NH (IH). In our proposal, by using the masses in Eq. (2.16), the solar Δm_{21}^2 and atmospheric Δm_{3l}^2 mass-squared differences up to first order in ε are expressed as

$$\begin{aligned} \Delta m_{21}^2 &= 3m_0^2(b^2 - b\varepsilon + 2ab - a\varepsilon), \\ \Delta m_{31}^2 &= 2m_0^2(b-c)(2a-2c+\varepsilon), \\ \Delta m_{32}^2 &= m_0^2(3a\varepsilon - 2ab - 2c\varepsilon - 3b^2 + 5b\varepsilon - 4c(a+b-c)). \end{aligned} \quad (2.19)$$

By using the mixing angles in Eq. (2.17), we show in the left panel (right panel) of Fig. 1 the correlation among the parameters $\sin \theta_{23}$, ε , and $\sin \theta_{13}$ for the NH case (IH case). The experimental inputs of the mass squared differences

Δm_{31}^2 (Δm_{32}^2) as well as their expressions given in Eq. (2.19) are taken into account. Before we discuss the ranges of the oscillation parameters, we should notice that the recent measurement of the atmospheric angle from the NOvA experiment disfavored the maximal value $\theta_{23} = 45^\circ$ [12], while experiments like T2K [5] and IceCube [34] still prefer maximal mixing. In the case of nonmaximal mixing, there are two different octants of θ_{23} ; the lower octant (LO) with $\theta_{23} < 45^\circ$ and the higher octant (HO) with $\theta_{23} > 45^\circ$. The NOvA experiment provided two degenerate ranges for the normal mass hierarchy [12]: $\sin^2 \theta_{23} = 0.404_{-0.022}^{+0.030}$ (LO) and $\sin^2 \theta_{23} = 0.624_{-0.030}^{+0.022}$ (HO). Back to Fig. 1, we observe that while the entire 3σ range of $\sin \theta_{13}$ is allowed, the ranges of the atmospheric angle become more restrained. In the left panel (normal hierarchy), we observe that both octants of the atmospheric angle are allowed and we have

$$\begin{aligned} 0.629 \lesssim \sin \theta_{23}(\text{LO}) \lesssim 0.637, \\ 0.776 \lesssim \sin \theta_{23}(\text{HO}) \lesssim 0.784. \end{aligned} \quad (2.20)$$

These intervals correspond to

$$0.03 \lesssim \varepsilon(\text{LO}) \leq 0.1, \quad 0.01 \lesssim \varepsilon(\text{HO}) \leq 0.1. \quad (2.21)$$

In the right panel (inverted hierarchy), we have for both octants of the atmospheric angle

$$\begin{aligned} 0.629 \lesssim \sin \theta_{23}(\text{LO}) \lesssim 0.637, \\ 0.777 \lesssim \sin \theta_{23}(\text{HO}) \lesssim 0.784, \end{aligned} \quad (2.22)$$

which correspond to the following intervals of the deviation parameter:

$$0.015 \lesssim \varepsilon(\text{LO}) \leq 0.1, \quad 0.013 \lesssim \varepsilon(\text{HO}) \leq 0.1. \quad (2.23)$$

In our proposal, it is clear that the maximal atmospheric angle, which corresponds to $\sin \theta_{23} \simeq 0.7$ in both panels of Fig. 1, is excluded. In fact, this is due to the contribution of the Higgs triplet $\Delta_d \in \mathbf{15}_{\Delta_d}$ (encoded in the parameter ε) which led to the Majorana mass matrix (2.13) via Type II seesaw mechanism, allowing us to explain the nonzero reactor angle $\theta_{13} \neq 0$ as well as providing a deviation of the atmospheric angle from its maximal value. All the allowed regions predicted in our model for $\sin \theta_{23}$ in the case of normal hierarchy are within the ranges of LO and HO provided by the NOvA experiment. To plot the above figures, we have taken $|a| \lesssim 1$ and $|b| \lesssim 1$ which is clear from Eq. (2.9) while the parameter c is allowed to vary freely. Moreover, as the parameter of deviation ε has to be small, we have taken its range to be around $\mathcal{O}(\frac{1}{10})$. We have also fixed m_0 in the range $[0, \frac{1}{10}]$ since it is well known that the mass of the right-handed neutrinos—proportional to m_R —lies at a scale beyond the reach of present

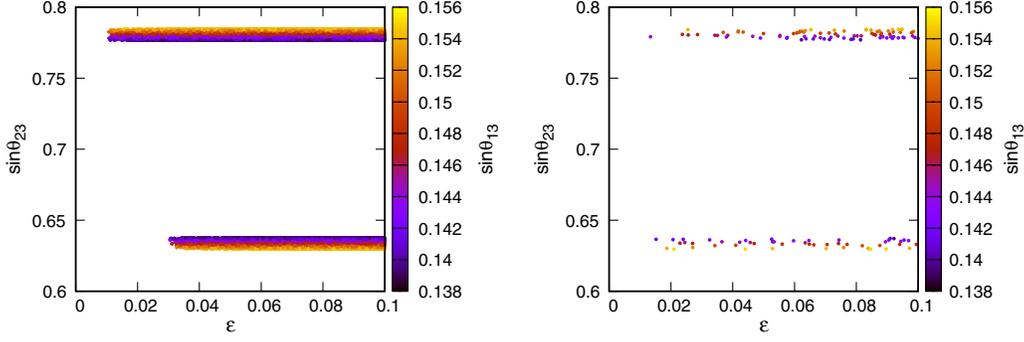


FIG. 1. $\sin \theta_{23}$ as a function of the parameter of deviation ε with $\sin \theta_{13}$ shown in the palette for the NH (left panel) and the IH (right panel).

