Gravitational waves in modified teleparallel theories

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(Received 26 April 2018; published 26 June 2018)

We investigate the gravitational waves and their properties in various modified teleparallel theories, such as $f(T)$, $f(T,B)$, and $f(T,T_G)$ gravities. We perform the perturbation analysis both around a Minkowski background and in the case where a cosmological constant is present, and for clarity we use both the metric and the tetrad languages. For $f(T)$ gravity we verify the result that no further polarization modes comparing to general relativity are present at first-order perturbation level, and we show that in order to see extra modes one should look at third-order perturbations. For nontrivial $f(T,B)$ gravity, by examining the geodesic deviation equations, we show that extra polarization models, namely the longitudinal and breathing modes, do appear at first-order perturbation level, and the reason for this behavior is the fact that although the first-order perturbation does not have any effect on T, it does affect the boundary term B. Finally, for $f(T,T_G)$ gravity we show that at first-order perturbations the gravitational waves exhibit the same behavior as those of $f(T)$ gravity. Since different modified teleparallel theories exhibit different gravitational wave properties, the advancing gravitational-wave astronomy would help to alleviate the degeneracy not only between curvature and torsional modified gravity but also between different subclasses of modified teleparallel gravities.

DOI: 10.1103/PhysRevD.97.124064

I. INTRODUCTION

The discovery of the late-time accelerating expansion of the Universe and the study of galactic rotation curves have generated a lot of interest and investigation, particularly in the direction of dark energy and dark matter [1–3]. Additionally, it has led to investigations into gravitational theories beyond general relativity (GR), with the most studied cases being modifications of the Einstein-Hilbert action, which is constructed from the Ricci scalar R. Amongst others one may have $f(R)$ gravity [4,5], theories with inclusion of other scalar invariants [for instance, $f(R,G)$ gravity where G is the Gauss-Bonnet term [6,7], and more generally Lovelock gravity [8,9]], theories with nonminimal curvature-matter couplings [e.g., $f(R,\mathcal{T})$ gravity, where $\mathcal{T}$ is the trace of the stress-energy tensor [10–12]], or more radical modifications such as massive gravity [13] and Hořava-Lifshitz [14]. The goal of all these endeavors is to consistently explain the aforementioned observational phenomena while also retaining GR as a particular limit [15].

Recently, there has been a significant rise in interest in a specific class of theories originally investigated by Einstein and Cartan. By considering gravitation to be described by torsion rather than curvature, gravitation can retain many of the features present in the original GR formalism [16,17]. This is most commonly referred to as teleparallel gravity. Furthermore, the fundamental dynamical quantity of the theory is not the metric tensor but the more subtle, so-called, tetrad field. In the simplest form of these theories the Lagrangian is just the torsion scalar T, constructed by contractions of the torsion tensor, and variation with respect to the tetrad gives rise to exactly the same equations with GR, which is why this theory was named "teleparallel equivalent of general relativity" (TEGR) [18–21]. The source of the above equivalence is a boundary quantity, B, which relates the two Lagrangians, namely the Ricci scalar of GR and the torsion scalar of TEGR:

$$R = -T + B, \tag{1}$$

where R is calculated with the regular Levi-Cività connection while T is calculated with the Weitzenböck connection.

Inspired by the gravitational modifications that are based on the curvature formulation of gravity, one can construct

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modified gravity theories starting from TEGR. The simplest such modified teleparallel theory is the $f(T)$ gravity, in which one generalizes T to a function $f(T)$ in the Lagrangian [22,23] (see [24] for a review). One can immediately see that due to relation (1) and in particular to the boundary term, $f(T)$ gravity is not equivalent to $f(R)$ gravity, and thus it is a novel gravitational modification. Additionally, the advantage of this theory is that the equations of motion are of second order, in contrast to the fourth-order equations of $f(R)$ gravity. These features led to many investigations in various fields of cosmology in this theory [25–33]. Furthermore, one may proceed in constructing other modifications and extensions of teleparallel theories, such as the $f(T, T_G)$ gravity, where T_G is the teleparallel equivalent of G [34–36], the $f(T, \mathcal{T})$ gravity, where $\mathcal{T}$ is the trace of the stress-energy tensor [37,38], or torsional gravities with higher-order derivatives [39]. Finally, one interesting class of torsional gravitational modification is the $f(T, B)$ gravity, in which one allows for the use of the boundary term B in the Lagrangian [40–42].

On the other hand, gravitational wave (GW) observations not only have confirmed the existence of gravitational waves as the mediator of gravitational information [43] but also have set bounds on the polarization modes of these waves from known sources [44], as well as on their speed, which is equal to the light speed with great accuracy [45–49]. These observations are very important for alternative theories of gravity, since in general one can obtain extra polarization modes or variant speed. Although there have been some works investigating gravitational waves in $f(T)$ gravity [50–52], the systematic study of gravitational waves in modified teleparallel gravities has not been performed.

In this work we are interested in looking at $f(T)$, $f(T, B)$, and $f(T, T_G)$ gravities in the realm of gravitational waves through detailed perturbation analysis. Our goals are to determine whether various teleparallel gravities predict extra modes and to investigate the strength of these modes in the scales where they arise. Although in GR there are two GW polarizations, namely the plus and cross polarizations, alternative and extended theories might yield more modes [as for instance in $f(R)$ gravity [53]]. As we will show, although in the case of $f(T)$ gravity the polarization models are identical to those of GR [50], this is not the case when an arbitrary boundary contribution is included, as for instance in $f(T, B)$ gravity.

The paper is organized as follows. In Sec. II we briefly review teleparallel gravity and its various modifications. In Sec. III we perform an analysis of the gravitational waves in the case of $f(T)$ gravity, both in the metric and tetrad languages, and for both zero and nonzero spin connections. In Sec. IV we investigate the gravitational waves in the case of $f(T, B)$ gravity, both around a Minkowski background, a case which is obtained in the absence of a cosmological constant, but also in the case where the presence of a cosmological constant changes the background around which the perturbations are realized. In Sec. V we examine the gravitational waves in $f(T, T_G)$. Finally, the work closes with a discussion and conclusion of results in Sec. VI.

II. MODIFIED TELEPARALLEL THEORIES OF GRAVITY

In teleparallel theories of gravity the fundamental dynamical variable is the tetrad (or vierbein) $e^a\mu$, which relates the standard coordinate frame $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ with an orthonormal and noncoordinate frame (e-frame). In general, non-coordinate frames are anholonomic, a property that is attributed to the existence of noninertial effects. The metric tensor $g_{\mu\nu}$ can be related to the tetrad through the Minkowski metric η_{ab} by

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^a{}_\nu, \tag{2}$$

where the point dependence is suppressed for brevity. In the whole manuscript Greek indices refer to the spacetime coordinates, while Latin indices refer to the tangent-space ones. The inverse tetrad is denoted by $E_a{}^\mu$ for transparency, and one can show that

$$e^a{}_\mu E_a{}^\nu = \delta^\nu_\mu, \qquad e^a{}_\mu E_b{}^\mu = \delta^a_b. \tag{3}$$

The connection used in the teleparallel theories of gravity is defined as a connection that has vanishing curvature. This connection is the so-called Weitzenböck connection, and the fact that it is torsionful makes the connection coefficients nonsymmetric in the lower indices in contrast with the Levi-Cività connection where the indices are symmetric. The tetrad enables us to relate to each Lorentz spin connection $\omega^a{}_{b\mu}$ the Weitzenböck connection via [54]

$$\hat{\Gamma}^\rho_{\nu\mu} \equiv E_a{}^\rho \partial_\mu e^a{}_\nu + E_a{}^\rho \omega^a{}_{b\mu} e^b{}_\nu. \tag{4}$$

The spin connection $\omega^a{}_{b\mu}$ does not represent any additional gravitational degrees of freedom (d.o.f.). If one switches over to the e-frame and applies the Weitzenböck covariant derivative to the basis vectors of the e-frame, assuming the so-called Weitzenböck condition where $\omega^a{}_{b\mu} = 0$, then the result will be zero. This phenomenon is called complete frame induced parallelism and in the physics literature is frequently called teleparallelism or absolute parallelism [21]. The Riemann and Ricci tensors calculated with the Weitzenböck connection are identically zero, while the torsion tensor is written as

$$T^a{}_{\mu\nu} \equiv \hat{\Gamma}^a_{\nu\mu} - \hat{\Gamma}^a_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu. \tag{5}$$

Moreover, one can define the *superpotential tensor* as

$$S_a{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_a + e_a{}^\mu T^{\alpha\nu}{}_\alpha - e_a{}^\nu T^{\alpha\mu}{}_\alpha), \tag{6}$$

where $K^{\mu\nu}{}_a$ is the *contorsion tensor* identified as

$$K^{\mu\nu}{}_a \equiv \frac{1}{2}(T_a{}^{\mu\nu} + T^{\nu\mu}{}_a - T^{\mu\nu}{}_a), \tag{7}$$

which represents the difference between the Levi-Civitá connection and the Weitzenböck connection. The Lagrangian of TEGR is the torsion scalar T, constructed by contractions of the torsion tensor, namely [54]

$$T \equiv S_a{}^{\mu\nu} T^a{}_{\mu\nu} = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \tag{8}$$

Therefore, the action of TEGR reads

$$S = \frac{1}{16\pi G} \int d^4x eT + \int d^4x e\mathcal{L}_m, \tag{9}$$

where $e = \det(e^a{}_\mu) = \sqrt{-g}$, with g being the determinant of the metric tensor, and where $\mathcal{L}_m$ is the matter Lagrangian and G is Newton's constant.

