

## Existence and construction of Galilean invariant $z \neq 2$ theories

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We prove a no-go theorem for the construction of a Galilean boost invariant and  $z \neq 2$  anisotropic scale invariant field theory with a finite dimensional basis of fields. Two point correlators in such theories, we show, grow unboundedly with spatial separation. Correlators of theories with an infinite dimensional basis of fields, for example, labeled by a continuous parameter, do not necessarily exhibit this bad behavior. Hence, such theories behave effectively as if in one extra dimension. Embedding the symmetry algebra into the conformal algebra of one higher dimension also reveals the existence of an internal continuous parameter. Consideration of isometries shows that the nonrelativistic holographic picture assumes a canonical form, where the bulk gravitational theory lives in a space-time with one extra dimension. This can be contrasted with the original proposal by Balasubramanian and McGreevy, and by Son, where the metric of a  $(d + 2)$ -dimensional space-time is proposed to be dual of a  $d$ -dimensional field theory. We provide explicit examples of theories living at fixed point with anisotropic scaling exponent  $z = \frac{2\ell}{\ell+1}$ ,  $\ell \in \mathbb{Z}$ .

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### I. INTRODUCTION

Gravity duals of nonrelativistic field theories have been proposed in [1,2]. It has been observed in Ref. [1] that one can consistently define an algebra with Galilean boost invariance and arbitrary anisotropic scaling exponent  $z$ . While the metric having isometry of this generalized Schrödinger group has been used with the holographic dictionary to construct correlators of a putative field theory [3–9], there is no explicit field theoretic realization of such a symmetry for  $z \neq 2$ .<sup>1</sup> One surprising feature, noted as a “strange aspect” in Ref. [1], is that, unlike in the canonical AdS/CFT correspondence, where the conformal field theory (CFT) in  $d$  dimensions is dual to the gravity in  $(d + 1)$  dimensions, in the nonrelativistic case the metric is of a space-time with two additional dimensions. The  $(d + 2)$ -dimensional metric, having isometries of the  $d$ -dimensional generalized Schrödinger group, is given by [1,2]

$$ds^2 = L^2 \left[ -\frac{dt^2}{r^{2z}} + \frac{2d\xi dt + dx^2}{r^2} + \frac{dr^2}{r^2} \right], \quad (1)$$

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<sup>1</sup>We note that just matching the isometries is necessary but not sufficient for the existence of a holographic description. Here we just seek a group invariant field theory, which may or may not have a gravity dual.

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where  $\xi$  is the extra dimension having no analogous appearance in the relativistic AdS-CFT correspondence. The metric is invariant under the required anisotropic scaling symmetry

$$x_i \rightarrow \lambda x_i, \quad t \rightarrow \lambda^z t, \quad r \rightarrow \lambda r, \quad \xi \rightarrow \lambda^{2-z} \xi, \quad (2)$$

and under Galilean boosts

$$x_i \rightarrow x_i + v_i t, \quad \xi \rightarrow \xi - \frac{1}{2}(2v_i x_i + v^2 t). \quad (3)$$

For  $z = 2$ , an explicit construction of Galilean boost invariant field theory in  $(d - 1) + 1$  dimensions has been known. Thus a question arises naturally as to whether one can get rid of the extra  $\xi$  direction and reduce the correspondence to a canonical correspondence between a  $d$ -dimensional quantum field theory on flat space and a  $(d + 1)$ -dimensional gravitational theory. This was answered positively in Ref. [10]. But for  $z \neq 2$  we do not know of any explicit  $d$ -dimensional field theoretic example having the generalized Schrödinger symmetry, nor do we know an example of a  $(d + 1)$ -dimensional metric having the same set of isometries. Thus the strange aspect of  $d$ - $(d + 2)$  correspondence appears to persist for  $z \neq 2$ .

In this paper, we initiate a field theoretic study of  $z \neq 2$  theories.<sup>2</sup> We prove a no-go theorem for the construction of a space-time translation invariant, rotation invariant,

<sup>2</sup>Theories with  $z = 2$  have been studied from a field theoretic point of view in many works; see, e.g., Refs. [11–14]. The  $z = \infty$  case without particle number symmetry has been explored in Refs. [15,16].

Galilean boost invariant,<sup>3</sup> and  $z \neq 2$  anisotropic scale invariant field theory with a finite number<sup>4</sup> of fields in  $d$  dimensions. Two point correlators in such theories, we show, grow unboundedly with spatial separation. By contrast, correlators of theories with an infinite number of fields, e.g., labeled by a continuous parameter, do not necessarily exhibit this bad behavior. Hence, such theories behave effectively as a  $(d + 1)$ -dimensional theory. In the context of holography, this explains the strange aspect; the  $z \neq 2$  theories indeed provide us with the possibility of a canonical realization of holography; i.e., a  $(d + 1)$ -dimensional theory is dual to a  $(d + 2)$ -dimensional geometry. The  $z = 2$  case is special in that respect since it is possible to obtain a  $d$ -dimensional theory with a finite number of fields such that the symmetries on the field theory side match the isometries of a  $(d + 2)$ -dimensional geometry. The special role of  $z = 2$  has been emphasized in the context of the holographic dictionary in Refs. [8,9]. For  $z = 2$ , the dual space-time can be made into a  $(d + 1)$ -dimensional one via Kaluza-Klein reduction of the  $(d + 2)$ -dimensional metric [10]. This is possible since, for  $z = 2$ , the extra direction  $\xi$  does not scale by the transformations given in Eq. (2). The scaling of  $\xi$  given in Eq. (2) can be verified on the field theory side of the duality by embedding the  $d$ -dimensional generalized Schrödinger group into the conformal group of one higher dimension, i.e.,  $SO(d, 2)$ . By contrast, since for  $z \neq 2$  the  $\xi$  direction does scales, any attempt to compactify the extra direction  $\xi$  is at odds with the continuous scaling symmetry. The no-go theorem that we have proved is consistent with the argument in Ref. [17], based on the consistency of the thermodynamic equation of state, that a perfect fluid with  $z \neq 2$  Schrödinger symmetry and discrete spectrum for the energy and particle number,  $H$  and  $N$ , cannot exist. In Sec. IV, we present some fixed point theories with  $z = \frac{2\ell}{\ell+1}$ , with  $\ell \in \mathbb{Z}$ .