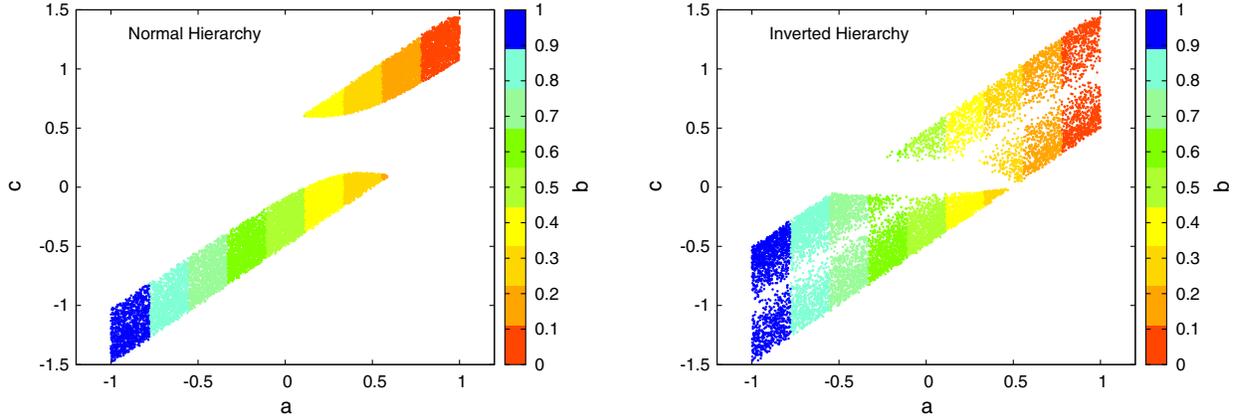


FIG. 2. Correlation among the parameters a , b , and c .

experiments, and it is usually taken at the GUT scale in grand unified theories. As a follow-up to the above discussion, it is clear that the intervals of the parameters a , b , and c —expressed as a function of $\alpha = (\lambda_2 v_\Phi / 3m_R)$ —are fixed according to Eq. (2.9). However, in order to find their restricted ranges compatible with the oscillation experiments, we plot in Fig. 2 the correlation among them by using the 3σ experimental values of the mixing angles and the mass-squared differences as well as Eqs. (2.17) and (2.19). Hence, we observe that for both mass hierarchies, the allowed ranges for the parameter b is around $[0, 0.9997]$, while the parameters a and c vary in the ranges $[-1, 1]$ and $[-1.47, 1.43]$, respectively. These new ranges will be used as inputs to perform a numerical study concerning the phenomenology of neutrino in the next section.

III. PHENOMENOLOGICAL IMPLICATIONS

In this section, by using the model parameters that are restricted by the 3σ experimental values of the mixing angles and the mass-squared differences, we show by means of scatter plots for both hierarchies the physical observables m_{ee} and m_β related respectively to neutrinoless double beta decay and tritium beta decay experiments, and

we also provide scatter plot predictions on the sum of neutrino masses as well as on the Dirac CP violating phase.

A. Neutrinoless double beta decay

One of the most known neutrino mass related experiments is the neutrinoless double beta decay ($0\nu\beta\beta$) process, which has not been observed yet. Its discovery would prove that neutrinos are Majorana particles, and it would also prove that the lepton number L is violated. The decay amplitude for the $0\nu\beta\beta$ process is proportional to the effective Majorana neutrino mass given by [35]

$$|m_{ee}| = \left| \sum_{i=1}^3 U_{ei}^2 m_i \right|, \quad (3.1)$$

where m_i are the three neutrino masses and U_{ei} are the elements of the first row of the PMNS matrix [36]. In our proposal, this mixing matrix is given by

$$\tilde{U}_d = \tilde{U} \cdot \text{diag}(1, e^{i\alpha}, e^{i\beta}), \quad (3.2)$$

where α and β are the Majorana CP violating phases and \tilde{U} is given in Eq. (2.15). Currently, the most recent bounds of

m_{ee} come from the KamLAND-Zen [37] and GERDA [38] experiments; they are respectively given by

$$|m_{ee}| < 0.061\text{--}0.165 \text{ eV}, \quad |m_{ee}| < 0.15\text{--}0.33 \text{ eV}. \quad (3.3)$$

To study the variation the effective Majorana mass m_{ee} with the lightest neutrino mass in our model for both hierarchies, we replace U_{ei} in Eq. (3.1) by the elements of the first row of \tilde{U}_d ; the effective Majorana mass takes the form

$$|m_{ee}| = \left| \frac{2m_1}{3} + \frac{m_2}{3} e^{2i\alpha} + \frac{m_3}{8} \frac{\epsilon^2}{(a-c)^2} e^{2i\beta} \right|. \quad (3.4)$$

Furthermore, for the NH case where m_1 is the lightest neutrino mass, we substitute m_2 by $\sqrt{\Delta m_{21}^2 + m_1^2}$ and m_3 by $\sqrt{\Delta m_{31}^2 + m_1^2}$, and for the IH case where m_3 is the lightest neutrino mass, we substitute m_2 by $\sqrt{m_3^2 - \Delta m_{32}^2}$ and m_1 by $\sqrt{m_3^2 - \Delta m_{32}^2 - \Delta m_{21}^2}$. The explicit forms of m_i and Δm_{ij}^2 as a function of parameter space of the model are as shown in Eqs. (2.16) and (2.19). By using the above definitions and the limits from experiments—see Eq. (3.3)—we plot in Fig. 3 m_{ee} as a function of the lightest neutrino mass for both mass hierarchies where the Majorana phases α and β are allowed to vary in the range $[0 - 2\pi]$; we find that the 3σ allowed regions for the effective Majorana mass are $m_{ee}(\text{eV}) \in [0.00017, 0.06084]$, which corresponds to $m_1(\text{eV}) \in [0.00012, 0.08267]$ for the normal hierarchy, and $m_{ee}(\text{eV}) \in [0.02286, 0.05878]$, which corresponds to $m_3(\text{eV}) \in [0.00144, 0.05879]$ for the inverted hierarchy. For both hierarchies, the obtained regions of m_{ee} are within the current experimental data and may be reached in future neutrinoless double-beta decay experiments [39]. In particular, the obtained ranges can be tested in future experiments like KamLAND-Zen, which plans to reach a sensitivity below 50 meV on $|m_{ee}|$, and thus, it will start to constrain the inverted mass hierarchy region [40].