As we mentioned in the Introduction, one can show that the Ricci scalar calculated with the Levi-Cività connection, and the torsion scalar calculated with the Weitzenböck connection, are related through

$$R = -T - 2\nabla^\mu T^\nu{}_{\mu\nu}, \tag{10}$$

and thus we can identify the boundary term $B \equiv -2\nabla^\mu T^\nu{}_{\mu\nu}$. Hence, one can immediately see that GR and TEGR will lead to exactly the same equations. However, this will not be the case if one uses $f(R)$ and $f(T)$ as the Lagrangian of the theory, which therefore correspond to different gravitational modifications.

A general class of modified teleparallel gravity would thus be composed by an arbitrary function of T and B, leading to $f(T,B)$ gravity [40], characterized by the action

$$S = \frac{1}{16\pi G} \int d^4x e f(T,B) + \int d^4x e\mathcal{L}_m. \tag{11}$$

By varying the action with respect to the vierbein we obtain the following field equations:

$$\begin{aligned} &E_a{}^\mu \Box f_B - E_a{}^\nu \nabla^\mu \nabla_\nu f_B + \frac{1}{2} B f_B E_a{}^\mu \\ &\quad + 2\partial_\nu (f_B + f_T) S_a{}^{\nu\mu} + 2e^{-1}\partial_\nu (e S_a{}^{\nu\mu}) f_T \\ &\quad - 2 f_T T^\alpha{}_{\nu a} S_\alpha{}^{\mu\nu} - \frac{1}{2} E_a{}^\mu f = 8\pi G \Theta_a{}^\mu, \end{aligned} \tag{12}$$

where $\Theta_a{}^\mu$ is the stress-energy tensor, which in terms of the matter Lagrangian is given by $\Theta_a{}^\mu = -\delta\mathcal{L}_\mathrm{m}/\delta e^a{}_\mu$. In the above equation we have defined that $f_T \equiv \partial f/\partial T$ and $f_B \equiv \partial f/\partial B$. Note that the derived equations are given for the zero spin connection case. Additionally, in terms of spacetime indices the equations of motion can take the form

$$\begin{aligned} &- f_T G_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f_B \\ &\quad + \frac{1}{2} g_{\mu\nu} (f_B B + f_T T - f) \\ &\quad + 2 S_\nu{}^\alpha{}_\mu \partial_\alpha (f_T + f_B) = 8\pi G \Theta_{\mu\nu}, \end{aligned} \tag{13}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor calculated with the Levi-Cività connection.

Before closing this section let us make some comments on the spin connection $\omega^a{}_{b\mu}$ that is present in the definition (4). In the traditional works of TEGR one usually sets it to zero for convenience, by choosing a suitable frame (specifically autoparallel orthonormal frame) [54]. Although this does not have any effect for TEGR, in the case of $f(T)$ gravity such a preferred frame choice should be used carefully. In particular, one is allowed to make such a choice in order to find cosmological solutions; however, one has to have in mind that in investigations which include questions on Lorentz transformations such a formulation is in general inadequate. In this case one should formulate $f(T)$ gravity in a fully covariant way, keeping a general nonzero spin connection [55]. In this way the theory becomes completely consistent with Lorentz invariance, nevertheless at the price of increased complication. In the largest part of this manuscript we will consider the zero spin connection case (which is a safe choice for cosmological applications), especially for the general $f(T,B)$ and $f(T,T_G)$ theories; however, in the simpler $f(T)$ theory, for completeness, we will also discuss the nonzero spin connection case.

III. GRAVITATIONAL WAVES IN $f(T)$ GRAVITY

Let us now start with the investigation of the gravitational waves in the case of the simplest modification to teleparallel gravity, namely $f(T)$ gravity. We first perform the analysis for first-order perturbations at a tetrad level, and then we proceed to higher-order examination in order to understand more transparently the potential deviations from GR.

A. Tetrad solutions for GWs in $f(T)$ gravity

We start by considering the tetrad form of the field equations (13) in the case of $f(T)$ gravity, namely

$$\begin{aligned} &e^{-1} f_T \partial_\nu (e S_a{}^{\mu\nu}) + f_{TT} S_a{}^{\mu\nu} \partial_\nu T \\ &\quad - f_T T^b{}_{\nu a} S_b{}^{\mu\nu} + \frac{1}{4} f(T) e^a{}_\mu = 0, \end{aligned} \tag{14}$$

where as before we neglect the matter sector. We consider a tetrad perturbation of the form

$$e^a{}_\mu = \gamma_\mu^{(0)a} + \gamma_\mu^{(1)a} + \mathcal{O}(\gamma_\mu^{(2)a}), \tag{15}$$

where $|\gamma_\mu^{(i)a}| \ll 1$ except for the zeroth-order contribution, and each successive order is much smaller than the preceding order, i.e., $|\gamma_\mu^{(2)a}| \ll |\gamma_\mu^{(1)a}|$. This last comment applies throughout to every successive order quantity. Throughout the work, superscripts with parentheses will represent the perturbative order of the quantity being presented. As usual, from (2) we obtain for the zeroth-order perturbation

$$\eta_{\mu\nu} = \eta_{ab}\gamma_\mu^{(0)a}\gamma_\nu^{(0)b}. \tag{16}$$

Thus, the torsion tensor (5), assuming for the moment zero spin connection, up to first order can be expressed as

$$T^a{}_{\mu\nu} = \partial_\mu\gamma_\nu^{(0)a} - \partial_\nu\gamma_\mu^{(0)a} + \partial_\mu\gamma_\nu^{(1)a} - \partial_\nu\gamma_\mu^{(1)a}. \tag{17}$$

This gives an impression that an arbitrary choice of $\gamma_\nu^{(0)a}$ will yield a zeroth-order contribution. However, as discussed in Ref. [55], if the gravitational strength, i.e., the gravitational constant, vanishes, one obtains the Minkowski background, in which the torsion tensor vanishes. In this perturbation regime $e^a{}_\mu|_{G\to 0} = \gamma_\mu^{(0)a}$, since this corresponds to the Minkowski background, while the higher-order perturbations are due to gravitational effects. Therefore

$$T^a{}_{\mu\nu}|_{G\to 0} = \partial_\mu\gamma_\nu^{(0)a} - \partial_\nu\gamma_\mu^{(0)a} = 0. \tag{18}$$

Hence, as we mentioned above, the torsion tensor is first order, and so are the contorsion and superpotential, which ultimately imply that the torsion scalar is second order at the level of perturbations. Finally, in order to handle the $f(T)$ term we will consider the Taylor expansion (59).

Inserting the above expressions into Eq. (14), order by order leads to

$$\gamma_a^{(0)\rho} f^{(0)} = 0, \tag{19}$$

$$f_T^{(0)} e^{-1(0)}\partial_\nu(e^{(0)}S_a{}^{(1)\mu\nu}) + \gamma_a^{(0)\rho}\frac{f^{(1)}}{4} + \gamma_a^{(1)\rho}\frac{f^{(0)}}{4} = 0. \tag{20}$$

As mentioned above, the first condition implies that no cosmological constant is present. We next identify that $e^{(0)} = e^{-1(0)} = 1$, and we assume $f^{(1)} = 0$, since T is a second-order quantity and thus its function cannot have first-order contributions. Last, we focus on the nontrivial case $f_T^{(0)} \neq 0$. Under these considerations Eq. (20) becomes

$$\partial_\nu S_a{}^{(1)\mu\nu} = 0. \tag{21}$$

We note here that these intermediate steps are different from the $f(T)$ GWs analysis carried out previously with the Einstein tensor, since we now follow the tetrad language. Moreover, notice that this equation appears in TEGR, too [56].

