Muonic hydrogen and the proton size

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We reexamine the structure of the $n = 2$ levels of muonic hydrogen using a two-body potential that includes all relativistic, recoil and one-loop corrections. The potential was originally derived from QED to describe the muonium atom and accounts for all contributions to order α^5. Since one-loop corrections are included, the anomalous magnetic moment contributions of the muon can be identified and replaced by the proton anomalous magnetic moment to describe muonic hydrogen with a pointlike proton. This serves as a convenient starting point to include the dominant electron vacuum polarization corrections to the spectrum and extract the proton's mean squared radius $r_p = \sqrt{\langle r^2 \rangle}$. Our results are consistent with other theoretical calculations that find that the muonic hydrogen value for r_p is smaller than the result obtained from electron scattering.

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I. INTRODUCTION

The muonic hydrogen experiments [1,2] measure both the $2^3S_{1/2} \leftrightarrow 2^5P_{3/2}$ and the $2^1S_{1/2} \leftrightarrow 2^3P_{3/2}$ transitions. The experimental results are, respectively, $49881.35(65)\ \mathrm{GHz} = 206.2925(3)\ \mathrm{meV}$ and $54611.16(1.05)\ \mathrm{GHz} = 225.8535(4)\ \mathrm{meV}$. These measurements have been compared with a variety of theoretical calculations [3–9] that include a dependence on the mean squared proton radius $\langle r^2 \rangle$. Our purpose here is to compare the contributions to the Lamb shift that are independent of the proton structure with those of previous calculations. We find a value of 205.980 meV, about 0.07 meV smaller than other theoretical calculations.

If proton structure corrections are included, the resulting values of r_p from muonic hydrogen are systematically smaller than those generally obtained from electron scattering data [10], leading to a disparity between the two approaches. Some of the disparity could be associated with uncertainties in the scattering data. The proton radius experiment at the Jefferson Laboratory is designed to address this issue [11]. A recent spectroscopic measurement of the Rydberg constant [12] reports a smaller value of r_p consistent with muonic hydrogen, but a new spectroscopic measurement of the $1S \to 3S$ transition in hydrogen [13] supports a larger value as do most other spectroscopic measurements.

Here, we reexamine the theoretical calculation from a slightly different starting point. Our approach is to modify the two-body potential originally derived from QED to describe the muonium atom [14]. This potential contains all relativistic, recoil and one-loop terms that contribute to order α^5. The inclusion of the one-loop corrections enables us to identify the muon anomalous magnetic moment and replace it by the proton's anomalous magnetic moment $\kappa = 1.79285$. The resulting potential can be used to calculate the fine structure, hyperfine structure, Lamb shift and recoil corrections for muonic hydrogen with a a pointlike proton. It also serves as the starting point to include the dominant electron vacuum polarization contributions. The resulting hyperfine, spin-orbit, tensor and spin-independent potentials are [15], respectively,

$$V_{HF} = \frac{4\pi\alpha}{m_1 m_2}\left[\frac{2}{3}(1+a_\mu)(1+\kappa) + \frac{\alpha}{\pi}\frac{m_1 m_2}{m_1^2 - m_2^2}\ln\left(\frac{m_2^2}{m_1^2}\right)\right] \times \vec{S}_1 \cdot \vec{S}_2 \delta(\vec{r}), \tag{1}$$

$$V_{LS} = \frac{\alpha}{r^3}\left[\frac{(1+2a_\mu)}{2m_1^2} + \frac{(1+a_\mu)}{m_1 m_2}\right]\vec{L}\cdot\vec{S}_1 + \frac{\alpha}{r^3}\left[\frac{(1+2\kappa)}{2m_2^2} + \frac{(1+\kappa)}{m_1 m_2}\right]\vec{L}\cdot\vec{S}_2, \tag{2}$$

$$V_T = \frac{\alpha(1+a_\mu)(1+\kappa)}{m_1 m_2 r^3}(3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r} - \vec{S}_1\cdot\vec{S}_2), \tag{3}$$

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$$V_{SI} = -\frac{\alpha}{m_1 m_2 r}\vec{p}^2 + \frac{\alpha\pi}{2\mu^2}\delta(\vec{r}) - \frac{1}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)(\vec{p}^2)^2 + \frac{\alpha^2\mu}{2m_1m_2r^2} + \alpha^2\left[\frac{4}{3}\left(\frac{1}{\mu^2}\ln\left(\frac{\mu}{\lambda_{IF}}\right) - \frac{1}{m_1^2}\ln(\eta_2) - \frac{1}{m_2^2}\ln(\eta_1)\right)\delta(\vec{r}) + \left(\frac{2(m_1^2\ln(\eta_1) - m_2^2\ln(\eta_2))}{m_1m_2(m_1^2-m_2^2)}\right)\delta(\vec{r}) + \frac{7}{6\pi m_1 m_2}\nabla^2\left(\frac{\ln(\mu r)+\gamma}{r}\right) - \frac{4}{15}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\delta(\vec{r})\right]. \quad (4)$$

Here, μ is the reduced mass, $\eta_i = m_i/(m_1+m_2)$, γ is Euler's constant and λ_{IF} is the infrared cutoff. The last term in Eq. (4) is the contribution from the muon and proton vacuum polarizations. In what follows, we use [16] $m_1 = 105.6583715(35)$ MeV, $m_2 = 938.272046(21)$ MeV, $\mu = 94.9645$ MeV, $m_e = 0.510998928(11)$ MeV, $\alpha = 1/137.035999074(44)$ and $a_\mu = \alpha/2\pi$.

Much of the reason for undertaking the following calculation is that Eq. (4) contains several terms that differ from those commonly used in determining the muonic hydrogen spectrum. In particular, the order α delta function term behaves as μ^{-2} whereas the corresponding term in [5] has an overall $(1/m_1^2 + 1/m_2^2)$ factor. The difference in the s-state contribution is several meV in the order α^4 correction. Also, the one-loop term $\mu\alpha^2/(2m_1m_2r^2)$ contributes at order α^4 at the few meV level. The μ^{-2} dependence of the order α^2 one-loop $\ln(\mu/\lambda_{IF})$ term results in a recoil correction to the Lamb shift of order $\mu^3\alpha^5/m_2^2$ that is larger than the order α^6 correction to this contribution given in [5]. We undertook the calculation to determine the implication of these differences.

II. ELECTRON VACUUM POLARIZATION EFFECTS

A. One-loop correction to the Coulomb potential

The dominant contributions to the $2P-2S$ splitting in muonic hydrogen are due to the electron vacuum polarization corrections to the photon propagator. These contributions can be included by using the dispersion representation for the photon propagator [17–19]

$$D(k^2) = \frac{1}{k^2} - \int_0^\infty \frac{d\lambda}{\lambda}\frac{\Delta(\lambda)}{\lambda - k^2}, \quad (5)$$

where $\Delta(q^2)$ is

$$\Delta(q^2) = \frac{(2\pi)^3}{3q^2}\sum_n \delta^{(4)}(q-q_n)\langle 0|j_\mu(0)|n\rangle\langle n|j^\mu(0)|0\rangle. \quad (6)$$

For the $e\bar{e}$ intermediate state, $\Delta^{(2)}(\lambda)$ is

$$\Delta^{(2)}(\lambda) = \frac{\alpha}{3\pi}(1+2m_e^2/\lambda)\sqrt{1-4m_e^2/\lambda}\,\theta(\lambda - 4m_e^2). \quad (7)$$

If we take k^2 to be spacelike, then the modified Coulomb interaction at the one-loop level is

$$V(\vec{k}^2) = -\frac{e^2}{\vec{k}^2} - e^2\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\frac{\Delta^{(2)}(\lambda)}{\vec{k}^2+\lambda} \quad (8)$$

$$= V_C(\vec{k}^2) + V_{VP}(\vec{k}^2). \quad (9)$$

The explicit form of $V_{VP}(\vec{k}^2)$ in momentum space can be obtained by integrating over λ which results in

$$V_{VP}(\vec{k}^2) = \frac{-e^2}{\vec{k}^2}\Pi_f^{(2)}(\vec{k}^2), \quad (10)$$

with

$$\Pi_f^{(2)}(\vec{k}^2) = \frac{2\alpha}{\pi}\left[\frac{1}{3}(1-2m_e^2/\vec{k}^2) \times\left(\sqrt{1+4m_e^2/\vec{k}^2}\,\mathrm{arcsinh}\left(\frac{|\vec{k}|}{2m_e}\right) - 1\right) + \frac{1}{18}\right]. \quad (11)$$

$\Pi_f^{(2)}(\vec{k}^2)$ is the electron one-loop vacuum polarization correction in the spacelike region.