B. Tritium beta decay

The tritium beta decay is the most sensitive direct way to measure the absolute neutrino mass scale ignoring the nature of neutrinos [41]. The limit (at 95% C.L.) from the Troitsk and Mainz experiments of the effective electron neutrino mass are, respectively, given by $m_\beta < 2.12 \text{ eV}$ and $m_\beta < 2.3 \text{ eV}$ [42,43], while the current generation of neutrino mass measurement comes from the KATRIN experiment with a sensitivity of $m_\beta < 0.2 \text{ eV}$ (at 90% C.L.) [44]. The quantity m_β (or m_{ν_e}) is defined in terms of the mass eigenvalues m_i and mixing matrix elements U_{ei} : $m_\beta^2 = \sum_{i=1}^3 U_{ei}^2 \cdot m_i^2$. In terms of our model parameters, it is expressed as

$$m_\beta = \left(\frac{2m_1^2}{3} + \frac{m_2^2}{3} + \frac{m_3^2}{8} \frac{\epsilon^2}{(a-c)^2} \right)^{\frac{1}{2}}. \quad (3.5)$$

Similar to the discussion of the effective Majorana neutrino mass, in the NH (IH) case, we use the same definitions for m_2 and m_3 (m_1 and m_2). Then, we plot in Fig. 4 the effective electron neutrino mass m_β as a function of the lightest neutrino mass m_i . The cyan region (green region) is obtained by varying all the input parameters in their 3σ ranges for NH (IH) while our model values are presented by the orange points (the red points). Hence, we find that the effective electron neutrino mass lies in the range $0.0214 \lesssim m_\beta(\text{eV}) \lesssim 0.0298$ for NH and $0.0488 \lesssim m_\beta(\text{eV}) \lesssim 0.0882$ for IH, while their corresponding lightest neutrino masses are constrained in the range $0.0206 \lesssim m_1(\text{eV}) \lesssim 0.0291$ for NH and $0.0058 \lesssim m_3(\text{eV}) \lesssim 0.0729$ for IH. The extracted ranges of m_β are compatible with the above mentioned experiments for both mass hierarchies. However, the expected future sensitivity from Project 8 [45] is as low as 0.04 eV, which means that only the range corresponding to NH is allowed.

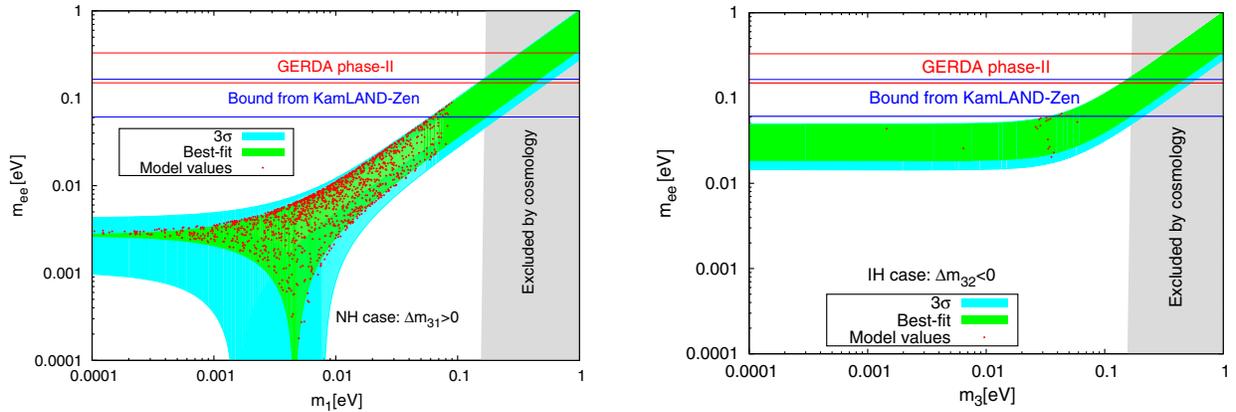


FIG. 3. Left: The effective Majorana mass as a function of the lightest neutrino mass for NH. Right: Same as in the left panel but for IH. The horizontal gray band in both panels indicates an upper limit on the sum of the three light neutrino masses from Planck Collaboration.

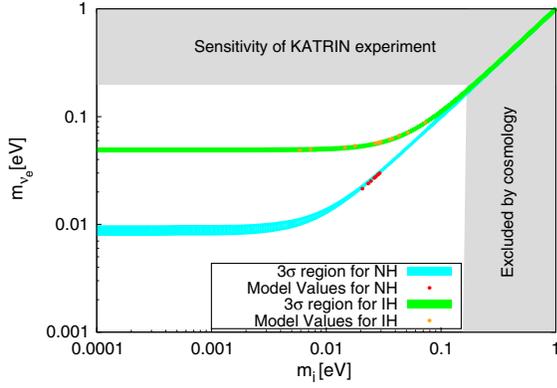


FIG. 4. m_{β} as a function of the lightest neutrino mass m_i for both mass hierarchies.