Before delving into a technical proof, we present a physical argument for our main result.<sup>5</sup> Consider a theory invariant under  $z = 2$  Schrödinger symmetry, where, under a boost [1],

$$\phi(\mathbf{x}, t) \mapsto \exp \left[ -in \left( \frac{1}{2} v^2 t + \mathbf{v} \cdot \mathbf{x} \right) \right] \phi(\mathbf{x} - \mathbf{v}t, t), \quad (4)$$

where  $[N, \phi] = n\phi$ . In turn, the state of a particle with momentum  $\mathbf{k} = 0$ , i.e.,  $\phi_{\mathbf{k}=0}^\dagger |0\rangle$ , transforms under the boost by  $\mathbf{v}$  as follows:

$$\begin{aligned} |\mathbf{v}\rangle &\equiv e^{-iK \cdot \mathbf{v}} \phi_{\mathbf{k}=0}^\dagger |0\rangle \\ &= \int d\mathbf{x} \exp \left[ in \left( \frac{1}{2} v^2 t + \mathbf{v} \cdot \mathbf{x} \right) \right] \phi^\dagger(\mathbf{x} - \mathbf{v}t, t) |0\rangle \\ &= \exp \left[ i \frac{nv^2}{2} t \right] \phi_{\mathbf{k}=n\mathbf{v}}^\dagger |0\rangle. \end{aligned} \quad (5)$$

This has the interpretation of having a boosted particle moving with momentum  $n\mathbf{v}$  and kinetic energy  $-\frac{1}{2}nv^2$ . A positive value of  $n$  results in decreasing energy with increasing boost. Therefore, negative semidefiniteness of  $n$  is required for stability. In case of more than a single species of particle, the matrix  $\mathbb{N}$  appearing in  $[N, \Phi^\dagger] = -\mathbb{N}\Phi^\dagger$  has to be negative semidefinite. As we will see, from the symmetry algebra it follows that for a theory with a finite number of fields with  $z \neq 2$  the trace of  $\mathbb{N}$  must vanish, spoiling the negative semidefiniteness and the stability in the sense discussed above; by contrast, for  $z = 2$  there is no constraint on the trace of  $\mathbb{N}$ . The above is merely a heuristic argument, giving intuition behind the technical result presented below.

## II. GENERALIZED SCHRÖDINGER ALGEBRA AND ITS REPRESENTATION

The Galilean algebra consists of generators corresponding to spatial translations,  $P_i$ , time translation,  $H$ , Galilean boosts,  $K_i$ , rotations,  $M_{ij}$ , along with a particle number generator,  $N$ , such that they satisfy the following commutation relations [11–14]:

$$\begin{aligned} [M_{ij}, N] &= [P_i, N] = [K_i, N] = [H, N] = 0, \\ [M_{ij}, P_k] &= i(\delta_{ik}P_j - \delta_{jk}P_i), \\ [M_{ij}, K_k] &= i(\delta_{ik}K_j - \delta_{jk}K_i), \\ [M_{ij}, M_{kl}] &= i(\delta_{ik}M_{jl} - \delta_{jk}M_{il} + \delta_{il}M_{kj} - \delta_{jl}M_{ki}), \\ [P_i, P_j] &= [K_i, K_j] = 0, \quad [K_i, P_j] = i\delta_{ij}N, \\ [H, N] &= [H, P_i] = [H, M_{ij}] = 0, \\ [H, K_i] &= -iP_i. \end{aligned} \quad (6)$$

The algebra can be enhanced by appending a dilatation generator  $D$ ,<sup>6</sup> which scales space and time separately, in the following way:

$$x_i \rightarrow \lambda x_i, \quad t \rightarrow \lambda^z t. \quad (7)$$

The commutators of  $D$  with the rest of the generators are given by

<sup>6</sup>This enhanced algebra corresponds to that of deformed ISIM(2) [18], with the following identification:  $H \mapsto P_+$ ,  $N \mapsto P_-$ ,  $K_i \mapsto M_{+i}$ , and  $D \mapsto -\frac{1}{b}N$  where  $b(z-1) = 1$ .

<sup>3</sup>Here by Galilean boost invariance, we mean invariance under both the boost and  $U(1)$  particle number symmetries. The  $U(1)$  naturally arises as a commutator of generators of boosts and translations.

<sup>4</sup>More precisely, a finite-dimensional basis of operators as defined below Eq. (12).

<sup>5</sup>We thank John McGreevy for discussions leading to this argument.

$$\begin{aligned} [D, P_i] &= \iota P_i, & [D, K_i] &= (1-z)\iota K_i, & [D, H] &= z\iota H, \\ [D, N] &= \iota(2-z)N, & [M_{ij}, D] &= 0. \end{aligned} \quad (8)$$

The physical interpretation of  $N$  is subtle. For  $z = 2$  it is usually thought of as a particle number symmetry generator. The subtlety in the context of holography has been explored in [10]. For the rest of this work, we take an agnostic viewpoint and treat  $N$  as a generator of symmetry without specifying its physical origin. This will enable us to explore all the possibilities, as allowed by symmetries. The case  $z = 2$  is very special in that one can append an additional generator  $C$  of special conformal transformations. Thus one can have the full Schrödinger algebra for  $z = 2$  [13,14,19–23]. When  $z \neq 2$ , the generator corresponding to special conformal transformation is not available.

In what follows, we will assume (unless otherwise specified) that the field theory lives in  $d = (d-1) + 1$  dimensions and that the vacuum is invariant under Galilean boosts, i.e.,  $K_i|0\rangle = \langle 0|K_i = 0$ .

The field representation is built by defining local operators  $\Phi$  such that  $H$  and  $P$  act canonically,

$$[H, \Phi] = -\iota\partial_t\Phi, \quad [P_i, \Phi] = \iota\partial_i\Phi. \quad (9)$$

We consider representations of the little group, generated by  $D, K_i, N$ , and  $M_{ij}$ , that keeps the origin,  $(\mathbf{0}, 0)$ , invariant. The fields  $\Phi$  have definite transformation properties under  $D, K_i$ , and  $N$ ,<sup>7</sup>

$$[D, \Phi(\mathbf{x} = \mathbf{0}, t = 0)] = \iota\mathcal{D}\Phi(\mathbf{x} = \mathbf{0}, t = 0), \quad (10)$$

$$[N, \Phi(\mathbf{x} = \mathbf{0}, t = 0)] = \mathcal{N}\Phi(\mathbf{x} = \mathbf{0}, t = 0), \quad (11)$$

$$[K_i, \Phi(\mathbf{x} = \mathbf{0}, t = 0)] = \mathcal{K}_i\Phi(\mathbf{x} = \mathbf{0}, t = 0), \quad (12)$$

where  $\mathcal{D}$ ,  $\mathcal{N}$ , and  $\mathcal{K}_i$  are linear operators. We refer to the smallest nontrivial irreducible representation in Eqs. (10)–(12) as “the basis of operators.”<sup>8</sup> For Lagrangian theories the basis of operators corresponds to the elementary fields from which the Lagrangian is constructed. Henceforth, we restrict our attention to the basis of operators, and we continue to denote by  $\mathcal{D}$ ,  $\mathcal{N}$ , and  $\mathcal{K}_i$  their linear representations. In the finite dimensional case, we denote these by finite dimensional matrices  $\mathbf{\Delta}$ ,  $\mathbb{N}$ , and  $\mathbb{K}_i$ , respectively.