Let us proceed by extracting explicit solutions. We first remark, however, that the above equation is not possible to solve in general. However, we can assume that GR gauge conditions on the perturbed metric, namely $h_{\mu\nu}^{(1)}$, can also be imposed here, specifically that it is traceless,

$$h^{(1)\mu}{}_\mu = 2\eta^{\mu\nu}\eta_{ab}\gamma_\mu^{(0)a}\gamma_\nu^{(1)b} = 0, \tag{22}$$

and satisfies the Lorenz gauge condition

$$0 = \partial^\mu h_{\mu\nu}^{(1)} = \partial_b\gamma_\nu^{(0)b} + \eta_{ab}[\gamma_\nu^{(1)b}\partial^\mu\gamma_\nu^{(0)b} + \gamma_\mu^{(1)a}\partial^\mu\gamma_\mu^{(0)a} + \gamma_\mu^{(1)a}\partial^\mu\gamma_\nu^{(0)b}]. \tag{23}$$

Together with the relation

$$\gamma_\nu^{(0)d} = \eta^{cd}\eta_{\mu\nu}\gamma_c^{(0)\mu} \tag{24}$$

and for simplicity we consider the case $\gamma_\mu^{(0)a} = \delta_\mu^a$, where Eq. (21) can solely be expressed in terms of $\gamma_\mu^{(1)a}$, namely

$$A_d^\mu \equiv \eta^{\mu\alpha}\eta_{df}\Box\gamma_\alpha^{(1)f} + \delta_a^\mu\delta_d^\rho\Box\gamma_\rho^{(1)a} = 0. \tag{25}$$

This yields the following system of equations:

$$A_0^0\colon\ \Box\gamma_0^{(1)0} = 0, \tag{26}$$

$$A_i^0 = -A_0^i\colon\ \Box(\gamma_i^{(1)0} - \gamma_0^{(1)i}) = 0, \tag{27}$$

$$A_j^i(i\neq j)\colon\ \Box(\gamma_i^{(1)j} + \gamma_j^{(1)i}) = 0, \tag{28}$$

$$A_m^i(i=m)\colon\ \Box\gamma_i^{(1)i} = 0, \tag{29}$$

where we have used the fact that $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $i,j = \{1,2,3\}$. Since we are working in the Minkowski metric Cartesian coordinate system, the indices $\{0,1,2,3\}$ correspond to $\{t,x,y,z\}$, respectively. Last, we can demand the extra gauge condition that the waves are transverse, i.e., $h_{0\mu}^{(1)} = 0$, which sets $\gamma_0^{(1)0} = 0$ and $\gamma_0^{(1)i} = \gamma_i^{(1)0}$. In summary, the full list of conditions and equations are found to be

$$\text{Traceless condition}\colon\ \gamma_i^{(1)i} = 0, \tag{30}$$

$$\text{Lorenz gauge condition}\colon\ \partial_j(\gamma_i^{(1)j} + \gamma_j^{(1)i}) = 0, \tag{31}$$

$$A_j^i(i\neq j)\colon\ \Box(\gamma_i^{(1)j} + \gamma_j^{(1)i}) = 0, \tag{32}$$

$$A_m^i(i=m)\colon\ \Box\gamma_i^{(1)i} = 0. \tag{33}$$

Without loss of generality we make the choice that the gravitational wave propagates in the z direction, and as usual we work in the Fourier space. The wave equations then imply that

$$\gamma_i^{(1)i} = A_i{}^i \exp(ik_\mu x^\mu),\quad i = \{1,2,3\} \quad \text{(fixed index)}, \tag{34}$$

$$\gamma_i^{(1)j} + \gamma_j^{(1)i} = B_i{}^j \exp(ip_\mu x^\mu),\quad i = \{1,2,3\}, \qquad i \neq j, \tag{35}$$

where k_μ and p_μ are wave vectors such that $k_\mu k^\mu = p_\mu p^\mu = 0$, and where $A_i{}^i$ and $B_i{}^j$ are coefficients such that $A_1{}^1 = -A_2{}^2$, $A_3{}^3 = 0$, and $B_1{}^3 = B_2{}^3 = 0$. Note that these conditions arise from the traceless and Lorenz gauge conditions. Therefore, the undetermined coefficients are $A_1{}^1$, $B_1{}^3$, $B_2{}^3$, and $B_1{}^2$, which leads to the perturbed tetrad solution

$$\gamma_\mu^{(1)a} = \begin{pmatrix} 0 & \gamma_0^{(1)1} & \gamma_0^{(1)2} & \gamma_0^{(1)3} \\ \gamma_0^{(1)1} & \gamma_1^{(1)1} & \gamma_1^{(1)2} & \gamma_1^{(1)3} \\ \gamma_0^{(1)2} & B_1{}^2\exp(ip_\mu x^\mu) - \gamma_1^{(1)2} & -\gamma_1^{(1)1} & \gamma_2^{(1)3} \\ \gamma_0^{(1)3} & -\gamma_1^{(1)3} & -\gamma_2^{(1)3} & 0 \end{pmatrix}. \tag{36}$$

Here the $\gamma_i^{(1)j}$ are undetermined tetrad components which are not constrained by the equations. We mention that the perturbed metric then takes the form

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\gamma_1^{(1)1} & B_1{}^2\exp(ip_\mu x^\mu) & 0 \\ 0 & B_1{}^2\exp(ip_\mu x^\mu) & -2\gamma_1^{(1)1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

Note that obtaining the perturbed tetrad is not a trivial task in general, since amongst the infinite choices of perturbed tetrad ansatzes corresponding to the same perturbed metric, one should use the appropriate ones in order to obtain consistency [25,52,57].

Observing the solution (37) we can easily identify the standard $+$ and $\times$ polarizations of GR by defining $h_+ \equiv 2\gamma_1^{(1)1}$ and $h_\times \equiv B_1{}^2 \exp(ip_\mu x^\mu)$. Therefore, the perturbed tetrad has 2 physical d.o.f. (h_+ and $h_\times$) and 6 arbitrary d.o.f. related to Lorentz transformations. Hence, through the explicit solutions we did verify the result obtained previously, namely that at first-order perturbation level there are not any new polarization modes in $f(T)$ gravity. Finally, as we mentioned above, in order to examine the 6 d.o.f. related to Lorentz transformations it is necessary to reformulate the theory in a fully covariant way, namely keeping an arbitrary spin connection $\omega^a_{b\mu}$. This is performed in the next subsection.

B. GWs in $f(T)$ gravity with nonzero spin connection

For completeness and transparency, in this subsection we perform the analysis of the previous subsection, but in the case of a general spin connection, i.e., for the fully covariant formulation of $f(T)$ gravity presented in [55].

We start by considering the tetrad perturbation (15); however, we insert it in the torsion tensor (5) maintaining an arbitrary spin connection, obtaining up to first order:

$$T^a{}_{\mu\nu} = \partial_\mu\gamma_\nu^{(0)a} - \partial_\nu\gamma_\mu^{(0)a} + \omega^a_{b\mu}\gamma_\nu^{(0)b} - \omega^a{}_{b\nu}\gamma_\mu^{(0)b} + \partial_\mu\gamma_\nu^{(1)a} - \partial_\nu\gamma_\mu^{(1)a} + \omega^a{}_{b\mu}\gamma_\nu^{(1)b} - \omega^a{}_{b\nu}\gamma_\mu^{(1)b}. \tag{38}$$

As discussed in [55], the purely inertial spin connection can be found by demanding that the torsion tensor is zero when the gravitational constant vanishes, namely $G \to 0$, which yields the expression

$$\omega^a{}_{b\mu} = \Gamma^a_{b\mu} - E_b{}^\nu \partial_\mu e^a{}_\nu|_{G\to 0}, \tag{39}$$

where $\Gamma^a_{b\mu}$ is the GR Levi-Cività connection. As before, $e^a{}_\mu|_{G\to0} = \gamma_\mu^{(0)a}$. Furthermore, $g_{\mu\nu}|_{G\to0} = \eta_{\mu\nu}$ and hence $\Gamma^a_{b\mu}|_{G\to0} = 0$. Therefore, the spin connection turns out to be

$$\omega^a{}_{b\mu} = -\gamma_b^{(0)\nu}\partial_\mu\gamma_\nu^{(0)a}. \tag{40}$$

Since the torsion tensor is zero when the gravitational constant is zero, then

$$T^a{}_{\mu\nu}|_{G\to0} = \partial_\mu\gamma_\nu^{(0)a} - \partial_\nu\gamma_\mu^{(0)a} + \omega^a{}_{b\mu}\gamma_\nu^{(0)b} - \omega^a{}_{b\nu}\gamma_\mu^{(0)b} = 0. \tag{41}$$

Thus, the torsion tensor is first order, and so are the contorsion and superpotential, which ultimately implies that the torsion scalar is second order at the level of perturbations.

Before investigating the field equations we make the following remark. The purely inertial spin connection is given by

$$\omega^a{}_{b\mu} = -\Lambda_b{}^c\partial_\mu\Lambda^a{}_c, \tag{42}$$

where $\Lambda_d{}^c$ is a Lorentz matrix with inverse $\Lambda^a{}_c$. Thus, under this formulation, we deduce that the zeroth-order tetrad perturbations $\gamma_\nu^{(0)a}$ are precisely the Lorentz matrices. This shall be considered in what follows.