C. Sum of neutrino masses

Although the absolute mass scale of the neutrinos remains unknown, the sum of the three light neutrino masses $\sum_{i=1}^3 |m_i|$ is constrained by a cosmological upper bound

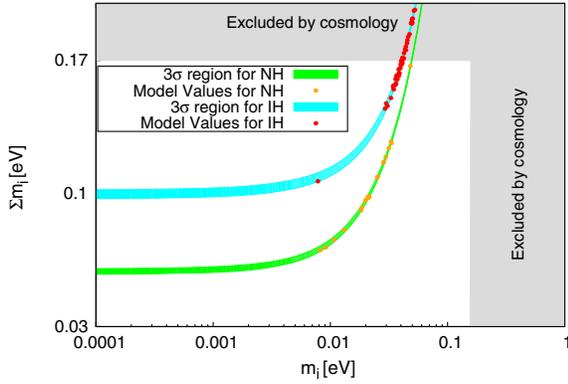


FIG. 5. Sum of neutrino masses as a function of the lightest neutrino mass m_i , present in both cases NH and IH, and the horizontal and vertical bands in gray correspond to the bounds excluded by cosmology.

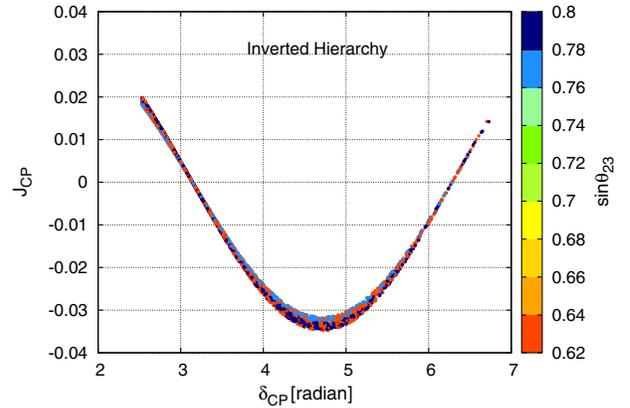
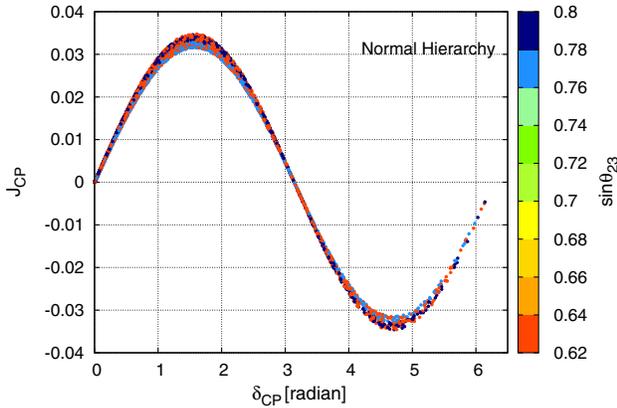


FIG. 6. Left: Scatter plot of $\sin\theta_{23}$ in the (J_{CP}, δ_{CP}) plane for normal hierarchy. Right: Same as in left panel for IH.

given by the Planck Collaboration's limit $\sum_{i=1}^3 |m_i| < 0.17$ eV [46]. In our model, the sum of neutrino masses is expressed in terms of the model parameters as

$$m_{\Sigma} = \sum_{i=1}^3 |m_i| = m_0(\varepsilon - 2c - a). \quad (3.6)$$

Using the 3σ ranges of mass-squared differences (2.18) and mixing angles (2.11), we show in Fig. 5 the dependence of the sum of the light neutrino masses as a function of the lightest neutrino mass for both mass hierarchies. The green region (cyan region) is obtained by varying all the input parameters in their 3σ ranges for normal hierarchy (inverted hierarchy) while our model values are presented by the orange points (the red points). Hence we find that the sum of the light neutrino masses lies in the range $0.0702 \lesssim m_{\Sigma}(\text{eV}) \lesssim 0.1670$ for NH and $0.1064 \lesssim m_{\Sigma}(\text{eV}) \lesssim 0.1698$ for IH, while their corresponding lightest neutrino masses are constrained in the range $0.0081 \lesssim m_1(\text{eV}) \lesssim 0.0480$ for NH and $0.0078 \lesssim m_3(\text{eV}) \lesssim 0.0406$ for IH. Thus, for both mass hierarchies, the sum of neutrino masses gets more restricted as compared to the Planck limit, and these ranges may be tested in future cosmological observations.

D. Dirac CP violation

The Dirac CP violating (CPV) phase δ_{CP} is one among the unknown quantities in the physics of neutrino, and its measurement becomes more important when recent experiments reported the nonzero value of the reactor angle θ_{13} as they are related in the PMNS matrix. Moreover, estimations on the CPV phase δ_{CP} can be obtained by considering the Jarlskog invariant quantity J_{CP} which is defined as $J_{CP} = \text{Im}\{U_{\mu 3}U_{e 3}^*U_{e 3}U_{\mu 3}^*\}$ and by using the PMNS matrix. It is expressed as [47]

$$J_{CP} = \cos\theta_{12} \sin\theta_{12} \cos\theta_{23} \sin\theta_{23} \cos^2\theta_{13} \sin\theta_{13} \sin\delta_{CP}, \quad (3.7)$$

TABLE II. Allowed ranges of J_{CP} for both mass hierarchies and both octants and their corresponding δ_{CP} and $\sin \theta_{23}$ ranges extracted from Fig. 6.