<sup>7</sup>The fields  $\Phi$  also have definite transformation properties under  $M_{ij}$ , but this will not play a role in the discussion below.

<sup>8</sup>For example, the free Schrödinger field theory is invariant under  $z = 2$  Schrödinger algebra and the single field  $\phi$  forms a one-dimensional irreducible representation of the little group, i.e.,  $[D, \phi(\mathbf{0}, 0)] = \iota\frac{d}{2}\phi(\mathbf{0}, 0)$ ,  $[N, \phi(\mathbf{0}, 0)] = N\phi(\mathbf{0}, 0)$ , and  $[K_i, \phi(\mathbf{0}, 0)] = 0$ .

Consider  $G_{\alpha\beta} \equiv \langle 0|\Phi_\alpha(x, t)\Phi_\beta(0, 0)|0\rangle$ . Using Eqs. (9), the commutator in (12) translates, in the finite dimensional case, to

$$[K_i, \Phi] = (-\iota t\partial_i\mathbf{I} + x_i\mathbb{N} + \mathbb{K}_i)\Phi, \quad (13)$$

where  $x_i = x^i$ . Galilean boost invariance of the vacuum,  $K_i|0\rangle = \langle 0|K_i = 0$ , then gives

$$\begin{aligned} \langle 0|[K_i, \Phi_\alpha(x, t)\Phi_\beta(0, 0)]|0\rangle &= 0 \\ \Rightarrow (-\iota t\partial_i\delta_{\alpha\sigma} + x_i\mathbb{N}_{\alpha\sigma} + \mathbb{K}_{i\alpha\sigma})G_{\sigma\beta} + \mathbb{K}_{i\beta\sigma}G_{\alpha\sigma} &= 0. \end{aligned}$$

Using the fact that  $[\mathbb{N}, \mathbb{K}_i] = 0$ , the solution to the above differential equation is given by

$$G_{\alpha\beta} = \left( e^{-\iota\frac{|\mathbf{x}|^2}{2t}\mathbb{N}} e^{-\iota\frac{\mathbf{x}\cdot\mathbf{K}}{t}} C(t) e^{-\iota\frac{\mathbf{x}\cdot\mathbf{K}}{t}} \right)_{\alpha\beta},$$

where  $C(t)$  is an as yet undetermined matrix function of  $t$ . The norm is defined as  $|\mathbf{x}|^2 \equiv \sum_i (x^i)^2$  while the dot product is defined as  $\mathbf{x} \cdot \mathbf{K} = \sum_i x_i K_i$ . Similarly, one can consider  $G'_{\alpha\beta} \equiv \langle 0|\Phi_\alpha(x, t)\Phi_\beta^\dagger(0, 0)|0\rangle$  which is given by

$$G'_{\alpha\beta} = \left( e^{-\iota\frac{|\mathbf{x}|^2}{2t}\mathbb{N}} e^{-\iota\frac{\mathbf{x}\cdot\mathbf{K}}{t}} C'(t) e^{\iota\frac{\mathbf{x}\cdot\mathbf{K}}{t}} \right)_{\alpha\beta},$$

where  $C'(t)$  is as yet undetermined.

Since  $[D, N] = \iota(2-z)N$ , we have

$$[\mathbf{\Delta}, \mathbb{N}] = (2-z)\mathbb{N},$$

and this leads to  $\text{Tr}(\mathbb{N}) = 0$ ; similarly, for  $z \neq 1$  we have  $\text{Tr}(\mathbb{K}_i) = 0$ . Now using Jordan-Chevalley decomposition, we can write

$$\begin{aligned} \mathbb{N} &= \mathbb{N}_1 + \mathbb{N}_2, & [\mathbb{N}_1, \mathbb{N}_2] &= 0, \\ \mathbb{K} &= \mathbb{K}_1 + \mathbb{K}_2, & [\mathbb{K}_1, \mathbb{K}_2] &= 0, \end{aligned}$$

where  $\mathbb{K}_1$  and  $\mathbb{N}_1$  are diagonalizable matrices while  $\mathbb{N}_2$  and  $\mathbb{K}_2$  are nilpotent matrices (here and below we suppress the vector index in  $\mathbb{K}$  and  $\mathbb{K}_{1,2}$  to avoid clutter). Let us define diagonal matrices  $D_{N_1}$  and  $D_{K_1}$  such that

$$P_N D_{N_1} P_N^{-1} = \mathbb{N}_1, \quad P_K D_{K_1} P_K^{-1} = \mathbb{K}_1, \quad (14)$$

where  $P_N$  and  $P_K$  diagonalize  $\mathbb{N}_1$  and  $\mathbb{K}_1$ , respectively. The zero trace condition leads to  $\text{Tr}(\mathbb{K}_1) = \text{Tr}(D_{K_1}) = 0$  and  $\text{Tr}(\mathbb{N}_1) = \text{Tr}(D_{N_1}) = 0$ , which in turn implies that either all the diagonal entries of  $D_{N_1}$  (or  $D_{K_1}$ ) are zero, in which case  $\mathbb{N}$  (or  $\mathbb{K}$ ) is a nilpotent matrix, or there has to be both positive and negative entries. We can then recast the correlators as follows:

$$G_{\alpha\beta} = \left( e^{\frac{|x|^2}{2t}\mathbb{N}_1} e^{\frac{|x|^2}{2t}\mathbb{N}_2} e^{\frac{x\cdot\mathbb{K}_1}{t}} e^{\frac{x\cdot\mathbb{K}_2}{t}} C(t) e^{\frac{x\cdot\mathbb{K}_1^T}{t}} e^{\frac{x\cdot\mathbb{K}_2^T}{t}} \right)_{\alpha\beta}, \quad (15)$$

$$G'_{\alpha\beta} = \left( e^{\frac{|x|^2}{2t}\mathbb{N}_1} e^{\frac{|x|^2}{2t}\mathbb{N}_2} e^{\frac{x\cdot\mathbb{K}_1}{t}} e^{\frac{x\cdot\mathbb{K}_2}{t}} C'(t) e^{-\frac{x\cdot\mathbb{K}_1^\dagger}{t}} e^{-\frac{x\cdot\mathbb{K}_2^\dagger}{t}} \right)_{\alpha\beta}. \quad (16)$$