The next step is to expand the field equations at a perturbation level. In this case, the field equations for $f(T)$ gravity with an arbitrary spin connection are given by

$$e^{-1}f_T\partial_\nu(eS_a{}^{\mu\nu}) + f_{TT}S_a{}^{\mu\nu}\partial_\nu T - f_T T^b{}_{\nu a}S_b{}^{\nu\mu} + f_T\omega^b{}_{a\nu}S_b{}^{\nu\mu} + \frac{1}{4}f(T)e^a{}_\mu = 0, \quad (43)$$

which under the Taylor expansion (59), expanding order by order we obtain

$$\gamma_a^{(0)\rho}f^{(0)} = 0, \quad (44)$$

$$f_T^{(0)}[e^{-1(0)}\partial_\nu(e^{(0)}S_a{}^{(1)\mu\nu}) + \omega^b{}_{a\nu}S_b{}^{(1)\nu\mu}] + \gamma_a^{(0)\rho}\frac{f^{(1)}}{4} + \gamma_a^{(1)\rho}\frac{f^{(0)}}{4} = 0, \quad (45)$$

which generalize (19) and (20) in the case of nonzero spin connection. As before, the first condition implies that no cosmological constant is present. Moreover, we choose $e^{(0)} = e^{-1(0)} = 1$, since $\det\gamma_\nu^{(0)a} = \det\Lambda^a{}_\nu = 1$, which is a property of Lorentz matrices; similar to the zero spin connection case we impose $f^{(1)} = 0$ and we assume $f_T^{(0)} \neq 0$. Hence, Eq. (45) becomes

$$\partial_\nu S_a^{(1)\mu\nu} + \omega^b{}_{a\nu}S_b{}^{(1)\nu\mu} = 0, \quad (46)$$

which is the generalization of (21) in the case of nonzero spin connection. Finally, using the definition of the spin connection $\omega^a{}_{b\mu} = -\gamma_b^{(0)\nu}\partial_\mu\gamma_\nu^{(0)a}$, Eq. (46) can be recast into the simpler form

$$\gamma_a^{(0)\alpha}\partial_\nu[\gamma_\alpha^{(0)b}S_b{}^{(1)\mu\nu}] = \partial_\nu S_a{}^{(1)\mu\nu} + \gamma_a^{(0)\alpha}\partial_\nu\gamma_\alpha^{(0)b}S_b{}^{(1)\mu\nu} = 0. \quad (47)$$

Similar to the spin zero case, solving the above equation is not possible in general. However, we will again assume that the GR gauge conditions on the perturbed metric being traceless and satisfying the Lorenz gauge condition can also be imposed here. Hence, together with the relation

$$\gamma_\nu^{(0)d} = \eta^{cd}\eta_{\mu\nu}\gamma_c^{(0)\mu}, \quad (48)$$

the equation of motion reduces to the following simplified expression:

$$\eta^{\mu\alpha}\eta_{\beta\rho}\gamma_d^{(0)\beta}\Box(\gamma_f^{(0)\rho}\gamma_\alpha^{(1)f}) - \gamma_d^{(0)\beta}\partial_b\gamma_c^{(0)\mu}\partial^c\gamma_\beta^{(1)b} + \gamma_d^{(0)\beta}\Box(\gamma_b^{(0)\mu}\gamma_\beta^{(1)b}) - \gamma_d^{(0)\beta}\partial_\nu\gamma_\beta^{(1)c}\partial^\mu\gamma_c^{(0)\nu} - \eta^{\mu\alpha}\partial_b\gamma_d^{(0)\beta}\partial_\beta\gamma_\alpha^{(1)b} + \eta^{\mu\alpha}\partial_d\gamma_b^{(0)\beta}\partial_\beta\gamma_\alpha^{(1)b} = 0, \quad (49)$$

which can alternatively be expressed in terms of the spin connection as

$$\eta^{\mu\alpha}\eta_{df}\Box\gamma_\alpha^{(1)f} + \gamma_d^{(0)\beta}\gamma_b^{(0)\mu}\Box\gamma_\beta^{(1)b} + 2\eta^{\mu\alpha}\omega_{df\nu}\partial^\nu\gamma_\alpha^{(0)f} + \eta^{\mu\alpha}\gamma_d^{(0)\beta}\gamma_\alpha^{(1)f}\partial_\nu\omega_{\beta f}{}^\nu - \gamma_d^{(0)\beta}\omega^\mu{}_{cb}\partial^c\gamma_\beta^{(1)b} + 2\gamma_d^{(0)\beta}\partial^\alpha\gamma_\beta^{(0)b}\omega^\mu{}_{b\alpha} + \gamma_d^{(0)\beta}\gamma_\beta^{(1)b}\partial_\alpha\omega^\mu{}_b{}^\alpha - \gamma_d^{(0)\beta}\partial_\nu\gamma_\beta^{(1)c}\omega^\nu{}_c{}^\mu - \eta^{\mu\alpha}\omega^\beta{}_{db}\partial_\beta\gamma_\alpha^{(1)b} + \eta^{\mu\alpha}\omega^\beta{}_{bd}\partial_\beta\gamma_\alpha^{(1)b} = 0. \quad (50)$$

Finally, note that if we choose our frame of reference to correspond to zero spin connection, then the above equation reduces to

$$\eta^{\mu\alpha}\eta_{df}\Box\gamma_\alpha^{(1)f} - \Box\gamma_d^{(1)\mu} = 0, \quad (51)$$

where we have used the fact that

$$\gamma_b^{(1)\nu} = -\gamma_a^{(0)\nu}\gamma_\mu^{(1)a}\gamma_b^{(0)\mu}, \quad (52)$$

which arises from (3).

In general, as it was mentioned in [55], in the case of nonzero spin connection it is hard even to extract the background solutions. Hence we can see that obtaining the perturbed solution seems very difficult, since the background tetrad affects the perturbed solution. The detailed examination of the perturbed solutions in the case of $f(T)$ gravity with nonzero spin connection lies beyond the scope of the present work.

We close this section by mentioning that the presented methodology can be extended to more general torsional modified gravitational theories, by defining appropriate gauge conditions on the tetrad, especially to theories in which the coordinate-indexed form of the field equations results in mixing between the metric and tetrad tensors. In this way, any information about the tetrad is not lost within the metric tensor, since the appropriate field equations are solved.

C. Higher-order metric perturbations

In this subsection we proceed to the analysis of higher-order perturbations, in order to understand more transparently the potential deviations from general relativity. The standard approach is to consider perturbations around a flat Minkowski background. This is achieved by perturbing the metric tensor in the following manner:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + \mathcal{O}(h_{\mu\nu}^{(4)}), \quad (53)$$

where $|h_{\mu\nu}^{(i)}| \ll 1$, which is retained up to third order in this instance. Since the fundamental variable in the torsional formulation is the tetrad, the above metric perturbation can be obtained by the tetrad perturbation

$$e^a{}_\mu = \delta_\mu^a + \gamma_\mu^{(1)a} + \gamma_\mu^{(2)a} + \gamma_\mu^{(3)a} + \mathcal{O}(\gamma_\mu^{(4)a}), \quad (54)$$

where $|\gamma_\mu^{(i)a}| \ll 1$. We remark that, in general, the zeroth-order part of the tetrad perturbation is determined by the background metric, which in linearized gravity is usually the Minkowski metric. Thus, in the current case the zeroth-order contribution to the tetrad perturbation turns out to be represented by the identity matrix, and that is why we introduced the Kronecker delta. However, we mention that this does not affect the obtained results, and the same conclusion is reached for other backgrounds, too.

By the definition of the metric tensor $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, we can relate the metric and tetrad perturbations through

$$h_{\mu\nu}^{(1)} = \eta_{ab}(\delta_\mu^a \gamma_\nu^{(1)b} + \gamma_\mu^{(1)a}\delta_\nu^b), \tag{55}$$

$$h_{\mu\nu}^{(2)} = \eta_{ab}(\delta_\mu^a \gamma_\nu^{(2)b} + \gamma_\mu^{(1)a}\gamma_\nu^{(1)b} + \gamma_\mu^{(2)a}\delta_\nu^b), \tag{56}$$

$$h_{\mu\nu}^{(3)} = \eta_{ab}(\delta_\mu^a \gamma_\nu^{(3)b} + \gamma_\mu^{(1)a}\gamma_\nu^{(2)b} + \gamma_\mu^{(2)a}\gamma_\nu^{(1)b} + \gamma_\mu^{(3)a}\delta_\nu^b). \tag{57}$$

Inserting these expressions into the definition of the torsion tensor (5), and assuming for the moment zero spin connection, we obtain

$$T^a{}_{\mu\nu} = \partial_\mu \gamma_\nu^{(1)a} - \partial_\nu \gamma_\mu^{(1)a} + \mathcal{O}(\gamma_\mu^{(2)a}), \tag{58}$$

from which we can see that the torsion tensor is at least of first order, with the zeroth-order contribution equal to zero. Consequently, from the definitions of the contorsion and superpotential tensors, namely relations (7) and (6), respectively, we deduce that they are both also at least of first order since their zeroth-order contributions are zero. Thus, the torsion scalar T, which is quadratic in the torsion tensor, becomes a second-order quantity. Finally, in order to handle the $f(T)$ term, for simplicity we assume that this function is Taylor expandable around $T = 0$, namely

$$f(T) = f(0) + f_T(0)T + \frac{1}{2!} f_{TT}(0)T^2 + \cdots. \tag{59}$$

Let us proceed by perturbing the equations of motion. According to Eq. (13), in the case of $f(T)$ gravity the field equations become

$$-f_T G_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(f_T T - f) + 2S_\nu{}^\alpha{}_\mu \partial_\alpha f_T = 0. \tag{60}$$

We mention that since we are interested in examining the properties of the gravitational waves, for simplicity we have neglected the contribution of the matter stress-energy tensor; namely we neglect quadrupole moments which arise from the stress-energy tensor.