	J_{CP}	δ_{CP}	$\sin \theta_{23}$	Color regions
NH(HO)	$[-0.034, 0.034]$	$[0, 1.91\pi]$	$[0.776, 0.784]$	Blue and dark blue
NH(LO)	$[-0.034, 0.034]$	$[0, 1.93\pi]$	$[0.629, 0.637]$	Dark orange
IH(HO)	$[-0.034, 0.019]$	$[0.8\pi - 2.08\pi]$	$[0.779, 0.784]$	Blue and dark blue
IH(LO)	$[-0.034, 0.019]$	$[0.8\pi - 2.14\pi]$	$[0.629, 0.637]$	Dark orange

where the allowed ranges at 3σ of $\sin \theta_{12}$, $\sin \theta_{23}$, and $\sin \theta_{13}$ are given in Eq. (2.11) while the allowed 3σ ranges of CPV phase δ_{CP} are given by [9]

$$0 \leq \delta_{CP} \leq 2\pi \text{ for NH, } 0.8\pi \leq \delta_{CP} \leq 2.17\pi \text{ for IH. } \quad (3.8)$$

We show in Fig. 6, the behavior of J_{CP} as a function of δ_{CP} with $\sin \theta_{23}$ presented in the palettes for NH (left panel) and IH (right panel). The ranges of J_{CP} and their corresponding Dirac CPV δ_{CP} as well as the ranges of $\sin \theta_{23}$ for both HO and LO are as shown in Table II. Therefore, the left panel shows that for the values around $\delta_{CP} = 0.5\pi$ and $\delta_{CP} = 1.5\pi$, the CP is maximally violated when the magnitude of J_{CP} is maximal ($J_{CP} \approx -0.034$ and $J_{CP} \approx 0.034$) while in the right panel it is maximally violated ($J_{CP} \approx -0.034$ and $J_{CP} \approx 0.019$) around the values $\delta_{CP} = 0.8\pi$ and $\delta_{CP} = 1.5\pi$.

IV. CONCLUSION

In this work, we have constructed a renormalizable hybrid seesaw neutrino model in the framework of SUSY $SU(5)$ GUT extended by a discrete \mathbb{A}_4 family symmetry. The dominant TBM pattern is obtained from Type I seesaw mechanism while Type II seesaw is responsible for a small deviation from TBM. Both seesaws are controlled by the action of the \mathbb{A}_4 flavor symmetry through its algebraic properties. We found that the predictions of our proposal concerning the mixing angles and masses are consistent with the recent measurements. In particular, we showed that the deviation by Type II seesaw leads to a nonmaximal atmospheric angle θ_{23} as reported recently by the NOvA experiment and a nonvanishing reactor angle θ_{13} . Thus, we made a full analysis depending on the octant of θ_{23} .

We also studied the phenomenological consequences of our proposal where we showed through scatter plots the allowed ranges for the physical observables and model parameters which we have restricted by using the 3σ ranges of the neutrino oscillation parameters for both mass hierarchies. We found also that the sum of neutrino masses and CPV phase are within the allowed experimental regions. Furthermore, we found that the ranges of the physical observables involving the effective Majorana neutrino mass $m_{\beta\beta}$ and the electron neutrino mass m_β are preferred in the case of normal mass hierarchy. For the

latter, the obtained range of m_β in the inverted mass hierarchy case is forbidden by future sensitivity from Project 8.

APPENDIX: CHARGED FERMION SECTORS AND PROTON DECAY

In this appendix, we provide a brief study of the charged lepton sector to show the possibility to use an \mathbb{A}_4 assignment for the remaining $SU(5)$ superfield content that does not affect the neutrino mixing. However, it is well known in GUTs that because the quarks and leptons are unified in the same group representations, the charged lepton and the down quark masses are derived from the same superpotential. Thus we also provide in this appendix a concise discussion of the quark sector fixing up the unwanted mass relations between down quarks and charged leptons

$$m_e = m_d, \quad m_\mu = m_s, \quad m_\tau = m_b. \quad (A1)$$

We begin by assigning the quantum numbers to the rest of the chiral superfields of our $SU(5) \times \mathbb{A}_4$ proposal. Thus, in addition to the superfields relevant for the neutrino sector—see Table I—the matter $10_m^i = (U_i^c, E_i^c, Q_i)$ of the three generations $i = 1, 2, 3$ live in the \mathbb{A}_4 representations $1_{(1,\omega)}$, $1_{(1,\omega^2)}$, and $1_{(1,1)}$, respectively. As discussed in the neutrino sector above, one flavon superfield is necessary to accommodate the observed neutrino oscillation parameters. Similarly, to generate appropriate masses for the three generations of up quarks and down quarks (as well as charged leptons), two extra flavons are needed in the superpotential of up quarks W_u ; these are denoted by χ and φ . On the other hand, three extra flavons are required in the superpotential of down quarks and charged leptons $W_{e,d}$; these are denoted as ρ , η , and σ . The \mathbb{A}_4 irreducible representations of these new flavons are as given in Table III. Furthermore, in order to achieve the correct mass hierarchy and to get rid of the unwanted couplings, we add an additional global $U(1)$ symmetry where its charge assignments for all the superfields in our proposal are as given in the last rows of Tables III and IV. In fact, these $U(1)$ quantum numbers are identified by taking into account the preexisting $SU(5) \times \mathbb{A}_4$ invariant Yukawa couplings in the neutrino sector. Indeed, the flavon Φ must carry a zero $U(1)$ charge in order to preserve both couplings given in the Majorana superpotential (2.5).

TABLE III. Flavon superfields needed in the quark and charged lepton sectors and their quantum numbers under $SU(5) \times \mathbb{A}_4 \times U(1)$.

Flavons	Φ	χ	φ	ρ	σ	η
SU(5)	1	1	1	1	1	1
\mathbb{A}_4	$3_{-1,0}$	$1_{(1,\omega)}$	$1_{(1,\omega^2)}$	$3_{-1,0}$	$3_{-1,0}$	$3_{-1,0}$
U(1)	0	-4	2	7	8	5

TABLE IV. Matter and Higgs content of the model and their quantum numbers under $SU(5) \times \mathbb{A}_4 \times U(1)$.