It follows that when  $\mathbb{N}_1 \neq 0$ ,  $e^{\frac{|x|^2}{2t}\mathbb{N}_1} = P_N e^{\frac{|x|^2}{2t}D_{N_1}} P_N^{-1}$  has exponential growth for imaginary time irrespective of how we do the analytical continuation of the correlator to imaginary time. This growth cannot be overcome by any of the other terms as nilpotency of  $\mathbb{N}_2$  guarantees that

$$e^{-t\frac{|x|^2}{2t}\mathbb{N}_2} = \sum_{\ell=0}^{\ell=M-1} \left( -t\frac{|x|^2}{2t} \right)^\ell \mathbb{N}_2^\ell, \quad (17)$$

where  $\mathbb{N}_2^M = 0$  for some integer  $M$ . Also, terms such as  $e^{\frac{x\cdot\mathbb{K}}{t}}$  cannot suppress the exponential growth arising from  $\mathbb{N}_1$ .

If instead  $\mathbb{N}_1 = 0$ , then  $e^{-t\frac{|x|^2}{2t}\mathbb{N}_2}$  gives polynomial growth with  $x$ . We employ the same technique to establish the effect of  $e^{\frac{x\cdot\mathbb{K}}{t}}$ . If  $\mathbb{K}_1 \neq 0$ , there will be exponential growth for some entries, while terms involving  $\mathbb{K}_2$  are polynomial in nature, giving exponential growth as a whole. Alternatively, if  $\mathbb{K}_1 = 0$ , then  $\mathbb{K}$  is nilpotent and we have polynomial growth.

We note that only when  $z = 2$  or the representation is infinite, we cannot implement the  $\text{Tr}(\mathbb{N}) = 0$  condition and the above argument fails. This is expected for  $z = 2$  since the two point correlator is well behaved in this case, which corresponds to the Schrödinger field theory [19–22]. We conclude that in the finite dimensional case for  $z \neq 2$  a quantum field theory with the symmetry of the algebra in Eqs. (6) and (8) is ill-behaved. For example, since correlators grow with spatial separation cluster decomposition fails. The same conclusion can be drawn via an independent argument in the case that  $\Delta$  is diagonal; see Appendix.

Therefore, for  $z \neq 2$  we are left to consider infinite dimensional representations. In this case we can display explicitly an example that does not obviously lead to problematic quantum field theories. To achieve this, we introduce fields  $\psi$  labeled by a new noncompact variable  $\xi$ , such that

$$[N, \psi] = t\partial_\xi \psi, \quad (18)$$

$$[D, \psi] = t(zt\partial_t + x^i\partial_i + (2-z)\xi\partial_\xi + \Delta_\psi)\psi, \quad (19)$$

$$[K_i, \psi] = (-it\partial_i + ix_i\partial_\xi)\psi. \quad (20)$$

Thus,  $D = (2-z)\xi\partial_\xi + \Delta_\psi$ ,  $\mathcal{N} = t\partial_\xi$ , and  $\mathcal{K}_i = 0$ . Note that  $\xi$  must be a noncompact variable, or else scaling symmetry is broken. To be concrete,

$$[\xi, \partial_\xi] = -1, \quad [\xi\partial_\xi, \partial_\xi] = -\partial_\xi \quad (21)$$

are well defined only when  $\xi$  is a noncompact variable. If we take a Fourier transform with respect to  $\xi$ , it becomes obvious that  $\mathcal{N}$  is diagonal while  $\mathcal{D}$  is not diagonal. This, however, is immaterial, since in terms of a new variable  $\xi' = \ln|\xi|$ ,  $\mathcal{N}$  is nondiagonal and  $\mathcal{D}$  is diagonal.

We say  $\psi$  is a *primary* operator if  $[K_i, \psi(x=0, t=0; \xi)] = 0$ , that is,  $\mathcal{K}_i = 0$ ; this was assumed in the commutation relations (20). Once again, one can invoke the Galilean boost invariance of the vacuum to obtain the form of the two point correlator of primaries  $\psi$  and  $\phi$ . This is most easily computed in terms of the Fourier transformed operators, e.g.,  $\psi(x, t, m_1) = \int d\xi \psi(x, t, \xi) e^{im\xi}$ ; we obtain

$$\begin{aligned} \langle 0 | \psi(x, t, m_1) \phi(0, 0, m_2) | 0 \rangle \\ = \begin{cases} h(t) \delta(m_1 + m_2) f(t^{2-z} m_1^z) \exp\left(\frac{im_1 |x|^2}{2t}\right), & z \neq 0 \\ h(t) \delta(m_1 + m_2) f(m_1) \exp\left(\frac{im_1 |x|^2}{2t}\right), & z = 0 \end{cases}, \end{aligned} \quad (22)$$

where  $h(t)$  is an as yet undetermined function of  $t$ . Evidently, Eq. (22) is consistent with the correlator of the  $z = 2$  theory [13,14]. For  $z \neq 2$ , rewriting in terms of  $\xi$ , we obtain

$$\begin{aligned} \langle 0 | \psi(x, t, \xi) \phi(0, 0, 0) | 0 \rangle \\ \propto \begin{cases} h(t) t^{1-2/z} \tilde{g}\left(\frac{|x|^2 - 2t\xi}{2t^{2/z}}\right), & z \neq 0 \\ \tilde{h}(t) \tilde{f}\left(\frac{|x|^2}{2t} - \xi\right) = \tilde{h}(t) \left(\frac{|x|^2}{2t} - \xi\right)^{-\Delta/2}, & z = 0 \end{cases}, \end{aligned} \quad (23)$$

where  $\tilde{g}(s) = \int dy e^{-ys} g(y)$ ,  $g(y) = f(y^z)$ , and  $y^z = m^z t^{2-z}$ . When  $z = 0$ , we use the fact that  $\tilde{f}$  has to scale covariantly under  $z = 0$  scaling, where  $\tilde{f}$  is the Fourier transform of  $f$ ; here  $h(t)$  must be a power law of  $t$  with  $t^{-\alpha}$  such that the scaling dimensions of  $\psi$  and  $\phi$  add up to  $\alpha z + (2-z)$  for  $z \neq 0$  and  $\Delta$  for  $z = 0$  with  $\tilde{h}(t)$  being any function of  $t$ .