Inserting the perturbed tetrad and metric in the field equations (60), and under the Taylor expansion (59), order by order we obtain

$$\eta_{\mu\nu} f(0) = 0, \tag{61}$$

$$f_T(0) G_{\mu\nu}^{(1)} = 0, \tag{62}$$

$$f_T(0) G_{\mu\nu}^{(2)} = 0, \tag{63}$$

$$f_T(0) G_{\mu\nu}^{(3)} + f_{TT}(0) T^{(2)} G_{\mu\nu}^{(1)} - 2 f_{TT}(0) S_\nu^{(1)\alpha}{}_\mu \partial_\alpha T^{(2)} = 0. \tag{64}$$

Considering only the nontrivial case $f_T(0) \neq 0$ (otherwise GR cannot be obtained at any limit) the perturbed field equations simplify further to

$$\eta_{\mu\nu} f(0) = 0, \tag{65}$$

$$G_{\mu\nu}^{(1)} = 0, \tag{66}$$

$$G_{\mu\nu}^{(2)} = 0, \tag{67}$$

$$G_{\mu\nu}^{(3)} = 2 \frac{f_{TT}(0)}{f_T(0)} S_\nu^{(1)\alpha}{}_\mu \partial_\alpha T^{(2)}. \tag{68}$$

As we observe, the zeroth-order equation (65) implies that no cosmological constant is present in the analysis, which was expected since the considered perturbations are around a Minkowski background and not around a cosmological constant one. The first- and second-order equations coincide with the standard GR perturbed equations in vacuum. However, the new information is that at the third-order equation (68) we find a deviation from the standard GR perturbation equation, with a contribution arising from the f_{TT} term. Thus, the $f(T)$ effect on the perturbation equations enters only at the higher than second order, and the reason behind this is that the torsion scalar is quadratic in the torsion tensor. This is a radical difference with the case of curvature-based modified gravity, where the effect of the modification becomes manifest from first-order perturbation already. These features will become more transparent in the next section, where we study the case of $f(T, B)$ gravity. Finally, note that in the GR limit, i.e., at $f_{TT}(0) = 0$, we reobtain the standard GR results.

In summary, as we showed, in order to see the effect of $f(T)$ gravity on the gravitational waves themselves, one should look at third-order perturbations (higher-order contributions in curvature gravity have been examined in literature; see e.g., [58,59]). Note that this concerns the effect on the "internal" properties of the gravitational waves, as for instance in their polarization modes, where it was known that no further polarization modes are present in $f(T)$ gravity at first-order perturbation levels [50,51]. However, we stress that in general the effect of $f(T)$ gravity on the cosmological gravitational wave propagation can be seen straightaway from the dispersion relation at first order, due to the effect of $f(T)$ gravity on the cosmological background itself [52].

IV. GRAVITATIONAL WAVES IN $f(T,B)$ GRAVITY

In this section we will investigate the gravitational waves in the case of $f(T,B)$ gravity with action (11). From now on we consider only the case of zero spin connection, and we focus on the case $f(T,B)\neq f(T)$ since $f(T)$ gravity was investigated in the previous section. Furthermore, for convenience, we first study the gravitational waves around a Minkowski background, i.e., in the case where a cosmological constant is absent from the $f(T,B)$ form, and then we proceed to the general investigation of the case where a cosmological constant is allowed.

A. GWs in $f(T,B)$ gravity in the absence of a cosmological constant

We start with the perturbed metric around a Minkowski background:

$$g_{\mu\nu}=\eta_{\mu\nu}+h^{(1)}_{\mu\nu}+\mathcal{O}(h^{(2)}_{\mu\nu}), \tag{69}$$

where $|h^{(i)}_{\mu\nu}|\ll 1$. This metric perturbation can be obtained from the perturbed tetrad

$$e^a{}_\mu=\delta^a_\mu+\gamma_\mu^{(1)a}+\mathcal{O}(\gamma_\mu^{(2)a}). \tag{70}$$

Using relation $g_{\mu\nu}=\eta_{ab}e^a{}_\mu e^b{}_\nu$ we acquire

$$h^{(1)}_{\mu\nu}=\eta_{ab}(\delta^a_\mu\gamma_\nu^{(1)b}+\gamma_\mu^{(1)a}\delta^b_\nu), \tag{71}$$

and thus for the perturbed torsion tensor we obtain

$$T^a{}_{\mu\nu}=\partial_\mu\gamma_\nu^{(1)a}-\partial_\nu\gamma_\mu^{(1)a}+\mathcal{O}(\gamma_\mu^{(2)a}). \tag{72}$$

As we mentioned earlier, the torsion tensor is at least first order, and thus the torsion scalar T is of second order in perturbations. This has a significant consequence, namely that relation (1), specifically $R=-T+B$, at first order becomes $R^{(1)}=B^{(1)}$ (we remind the reader that R is calculated using the Levi-Cività connection while T and B are calculated with the Weitzenböck connection). Indeed, the Ricci scalar at first order is given to be

$$R^{(1)}=\eta^{\mu\nu}\partial_\rho\partial_\nu h^{(1)\rho}{}_\mu-\Box h^{(1)}, \tag{73}$$

where indices are raised with respect to the Minkowski metric, $h^{(1)}\equiv h^{(1)\mu}{}_\mu$ and $\Box\equiv\partial_\mu\partial^\mu$. Expanding in terms of tetrads yields

$$R^{(1)}=2\delta^\rho_b(\eta^{\mu\nu}\partial_\nu\partial_\rho\gamma_\mu^{(1)b}-\Box\gamma_\rho^{(1)b}). \tag{74}$$

On the one hand, expanding the boundary term at first order yields

$$B^{(1)}=-2(\nabla^\mu T^\nu{}_{\mu\nu})^{(1)}=-2\eta^{\mu\rho}\partial_\rho T^{(1)\nu}{}_{\mu\nu} \\ =2\delta^\rho_b(\eta^{\mu\nu}\partial_\nu\partial_\rho\gamma_\mu^{(1)b}-\Box\gamma_\rho^{(1)b}). \tag{75}$$

Thus, we can immediately see that at this order it is equal to the Ricci scalar.

In order to handle the $f(T,B)$ term for simplicity we assume that its form is Taylor expandable around the current values T_0 and B_0, namely

$$\begin{aligned}f(T,B)=&f(T_0,B_0)+f_T(T_0,B_0)(T-T_0)\\&+f_B(T_0,B_0)(B-B_0)\\&+\frac{1}{2!}f_{TT}(T_0,B_0)(T-T_0)^2\\&+\frac{1}{2!}f_{BB}(T_0,B_0)(B-B_0)^2\\&+f_{TB}(T_0,B_0)(T-T_0)(B-B_0)+\cdots.\end{aligned} \tag{76}$$

Furthermore, since we are only examining the properties of the gravitational waves, we neglect the matter sector.

Inserting all the above into the field equations of $f(T,B)$ gravity, namely Eq. (13), order by order we obtain

$$\eta_{\mu\nu}f(0,0)=0, \tag{77}$$

$$-f_T(0,0)G^{(1)}_{\mu\nu}+f_{BB}(0,0)(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)R^{(1)}=0, \tag{78}$$

where we have used the fact that $R^{(1)}=B^{(1)}$, and that $f(0,0)=0$ from the zeroth-order condition. The latter condition is another statement for the fact that the arbitrary Lagrangian function does not include a cosmological constant.

We proceed following [51], and we define an effective mass by considering the trace of the first-order equation. This is also similar to the $f(R)$ gravity case. However, our effective mass is different from that of Ref. [51]. Indeed, by taking the trace

$$f_T(0,0)R^{(1)}+3f_{BB}(0,0)\Box R^{(1)}=0, \tag{79}$$

we identify the effective mass m by bringing the equation in the form $(\Box-m^2)R^{(1)}=0$, which turns out to be

$$m^2\equiv-\frac{f_T(0,0)}{3f_{BB}(0,0)}. \tag{80}$$

We remark that in the $|m^2|\to\infty$ limit [for instance when $f_{BB}(0,0)=0$ and $f_T(0,0)\neq 0$], the equation reduces to that of GR. Since it is known that $f(T)$ gravity yields no further gravitational wave modes [50], as we verified in the previous section, this special condition leads to a broader class of theories in which at first order yields the gravitational wave solutions.