Fields	F_i	N_i	H_5	$H_{\bar{5}}$	H_{15}	T_1	T_2	T_3	$H_{\bar{45}}$
SU(5)	$\bar{5}_m^i$	1_ν^i	5_{H_u}	5_{H_d}	15_{Δ_d}	10_m^1	10_m^2	10_m^3	$\bar{45}_H$
\mathbb{A}_4	$3_{-1,0}$	$3_{-1,0}$	$1_{(1,1)}$	$1_{(1,\omega)}$	$1_{(1,\omega^2)}$	$1_{(1,\omega)}$	$1_{(1,\omega^2)}$	$1_{(1,1)}$	$3_{-1,0}$
U(1)	-2	0	2	-4	4	1	-2	-1	4

However, since the nonrenormalizable terms up to order $\mathcal{O}(1/\Lambda^2)$ are needed in the charged fermion sectors as we will see below, this zero $U(1)$ charge for the flavon Φ enables its coupling with the operators $F_i F_i H_{\bar{15}}$ and $N_i F_i H_5$ via the following higher dimensional operators:

$$F_i F_i H_{\bar{15}} \left(\frac{\Phi}{\Lambda} \right), \quad N_i F_i H_5 \left(\frac{\Phi}{\Lambda} \right). \quad (\text{A2})$$

These couplings which destroy the form of neutrino mass matrix (2.14) that led to the desired oscillation parameters must be suppressed. This is possible if we assume that $v_\Phi \ll \Lambda$, which is acceptable according to Eqs. (2.7) and (2.9). On the other hand, even if the VEV of the flavon Φ is around the cutoff scale—say $\Phi \simeq \Lambda$ —this would just give terms that are relative to the leading ones: $F_i F_i H_{\bar{15}}$ and $N_i F_i H_5$. Moreover, it is well known that the $SU(5)$ GUT predicts the mass relations in Eq. (A1), which are not acceptable for the first and second generations due to their disagreement with the experimental data [35]. Nevertheless, the well known Georgi-Jarlskog (GJ) mechanism [48] overcomes this issue by introducing an additional Higgs in the 45-dimensional $SU(5)$ representation leading to the mass relations

$$3m_e = m_d, \quad m_\mu = 3m_s, \quad m_\tau = m_b. \quad (\text{A3})$$

In our proposal, these 45 Higgs denoted as $H_{\bar{45}}$ are placed in an A_4 triplet¹— $\tilde{H}_{\bar{45}} = (H_{\bar{45}}, 0, 0)^T$ —while its charge under the additional $U(1)$ symmetry is $q_{U(1)} = -5$. Recall that this Higgs $H_{\bar{45}}$ is antisymmetric in $SU(5)$ indices and satisfies the following relations [48]:

¹Notice that the choice of putting the 45-dimensional Higgs $H_{\bar{45}}$ in an \mathbb{A}_4 triplet is to ensure the invariance of its coupling with $T_2 F_i$ without having to add other flavon triplets.

$$(H_{\bar{45}}^i)^{ab} = -(H_{\bar{45}}^j)^{ba}, \quad (H_{\bar{45}}^i)^{aa} = 0, \\ \langle (H_{\bar{45}}^i)^{i5} \rangle = v_{45}, \quad i = 1, 2, 3, \quad \langle (H_{\bar{45}}^i)^{45} \rangle = -3v_{45}. \quad (\text{A4})$$

With the $\mathbb{A}_4 \times U(1)$ charge assignments shown in Table IV, the usual renormalizable Yukawa couplings $Y_1 T_1 F_i H_{\bar{5}}$, $Y_2 T_2 F_i H_{\bar{5}}$, and $Y_3 T_3 F_i H_{\bar{5}}$ are not invariant under \mathbb{A}_4 flavor symmetry and they are carrying the $U(1)$ charges -5 , -8 , and -7 , respectively. Thus, to restore the invariance under the $\mathbb{A}_4 \times U(1)$ symmetry, each one of these couplings requires a different A_4 triplet flavon superfield, namely η , σ , and ρ with $U(1)$ charges 5, 8, and 7, respectively. Therefore, the $\mathbb{A}_4 \times U(1)$ invariant superpotential of the down quarks and charged leptons involving the three flavons η , σ , and ρ as well the Higgs $H_{\bar{45}}$ is given by

$$W_{d,e} = \frac{Y_1}{\Lambda} T_1 (F_i \eta) H_{\bar{5}} + \frac{Y_2}{\Lambda} T_2 (F_i \sigma) H_{\bar{5}} + \frac{Y_3}{\Lambda} T_3 (F_i \rho) H_{\bar{5}} \\ + Y_{45} T_2 F_i \tilde{H}_{\bar{45}}, \quad (\text{A5})$$

where Y_1 , Y_2 , Y_3 , and Y_{45} are the Yukawa mass matrices and Λ represents the cutoff scale of the model. Notice that the coupling $T_2 F_i \tilde{H}_{\bar{45}} (\frac{\Phi}{\Lambda})$ is also allowed by the symmetries of the model, but again its suppression is guaranteed by the condition $v_\Phi \ll \Lambda$. Using \mathbb{A}_4 tensor products, the superpotential $W_{d,e}$ develops into

$$W_{d,e} = \frac{Y_1}{\Lambda} T_1 F_2 \eta H_{\bar{5}} + \frac{Y_2}{\Lambda} T_2 F_1 \sigma H_{\bar{5}} + \frac{Y_3}{\Lambda} T_3 (F_3 \rho) H_{\bar{5}} \\ + Y_{45} T_2 F_2 H_{\bar{45}}. \quad (\text{A6})$$