Transforming to coordinate space

$$V_{VP}(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k V_{VP}(\vec{k})e^{i\vec{k}\cdot\vec{r}}, \quad (12)$$

$$V_{VP}(\vec{r}) = -\frac{\alpha}{r}\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\Delta^{(2)}(\lambda)e^{-\sqrt{\lambda}r}. \quad (13)$$

To compute the effect of V_{VP}, we need to calculate the difference between $\langle 21m|V_{VP}|21m\rangle$ and $\langle 200|V_{VP}|200\rangle$. For example, $\langle 21m|V_{VP}|21m\rangle$ is

$$\langle 21m|V_{VP}|21m\rangle = -\frac{\alpha}{24a}\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\Delta^{(2)}(\lambda)\int_0^\infty dt t^3 e^{-(1+a\sqrt{\lambda})t},$$

$$\langle 21m|V_{VP}|21m\rangle = -\frac{\alpha}{4a}\int_4^\infty \frac{dx}{x}\frac{\Delta^{(2)}(x)}{(1+m_e a\sqrt{x})^4}, \quad (14)$$

with $a = 1/\mu\alpha$ and $\lambda = m_e^2 x$. A similar calculation gives

$$\langle 200|V_{VP}|200\rangle = -\frac{\alpha}{4a}\int_4^\infty \frac{dx}{x}\frac{\Delta^{(2)}(x)(1+2m_e^2a^2x)}{(1+m_ea\sqrt{x})^4}. \quad (15)$$

The difference between Eqs. (14) and (15) is

$$EE_{VP} = \Delta(2P-2S) = \frac{\mu\alpha^3}{6\pi}\int_4^\infty \frac{dx(1+2/x)\sqrt{1-4/x}m_e^2a^2}{(1+m_ea\sqrt{x})^4} = 205.007\ \mathrm{meV}. \quad (16)$$

This agrees with [5]. It is worth noting that the muon contribution from Eq. (16) is 0.0167 meV which is virtually the same as the 0.0168 meV result coming from the m_1^{-2} muon contribution in the last term of Eq. (4). The proton vacuum contribution from the m_2^{-2} term in Eq. (4) is indistinguishable from the result obtained using Eq. (16).

B. Two-loop correction to the Coulomb potential

The two-loop contributions to $\Delta(\lambda)$, $\Delta^{(4)}(\lambda)$, have been calculated by Källen and Sabry [19]. They contain both the reducible double electron bubble diagram and the irreducible fourth-order term $\Pi_f^{(4)}(\vec{k}^2)$. These corrections can be expressed as a correction to the Coulomb potential in the form [20]

$$V_{VP2}(r) = -\frac{\alpha}{r}F(r), \quad (17)$$

where

$$F(r) = -\frac{\alpha^2}{\pi^2}\int_1^\infty dt e^{-2tr}\Bigg[\left(\frac{13}{54t^2}+\frac{7}{108t^4}+\frac{2}{9t^6}\right)\sqrt{t^2-1} + \left(\frac{4}{3t^2}+\frac{2}{3t^4}\right)\sqrt{t^2-1}\log(8t(t^2-1)) + \left(-\frac{44}{9t}+\frac{2}{3t^3}+\frac{5}{4t^5}+\frac{2}{9t^7}\right)\mathrm{arccosh}(t) + \left(-\frac{8}{3t}+\frac{2}{3t^5}\right)\int_t^\infty dx f(x)\Bigg], \quad (18)$$

with

$$f(x) = \frac{3x^2-1}{x(x^2-1)}\mathrm{arccosh}(x) - \frac{1}{\sqrt{x^2-1}}\log(8x(x^2-1)). \quad (19)$$

By transforming this potential to momentum space and comparing it with Eq. (5), λ can be identified with $4t^2$ and the second-order energy shift expressed as

$$EE_{VP2} = -\frac{\mu\alpha^4}{4\pi^2}\int_4^\infty dx \frac{\Delta^{(4)}(\sqrt{x}/2)m_e^2a^2}{(1+m_ea\sqrt{x})^4} = 1.508\ \mathrm{meV}, \quad (20)$$

where $\Delta^{(4)}(x)$ is

$$\Delta^{(4)}(x) = \left(\frac{13}{54x}+\frac{7}{108x^3}+\frac{2}{9x^5}\right)\sqrt{x^2-1} + \left(\frac{4}{3x}+\frac{2}{3x^3}\right)\sqrt{x^2-1}\log(8x(x^2-1)) + \left(-\frac{44}{9}+\frac{2}{3x^2}+\frac{5}{4x^4}+\frac{2}{9x^6}\right)\mathrm{arccosh}(x) + \left(-\frac{8}{3}+\frac{2}{3x^4}\right)\Bigg[\frac{2\pi^2}{3} - \mathrm{arccosh}^2(x) - \log(8x(x^2-1))\mathrm{arccosh}(x) - 2\mathrm{Re}[\mathrm{Li}_2((x+\sqrt{x^2-1})^2)] + \mathrm{Li}_2(-(x-\sqrt{x^2-1})^2)\Bigg]. \quad (21)$$

The terms in the large square brackets result from evaluating the last integral in Eq. (18) and $\mathrm{Li}_2(z)$ is the Spence function

$$\mathrm{Li}_2(z) = -\int_0^1 \frac{dt}{t}\log(1-zt).$$

C. Three-loop correction to the Coulomb potential

The three-loop contribution to $\Delta(\lambda)$, $\Delta^{(6)}(\lambda)$, is not available in the literature. These contributions have been calculated by Kinoshita and Nio [21]. They note that the three-loop correction is comprised of two reducible diagrams and one irreducible diagram that can be represented as

$$(\Pi_f^{(2)}(\vec{k}^2))^3 + 2\Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(4)}(\vec{k}^2) + \Pi_f^{(6)}(\vec{k}^2). \quad (22)$$

Remembering the reducible fourth-order dispersion result

$$-e^2\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\frac{\Delta^{(4)}(\lambda)}{\vec{k}^2+\lambda} = -\frac{e^2}{\vec{k}^2}\Pi^{(4)}(\vec{k}^2) \quad (23)$$

and

$$\Pi^{(4)}(\vec{k}^2) = (\Pi_f^{(2)}(\vec{k}^2))^2 + \Pi_f^{(4)}(\vec{k}^2), \quad (24)$$

it is possible to verify the contributions of the reducible terms. They are 0.000396 and 0.0029312 meV, respectively. The numerical evaluation of the $\Pi_f^{(6)}(\vec{k}^2)$ contribution is [21] 0.001103 meV, giving a total three-loop correction to the Coulomb potential of 0.004431 meV.

D. α^4 second-order nonrelativistic perturbation correction

The large size of the one-loop correction suggests that the contribution of $V_{VP}(\vec{r})$ in the second-order nonrelativistic perturbation theory is not negligible. Evaluating this correction necessitates using the radial portions of the Coulomb Green's function for $n=2$ and $\ell=0,\ 1$, expressed as $\mu^2\alpha g_{2\ell}(x,x')$. General expressions for these Green's functions were derived by Hostler [22,23] and explicit expressions for small values of n are contained in [24,25]. Due to some typographical errors in the latter papers [g_{20} in Ref. [24] and Eq. (2.18) in Ref. [25]], the expressions for $g_{20}(x,x')$ and $g_{21}(x,x')$ ($x=r/a$, $x'=r'/a$) are given here:

$$g_{20}(x,x') = e^{-(x+x')/2}\left[(2-x)(2-x')\left(\ln(x)+\ln(x') + \frac{(x+x')}{4} + 2\gamma - \frac{15}{4} + \mathrm{Ei}(x_<)\right) + 12 - 2x - 2x' - \frac{2}{x} - \frac{2}{x'} + \frac{x}{x'} + \frac{x'}{x} - xx' + (2-x_>)e^{x_<}\left(\frac{1}{x_<}-1\right)\right]\frac{1}{4\pi}, \tag{25a}$$

$$g_{21}(x,x') = \frac{xx'}{3}e^{-(x+x')/2}\left[\ln(x)+\ln(x') + \frac{(x+x')}{4} + 2\gamma - \frac{49}{12} - \frac{3}{x} - \frac{3}{x^2} - \frac{2}{x^3} - \frac{3}{x'} - \frac{3}{x'^2} - \frac{2}{x'^3} - \mathrm{Ei}(x_<) + \left(\frac{1}{x_<} + \frac{1}{x_<^2} + \frac{2}{x_<^3}\right)e^{x_<}\right]\frac{3\hat{x}\cdot\hat{x}'}{4\pi}. \tag{25b}$$

Here, $\mathrm{Ei}(x)$ is

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} dt\frac{e^t}{t}, \tag{26}$$

and

$$x_< = x\theta(x'-x) + x'\theta(x-x'),$$
$$x_> = x'\theta(x'-x) + x\theta(x-x'). \tag{27}$$

The contributions take the form

$$\mu^2\alpha\int_0^\infty dr r^2 R_{n\ell}(r)V_{VP}(r)\int_0^\infty dr' r'^2 g_{n\ell}(x,x')V_{VP}(r')R_{n\ell}(r'), \tag{28}$$

for $n=2$ and $\ell=0,\ 1$. The integrations over r, r' can be evaluated exactly using *Mathematica*, and integrations over propagator parameters λ and λ' can then be calculated numerically. The results are $EE^{(2)}2P_{VP} = -0.0022671$ meV and $EE^{(2)}2S_{VP} = -0.153164$ meV for a net contribution of [5]

$$EE_{VP}^{(2)} = 0.1509 \text{ meV}. \tag{29}$$

E. α^5 second-order nonrelativistic perturbation correction

There is also a second-order nonrelativistic perturbative contribution from the combination of a one-loop vacuum polarization correction and a two-loop vacuum polarization correction. The calculation is similar to the one-loop second-order calculation and results in a 0.00215 meV contribution. In addition, there is a third-order nonrelativistic perturbative contribution from three one-loop vacuum polarization corrections [21,26] which gives 0.00007 meV.