### III. NULL REDUCTION AND EMBEDDING INTO CONFORMAL GROUP $SO(d, 2)$

A standard trick to obtain a  $d$ -dimensional  $z = 2$  Schrödinger invariant theory is to start with a conformal field theory in  $d + 1$  dimensions and perform a null cone reduction [24–29]. This is possible because the Schrödinger group,  $\text{Sch}(d)$ , can be embedded into  $SO(d, 2)$ . Next we show that the generalized Schrödinger group can also be embedded into  $SO(d, 2)$ . A similar embedding has been considered in [18] in the context of the Lie algebra of the deformed ISIM(2) group.

If the generators of  $SO(d, 2)$  are given by  $P_\mu^{(r)}$ ,  $M_{\mu\nu}^{(r)}$ ,  $D^{(r)}$ , and  $C_\mu^{(r)}$  where  $P_\mu^{(r)}$  are translation generators,  $M_{\mu\nu}^{(r)}$

are Lorentz generators,  $D^{(r)}$  is the relativistic scaling generator, and  $C_\mu^{(r)}$  are special conformal generators [here the superscript “(r)” denotes the relativistic generators], then the following generators generate the generalized Schrödinger algebra:

$$K_i = M_{i-}^{(r)}, \quad H = P_+^{(r)}, \quad N = P_-^{(r)}, \quad (24)$$

$$M_{ij} = M_{ij}^{(r)}, \quad P_i = P_i^{(r)}, \quad (25)$$

$$D = D^{(r)} + (1-z)M_{+-}^{(r)}. \quad (26)$$

It is straightforward to verify that  $D$  scales  $x^- \rightarrow \lambda^{2-z}x^-$ . Only for  $z = 2$  does  $x^-$  not scale and one is able to do a null cone reduction via compactification in the  $x^-$  direction, yielding a discrete spectra for  $N$ . On the other hand, for  $z \neq 2$ , even via null cone reduction one cannot truly get rid of the  $x^-$  direction since any compactification in the  $x^-$  direction would spoil the scaling symmetry. As a result, for  $z \neq 2$  the null reduction always leaves a continuous spectra for the generator  $N$ .

#### IV. EXPLICIT $(d+1)$ -DIMENSIONAL EXAMPLES

##### A. $z = 0$

Here we provide an explicit example of a generalized Schrödinger invariant theory in  $(d-1) + 1$  dimensions with  $z = 0$  and verify that the two point correlator indeed conforms to the general form given in Eq. (23).

We consider a Lagrangian model given by

$$\mathcal{L} = \phi^\dagger (2\partial_t \partial_\xi - \nabla^2 + 2i\partial_\xi) \phi, \quad (27)$$

and the two point correlator is given by<sup>9</sup>

$$\langle \phi \phi^\dagger \rangle \propto \left( \frac{1}{t} \right)^{\frac{d-1}{2}} \exp[-it] \left( \frac{|\mathbf{x}|^2}{2t} - \xi \right)^{-\frac{d-1}{2}}. \quad (28)$$

In  $d+1$  dimensions,  $\frac{d-1}{2} = \frac{(d+1)-2}{2}$  is precisely the dimension of a free relativistic scalar. This is because the generalized Schrödinger algebra can be embedded into the conformal group of one higher dimension, as mentioned in Sec. III.

For  $z = 0$ ,  $t$  does not scale. One may contemplate perturbing the Gaussian fixed point by marginal operators constructed out of powers of  $\partial_t$ , for example,  $\phi^\dagger \exp(i\partial_t) \partial_\xi \phi$ . However, Galilean boost invariance requires that  $\partial_t$  appears in the combination with other derivatives shown in Eq. (27). By contrast, in the models presented in Refs. [15,16], where  $N = 0$  and the Lagrangian

<sup>9</sup>The correlator in (28) follows from (27) only after restricting the field  $\phi$  to positive  $\xi$ -Fourier modes; see the footnote below Eq. (30) for more details.

is invariant under  $\mathbf{x} \rightarrow \mathbf{x}$  and  $t \rightarrow \lambda t$ , arbitrary powers of spatial derivatives are allowed.

$$\mathbf{B}. z = \frac{2\ell}{\ell+1}, \quad \ell \in \mathbf{Z}, \quad \ell \geq 1$$

These series of examples are given by the following Lagrangian:

$$\mathcal{L}_\ell = \phi^\dagger (2\partial_t \partial_\xi - \nabla^2 + 2g(i\partial_\xi)^{\ell+1}) \phi \quad (29)$$

The two point correlators, after partial Fourier transformation, are given by

$$G(\mathbf{x}, t, m) \propto t^{-\frac{d-1}{2}} m^{\frac{d-3}{2}} \exp \left[ i \left( \frac{m|\mathbf{x}|^2}{2t} - gm^\ell t \right) \right], \quad (30)$$

where  $z = \frac{2\ell}{\ell+1}$ . One can Fourier transform<sup>10</sup> to obtain the correlator in position space-time only depending on the analytical ease to do so. For  $d = 3$ ,  $\ell = 2$ , i.e.,  $z = \frac{4}{3}$ , we have

$$G(\mathbf{x}, t, \xi) \propto t^{-1} \frac{1}{\sqrt{gt}} \exp \left[ \frac{i(x^2 - 2\xi t)^2}{16gt^3} \right], \quad (32)$$

which is consistent with Eq. (23) for  $z \neq 0$ . After performing a Euclidean rotation,  $t \rightarrow -i\tau$ ,  $\xi \rightarrow i\xi$ , one finds good behavior of this correlator at large spatial separation (as long as  $g < 0$ ).