The masses arise from the breaking of $\mathbb{A}_4 \times U(1)$ family symmetry as well as the breaking of the electroweak symmetry. Therefore, by taking the flavon triplet VEVs along the directions

$$\langle \sigma \rangle = v_\sigma (1, 0, 0)^T, \quad \langle \rho \rangle = v_\rho (1, 0, 0)^T, \quad \langle \eta \rangle = v_\eta (1, 0, 0)^T, \quad (\text{A7})$$

the Higgs doublet H_d responsible for the electroweak symmetry breaking as usual $\langle H_d \rangle = v_d$, and the Higgs 45 as in Eq. (A4), we obtain the mass matrices for down-type quarks M_d and charged leptons M_e

$$M_d = \begin{pmatrix} 0 & Y_1 r & 0 \\ Y_2 h & Y_{45} v_{45} & 0 \\ 0 & 0 & Y_3 t \end{pmatrix}, \\ M_e = \begin{pmatrix} 0 & Y_2 h & 0 \\ Y_1 r & -3Y_{45} v_{45} & 0 \\ 0 & 0 & Y_3 t \end{pmatrix}, \quad (\text{A8})$$

where $r = v_d v_\eta / \Lambda$, $h = v_d v_\sigma / \Lambda$, and $t = v_d v_\rho / \Lambda$. By assuming $Y_{45} v_{45} \gg Y_1 r \approx Y_2 h$, we diagonalize the mass

matrices M_d and M_e where we find that the masses of down-type quarks and charged leptons are respectively given by

$$m_d = \left| \frac{Y_1^2 r^2}{Y_{45} v_{45}} \right|, \quad m_s = \left| Y_{45} v_{45} + \frac{Y_1^2 r^2}{Y_{45} v_{45}} \right|, \quad m_\tau = |Y_3 t|,$$

$$m_e = \left| \frac{Y_1^2 r^2}{3Y_{45} v_{45}} \right|, \quad m_\mu = \left| 3Y_{45} v_{45} + \frac{Y_1^2 r^2}{3Y_{45} v_{45}} \right|, \quad m_b = |Y_3 t|,$$
(A9)

where these masses imply the Georgi-Jarlskog relations given in Eq. (A3). Notice that these mass relations are admissible at the GUT scale at leading order and can be improved assuming the SUSY threshold corrections and appropriate values of $\tan\beta = \frac{v_u}{v_d}$; for more details on the SUSY threshold corrections procedure see Refs. [49,50]. On the other hand, an alternative way to go beyond the $b - \tau$ unification in GJ predictions at high scale is through higher dimensional effective operators [49,51]. These operators involve additional Higgses in 24_H or 75_H and nontrivial $SU(5)$ messenger fields X and \bar{X} allowing for relations such as $m_\tau = \frac{3}{2}m_b$. All possible relations between down-quark and the charged lepton masses are listed in Table 1 of Ref. [49] and Table 2 of Ref. [51]. One of these GUT scale relations using fermion and scalar messenger fields is studied in the framework of $SU(5) \times A_4$ in Ref. [52].

Regarding the up-type quark sector, besides the top quark mass which is preferred to arise from a renormalizable coupling, the remaining up and charm quark masses are derived from higher dimensional Yukawa couplings involving flavon superfields. Indeed, in our model, two different flavons χ and φ couple to the first and second generations, respectively. Thus, the superpotential of the up-type quarks respecting gauge and flavor symmetries takes the form

$$W_u = \frac{Y^u}{\Lambda} T_1 T_1 H_5 \chi + \frac{Y^c}{\Lambda} T_2 T_2 H_5 \varphi + Y^t T_3 T_3 H_5, \quad (\text{A10})$$

where y^u , y^c , and y^t are the Yukawa coupling constants for up-, charm-, and top-type quarks. As usual, the up-type quark masses arise from the breaking of the flavor and electroweak symmetries. Thus, when the flavons φ and χ and the Higgs H_u develop their VEVs as

$$\langle \varphi \rangle = v_\varphi, \quad \langle \chi \rangle = v_\chi, \quad \langle H_u \rangle = v_u, \quad (\text{A11})$$

we obtain a diagonal mass matrix of the up-type quarks given by

$$M_{up} = v_u \begin{pmatrix} Y^u v_\chi / \Lambda & 0 & 0 \\ 0 & Y^c v_\varphi / \Lambda & 0 \\ 0 & 0 & Y^t \end{pmatrix} \quad (\text{A12})$$

with the mass eigenvalues as

$$m_u = Y^u \frac{v_\chi}{\Lambda} v_u, \quad m_c = Y^c \frac{v_\varphi}{\Lambda} v_u, \quad m_t = Y^t v_u. \quad (\text{A13})$$

The large mass of the top quark is obtained at tree level, while the mass hierarchy among the first two generations of up-type quarks can be obtained by assuming a hierarchy between the VEVs of the flavons χ and φ .