The remaining corrections to the energy levels come from the potentials in Eqs. (1)–(4). In what follows, we use simultaneous eigenstates of $\vec{F}^2$, F_z, $\vec{J}^2$, $\vec{S}_2{}^2$, where $\vec{F} = \vec{L} + \vec{S}_1 + \vec{S}_2$, $\vec{J} = \vec{L} + \vec{S}_1$.

III. SPIN-INDEPENDENT TERMS

A. Order α^4 terms

The terms in the first line of Eq. (4) contribute to the s and p levels in order α^4. Their expectation values are (in meV)

$$\left\langle -\frac{\alpha}{m_1m_2r}\vec{p}^2\right\rangle = -\frac{\mu^3\alpha^4}{n^3m_1m_2}\left(\frac{4}{2\ell+1} - \frac{1}{n}\right)10^9, \tag{30a}$$

$$\left\langle \frac{\alpha\pi}{2\mu^2}\delta(\vec{r})\right\rangle = \frac{\mu\alpha^4}{2n^3}\delta_{\ell 0}10^9, \tag{30b}$$

$$\left\langle -\frac{1}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)(\vec{p}^2)^2\right\rangle = -\frac{\mu\alpha^4}{8n^3}\left(\frac{\mu^3}{m_1^3}+\frac{\mu^3}{m_2^3}\right)\left(\frac{8}{2\ell+1}-\frac{3}{n}\right)10^9, \tag{30c}$$

$$\left\langle \frac{\alpha^2\mu}{2m_1m_2r^2}\right\rangle = \frac{\mu^3\alpha^4}{n^3m_1m_2}\frac{1}{2\ell+1}10^9, \tag{30d}$$

and it should be noted that the $1/r^2$ term is part of the one-loop correction. For the $n=2$ state, the contributions are

$$E_2(2S) = -10.711 \text{ meV}, \tag{31a}$$

$$E_2(2P) = -5.100 \text{ meV}, \tag{31b}$$

$$E_2 = E_2(2P) - E_2(2S) = 5.610 \text{ meV}. \tag{31c}$$

B. Order α^5 terms

The remainder of the terms in Eq. (4) are of order α^5 or $\alpha^5\ln(\alpha)$. There are two issues to address when evaluating these terms. The first is the elimination of the photon mass dependence. This is accomplished by using the "Bethe logarithm" technique, which amounts to the replacement of $\ln(\mu/\lambda_{IF})$ by [27]

$$\left(\ln\frac{R_\infty}{\alpha^2 k_0(n,0)}+\frac{5}{6}\right)\delta_{\ell 0}+\ln\frac{R_\infty}{k_0(n,\ell)}(1-\delta_{\ell 0}). \quad (32)$$

The other is the matrix element of $\nabla^2[(\ln(\mu r)+\gamma)/r]$. For states with $\ell > 0$, this reduces to $-1/r^3$. When $\ell = 0$, the result is

$$\langle n0|\frac{1}{2\pi}\nabla^2\left[\frac{\ln(\mu r)+\gamma}{r}\right]|n0\rangle = \frac{2\mu^3\alpha^3}{\pi n^3}\left[\ln\frac{2\alpha}{n}+\frac{n-1}{2n}+\sum_{k=1}^{n}\frac{1}{k}\right]. \quad (33)$$

Using

$$\ln k_0(2S) = 2.8117699, \quad (34)$$

$$\ln k_0(2P) = -0.0300167 \quad (35)$$

and denoting the expectation values of the order α^5 as $E_4(n\ell)$, the results are

$$E_4(2S) = 0.7077\ \text{meV}, \quad (36a)$$

$$E_4(2P) = 0.0004\ \text{meV}, \quad (36b)$$

$$E_4 = E_4(2P) - E_4(2S) = -0.7073\ \text{meV}. \quad (36c)$$

IV. SPIN-DEPENDENT TERMS

A. $\ell = 0$ hyperfine

The expectation value of V_{HF} affects only s states and is

$$E_{HF}(nS_{1/2}) = \frac{\mu^3\alpha^4}{n^3 m_1 m_2}\left[\frac{2}{3}(1+a_\mu)(1+\kappa) + \frac{\alpha}{\pi}\frac{m_1 m_2}{m_1^2-m_2^2}\ln\left(\frac{m_2^2}{m_1^2}\right)\right](2s(s+1)-3), \quad (37a)$$

$$E_{HF}(^3S_{1/2}) = 5.704\ \text{meV}, \quad (37b)$$

$$E_{HF}(^1S_{1/2}) = -17.113\ \text{meV}, \quad (37c)$$

$$\Delta E_{HF}(S_{1/2}) = 22.817\ \text{meV}. \quad (37d)$$

B. Spin-orbit and tensor terms

The largest contribution of V_{LS} is that associated with the $\vec{L}\cdot\vec{S}_1$ term. It accounts for the fine structure splitting between the $P_{3/2}$ and the $P_{1/2}$ states. The contribution is

$$E_{FS}(P_{j_1}) = \alpha\left[\frac{(1+2a_\mu)}{2m_1^2}+\frac{(1+a_\mu)}{m_1 m_2}\right]\left\langle\frac{1}{r^3}\right\rangle\langle\vec{L}\cdot\vec{S}_1\rangle. \quad (38)$$

The expectation value of r^{-3} is

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{2\mu^3\alpha^3}{n^3\ell(\ell+1)(2\ell+1)} \to \frac{\mu^3\alpha^3}{24}. \quad (39)$$

Since the eigenstates we are using are eigenstates of $\vec{J}^2$,

$$2\vec{L}\cdot\vec{S}_1 = \vec{J}^2 - \vec{L}^2 - \vec{S}_1^2 = j(j+1) - 2 - 3/4. \quad (40)$$

Then,

$$E_{FS}(P_j) = \frac{\mu^3\alpha^4}{48 m_1^2}\left[(1+2a_\mu)+\frac{2m_1}{m_2}(1+a_\mu)\right]\langle\vec{L}\cdot\vec{S}_1\rangle, \quad (41)$$

and

$$E_{FS}(P_{3/2}) = 2.782\ \text{meV}, \quad (42a)$$

$$E_{FS}(P_{1/2}) = -5.564\ \text{meV}, \quad (42b)$$

$$\Delta E_{FS} = E_{FS}(P_{3/2}) - E_{FS}(P_{1/2}) = 8.347\ \text{meV}. \quad (42c)$$

The remaining spin-dependent terms are the $\vec{L}\cdot\vec{S}_2$ portion of Eq. (2) (call it V'_{LS}) and V_T. Their matrix elements for a generic $2P$ state are

$$\langle 2P|V'_{LS}|2P\rangle = \frac{\mu^3\alpha^4(1+\kappa)}{24 m_1 m_2}\left[1+\frac{m_1}{2m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right]\langle\vec{L}\cdot\vec{S}_2\rangle, \quad (43a)$$

$$\langle 2P|V'_T|2P\rangle = \frac{\mu^3\alpha^4(1+a_\mu)(1+\kappa)}{24 m_1 m_2}\langle 3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r} - \vec{S}_1\cdot\vec{S}_2\rangle. \quad (43b)$$

For V'_{LS}, we can use the fact that $\vec{F}^2$ and $\vec{J}^2$ are diagonal, so $\vec{F}^2 = (\vec{J}+\vec{S}_2)^2$ and we can obtain the relation

$$2\vec{L}\cdot\vec{S}_2 = \vec{F}^2 - \vec{J}^2 - 2\vec{S}_1\cdot\vec{S}_2 - 3/4 = \vec{F}^2 - \vec{J}^2 + 3/4 - \vec{S}^2. \quad (44)$$

Both the $^5P_{3/2}$ and $^1P_{1/2}$ states are eigenstates of $\vec{S}^2$ with eigenvalue 2, so $\langle ^5P_{3/2}|\vec{L}\cdot\vec{S}_2|^5P_{3/2}\rangle = 1/2$ and $\langle ^1P_{1/2}|\vec{L}\cdot\vec{S}_2|^1P_{1/2}\rangle = -1$:

$$\langle {}^1P_{1/2}|V'_{LS}|{}^1P_{1/2}\rangle = -\frac{\mu^3\alpha^4(1+\kappa)}{24m_1m_2}\left[1+\frac{1}{2}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right], \tag{45a}$$

$$\langle {}^5P_{3/2}|V'_{LS}|{}^5P_{3/2}\rangle = \frac{\mu^3\alpha^4(1+\kappa)}{48m_1m_2}\left[1+\frac{1}{2}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right]. \tag{45b}$$

The matrix elements of V_T for these states are

$$\langle {}^1P_{1/2}|V_T|{}^1P_{1/2}\rangle = -\frac{\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\frac{(1+a_\mu)}{8}, \tag{46a}$$

$$\langle {}^5P_{3/2}|V_T|{}^5P_{3/2}\rangle = -\frac{\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\frac{(1+a_\mu)}{80}. \tag{46b}$$