One can add classically marginal interactions to the model in (29). For example, one may add  $(\phi\phi^\dagger)^{n-1} \phi (i\partial_\xi)^k \phi^\dagger$  with  $k = (\ell+1)[(d-1)\beta + d-2]$  and  $n = 2\beta + 3$ , where  $\beta$  is a non-negative integer. Furthermore, one can have supersymmetric generalizations of  $z \neq 2$  theories, much as the  $z = 2$  case presented in [31] where supersymmetry is an internal symmetry exchanging fermionic and bosonic fields.

<sup>10</sup>Care is needed regarding the allowed values of  $m$ . The correlator in (30) is most readily obtained by Fourier transform of

$$G(\mathbf{k}, t, m) \propto \exp \left[ -it \left( \frac{|\mathbf{k}|^2}{2m} + gm^\ell \right) \right]. \quad (31)$$

For even, positive  $\ell$ , the integral over  $\mathbf{k}$  is well defined only for  $\text{Im}(t/m) < 0$ , and the result can be analytically continued to all values of  $t/m$ . The integral over  $m$  requires  $\text{Im}(t) < 0$  (for  $g > 0$ ), and again one analytically continues to all values of  $t$ . For odd (and positive)  $\ell$ , the Fourier transform with respect to  $m$  is ill-behaved for any value of  $t$ , because there is no deformation of the contour of integration that can render the integral of  $\exp \left[ i \left( \frac{m|\mathbf{k}|^2}{2t} - gm^\ell t \right) \right]$  over  $m$  finite. Both for  $\ell$  odd and for  $\ell = 0$ , a sensible way to make this integral well defined is to restrict it to  $m > 0$ . This is, in fact, how we obtained the correlator for the  $z = \ell = 0$  in Eq. (28). Strictly speaking, these are not Lagrangian theories; these systems are close analogues of the chiral boson, where the Fourier modes are restricted [30].

## V. CONCLUSION

The most natural way to realize the Schrödinger algebra and its  $z \neq 2$  avatar in a gravity dual of a  $d$ -dimensional nonrelativistic field theory with Galilean boost and scale invariance with dynamical exponent  $z$  is via isometries of the bulk metric. As it turns out, the dual metric is of a  $(d+2)$ -dimensional space-time [1,2]. By contrast, for the canonical notion of gauge-gravity duality the bulk gravitational theory lives in one extra dimensional space-time. Above we have expounded the presence of the two extra dimensions in the duality. We showed that on the field theory side of the duality, for  $z \neq 2$ , one needs to have an internal continuous parameter, effectively making the field theory  $(d+1)$ -dimensional. Any attempt to construct a  $z \neq 2$  nonrelativistic field theory with Galilean boost and scale invariance with a finite number of fields is bound to run into trouble, since correlators will grow with separation and will fail to exhibit cluster decomposition. This result follows solely from constraints that the symmetry algebra places on two point correlators. It is important to have the particle number symmetry for the no-go theorem. Without particle number symmetry, there are indeed examples of Galilean boost invariant  $z \neq 2$  theories [32]. Examples of theories with  $z = \infty$  anisotropic scaling symmetry based on warped conformal field theories are discussed in Refs. [15,16].

Only for  $z=2$  is a consistent  $d$ -dimensional field theoretic realization of the symmetry, with a finite number of fields, possible, and therefore a conventional  $(d+1)$ -dimensional gravity dual is available. On the gravity side, the metric dual to a  $z=2$  Schrödinger theory has a direction  $\xi$  which does not scale, and can therefore be compactified. The Kaluza-Klein reduction of the momentum conjugate to  $\xi$  generates a discrete spectrum for  $N$  that matches onto a  $d$ -dimensional field theory. The  $\xi$  direction for  $z \neq 2$  duals scales, forbidding any such compactification. One can also see this by embedding the generalized Schrödinger group into  $SO(d,2)$ ; see Sec. III.

That there is no impediment to constructing a sensible  $z \neq 2$  nonrelativistic field theory with Galilean boost and scale invariance for an infinite number of fields is most easily established by giving explicit examples. Above we presented explicit examples of Galilean boost invariant theories, with  $z = \frac{2\ell}{\ell+1}$ .

Given that we have explicit examples and the generic form of the correlator, several new questions come to mind. One can ask how one may couple these theories to gravity. Nonrelativistic theory coupled to gravity gives a natural framework to study Ward identity anomalies and scale anomalies [25,28,33–42]. Since these theories are intrinsically  $(d+1)$ -dimensional, the use of Newton-Cartan geometry is not a natural choice. It would also be interesting to understand the dispersion relation of Goldstone bosons, arising from spontaneous breaking of  $z \neq 2$  scale symmetry; the  $z=2$  case has been studied in [43].

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## APPENDIX: DIAGONALIZABLE AND FINITE DIMENSIONAL DILATATION GENERATOR

In this appendix, we rederive some of the results in Sec. II under the stronger assumption that the matrix  $\Delta$  is both diagonal and finite dimensional. This discussion is intended for clarity, since it is less abstract than the one presented in the main text.

We recall that

$$[D, \tilde{\Phi}(\mathbf{x} = \mathbf{0}, t = 0)] = \iota D \tilde{\Phi}(\mathbf{x} = \mathbf{0}, t = 0) \quad (\text{A1})$$

and  $\mathcal{D}$  is renamed as  $\Delta$  in the finite dimensional case.

To warm up, we show that both  $\mathcal{D}$  and  $\mathcal{N}$  are Hermitian only if  $z=2$  or  $\mathcal{N}=0$ . From  $[D, N] = \iota(2-z)N$ , it follows that

$$[\mathcal{D}, \mathcal{N}] = (2-z)\mathcal{N}. \quad (\text{A2})$$

Since  $\mathcal{D}$  and  $\mathcal{N}$  are assumed Hermitian,  $[\mathcal{D}, \mathcal{N}]^\dagger = -[\mathcal{D}, \mathcal{N}] = -(2-z)\mathcal{N}$ . Hence

$$-(2-z)\mathcal{N} = [\mathcal{D}, \mathcal{N}]^\dagger = (2-z)\mathcal{N}^\dagger = (2-z)\mathcal{N},$$

which can only hold for  $z=2$  or  $\mathcal{N}=0$ . If one assumes  $\mathcal{N} \neq 0$  for some field, then  $z=2$ . One can have  $z \neq 2$  and Hermiticity if  $\mathcal{N}=0$  for all fields. In this case the generator  $N$  is superfluous, and one can extend the algebra by including the generator of special conformal transformations.<sup>11</sup> Below we assume  $N$  does not identically vanish. Similarly, both  $\mathcal{D}$  and  $\mathcal{K}$  are Hermitian only if  $z=1$  or  $\mathcal{K}=0$ .