As for the mixing in the quark sector, it is defined as $|U_Q| = |U_{up}^\dagger U_d|$ where U_d is the matrix that diagonalizes the mass matrix of down quarks M_d while U_{up} is the one that diagonalizes the mass matrix of up quarks M_{up} . Since this latter is diagonal (A12), U_{up} is just the identity matrix, and thus, the total mixing matrix is the one that diagonalizes the mass matrix of the down quarks M_d (A8); we find

$$U_Q = U_d = \begin{pmatrix} \frac{-Y_{45} v_{45} - F}{ZY_{1r}} & \frac{-Y_{45} v_{45} + F}{EY_{1r}} & 0 \\ \frac{Z}{Z} & \frac{Z}{E} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A14})$$

with

$$F = \sqrt{Y_{45}^2 v_{45}^2 + 4Y_1^2 r^2}, \quad Z = \sqrt{4 + \left(\frac{Y_{45} v_{45} + F}{Y_1 r} \right)^2},$$

$$E = \sqrt{4 + \left(\frac{Y_{45} v_{45} - F}{Y_1 r} \right)^2}. \quad (\text{A15})$$

Notice that the zero entries in the mixing matrix (A14) can be seen to be a first approximation to the mixing matrix V_{CKM} of the quark sector [35]. The nonzero values of this entries can be obtained by considering higher dimensional operators involving flavon superfields in the quark sector. As for the mixing in the charged lepton sector, the diagonalization of the mass matrix M_e in Eq. (A8) is given by

$$U_e \simeq \begin{pmatrix} \frac{3Y_{45} v_{45} - L}{GY_{1r}} & \frac{3Y_{45} v_{45} + L}{KY_{1r}} & 0 \\ \frac{Z}{G} & \frac{Z}{K} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A16})$$

with

$$L = \sqrt{9Y_{45}^2 v_{45}^2 + 4Y_1^2 r^2}, \quad G = \sqrt{4 + \left(\frac{-3Y_{45} v_{45} + L}{Y_1 r} \right)^2},$$

$$K = \sqrt{4 + \left(\frac{3Y_{45} v_{45} + L}{Y_1 r} \right)^2}. \quad (\text{A17})$$

From this matrix, it is clear that the charged lepton mixing angles θ'_{13} and θ'_{23} are both equal to zero; thus in our model the mixing from the charged lepton sector does not affect the

mixing angles of the neutrino sector given in Eq. (2.17). Notice by the way that the total mixing in the lepton sector $U_{\text{PMNS}} = U_e^\dagger \tilde{U}$ is proportional to \tilde{U} with a small shift of the mixing angle θ_{12}^l .

We end this appendix by giving comments concerning the well-known four- and five-dimensional operators that contribute to fast proton decay in supersymmetric $SU(5)$ GUT models. In this respect, the dangerous proton decay terms arise from the dimension four $\lambda^{ijk} 10_m^i \bar{5}_m^j \bar{5}_m^k$ and dimension five $\lambda^{ij} \lambda^{kl} 10_m^i 10_m^j 10_m^k \bar{5}_m^l$ operators. These operators are dangerous in the sense that they lead to proton decay rates far larger than the experimental limits. As regards to the former operators, they contribute to the proton decay through the term violating baryon number ($U_1^c D_1^c D_k^c$) combined with the term ($Q_i L_j D_k^c$) that violates the lepton number with family indices as $i, j = 1, 2$ and $k = 2, 3$. In fact, these operators which are renormalizable can be avoided by imposing the usual R parity as in the case of the MSSM [53]. However, in our $SU(5) \times \mathbb{A}_4 \times U(1)$ proposal, these four-dimensional operators that are given by

$$10_m^i \bar{5}_m^j \bar{5}_m^k \rightarrow T_1 F_j F_j + T_2 F_j F_j + T_3 F_j F_j \quad (\text{A18})$$

are prevented by the additional $U(1)$ symmetry. On the other hand, in flavor symmetries based models there are additional nonrenormalizable couplings which involve flavon fields and can generate proton decay operators. In our model, these nonrenormalizable operators up to order $\mathcal{O}(1/\Lambda^2)$ look like $(1/\Lambda) 10_m^i \bar{5}_m^j \bar{5}_m^k \Omega$ with $\Omega = \varphi, \chi, \rho, \sigma, \eta$ as the various flavon superfields used throughout the different sectors studied in this work. It is easy to check from Tables III and IV that these couplings are also not allowed as they are not invariant under the $U(1)$ symmetry.

Regarding the five-dimensional couplings $\lambda^{ijkl} 10_m^i \times 10_m^j 10_m^k \bar{5}_m^l$, they are mediated by the heavy color triplet Higgsino, and it is well known that their dressing diagrams² to form six-dimensional operators are the most disturbing operators that lead to fast proton decay in SUSY $SU(5)$ models [55,56]. These operators that are derived from the renormalizable up and down Yukawa couplings λTTH_5 and $\lambda' TFH_5$ are absent in our model since they behave, respectively, as nontrivial singlets and triplet under the \mathbb{A}_4 flavor symmetry. However, the last couplings—which are required to generate masses for the charged fermions—are allowed through their interactions with the flavon superfields as given in the Yukawa couplings (A10) and (A6). Thus, our model contains higher order operators of the kind $\frac{1}{M_T} TTTT (\frac{\Omega}{\Lambda})^n$ where M_T is the mass of the colored Higgs triplet and $n = 1, 2$; for $n = 1$ we have $\Omega = \eta$, and for $n = 2$ the relevant combinations are $\Omega^2 = \sigma\chi, \rho\chi, \eta\varphi$. Hence, the suppression of these operators compared to the usual five-dimensional couplings $TTTT$ is now enhanced by the factors $(\frac{\Omega}{\Lambda})^n$ coming from the flavon superfields required by \mathbb{A}_4 invariance, thus leading to highly suppressed proton decay. We should note, however, that to provide precise predictions for the proton decay rate, the renormalization group equations (RGEs) for the gauge couplings at one loop must be taken into account [57]; this clearly goes beyond the scope of this paper.

²The dressing procedure of five-dimensional operators consists of converting two scalars (sfermions) in the $TTTT$ couplings to two fermions by a loop diagram through the exchange of winos and Higgsinos—these are the dominant contributions to the operators $qqql$ and $u^c u^c d^c e^c$, respectively. For more details and examples on such diagrams see, for instance, Ref. [54].

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