Combining these two contributions gives

$$E_{HFS}({}^1P_{1/2}) = -\frac{\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\left[\frac{1}{4}+\frac{a_\mu}{8}+\frac{1}{16}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right] \to -5.968\ \text{meV}, \tag{47a}$$

$$E_{HFS}({}^5P_{3/2}) = \frac{\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\left[\frac{1}{20}-\frac{a_\mu}{80}+\frac{1}{32}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right] \to 1.272\ \text{meV}. \tag{47b}$$

The ${}^3P_{3/2}$ and ${}^3P_{1/2}$ states are mixed by the V'_{LS} and V_T potentials. This results in

$$\begin{aligned}\langle {}^3P_{1/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{1/2}\rangle &= 1/3,\\ \langle {}^3P_{3/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{3/2}\rangle &= -5/6,\\ \langle {}^3P_{3/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{1/2}\rangle &= -\sqrt{2}/3\end{aligned} \tag{48}$$

and

$$\begin{aligned}\langle {}^3P_{1/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{1/2}\rangle &= 1/3,\\ \langle {}^3P_{3/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{3/2}\rangle &= 1/6,\\ \langle {}^3P_{3/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{1/2}\rangle &= \sqrt{2}/6.\end{aligned} \tag{49}$$

The matrix elements of V'_{LS} are

$$\langle {}^3P_{1/2}|V'_{LS}|{}^3P_{1/2}\rangle = \frac{\mu^3\alpha^4(1+\kappa)}{72m_1m_2}\left[1+\frac{1}{2}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right], \tag{50a}$$

$$\langle {}^3P_{3/2}|V'_{LS}|{}^3P_{3/2}\rangle = -\frac{5\mu^3\alpha^4(1+\kappa)}{144m_1m_2}\left[1+\frac{1}{2}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right], \tag{50b}$$

$$\langle {}^3P_{3/2}|V'_{LS}|{}^3P_{1/2}\rangle = -\frac{\mu^3\alpha^4(1+\kappa)}{72m_1m_2}\left[1+\frac{1}{2}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right]\sqrt{2}, \tag{50c}$$

and those of V_T are

$$\langle {}^3P_{1/2}|V_T|{}^3P_{1/2}\rangle = \frac{\mu^3\alpha^4}{72m_1m_2}(1+\kappa)(1+a_\mu), \tag{51a}$$

$$\langle {}^3P_{3/2}|V_T|{}^3P_{3/2}\rangle = \frac{\mu^3\alpha^4}{144m_1m_2}(1+\kappa)(1+a_\mu), \tag{51b}$$

$$\langle {}^3P_{3/2}|V_T|{}^3P_{1/2}\rangle = \frac{\mu^3\alpha^4}{144m_1m_2}(1+\kappa)(1+a_\mu)\sqrt{2}. \tag{51c}$$

The V'_{LS} and V_T contributions can be combined to give

$$E_{HFS}({}^3P_{1/2}) = \frac{\mu^3\alpha^4(1+\kappa)}{9m_1m_2}\left[\frac{1}{4}+\frac{a_\mu}{8}+\frac{1}{16}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right] \to 1.989\ \text{meV}, \tag{52a}$$

$$E_{HFS}({}^3P_{3/2}) = -\frac{5\mu^3\alpha^4(1+\kappa)}{18m_1m_2}\left[\frac{1}{10}-\frac{a_\mu}{40}+\frac{1}{16}\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right] \to -2.120\ \text{meV}, \tag{52b}$$

$$E_{MIX}({}^3P_{1/2}-{}^3P_{3/2}) = -\frac{\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\left[1+\frac{m_1}{m_2}\frac{(1+2\kappa)}{(1+\kappa)}\right]\frac{\sqrt{2}}{48} \to -0.796\ \text{meV}. \tag{52c}$$

The expression for E_{MIX} omits the a_μ contribution and all these results agree with Ref. [5].

Diagonalizing the triplet P mixing matrix

$$\begin{pmatrix} E_{HFS}({}^3P_{1/2}) & E_{MIX} \\ E_{MIX} & E_{HFS}({}^3P_{3/2})+\Delta E_{FS}\end{pmatrix} \tag{53}$$

has the effect of shifting the ${}^3P_{3/2}$ level up by $\Delta = 0.1447$ meV and the ${}^3P_{1/2}$ level down by the same amount.

There are small electron vacuum polarization corrections to all of the terms in the potential that contribute to order α^4. These are computed in Appendix A and included in the results that are compared with the experiment.

V. RESULTS AND CONCLUSIONS

Relative to the $n = 2$ Bohr level, the energies of the various $n = 2$ states, including the small corrections calculated in Appendix A, are

$$E(^5P_{3/2}) = EE_{VP} + E_2 + E_4 + E_{FS}(P_{3/2}) + E_{HFS}(^5P_{3/2}) = 215.609 \text{ meV}, \quad (54a)$$

$$E(^3P_{3/2}) = EE_{VP} + E_2 + E_4 + E_{FS}(P_{3/2}) + E_{HFS}(^3P_{3/2}) + \Delta = 212.360 \text{ meV}, \quad (54b)$$

$$E(^3P_{1/2}) = EE_{VP} + E_2 + E_4 + E_{FS}(P_{1/2}) + E_{HFS}(^3P_{1/2}) - \Delta = 207.826 \text{ meV}, \quad (54c)$$

$$E(^1P_{1/2}) = EE_{VP} + E_2 + E_4 + E_{FS}(P_{1/2}) + E_{HFS}(^1P_{1/2}) = 200.006 \text{ meV}, \quad (54d)$$

$$E(^3S_{1/2}) = E_{HF}(^3S_{1/2}) = 5.7351 \text{ meV}, \quad (54e)$$

$$E(^1S_{1/2}) = E_{HF}(^1S_{1/2}) = -17.2054 \text{ meV}, \quad (54f)$$

$$E(P_{1/2}) - E(S_{1/2}) = EE_{VP} + E_2 + E_4 + E_{FS}(P_{1/2}) = 205.980 \text{ meV}. \quad (54g)$$

Here, EE_{VP} is the sum of the first seven rows in Table I, E_2 is the eighth row, E_4 the ninth row, E_{FS} the tenth row, the s-state hyperfine splitting the eleventh row and the p-state hyperfine splittings the twelfth row. The additional small corrections in Appendix A include the spin-independent terms in subsection I, the $\ell = 0$ hyperfine splitting in subsection IV, the fine structure splittings in subsection 2, and the $\ell = 1$ hyperfine splitting in subsections 3, 4 and 5. Equation (54g) gives our value for the Lamb shift excluding proton structure corrections.

We have used nonrelativistic wave functions throughout because our potential contains the leading-order relativistic, recoil and one-loop corrections. However, if we use the solutions to the Dirac equation given in Rose [28], the value of the electron one-loop vacuum polarization changes from 205.007 to 205.028 meV for the $2P_{1/2} - 2S_{1/2}$ interval and from 205.007 to 205.033 meV for the $2P_{3/2} - 2S_{1/2}$ interval [4]. We have also calculated the relativistic corrections to the two-loop vacuum polarization contribution using the approach of [21]. In this order, the change in the $2P_{1/2} - 2S_{1/2}$ interval is 0.0001 meV and the corresponding change in the $2P_{3/2} - 2S_{1/2}$ interval is 0.0002 meV.

A more fundamental way to calculate the relativistic corrections to the dominant electron vacuum polarization contribution would be to use the solutions to the Salpeter equation with an instantaneous Coulomb kernel. Estimates of this correction using the scalar Salpeter equation have been made [29] and the results are small. Unfortunately, analytic solutions for the spin $1/2$ Salpeter wave functions with unequal masses are not available.

Finally, one might wonder how the relatively large contributions from the mass dependence of the delta function term and the one-loop α^2/r^2 term mentioned in the introduction can still lead to a Lamb shift value that is in agreement with other calculations. The answer is that there are two versions of the spin-independent fine structure Hamiltonian that contribute order α^4 corrections to the $2P_{1/2} - 2S_{1/2}$ energy difference. One is the Breit-Pauli version [7] H'_{B-P} given by

TABLE I. The entries summarize the various corrections (in meV) to the $n = 2$ states of muonic hydrogen calculated using Eqs. (1)–(4).