In the case where $f_{BB}(0,0)\neq 0$ we can follow the procedure of $f(R)$ gravity [60–64] [note that $f(R)$ is a particular subclass of $f(T,B)$ gravity, namely $f(-T+B)$ gravity]. First, we introduce the tensor $\bar{h}^{(1)}_{\mu\nu}$ to be

$$h^{(1)}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \frac{1}{2}\bar{h}^{(1)}\eta_{\mu\nu} + \frac{f_{BB}(0,0)}{f_T(0,0)}\eta_{\mu\nu}R^{(1)}, \quad (81)$$

where $\bar{h}^{(1)}$ represents the trace of $\bar{h}^{(1)}_{\mu\nu}$. Similar to the previous section we consider the nontrivial case of $f_T(0,0) \neq 0$ (otherwise GR cannot be obtained at any limit). This simplifies Eq. (78) to

$$\partial^\rho\partial_\nu\bar{h}^{(1)}_{\rho\mu} + \partial^\rho\partial_\mu\bar{h}^{(1)}_{\nu\rho} - \eta_{\mu\nu}\partial^\rho\partial^\alpha\bar{h}^{(1)}_{\rho\alpha} - \Box\bar{h}^{(1)}_{\mu\nu} = 0. \quad (82)$$

As shown in [60], it is possible to consider the Lorenz gauge condition $\partial^\mu\bar{h}^{(1)}_{\mu\nu} = 0$, which simplifies the wave equation to

$$\Box\bar{h}^{(1)}_{\mu\nu} = 0, \quad (83)$$

as well as the traceless condition $\bar{h}^{(1)} = 0$. This allows for the solution

$$\bar{h}^{(1)}_{\mu\nu} = A_{\mu\nu}\exp(ik_\rho x^\rho), \quad (84)$$

where k_ρ is the four-wave vector, $A_{\mu\nu}$ are constant coefficients, $k_\rho k^\rho = 0$, $k^\mu A_{\mu\nu} = 0$, and $A^\mu{}_\mu = 0$. The last conditions are the Lorenz gauge and traceless conditions, respectively. On the other hand, the solution for (79) is

$$R^{(1)} = F\exp(ip_\mu x^\mu), \quad (85)$$

where F is a constant and p_μ is another four-wave vector such that $p_\mu p^\mu = -m^2$. Hence, the full solution for $h^{(1)}_{\mu\nu}$ is constructed as

$$h^{(1)}_{\mu\nu} = A_{\mu\nu}\exp(ik_\rho x^\rho) + \frac{f_{BB}(0,0)}{f_T(0,0)}\eta_{\mu\nu}F\exp(ip_\mu x^\mu). \quad (86)$$

Note that from (78) and (79), the Ricci tensor is found to be

$$R^{(1)}_{\mu\nu} = \frac{1}{6}\eta_{\mu\nu}R^{(1)} - \frac{f_{BB}(0,0)}{f_T(0,0)}\partial_\mu\partial_\nu R^{(1)}, \quad (87)$$

from which the solution of the Ricci scalar (85) simplifies to

$$R^{(1)}_{\mu\nu} = \left(\frac{1}{6}\eta_{\mu\nu} - \frac{1}{3m^2}p_\mu p_\nu\right)R^{(1)}. \quad (88)$$

Hence, it is trivial to verify that taking the trace yields a consistent relation for the Ricci scalar, as expected.

We proceed by analyzing the polarization states of the gravitational waves. As usual we consider the geodesic deviation as in Ref. [60]. We remark that although in teleparallel theories the particle motion is not described in terms of geodesics, mathematically one may still use the geodesic deviation formula, having in mind that all curvature quantities should obviously be calculated using the Levi-Cività connection [54] [for instance see [65] for the geodesic deviation in $f(T)$ gravity]. Hence, we start from the geodesic deviation formula [66]

$$\ddot{x}_i = -R_{i0j0}x^j, \quad (89)$$

where dots represent coordinate time derivatives, $R_{\mu\nu\lambda\rho}$ is the Riemann tensor calculated with the Levi-Cività connection, $(t,x,y,z) = (0,1,2,3)$, $i = \{1,2,3\}$, and $x^j = (x,y,z)$. Moreover, we consider the signature $(+,-,-,-)$, and for simplicity we assume that the wave propagates in the z direction.

From the perturbation analysis presented above, we find that

$$R_{i0j0} = \frac{1}{2}k_0^2\bar{h}^{(1)}_{ij} - \frac{1}{6m^2}[\eta_{ij}p_0^2R^{(1)} + p_ip_jR^{(1)}]. \quad (90)$$

Therefore, the geodesic deviation becomes

$$\ddot{x} = \left[\frac{1}{2}k_0^2\bar{h}^{(1)}_+ + \frac{1}{6m^2}p_0^2R^{(1)}\right]x + \frac{1}{2}k_0^2\bar{h}^{(1)}_\times y, \quad (91)$$

$$\ddot{y} = \left[-\frac{1}{2}k_0^2\bar{h}^{(1)}_+ + \frac{1}{6m^2}p_0^2R^{(1)}\right]y + \frac{1}{2}k_0^2\bar{h}^{(1)}_\times x, \quad (92)$$

$$\ddot{z} = \frac{1}{6m^2}(p_0^2 - p_3^2)R^{(1)}z = -\frac{1}{6}R^{(1)}z, \quad (93)$$

where in the last equation we have used that $p_\mu p^\mu = -m^2$. Additionally, since the wave propagates in the z direction, we have used and defined $\bar{h}^{(1)}_{11} = -\bar{h}^{(1)}_{22} \equiv \bar{h}^{(1)}_+$ and $\bar{h}^{(1)}_{12} = \bar{h}^{(1)}_{21} \equiv \bar{h}^{(1)}_\times$, which represent the massless $+$ and $\times$ polarizations.

As we observe, in the TEGR limit, namely at $|m^2| \to \infty$ and $R^{(1)} \to 0$, the remaining modes are the $+$ and $\times$ polarizations as expected. However, in the case $|m^2| < \infty$ we find the presence of the longitudinal and breathing modes in the geodesic deviation equations. This is one of the main results of the present work, namely that $f(T,B)$ gravity, in the case where $f(T,B) \neq f(T)$, does have further polarization modes at first-order perturbation, in contrast to the case of $f(T)$ gravity. The reason for this behavior is the fact that although the first-order perturbation does not have any effect on T, it does affect the boundary term B.

B. Tetrad solutions for GWs in $f(T,B)$ gravity

In the previous subsection we analyzed the gravitational waves in $f(T,B)$ gravity from the metric perturbation side. We now proceed to their examination from the tetrad perturbation side. In order to do this we start from the perturbed tetrad (70), and we insert it into the tetrad form of the $f(T,B)$ field equations, namely into Eq. (12). Neglecting the matter sector, order by order we obtain

$$\delta_a^\mu f^{(0)} = 0, \tag{94}$$

$$\delta_a^\mu \Box f_B^{(1)} - \delta_a^\nu \partial^\mu \partial_\nu f_B^{(1)} + \frac{1}{2} B^{(1)} f_B^{(0)} \delta_a^\mu + 2\partial_\nu S_a{}^{(1)\nu\mu} f_T^{(0)} - \frac{1}{2}\delta_a^\mu f^{(1)} = 0. \tag{95}$$

As before, the zeroth-order condition is a verification that there is no cosmological constant present, i.e., that the perturbation is performed around the Minkowski background. In order to simplify the first-order equation we remark that $f^{(1)} = f_B^{(0)} B^{(1)}$ and $f_B^{(1)} = f_{BB}^{(0)} B^{(1)}$, and as usual we consider the nontrivial case $f_T^{(0)} \neq 0$. Therefore, Eq. (95) reduces to

$$\partial_\nu S_a{}^{(1)\nu\mu} + \frac{f_{BB}^{(0)}}{2f_T^{(0)}}[\delta_a^\mu \Box B^{(1)} - \delta_a^\nu \partial^\mu \partial_\nu B^{(1)}] = 0. \tag{96}$$

Due to the introduction of the $B^{(1)}$ terms in the above equation, the traceless and Lorenz conditions used for the simple case of $f(T)$ gravity in Sec. III A need to be modified to accommodate a more suitable gauge choice. From the metric approach of the previous subsection we instead have the "trace-reversed" metric $\bar{h}_{\mu\nu}$ in (81), which satisfies the traceless and Lorenz gauge conditions $\bar{h} = 0$, and $\partial^\mu \bar{h}_{\mu\nu} = 0$. Therefore, the "traceless" condition becomes

$$\delta_b^\nu \gamma_\nu^{(1)b} = \frac{2f_{BB}^{(0)} B^{(1)}}{f_T^{(0)}}, \tag{97}$$

while the "Lorenz condition" reads as

$$\eta_{ab}(\partial^a \gamma_\nu^{(1)b} + \delta_\nu^b \partial^\mu \gamma_\mu^{(1)a}) = \frac{f_{BB}^{(0)} \partial_\nu B^{(1)}}{f_T^{(0)}}. \tag{98}$$