Now we consider the finite dimensional case where  $\Delta$  is diagonal. Alternatively, one can consider the case that  $\Delta$  is Hermitian (and therefore, as just proved,  $\mathbb{N}$  is not Hermitian). In the finite dimensional, Hermitian case, one can always choose to diagonalize  $\Delta$ . Since  $\Delta$  is diagonal,  $[\Delta, \mathbb{N}] = (2-z)\mathbb{N}$  implies that  $(\Delta_{\alpha\alpha} - \Delta_{\beta\beta} + z - 2)\mathbb{N}_{\alpha\beta} = 0$  (no summation over indices  $\alpha, \beta$  is implicit), which, in turn, for  $z \neq 2$  implies that  $\mathbb{N}_{\alpha\alpha} = 0$  and at least one of  $\mathbb{N}_{\alpha\beta}$  and  $\mathbb{N}_{\beta\alpha}$  vanish. This implies that  $\mathbb{N}$  is nilpotent,

$$\mathbb{N}^M = 0, \quad (\text{A3})$$

for some integer  $M$  no larger than the dimension of the representation. One can show this, without loss of generality,

<sup>11</sup>There are indeed examples of  $z \neq 2$  theories without particle number symmetry; see, for example, Refs. [15,16,32].

by arranging the components of the fields  $\tilde{\Phi}_\alpha$  so that  $\mathbb{N}$  is an upper triangular matrix. This result will play a pivotal role below.

Similarly, we assume that the field  $\tilde{\Phi}(\mathbf{0}, 0)$  has the following commutation relation:

$$[K_i, \tilde{\Phi}(\mathbf{x} = \mathbf{0}, t = 0)] = \mathcal{K}_i \tilde{\Phi}(\mathbf{x} = \mathbf{0}, t = 0). \quad (\text{A4})$$

For a finite dimensional case, we denote  $\mathcal{K}_i$  by  $\mathbb{K}_i$ . By the same argument as above, one can show that either  $z = 1$  or  $\mathbb{K}_i = 0$  or  $\mathbb{K}_i$  is nilpotent. Thus we have

$$\mathbb{K}_i^{L_i} = 0, \quad (\text{A5})$$

for some integer  $L_i$  no larger than the dimension of the representation. One can consider the operator

$$\Phi \equiv \prod_{i=1}^{i=d-1} K_i^{L_i-1}(\tilde{\Phi}), \quad (\text{A6})$$

where for any operator  $A$  and  $B$ , the action of the operator on the field is defined via

$$A(\tilde{\Phi}) \equiv [A, \tilde{\Phi}], \quad (\text{A7})$$

$$BA(\tilde{\Phi}) \equiv B(A(\tilde{\Phi})). \quad (\text{A8})$$

It can easily be verified that

$$[K_i, \Phi(\mathbf{x} = \mathbf{0}, t = 0)] = 0, \quad (\text{A9})$$

$$[D, \Phi(\mathbf{x} = \mathbf{0}, t = 0)] = \iota \Delta' \Phi(\mathbf{x} = \mathbf{0}, t = 0), \quad (\text{A10})$$

$$[N, \Phi(\mathbf{x}, t)] = \mathbb{N} \Phi(\mathbf{x}, t), \quad (\text{A11})$$

where  $\Delta' = (\Delta - (z-1)\sum_i(L_i-1))$ . We call ‘‘primary operators’’ those that satisfy (A9). One could have considered operators obtained from these by analogous operations as above, i.e., operators of the form  $[N^{M-1}, \Phi]$ , but that would not suffice to reveal the problems associated with finite dimensional representations.

Consider the two point correlator of primary operators in such a realization of the algebra,  $G_{\alpha\beta} \equiv \langle 0 | \Phi_\alpha(x, t) \Phi_\beta(0, 0) | 0 \rangle$ . Using Eqs. (9), the commutator in (A9) translates to

$$[K_i, \Phi] = (-\iota t \partial_i \mathbf{I} + x_i \mathbb{N}) \Phi. \quad (\text{A12})$$

Galilean boost invariance of the vacuum,  $K_i | 0 \rangle = \langle 0 | K_i = 0$ , then gives

$$\begin{aligned} \langle 0 | [K, \Phi_\alpha(x, t) \Phi_\beta(0, 0)] | 0 \rangle &= 0 \\ \Rightarrow (-\iota t \partial_i \delta_{\alpha\sigma} + x_i \mathbb{N}_{\alpha\sigma}) G_{\sigma\beta} &= 0. \end{aligned}$$

The solution to the above differential equation is given by

$$G_{\alpha\beta} = [e^{-\iota \frac{|\mathbf{x}|^2}{2t} \mathbb{N}}]_{\alpha\gamma} C_{\gamma\beta}(t) = \sum_{\ell=0}^{M-1} \frac{1}{\ell!} \left( -\iota \frac{|\mathbf{x}|^2}{2t} \right)^\ell (\mathbb{N}^\ell C(t))_{\alpha\beta}, \quad (\text{A13})$$

where  $|\mathbf{x}|^2 = \sum_i (x^i)^2$ ,  $C$  is an as yet undetermined matrix function of  $t$  alone; the scaling symmetry implies that  $C_{\alpha\beta}$  has a power law dependence on  $t$ . The above correlator (A13) is consistent with the one given in (15) with  $\mathbb{N}_1 = \mathbb{K}_1 = \mathbb{K}_2 = 0$ . The exponential becomes a finite degree polynomial because  $\mathbb{N}$  is nilpotent, and this is very specific to  $z \neq 2$  theories. As explained above, the correlators are badly behaved: polynomial rather than exponential dependence on  $|\mathbf{x}|$  leads to growth with spatial separation (and hence, cluster decomposition fails). In contrast, for  $z = 2$  the matrix  $\mathbb{N}$  is diagonal and there is no truncation of the expansion of the exponential. An additional constraint on the correlator follows from requiring that  $\langle 0 | [N, \Phi_\alpha(x, t) \Phi_\beta(0, 0)] | 0 \rangle = 0$ , which implies  $NG + GN^T = 0$ .