	$^1S_{1/2}$	$^3S_{1/2}$	$^1P_{1/2}$	$^3P_{1/2}$	$^3P_{3/2}$	$^5P_{3/2}$	$^3P_{1/2} \leftrightarrow {}^3P_{3/2}$
One-loop vacuum polarization			205.007	205.007	205.007	205.007	
Relativistic one-loop correction			0.021	0.021	0.026	0.026	
Two-loop vacuum polarization			1.508	1.508	1.508	1.508	
Relativistic two-loop correction			0.0001	0.0001	0.0002	0.0002	
Three-loop vacuum polarization			0.0044	0.0044	0.0044	0.0044	
NR 2nd-order α^4 vacuum polarization			0.1509	0.1509	0.1509	0.1509	
NR 2nd-order α^5 vacuum polarization			0.0022	0.0022	0.0022	0.0022	
Spin-independent α^4			5.6102	5.6102	5.6102	5.6102	
Spin-independent α^5			−0.7073	−0.7073	−0.7073	−0.7073	
Fine structure			−5.5645	−5.5645	2.7823	2.7823	
Hyperfine $\ell = 0$	−17.113	5.7045					
Hyperfine $\ell = 1$			−5.9682	1.9894	−2.1195	1.2717	
Mixing matrix element							−0.79615
			Vacuum polarization corrections to $V_2(\vec{k})$				
	−0.0920	0.0307	−0.0532	−0.0464	−0.0446	−0.0418	−0.0003
Total	−17.205	5.735	200.006	207.971	212.215	215.609	−0.7965
Result of mixing				207.826	212.360		

$$H'_{B-P} = \frac{\pi\alpha}{2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\delta(\vec{r}) - \frac{\alpha}{2m_1m_2} p_i \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) p_j \tag{55}$$

$$= \frac{\pi\alpha}{2\mu^2}\delta(\vec{r}) - \frac{\pi\alpha}{m_1m_2}\delta(\vec{r}) - \frac{\alpha}{2m_1m_2} p_i \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) p_j. \tag{56}$$

The other is H'_{GRS} of Ref. [14], which has the form

$$H'_{GRS} = \frac{\pi\alpha}{2\mu^2}\delta(\vec{r}) - \frac{\alpha}{m_1m_2}\frac{1}{r}\vec{p}^2 + \frac{\mu\alpha^2}{2m_1m_2r^2}, \tag{57}$$

where the last term arises in the calculation of the one-loop corrections. The last two terms of Eqs. (56) and (57) give identical contributions to the $2P_{1/2} - 2S_{1/2}$ splitting, namely,

$$\Delta E(2P - 2S) = \frac{\mu^3\alpha^4}{4m_1m_2}. \tag{58}$$

All the spin-dependent fine structure terms of the two versions are the same so, apart from minor differences in some recoil terms, the Lamb shift values agree. This implies that there should be no α^2/r^2 term associated with the one-loop corrections to Breit-Pauli Hamiltonian. In Appendix B this is shown to be the case.

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APPENDIX A: ELECTRON VACUUM POLARIZATION CORRECTIONS TO ORDER α^4 TERMS

The electron vacuum polarization corrections to the leading-order α^4 terms in the potential can be obtained from the momentum space representation of the potential, which is

$$V_2(\vec{k}) = -\frac{e^2}{\vec{k}^2}\Bigg[1 - \frac{\vec{k}^2}{8\mu^2} + \frac{i}{4m_1m_2}\left(\left(2 + \frac{m_2}{m_1}\right)(\vec{k}\times\vec{p})\cdot\vec{\sigma}_1 + (1+\kappa)\left(2 + \frac{1+2\kappa}{1+\kappa}\frac{m_1}{m_2}\right)(\vec{k}\times\vec{p})\cdot\vec{\sigma}_2\right) - \frac{(1+\kappa)}{6m_1m_2}\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 + \frac{(1+\kappa)}{4m_1m_2}\left(\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2 - \frac{1}{3}\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2\right) + \frac{\vec{p}^2}{m_1m_2} - \frac{\alpha\mu\pi k}{4m_1m_2} - \frac{1}{6}\langle r^2\rangle\vec{k}^2\Bigg], \tag{A1}$$

where the next to last term is a one-loop correction that contributes at order α^4. The electron vacuum polarization correction is obtained by making the replacement

$$-\frac{e^2}{\vec{k}^2} \to -e^2\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\frac{\Delta^{(2)}(\lambda)}{\vec{k}^2 + \lambda}. \tag{A2}$$

In addition, there is a second-order perturbative correction containing $V_{VP}(r)$ for each of these terms as well as for the relativistic kinetic energy terms. In these calculations, all integrals except those over λ can be evaluated analytically using *Mathematica*. The integrals over λ are performed using the *Mathematica* NIntegrate routine.

1. Spin-independent terms

a. $e^2\vec{k}^2/8\mu^2$

This leads to the expression

$$\frac{\pi\alpha}{2\mu^2}\int_{4m_e^2}^\infty \frac{d\lambda\Pi_e(\lambda)}{\lambda}\frac{\vec{k}^2}{\vec{k}^2 + \lambda}, \tag{A3}$$

and transforming to coordinate space gives

$$\frac{\pi\alpha}{2\mu^2}\int_{4m_e^2}^\infty \frac{d\lambda\Pi_e(\lambda)}{\lambda}\left(\delta(\vec{r}) - \frac{\lambda}{4\pi}\frac{1}{r}e^{-\sqrt{\lambda}r}\right). \tag{A4}$$

The delta function only contributes to $\ell = 0$ and gives

$$\Delta E(\delta) = \frac{\mu\alpha^5}{48\pi}\int_4^\infty \frac{dx}{x}(1 + 2/x)\sqrt{1 - 4/x}. \tag{A5}$$

The integral diverges, but the $\ell = 0$ contribution of the remaining term cancels this divergence. Using Eq. (14) and the extra factor of $\lambda = m_e^2 x$, the remaining contribution to the $E(2P) - E(2S)$ interval is

$$\Delta(E(2P) - E(2S)) = \frac{\mu\alpha^5}{48\pi}\int_4^\infty \frac{dx m_e^4 a^4 x^2(1 + 2/x)\sqrt{1 - 4/x}}{x(1 + m_e a\sqrt{x})^4}. \tag{A6}$$

The vacuum polarization correction to this term is

$$\Delta E_1(e^2\vec{k}^2/8\mu^2) = \frac{\mu\alpha^5}{48\pi}\int_4^\infty \frac{dx}{x}(1 + 2/x)\sqrt{1 - 4/x} \times \left[\frac{m_e^4 a^4 x^2}{(1 + m_e a\sqrt{x})^4} - 1\right] = -0.0363\ \text{meV}. \tag{A7}$$

The second-order correction is

$$\begin{aligned}\Delta E_2(e^2\delta(\vec{r})/8\mu^2) &= 2\left(\frac{\pi\alpha}{2\mu^2}\right)\int d^3r\psi_{20}(\vec{r})V_{VP}(r)\int dr'^3 g_{20}(r,r')\delta(\vec{r})\psi_{20}(\vec{r})\\ &= -\left(\frac{\pi\alpha^2}{\mu^2}\right)\int d^3r\frac{R_{20}(r)}{8\pi a^3}\frac{1}{r}e^{-\sqrt{\lambda}r}g_{20}(r/a,0)\\ &= -\left(\frac{\pi\alpha^2}{2\mu^2a^3}\right)\int_0^\infty drr\left(1-\frac{r}{2a}\right)e^{-r/2a}e^{-\sqrt{\lambda}r}g_{20}(r/a,0),\\ \Delta E_2(e^2\delta(\vec{r})/8\mu^2) &= -\left(\frac{\pi\alpha^3}{2\mu}\right)\int_0^\infty dxx(1-x/2)e^{-x/2}e^{-\sqrt{\lambda}ax}g_{20}(x,0),\end{aligned}\tag{A8}$$

where the integration over $d\lambda$ has been suppressed. The Greens function $g_{20}(x,0)$ is

$$g_{20}(x,0)=\frac{\mu^2\alpha}{4\pi}\frac{e^{-x/2}}{2x}[-4+(8\gamma-6)x+(13-4\gamma)x^2-x^3-4x(x-2)\ln(x)]=\frac{\mu^2\alpha}{4\pi}\mathcal{G}_{20}(x).\tag{A9}$$

The final expression for the $2p-2s$ splitting is

$$-\Delta E_2(e^2\delta(\vec{r})/8\mu^2)=\frac{\mu\alpha^4}{8}\frac{\alpha}{3\pi}\int_4^\infty\frac{dx}{x}(1+2/x)\sqrt{1-4/x}F(\beta\sqrt{x})=-0.0549\text{ meV},\tag{A10}$$

where

$$\begin{aligned}F(\beta\sqrt{x})&=\int_0^\infty dyy(1-y/2)e^{-y/2}e^{-\beta\sqrt{x}y}\mathcal{G}_{20}(y),\\ F(\beta\sqrt{x})&=-\frac{3+11\beta\sqrt{x}+4\beta^2x+12\beta^3x^{3/2}+4\beta^4x^2+4(1+\beta\sqrt{x})(1+2\beta^2x)\ln(1+\beta\sqrt{x})}{2(1+\beta\sqrt{x})^5},\end{aligned}\tag{A11}$$

and $\beta=m_ea$.

b. $-e^2\vec{p}^2/rm_1m_2$

Here, the expression is

$$\frac{-e^2}{m_1m_2}\frac{1}{(2\pi)^3}\int\frac{d^3ke^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+\lambda}\vec{p}^2=-\frac{\alpha}{m_1m_2r}e^{-\sqrt{\lambda}r}\vec{p}^2=-\frac{2\mu\alpha^2}{m_1m_2r^2}\left(1-\frac{r}{8a}\right)e^{-\sqrt{\lambda}r}.\tag{A12}$$