In this way, the field equations simplify to

$$A_a^\mu \equiv \eta^{\mu\alpha}\eta_{ab}\Box\gamma_\alpha^{(1)b} + \delta_b^\mu \delta_a^\rho \Box\gamma_\rho^{(1)b} - \delta_a^\mu \frac{f_{BB}^{(0)}\Box B^{(1)}}{f_T^{(0)}} = 0, \tag{99}$$

which yields the following system of field equations:

$$A_0^0\colon \Box\left(\gamma_0^{(1)0} - \frac{f_{BB}^{(0)} B^{(1)}}{2f_T^{(0)}}\right) = 0, \tag{100}$$

$$A_i^0 = -A_0^i\colon \Box(\gamma_i^{(1)0} - \gamma_0^{(1)i}) = 0, \tag{101}$$

$$A_j^i (i \neq j)\colon \Box(\gamma_i^{(1)j} + \gamma_j^{(1)i}) = 0, \tag{102}$$

$$A_m^i (i = m)\colon \Box\left(\gamma_i^{(1)i} - \frac{f_{BB}^{(0)} B^{(1)}}{2f_T^{(0)}}\right) = 0, \tag{103}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $i,j = \{1,2,3\}$. Moreover, since we are working in the Minkowski metric Cartesian coordinate system, the indices $\{0,1,2,3\}$ correspond to $\{t,x,y,z\}$, respectively.

The above equations are standard wave equations, and thus we assume a plane-wave solution by working in Fourier space. Without loss of generality, we shall assume that the waves propagate in the z direction. Hence, the solution for the perturbed tetrad is

$$\gamma_\mu^{(1)a} = \begin{pmatrix} A\exp(ik_\mu x^\mu) + \frac{f_{BB}^{(0)}B^{(1)}}{2f_T^{(0)}} & \gamma_0^{(1)1} & \gamma_0^{(1)2} & \gamma_0^{(1)3} \\ B_1\exp(ik_\mu x^\mu) + \gamma_0^{(1)1} & D\exp(ik_\mu x^\mu) + \frac{f_{BB}^{(0)}B^{(1)}}{2f_T^{(0)}} & \gamma_1^{(1)2} & \gamma_1^{(1)3} \\ B_2\exp(ik_\mu x^\mu) + \gamma_0^{(1)2} & C\exp(ik_\mu x^\mu) - \gamma_1^{(1)2} & -D\exp(ik_\mu x^\mu) + \frac{f_{BB}^{(0)}B^{(1)}}{2f_T^{(0)}} & \gamma_2^{(1)3} \\ \gamma_0^{(1)3} - 2A\exp(ik_\mu x^\mu) & B_1\exp(ik_\mu x^\mu) - \gamma_1^{(1)3} & B_2\exp(ik_\mu x^\mu) - \gamma_2^{(1)3} & -A\exp(ik_\mu x^\mu) + \frac{f_{BB}^{(0)}B^{(1)}}{2f_T^{(0)}} \end{pmatrix}. \tag{104}$$

Here, the $\gamma_i^{(1)j}$ are undetermined tetrad components, not constrained by the equations, A, $B_{1,2}$, C, and D are constants, and k_μ is the wave vector such that $k_\mu k^\mu = 0$.

As we observe, in the $f_{BB}^{(0)} \to 0$ limit the first-order perturbed equation (99) reduces to that of GR and $f(T)$ gravity, and hence the tetrad solution should describe the same solution. Comparing with the tetrad solution obtained in the case of simple $f(T)$ gravity, namely solution (36), we deduce that this is obtained by setting $A = B_{1,2} = 0$, implying that these constants reflect the transverse property of the $h_+ \equiv 2D\exp(ik_\mu x^\mu)$ and $h_\times \equiv C\exp(ik_\mu x^\mu)$ polarizations and hence can be removed. This can also be identified from the metric tensor solution corresponding to (104), namely

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} -2A\exp(ik_\mu x^\mu) - \frac{f_{BB}^{(0)}B^{(1)}}{f_T^{(0)}} & B_1\exp(ik_\mu x^\mu) & B_2\exp(ik_\mu x^\mu) & -2A\exp(ik_\mu x^\mu) \\ B_1\exp(ik_\mu x^\mu) & h_+ + \frac{f_{BB}^{(0)}B^{(1)}}{f_T^{(0)}} & h_\times & B_1\exp(ik_\mu x^\mu) \\ B_2\exp(ik_\mu x^\mu) & h_\times & -h_+ + \frac{f_{BB}^{(0)}B^{(1)}}{f_T^{(0)}} & B_2\exp(ik_\mu x^\mu) \\ -2A\exp(ik_\mu x^\mu) & B_1\exp(ik_\mu x^\mu) & B_2\exp(ik_\mu x^\mu) & -2A\exp(ik_\mu x^\mu) + \frac{f_{BB}^{(0)}B^{(1)}}{f_T^{(0)}} \end{pmatrix}, \quad (105)$$

which in the $A, B_{1,2}, f_{BB}^{(0)} \to 0$ limit reduces to the standard perturbed metric solution for waves traveling in the z direction.

C. GWs in $f(T,B)$ gravity in the presence of a cosmological constant

In the previous subsection we investigated the gravitational waves in $f(T,B)$ gravity through a perturbation around a Minkowski background in vacuum, i.e., in the absence of a cosmological constant in the form of $f(T,B)$. In the present subsection we examine the contribution of a cosmological constant to the gravitational waves following the procedure of [67,68]. Implications of the cosmological constant onto gravitational waves (for instance quadrupole contributions and effects during the cosmological history), in the case of other scenarios, have been investigated in [69–73].

Let Λ denote the cosmological constant. Similar to the previous analysis we shall again consider a linearized gravity approach; however, since the cosmological constant also affects the background metric, the perturbations will not be performed around a Minkowski background but around a background that incorporates the contributions of Λ. This is achieved through

$$g_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)} + \mathcal{O}(h_{\mu\nu}^{(2)}), \quad (106)$$

where $|h_{\mu\nu}^{(i)}| \ll 1$, and

$$h_{\mu\nu}^{(0)} = \eta_{\mu\nu} + \Lambda h_{\mu\nu}^{(0)\Lambda} + \mathcal{O}(\Lambda^2), \quad (107)$$

$$h_{\mu\nu}^{(1)} = h_{\mu\nu}^{(1)\,\mathrm{GW}} + \Lambda h_{\mu\nu}^{(1)\,\mathrm{GW}\Lambda} + \mathcal{O}(\Lambda^2), \quad (108)$$

where $h_{\mu\nu}^{(0)\Lambda}$ refers to the first-order contribution of Λ to the background metric, $h_{\mu\nu}^{(1)\,\mathrm{GW}}$ is the gravitational wave perturbation without the effect of Λ, and $h_{\mu\nu}^{(1)\,\mathrm{GW}\Lambda}$ is the contribution of the cosmological constant to the gravitational wave perturbation. Correspondingly, the tetrad perturbation can be constructed as

$$e^a{}_\mu = \gamma_\mu^{(0)a} + \gamma_\mu^{(1)a} + \mathcal{O}(\gamma_\mu^{(2)a}), \quad (109)$$

where $|\gamma_\mu^{(i)a}| \ll 1$, and

$$\gamma_\mu^{(0)a} = \delta_\mu^a + \Lambda\gamma_\mu^{(0,\Lambda)a} + \mathcal{O}(\Lambda^2), \quad (110)$$

$$\gamma_\mu^{(1)a} = \gamma_\mu^{(1,\mathrm{GW})a} + \Lambda\gamma_\mu^{(1,\mathrm{GW}\Lambda)a} + \mathcal{O}(\Lambda^2), \quad (111)$$

where $\gamma_\mu^{(0,\Lambda)a}$ refers to the first-order contribution of Λ to the background tetrad, $\gamma_\mu^{(1,\mathrm{GW})a}$ is the gravitational wave perturbation without the effect of Λ, and $\gamma_\mu^{(1,\mathrm{GW}\Lambda)a}$ is the contribution of the cosmological constant to the gravitational wave perturbation. Therefore, the metric perturbations are related to the tetrad perturbations through

$$h_{\mu\nu}^{(0)\Lambda} = \eta_{ab}(\delta_\mu^a\gamma_\nu^{(0,\Lambda)b} + \gamma_\mu^{(0,\Lambda)a}\delta_\nu^b), \quad (112)$$

$$h_{\mu\nu}^{(1)\,\mathrm{GW}} = \eta_{ab}(\delta_\mu^a\gamma_\nu^{(1,\mathrm{GW})b} + \gamma_\nu^{(1,\mathrm{GW})a}\delta_\mu^b), \quad (113)$$

$$h_{\mu\nu}^{(1)\,\mathrm{GW}\Lambda} = \eta_{ab}(\delta_\mu^a\gamma_\nu^{(1,\mathrm{GW}\Lambda)b} + 2\gamma_\mu^{(1,\mathrm{GW})a}\gamma_\nu^{(0,\Lambda)b} + \gamma_\mu^{(1,\mathrm{GW}\Lambda)a}\delta_\nu^b). \quad (114)$$

To facilitate the perturbation analysis, we again assume that the function $f(T,B)$ is Taylor expandable about $T = B = 0$, namely

$$f(T,B) = f(0,0) + f_T(0,0)T + f_B(0,0)B + \frac{1}{2!}f_{TT}(0,0)T^2 + \frac{1}{2!}f_{BB}(0,0)B^2 + f_{TB}(0,0)TB + \cdots. \quad (115)$$

Hence, we can identify $f(0,0) \equiv 2\Lambda$, since this term behaves as a cosmological constant, which can be seen when it is substituted in the equations of motion or in the action. The choice of this value is in order to ease the comparisons with the cosmological constant present in GR.