Consider next  $G'_{\alpha\beta} = \langle 0 | \Phi_\alpha(x, t) \Phi_\beta^\dagger(0, 0) | 0 \rangle$ . This is similarly given by

$$G'_{\alpha\beta} = \left[ \exp \left( -\iota \frac{|\mathbf{x}|^2}{2t} \mathbb{N} \right) C'(t) \right]_{\alpha\beta} \quad (\text{A14})$$

for some undetermined matrix  $C'$  such that  $C'_{\alpha\beta}$  is a function of  $t$  alone. Invariance under  $N$  implies that  $NG' - G'N^\dagger = 0$ . Notice that the condition on  $G'$  is different from that on  $G$ ; one may have nontrivial solutions to one but not the other. For example, one can consider the two component field,  $\Phi_{\alpha=1,2}$ , characterized by

$$\begin{aligned} \mathbb{N} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & C' &= g(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ g(t) &= t^{-(\Delta_{11} + \Delta_{22})/z} \\ \Delta &= \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}, & \Delta_{22} - \Delta_{11} &= (2-z), \\ G' &= t^{-(\Delta_{11} + \Delta_{22})/z} \begin{bmatrix} 1 & -\iota \frac{|\mathbf{x}|^2}{2t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= t^{-(\Delta_{11} + \Delta_{22})/z} \begin{bmatrix} -\iota \frac{|\mathbf{x}|^2}{2t} & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Note that for this example  $G_{\alpha\beta} = 0$  so consideration of the long distance behavior of this correlator alone does not, by itself, suggest the theory is ill-behaved.

- [1] K. Balasubramanian and J. McGreevy, *Phys. Rev. Lett.* **101**, 061601 (2008).
- [2] D. T. Son, *Phys. Rev. D* **78**, 046003 (2008).
- [3] S. F. Ross and O. Saremi, *J. High Energy Phys.* **09** (2009) 009.
- [4] M. Guica, K. Skenderis, M. Taylor, and B. C. van Rees, *J. High Energy Phys.* **02** (2011) 056.
- [5] R. N. Caldeira Costa and M. Taylor, *J. High Energy Phys.* **02** (2011) 082.
- [6] M. Guica, *J. High Energy Phys.* **12** (2012) 084.
- [7] B. C. van Rees, [arXiv:1206.6507](https://arxiv.org/abs/1206.6507).
- [8] T. Andrade, C. Keeler, A. Peach, and S. F. Ross, *Classical Quantum Gravity* **32**, 035015 (2015).
- [9] T. Andrade, C. Keeler, A. Peach, and S. F. Ross, *Classical Quantum Gravity* **32**, 085006 (2015).
- [10] K. Balasubramanian and J. McGreevy, *J. High Energy Phys.* **01** (2011) 137.
- [11] Y. Nishida and D. T. Son, *Phys. Rev. D* **76**, 086004 (2007).
- [12] Y. Nishida and D. T. Son, *Lect. Notes Phys.* **836**, 233 (2012).
- [13] W. D. Goldberger, Z. U. Khandker, and S. Prabhu, *J. High Energy Phys.* **12** (2015) 1.
- [14] S. Golkar and D. T. Son, *J. High Energy Phys.* **12** (2014) 063.
- [15] D. M. Hofman and B. Rollier, *Nucl. Phys.* **B897**, 1 (2015).
- [16] K. Jensen, *J. High Energy Phys.* **12** (2017) 111.
- [17] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, [arXiv:1710.04708](https://arxiv.org/abs/1710.04708).
- [18] G. W. Gibbons, J. Gomis, and C. N. Pope, *Phys. Rev. D* **76**, 081701 (2007).
- [19] C. R. Hagen, *Phys. Rev. D* **5**, 377 (1972).
- [20] U. Niederer, *Helv. Phys. Acta* **45**, 802 (1972).
- [21] T. Mehen, I. W. Stewart, and M. B. Wise, *Phys. Lett. B* **474**, 145 (2000).
- [22] M. Henkel and J. Unterberger, *Nucl. Phys.* **B660**, 407 (2003).
- [23] S. Pal, *Phys. Rev. D* **97** 105031 (2018).
- [24] J. Maldacena, D. Martelli, and Y. Tachikawa, *J. High Energy Phys.* **10** (2008) 072.
- [25] K. Jensen, [arXiv:1412.7750](https://arxiv.org/abs/1412.7750).
- [26] N. Banerjee, S. Dutta, and A. Jain, *Phys. Rev. D* **92**, 081701 (2015).
- [27] N. Banerjee, S. Dutta, and A. Jain, *Phys. Rev. D* **93**, 105020 (2016).
- [28] S. Pal and B. Grinstein, *Phys. Rev. D* **96**, 125001 (2017).
- [29] K. T. Grosvenor, J. Hartong, C. Keeler, and N. A. Obers, [arXiv:1712.03980](https://arxiv.org/abs/1712.03980).
- [30] C. Imbimbo and A. Schwimmer, *Phys. Lett. B* **193**, 455 (1987).
- [31] A. Meyer, Y. Oz, and A. Raviv-Moshe, *J. High Energy Phys.* **06** (2017) 128.
- [32] P. C. Stichel and W. J. Zakrzewski, *Eur. Phys. J. C* **70**, 713 (2010).
- [33] R. Banerjee, A. Mitra, and P. Mukherjee, *Phys. Lett. B* **737**, 369 (2014).
- [34] R. Banerjee, A. Mitra, and P. Mukherjee, *Classical Quantum Gravity* **32**, 045010 (2015).
- [35] K. Jensen, [arXiv:1408.6855](https://arxiv.org/abs/1408.6855).
- [36] M. Geracie, D. T. Son, C. Wu, and S.-F. Wu, *Phys. Rev. D* **91**, 045030 (2015).
- [37] D. T. Son, [arXiv:1306.0638](https://arxiv.org/abs/1306.0638).
- [38] M. Baggio, J. de Boer, and K. Holsheimer, *J. High Energy Phys.* **07** (2012) 099.
- [39] I. Arav, S. Chapman, and Y. Oz, *J. High Energy Phys.* **02** (2015) 078.
- [40] I. Arav, S. Chapman, and Y. Oz, *J. High Energy Phys.* **06** (2016) 158.
- [41] S. Pal and B. Grinstein, *J. High Energy Phys.* **12** (2016) 012.
- [42] R. Auzzi, S. Baiguera, F. Filippini, and G. Nardelli, *J. High Energy Phys.* **11** (2016) 163.
- [43] I. Arav, I. Hason, and Y. Oz, *J. High Energy Phys.* **10** (2017) 063.