For the $\ell=1$ state, the result of integrating over r is

$$\frac{\mu^3\alpha^4}{12m_1m_2}\frac{1}{(1+a\sqrt{\lambda})^4}\left(\frac{3}{4}-2(1+a\sqrt{\lambda})\right).\tag{A13}$$

The $\ell=0$ state when integrated over r gives

$$\frac{\mu^3\alpha^4}{m_1m_2}\frac{1}{(1+a\sqrt{\lambda})^4}\left(-(1+a\sqrt{\lambda})^3+\frac{9}{8}(1+a\sqrt{\lambda})^2-\frac{3}{4}(1+a\sqrt{\lambda})+\frac{3}{16}\right).\tag{A14}$$

Integrating the difference of these two results over λ gives

$$\Delta E_1(-e^2\vec{p}^2/rm_1m_2)=0.0142\text{ meV}.\tag{A15}$$

The expression for the second-order correction is (again, suppressing the integral over $d\lambda$ and including the factor of 2)

$$\Delta E_2(-e^2\vec{p}^2/rm_1m_2)=\frac{4\mu^3\alpha^4}{m_1m_2}\int_0^\infty drrR_{2\ell}(r)e^{-\sqrt{\lambda}r}\int_0^\infty dr'g_{2\ell}(r/a,r'/a)(1-r'/8a)R_{2\ell}(r').\tag{A16}$$

This expression reduces to

$$\frac{\mu^3\alpha^4}{6m_1m_2}\int_0^\infty dxx^2e^{-x/2}e^{-\sqrt{\lambda}ax}\int_0^\infty dx' g_{21}(x,x')x'e^{-x'/2}(1-x'/8) \tag{A17}$$

for $\ell = 1$ and

$$\frac{2\mu^3\alpha^4}{m_1m_2}\int_0^\infty dxx(1-x/2)e^{-x/2}e^{-\sqrt{\lambda}ax}\int_0^\infty dx' g_{20}(x,x')(1-x'/2)e^{-x'/2}(1-x'/8) \tag{A18}$$

for $\ell = 0$. The integrals are, respectively,

$$-\frac{13+33a\sqrt{\lambda}+32(1+a\sqrt{\lambda})\ln(1+a\sqrt{\lambda})}{8(1+a\sqrt{\lambda})^5} \tag{A19}$$

and

$$-\frac{30+86a\sqrt{\lambda}-60a^2\lambda+52a^3\lambda^{3/2}+64(1+a\sqrt{\lambda})(1+2a^2\lambda)\ln(1+a\sqrt{\lambda})}{64(1+a\sqrt{\lambda})^5}. \tag{A20}$$

Integrating the difference over $d\lambda$ gives a $2p-2s$ contribution of

$$\Delta E_2(-e^2\vec{p}^2/rm_1m_2) = 0.01459\ \text{meV}. \tag{A21}$$

c. $-(\vec{p}^2)^2/8m_1^3$

When integrated over $d\lambda$, the second-order correction to the relativistic kinetic energy contribution is

$$\Delta E_2(-(\vec{p}^2)^2/8m_1^3) = \frac{4\mu^4\alpha^4}{8m_1^3}\int_0^\infty drrR_{2\ell}(r)e^{-\sqrt{\lambda}r}\int_0^\infty dr' g_{2\ell}(r/a,r'/a)(1-r'/8a)^2R_{2\ell}(r'). \tag{A22}$$

This correction is negligible for $\ell = 1$. After completion of the integrals above, the contribution from the muon to the $2p-2s$ splitting for $\ell = 0$ is

$$\Delta E_2(-(\vec{p}^2)^2/8m_1^3) = \frac{\mu^4\alpha^4}{2m_1^3}\frac{\alpha}{3\pi}\int_4^\infty \frac{dx}{x}(1+2/x)\sqrt{1-4/x}G(\beta\sqrt{x}) = 0.02237\ \text{meV}, \tag{A23}$$

with

$$G(\beta\sqrt{x}) = -\frac{7+\beta\sqrt{x}(19+2\beta\sqrt{x}(-7+5\beta\sqrt{x}))+16(1+\beta\sqrt{x})(1+2\beta^2x)\ln(1+\beta\sqrt{x})}{16(1+\beta\sqrt{x})^5}. \tag{A24}$$

d. $\alpha\mu\pi k/4m_1m_2$

Here, the momentum space integral is

$$\frac{\mu\alpha^2\pi^2}{m_1m_2}\frac{1}{(2\pi)^3}\int\frac{d^3kke^{i\vec{k}\cdot\vec{r}}e^{-\varepsilon k}}{\vec{k}^2+\lambda} = \frac{\mu\alpha^2}{2m_1m_2r^2}\left(1-\sqrt{\lambda}r\int_0^\infty\frac{dt}{t^2+1}\sin(\sqrt{\lambda}tr)\right), \tag{A25}$$

where ε is taken to 0 after the integral is evaluated. The dt integral can be evaluated at this point, but it is more convenient to first integrate over dr with the integrand multiplied by $R_{2\ell}^2(r)$. For the $\ell = 1$, the result is

$$\frac{1}{24a^3}\int_0^\infty dr\left(1-\sqrt{\lambda}\int_0^\infty\frac{dt}{t^2+1}r\sin(\sqrt{\lambda}tr)\right)\frac{r^2}{a^2}e^{-r/a} = \frac{1}{24a^3}\left[2+\int_0^\infty\frac{dt}{t^2+1}\frac{24\lambda a^2t(\lambda a^2t^2-1)}{(1+\lambda a^2t^2)^4}\right]. \tag{A26}$$

The integration over dt then gives

$$\frac{1}{12a^2}\left[1-\frac{\lambda a^2(\lambda^3a^6-3\lambda^2a^4+15\lambda a^2-6(\lambda a^2+1)\ln(\lambda a^2)-13)}{(1-\lambda a^2)^4}\right]. \tag{A27}$$

The calculation for $\ell = 0$ is similar. Taking the difference and integrating over λ results in the expression

$$\Delta E_1(e^2\alpha\mu\pi k/4m_1m_2) = \frac{\mu^3\alpha^5}{72\pi m_1m_2}\int_4^\infty \frac{dx}{x}\frac{(1+2/x)\sqrt{1-4/x}}{(1-\beta^2x)^4}[26\beta^6x^3 - 30\beta^4x^2 + 6\beta^2x - 24\beta^4x^2(\beta^2x+1)\ln(\beta\sqrt{x}) - 2] = -0.0030\ \text{meV}, \quad (A28)$$

where $\beta = m_e a$.

The second-order correction for $\mu\alpha^2/2m_1m_2r^2$ is obtained by integrating

$$\Delta E_2(\ell) = -\frac{\mu^3\alpha^4a^3}{2m_1m_2}\int_0^\infty dx x R_{2\ell}(x)e^{-\sqrt{\lambda}ax}\int_0^\infty dx' g_{2\ell}(x,x')R_{2\ell}(x') \quad (A29)$$

over λ. The results are

$$\Delta E_2(P) = 0.00003\ \text{meV}, \qquad \Delta E_2(S) = 0.00154\ \text{meV}, \quad (A30)$$

for a $2p - 2s$ splitting contribution of

$$\Delta E_2(e^2\alpha\mu\pi k/4m_1m_2) = -0.0015\ \text{meV}. \quad (A31)$$

2. Fine structure

a. $-e^2ik_i/4m_1m_2$

The integral to be evaluated is

$$\frac{1}{(2\pi)^3}\int\frac{dk^3k_ie^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+\lambda} = i\frac{x_i}{4\pi r^3}(1+\sqrt{\lambda}r)e^{-\sqrt{\lambda}r}. \quad (A32)$$

This leads to the spin-orbit contributions

$$\frac{\alpha}{2m_1m_2r^3}\left[\left(2+\frac{m_2}{m_1}\right)\vec{L}\cdot\vec{S}_1 + (1+\kappa)\left(2+\frac{(1+2\kappa)m_1}{(1+\kappa)m_2}\right)\vec{L}\cdot\vec{S}_2\right](1+\sqrt{\lambda}r)e^{-\sqrt{\lambda}r}. \quad (A33)$$

Only the p state is affected and we have

$$\langle 2p|\frac{1}{r^3}(1+\sqrt{\lambda}r)e^{-\sqrt{\lambda}r}|2p\rangle = \frac{1}{24a^3}\frac{1+3a\sqrt{\lambda}}{(1+a\sqrt{\lambda})^3}. \quad (A34)$$

For the $\vec{L}\cdot\vec{S}_1$ term, $\langle\vec{L}\cdot\vec{S}_1\rangle$ is $1/2$ for the $P_{3/2}$ states and -1 for the $P_{1/2}$ states. Thus, their (fine structure) contributions are

$$\frac{\mu^3\alpha^5}{144\pi m_1^2}(1+2m_1/m_2)\langle\vec{L}\cdot\vec{S}_1\rangle\int_4^\infty\frac{dx}{x}\frac{(1+2/x)\sqrt{1-4/x}(1+3\beta\sqrt{x})}{(1+\beta\sqrt{x})^3} = \begin{cases}\Delta E_{1\ FS}(P_{3/2}) = 0.001\ \text{meV},\\ \Delta E_{1\ FS}(P_{1/2}) = -0.002\ \text{meV}.\end{cases} \quad (A35)$$

The second-order contribution can be obtained from the expression

$$\Delta E_{2FS}(P) = -\frac{\mu^3\alpha^4}{24m_1^2}(1+2m_1/m_2)\langle\vec{L}\cdot\vec{S}_1\rangle\int_0^\infty dt t^2e^{-t/2}e^{-\sqrt{\lambda}at}\int_0^\infty dt' g_{21}(t,t')e^{-t'/2}, \quad (A36)$$

integrated over λ. The result is

$$-\frac{\mu^3\alpha^5}{72m_1^2}(1+2m_1/m_2)\langle\vec{L}\cdot\vec{S}_1\rangle\int_4^\infty\frac{dx}{x}(1+2/x)\sqrt{1-4/x}H(\beta) = \begin{cases}\Delta E_{2\ FS}(P_{3/2}) = 0.0006\ \text{meV},\\ \Delta E_{2\ FS}(P_{1/2}) = -0.0012\ \text{meV},\end{cases} \quad (A37)$$

with

$$H(\beta\sqrt{x}) = -\frac{3+\beta\sqrt{x}(11+4\beta\sqrt{x}) + 4(1+\beta\sqrt{x})\ln(1+\beta\sqrt{x})}{2(1+\beta\sqrt{x})^5}. \quad (A38)$$

3. p state hyperfine splitting

The $\vec{L}\cdot\vec{S}_2$ term gives corrections to the hyperfine splitting. The values of these corrections are obtained from Eq. (A35) by changing the coefficient of the integral to

$$\frac{\mu^3\alpha^5}{144\pi m_1 m_2}(1+\kappa)\left(2+\frac{(1+2\kappa)}{(1+\kappa)}\frac{m_1}{m_2}\right)\langle\vec{L}\cdot\vec{S}_2\rangle \tag{A39}$$

and using

$$\langle{}^5P_{3/2}|\vec{L}\cdot\vec{S}_2|{}^5P_{3/2}\rangle=\frac{1}{2},\quad \langle{}^3P_{3/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{3/2}\rangle=-\frac{5}{6}, \tag{A40a}$$

$$\langle{}^3P_{1/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{1/2}\rangle=\frac{1}{3},\quad \langle{}^1P_{1/2}|\vec{L}\cdot\vec{S}_2|{}^1P_{1/2}\rangle=-1, \tag{A40b}$$

$$\langle{}^3P_{3/2}|\vec{L}\cdot\vec{S}_2|{}^3P_{1/2}\rangle=-\sqrt{2}/3. \tag{A40c}$$

The corresponding energy corrections are

$$\Delta E_{1\,HF}({}^5P_{3/2})=0.0006\ \text{meV},\quad \Delta E_{1\,HF}({}^3P_{3/2})=-0.0009\ \text{meV}, \tag{A41a}$$

$$\Delta E_{1\,HF}({}^3P_{1/2})=0.0004\ \text{meV},\quad \Delta E_{1\,HF}({}^1P_{1/2})=-0.0011\ \text{meV}, \tag{A41b}$$

$$\Delta E_{1\,HF}(Mix)=-0.0005\ \text{meV}. \tag{A41c}$$

The second-order perturbative hyperfine corrections can be obtained from Eq. (A37) by replacing its coefficient with

$$-\frac{\mu^3\alpha^5(1+\kappa)}{72m_1m_2}\left(2+\frac{(1+2\kappa)}{(1+\kappa)}\right)\langle\vec{L}\cdot\vec{S}_2\rangle. \tag{A42}$$

The results are

$$\Delta E_{2\,HF}({}^5P_{3/2})=0.0004\ \text{meV},\quad \Delta E_{2\,HF}({}^3P_{3/2})=-0.0006\ \text{meV}, \tag{A43a}$$

$$\Delta E_{2\,HF}({}^3P_{1/2})=0.0002\ \text{meV},\quad \Delta E_{2\,HF}({}^1P_{1/2})=-0.0007\ \text{meV}, \tag{A43b}$$

$$\Delta E_{2\,HF}(Mix)=-0.0003\ \text{meV}. \tag{A43c}$$

4. Hyperfine splitting

a. $e^2(1+\kappa)\vec{k}^2\sigma_1\cdot\sigma_2/6m_1m_2$

This term is very similar to the spin-independent $\vec{k}^2$ contribution treated above. The spin dependence means that the s states and p states must be treated separately. For the s state, we have

$$\Delta VS_{HF}(r)=\frac{\alpha(1+\kappa)}{6m_1m_2}\int_{4m_e^2}^{\infty}\frac{d\lambda}{\lambda}\Pi_e(\lambda)\left[4\pi\delta(\vec{r})-\frac{\lambda}{r}e^{-\sqrt{\lambda}r}\right]\sigma_1\cdot\sigma_2. \tag{A44}$$

The delta function gives

$$\frac{\mu^3\alpha^4(1+\kappa)}{12m_1m_2}\sigma_1\cdot\sigma_2\int_{4m_e^2}^{\infty}\frac{d\lambda}{\lambda}\Pi_e(\lambda), \tag{A45}$$

and the remaining term gives

$$-\frac{\mu^3\alpha^4(1+\kappa)}{12m_1m_2}\sigma_1\cdot\sigma_2\int_{4m_e^2}^{\infty}\frac{d\lambda}{\lambda}\Pi_e(\lambda)\left[\frac{a^4\lambda^2+a^2\lambda/2}{(1+a\sqrt{\lambda})^4}\right]. \tag{A46}$$

The energy corrections are

$$\Delta E_{1\,HF}(S)=\frac{\mu^3\alpha^5(1+\kappa)}{9\pi m_1m_2}\langle\vec{S}_1\cdot\vec{S}_2\rangle\int_4^{\infty}\frac{dx}{x}\frac{(1+2/x)\sqrt{1-4/x}(1+4\beta\sqrt{x}+11\beta^2x/2+4\beta^3x^{3/2})}{(1+\beta\sqrt{x})^4}$$
$$=\begin{cases}\Delta E_{1\,HF}({}^3S_{1/2})=0.0121\ \text{meV},\\ \Delta E_{1\,HF}({}^1S_{1/2})=-0.0362\ \text{meV}.\end{cases} \tag{A47}$$

The second-order contribution is given by

$$E_{2\,HF}(S)=-\frac{2\mu^3\alpha^4(1+\kappa)}{3m_1m_2}\langle\vec{S}_1\cdot\vec{S}_2\rangle\frac{\alpha}{3\pi}\int_4^{\infty}\frac{dx}{x}(1+2x)\sqrt{1-4/x}F(\beta\sqrt{x})$$
$$=\begin{cases}\Delta E_{2\,HF}({}^3S_{1/2})=0.01859\ \text{meV},\\ \Delta E_{2\,HF}({}^1S_{1/2})=-0.05579\ \text{meV},\end{cases} \tag{A48}$$

where $F(\beta\sqrt{x})$ is given by Eq. (A11). (See [4].)

For the p state,

$$\Delta E_{HF}(P) = -\frac{\mu^3\alpha^4(1+\kappa)}{6m_1m_2}\int_4^\infty \frac{dx}{x}\frac{\Pi_e(x)\beta^2 x}{(1+\beta\sqrt{x})^4}\langle\vec{S}_1\cdot\vec{S}_2\rangle. \tag{A49}$$

Using the expectation values

$$\langle {}^5P_{3/2}|\vec{S}_1\cdot\vec{S}_2|{}^5P_{3/2}\rangle = \frac{1}{4},$$
$$\langle {}^3P_{3/2}|\vec{S}_1\cdot\vec{S}_2|{}^3P_{3/2}\rangle = -\frac{5}{12}, \tag{A50a}$$
$$\langle {}^3P_{1/2}|\vec{S}_1\cdot\vec{S}_2|{}^3P_{1/2}\rangle = -\frac{1}{12},$$
$$\langle {}^1P_{1/2}|\vec{S}_1\cdot\vec{S}_2|{}^1P_{1/2}\rangle = \frac{1}{4}, \tag{A50b}$$

the corresponding energy corrections are

$$\Delta E_{HFP}({}^5P_{3/2}) = 0.0002\ \text{meV},$$
$$\Delta E_{HFP}({}^3P_{3/2}) = -0.0004\ \text{meV}, \tag{A51a}$$
$$\Delta E_{HFP}({}^3P_{1/2}) = -0.0001\ \text{meV},$$
$$\Delta E_{HFP}({}^1P_{1/2}) = 0.0002\ \text{meV}. \tag{A51b}$$

5. Tensor splitting

a. $e^2(1+\kappa)(k_ik_j-\delta_{ij}\vec{k}^2/3)/4m_1m_2$

The relevant integral in this case is

$$\frac{1}{(2\pi)^3}\int\frac{dk^3(k_ik_j-\vec{k}^2\delta_{ij}/3)e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+\lambda} = \frac{1}{4\pi r^3}(\delta_{ij}-3\hat{x}_i\hat{x}_j)\left(1+\sqrt{\lambda}r+\frac{\lambda r^2}{3}\right)e^{-\sqrt{\lambda}r}. \tag{A52}$$

The expression to be integrated over λ is

$$\langle 2P|\Delta V_T(\vec{r})|2P\rangle = \frac{\alpha(1+\kappa)}{m_1m_2}\left\langle\frac{1}{r^3}\left(1+\sqrt{\lambda}r+\frac{\lambda r^2}{3}\right)e^{-\sqrt{\lambda}r}\right\rangle \times\langle 3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2\rangle. \tag{A53}$$

Now,

$$\left\langle\frac{1}{r^3}\left(1+\sqrt{\lambda}r+\frac{\lambda r^2}{3}\right)e^{-\sqrt{\lambda}r}\right\rangle = \frac{1}{24a^3}\frac{(1+4a\sqrt{\lambda}+5a^2\lambda)}{(1+a\sqrt{\lambda})^4}, \tag{A54}$$

so

$$\langle 2P|\Delta V_T|2P\rangle = \frac{\mu^3\alpha^5(1+\kappa)}{72\pi m_1m_2}\int_4^\infty\frac{dx}{x}\frac{(1+2/x)\sqrt{1-4/x}(1+4\beta\sqrt{x}+5\beta^2x)}{(1+\beta\sqrt{x})^4}\langle 3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2\rangle. \tag{A55}$$

Using

$$\langle {}^5P_{3/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^5P_{3/2}\rangle = -\frac{1}{5},$$
$$\langle {}^3P_{3/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{3/2}\rangle = \frac{1}{6}, \tag{A56a}$$

$$\langle {}^3P_{1/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{1/2}\rangle = \frac{1}{3},$$
$$\langle {}^1P_{1/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^1P_{1/2}\rangle = -2, \tag{A56b}$$

$$\langle {}^3P_{3/2}|3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2|{}^3P_{1/2}\rangle = \sqrt{2}/6, \tag{A56c}$$

the corrections are

$$\Delta E_{1\,T}({}^5P_{3/2}) = -0.0003\ \text{meV},$$
$$\Delta E_{1\,T}({}^3P_{3/2}) = 0.0003\ \text{meV}, \tag{A57a}$$

$$\Delta E_{1\,T}({}^3P_{1/2}) = 0.0005\ \text{meV},$$
$$\Delta E_{1\,T}({}^1P_{1/2}) = -0.0030\ \text{meV}, \tag{A57b}$$

$$\Delta E_{1\,T}(Mix) = 0.00035\ \text{meV}. \tag{A57c}$$

The perturbative second-order corrections are obtained using Eq. (A38) and

$$\Delta E_T = -\frac{\mu^3\alpha^5(1+\kappa)}{36\pi m_1m_2}\int_4^\infty\frac{dx}{x}(1+2/x)\sqrt{1-4/x}H(\beta\sqrt{x}) \times\langle 3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\vec{S}_1\cdot\vec{S}_2\rangle. \tag{A58}$$

The results are

$$\Delta E_{2\,T}({}^5P_{3/2}) = -0.0001\ \text{meV},$$
$$\Delta E_{2\,T}({}^3P_{3/2}) = 0.0001\ \text{meV}, \tag{A59a}$$

$$\Delta E_{2\,T}({}^3P_{1/2}) = 0.0002\ \text{meV},$$
$$\Delta E_{2\,T}({}^1P_{1/2}) = -0.0013\ \text{meV}, \tag{A59b}$$

$$\Delta E_{1\,T}(Mix) = 0.00015\ \text{meV}. \tag{A59c}$$

APPENDIX B: ONE-LOOP EFFECTIVE POTENTIAL

To obtain the full one-loop effective potential V_4, one has to evaluate the one-loop corrections to the single photon exchange potential V_2 and calculate

$$V_4(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot\vec{r}}\left(\sum_i V_{4i}(\vec{k}) - \delta(V_2, V_2)\right), \quad \text{(B1)}$$

where i is the sum over all one-loop diagrams and

$$\delta(V_2, V_2) = \frac{1}{(2\pi)^3}\int d^3p'' \frac{V_2(\vec{p}', \vec{p}'')V_2(\vec{p}'', \vec{p})}{E_1(\vec{p}) + E_2(\vec{p}) - E_1(\vec{p}'') - E_2(\vec{p}'')}. \quad \text{(B2)}$$

The subtraction is necessary to prevent double counting of the Coulomb exchange in the box diagram.

Using our formulation, the α^2/r^2 term in Eq. (57) arises from the $\delta(V_2, V_2)$ subtraction term in Eq. (B1). In momentum space this term behaves as $|\vec{k}|^{-1}$. As can be seen in Eqs. (2.3)–(2.7) of Ref. [14], the only term of this type that survives is the one in $\delta(V_2, V_2)$, with $V_2(\vec{p}', \vec{p})$ given by

$$V_2(\vec{p}', \vec{p}) = \frac{-e^2}{(\vec{p}' - \vec{p})^2 + \lambda^2}\left(1 - \frac{(\vec{p}' - \vec{p})^2}{8\mu^2} + \frac{\vec{p}^2}{m_1 m_2} + \cdots\right), \quad \text{(B3)}$$

where the dots denote spin-dependent terms that are not relevant.

If one calculates the one-loop effective potential using the Breit-Pauli equation as a starting point, the corresponding subtraction term will involve a $V_2(\vec{p}', \vec{p})$ of the form

$$V_2(\vec{p}', \vec{p}) = \frac{-e^2}{(\vec{p}' - \vec{p})^2 + \lambda^2}\left(1 - \frac{1}{8}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)(\vec{p}' - \vec{p})^2 + \frac{\vec{p}^2}{m_1 m_2} + \frac{\vec{p}\cdot(\vec{p}' - \vec{p})\vec{p}\cdot(\vec{p}' - \vec{p})}{m_1 m_2 (\vec{p}' - \vec{p})^2} + \cdots\right). \quad \text{(B4)}$$

In this case, the last term in Eq. (B4) exactly cancels the coefficient of the $|\vec{k}|^{-1}$ term produced by the second and third terms when $\delta(V_2, V_2)$ is evaluated. Consequently, there is no α^2/r^2 term in the one-loop corrections to the Breit-Pauli potential.

[1] R. Pohl *et al.*, Nature (London) **466**, 213 (2010).
[2] A. Antognini *et al.*, Science **339**, 417 (2013).
[3] E. Borie and G. A. Rinker, Rev. Mod. Phys. **54**, 67 (1982).
[4] E. Borie, Phys. Rev. A **71**, 032508 (2005).
[5] K. Pachucki, Phys. Rev. A **53**, 2092 (1996).
[6] K. Pachucki, Phys. Rev. A **60**, 3593 (1999).
[7] R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, Annu. Rev. Nucl. Part. Sci. **63**, 175 (2013).
[8] A. Antognini, F. Kottmann, F. Biraben, P. Indelicato, F. Nez, and R. Pohl, Ann. Phys. (Amsterdam) **331**, 127 (2013).
[9] C. Peset and A. Pineda, Eur. Phys. J. A **51**, 156 (2015).
[10] P. J. Mohr, B. N. Taylor, and D. B. Newell, Rev. Mod. Phys. **84**, 1527 (2012), and Ref. [9].
[11] https://www.jlab.org/prad/collaboration.html.
[12] A. Beyer *et al.*, Science **358**, 79 (2017).
[13] H. Fleurbaey, S. Galtier, S. Thomas, M. Bonnaud, L. Julien, F. Biraben, F. Nez, M. Abgrall, and J. Guéna, Phys. Rev. Lett. **120**, 183001 (2018).
[14] S. N. Gupta, W. W. Repko, and C. J. Suchyta III, Phys. Rev. D **40**, 4100 (1989).
[15] See the related work: S. N. Gupta, Nucl. Phys. **57**, 19 (1964); S. N. Gupta and S. F. Radford, Phys. Rev. D **21**, 2213 (1980).
[16] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).
[17] G. Källen, Helv. Phys. Acta **25**, 417 (1952).
[18] H. Lehmann, Nuovo Cimento **11**, 342 (1954).
[19] G. Källen and A. Sabry, Dan. Mat. Fys. Medd. **29**, 1 (1955).
[20] J. Blomqvist, Nucl. Phys. **B48**, 95 (1972).
[21] T. Kinoshita and M. Nio, Phys. Rev. Lett. **82**, 3240 (1999).
[22] L. Hostler and R. H. Pratt, Phys. Rev. Lett. **10**, 469 (1963).
[23] L. Hostler, Phys. Rev. **178**, 126 (1969); J. Math. Phys. **16**, 1585 (1975).
[24] B. J. Laurenzi and A. Flamberg, Int. J. Quantum Chem. **XI**, 869 (1971).
[25] B. R. Johnson and J. O. Hirschfelder, J. Math. Phys. **20**, 2484 (1979).
[26] V. G. Ivanov, E. Y. Korzinin, and S. G. Karshenboim, Phys. Rev. D **80**, 027702 (2009).
[27] G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) **35**, 271 (1965).
[28] M. E. Rose, *Relativistic Electron Theory* (Wiley, New York, 1961), p. 178.
[29] U. D. Jentschura, Phys. Rev. A **84**, 012505 (2011).