Inserting the above into the field equations (13) and neglecting the matter sector, for the zeroth-order perturbation equation we obtain

$$-f_T^{(0)}G_{\mu\nu}^{(0)} + \frac{1}{2}h_{\mu\nu}^{(0)}(f_B^{(0)}B^{(0)} + f_T^{(0)}T^{(0)} - f^{(0)}) = 0, \quad (116)$$

which when expanded over the Λ order yields

$$f_T(0,0)G_{\mu\nu}^{(0)\Lambda} + \frac{1}{2}\eta_{\mu\nu}f(0,0) = 0, \quad (117)$$

where the superscript Λ refers to the cosmological constant contribution of the function (from here onwards, this shall be assumed for all symbols). Taking the trace yields

$$f_T(0,0)R^{(0)\Lambda} = 2f(0,0). \tag{118}$$

In the case of TEGR with a cosmological constant, i.e., for $f(T,B) = T + 2\Lambda$, the above expression yields precisely the standard result for GR with a cosmological constant, namely $R = 4\Lambda$. Note that the first derivative $f_T(0,0)$ rescales this value of the Ricci scalar.

Additionally, the first-order perturbation becomes

$$\begin{aligned}&-f_T^{(0)}G_{\mu\nu}^{(1)} - f_T^{(1)}G_{\mu\nu}^{(0)} + [h_{\mu\nu}^{(0)}\Box^{(0)} - \nabla_\mu^{(0)}\nabla_\nu^{(0)}]f_B^{(1)}\\ &+\frac{1}{2}h_{\mu\nu}^{(0)}[f_B^{(0)}B^{(1)} + f_B^{(1)}B^{(0)} + f_T^{(0)}T^{(1)} + f_T^{(1)}T^{(0)} - f^{(1)}]\\ &+\frac{1}{2}h_{\mu\nu}^{(1)}[f_B^{(0)}B^{(0)} + f_T^{(0)}T^{(0)} - f^{(0)}]\\ &+2S_\nu^{(0)\alpha}{}_\mu\partial_\alpha(f_T^{(1)} + f_B^{(1)}) + 2S_\nu^{(1)\alpha}{}_\mu\partial_\alpha(f_T^{(0)} + f_B^{(0)}) = 0.\end{aligned} \tag{119}$$

Expanding over the Λ order yields

$$f_T(0,0) - G^{(1)\,\mathrm{GW}} + f_{BB}(0,0)(\eta_{\mu\nu}\Box^{(0)\mathrm{BG}} - \partial_\mu\partial_\nu)B^{(1)\,\mathrm{GW}} = 0, \tag{120}$$

and

$$\begin{aligned}&-f_T(0,0)G^{(1)\,\mathrm{GW}\Lambda} - G^{(1)\,\mathrm{GW}}f_{TB}(0,0)B^{(0)\Lambda} - f_{TB}(0,0)B^{(1)\,\mathrm{GW}}G_{\mu\nu}^{(0)\Lambda} + f_{BB}(0,0)[h_{\mu\nu}^{(0)\Lambda}\Box^{(0)\mathrm{BG}} + \eta_{\mu\nu}\Box^{(0)\Lambda} - \nabla_\mu^{(0)\Lambda}\partial_\nu]B^{(1)\,\mathrm{GW}}\\ &+(\eta_{\mu\nu}\Box^{(0)} - \partial_\mu\partial_\nu)[f_{BB}(0,0)B^{(1)\,\mathrm{GW}\Lambda} + f_{TB}(0,0)T^{(1)\,\mathrm{GW}\Lambda}] + \frac{1}{2}\eta_{\mu\nu}f_{BB}(0,0)B^{(0)\Lambda}B^{(1)\,\mathrm{GW}} - \frac{1}{2}h_{\mu\nu}^{(1)\,\mathrm{GW}}f(0,0)\\ &+2[f_{TB}(0,0) + f_{BB}(0,0)]S_\nu^{(0,\Lambda)\alpha}{}_\mu\partial_\alpha B^{(1)\,\mathrm{GW}} + 2[f_{TB}(0,0) + f_{BB}(0,0)]S_\nu^{(1,\mathrm{GW})\alpha}{}_\mu\partial_\alpha B^{(0)\Lambda} = 0,\end{aligned} \tag{121}$$

where the superscript BG refers to the Minkowski metric contribution. It is clear that Eq. (120) is exactly the same as (78) (recalling that $R^{(1)} = B^{(1)}$). On the other hand, Eq. (121) describes the relation between the standard gravitational waves and the cosmological constant contribution to them, which is the main result of the present subsection.

Finally, it is interesting to note that for the case of simple $f(T)$ gravity, namely for $f(T,B) = f(T)$, Eqs. (120) and (121) reduce to

$$f_T(0)G^{(1)\,\mathrm{GW}} = 0, \tag{122}$$

$$f_T(0)G^{(1)\,\mathrm{GW}\Lambda} + \frac{1}{2}h_{\mu\nu}^{(1)\,\mathrm{GW}}f(0) = 0. \tag{123}$$

Thus, for $f(T)$ gravity at first order, the effect of the cosmological constant on the gravitational waves is also affected by the value of $f_T(0)$, as was also found in [52]. Last, in the case of TEGR these equations match identically to those of GR, as expected.

V. GRAVITATIONAL WAVES IN $f(T,T_G)$ GRAVITY

In this section we proceed to the investigation of the gravitational waves in another class of modified teleparallel gravity, namely $f(T,T_G)$ gravity, in which one uses in the Lagrangian the teleparallel equivalent of the Gauss-Bonnet combination T_G. In particular, in curvature-based modified gravity one may add in the Lagrangian functions of the higher-order Gauss-Bonnet invariant, defined as

$$G = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{124}$$

and construct $f(R,G)$ theories [6,7]. Correspondingly, one can construct its teleparallel equivalent T_G, which reads as [34]

$$\begin{aligned}T_G = (&K^\alpha{}_{\gamma\beta}K^{\gamma\lambda}{}_\rho K^\mu{}_{\epsilon\sigma}K^{\epsilon\nu}{}_\varphi - 2K^{\alpha\lambda}{}_\beta K^\mu{}_{\gamma\rho}K^\gamma{}_{\epsilon\sigma}K^{\epsilon\nu}{}_\varphi\\ &+2K^{\alpha\lambda}{}_\beta K^\mu{}_{\gamma\rho}K^{\gamma\nu}{}_\epsilon K^\epsilon{}_{\sigma\varphi} + 2K^{\alpha\lambda}{}_\beta K^\mu{}_{\gamma\rho}K^{\gamma\nu}{}_{\sigma,\varphi})\delta^{\beta\rho\sigma\varphi}_{\alpha\lambda\mu\nu},\end{aligned} \tag{125}$$

and use it as the Lagrangian, resulting in the so-called $f(T,T_G)$ gravity. The field equations of $f(T,T_G)$ gravity are written as [34,35]

$$2(H^{[ac]b}+H^{[ba]c}-H^{[cb]a})_{,c}+2(H^{[ac]b}+H^{[ba]c}-H^{[cb]a})C^{d}{}_{dc}+(2H^{[ac]d}+H^{dca})C^{b}{}_{cd}$$
$$+4H^{[db]c}C_{(dc)}{}^{a}+T^{a}{}_{cd}H^{cdb}-h^{ab}+(f-Tf_{T}-T_{G}f_{T_{G}})\eta^{ab}=0, \tag{126}$$

where

$$H^{abc}=f_{T}(\eta^{ac}K^{bd}{}_{d}-K^{bca})+\epsilon^{cprt}\epsilon^{a}{}_{kdf}\left[(f_{T_{G}}K^{bk}{}_{p}K^{df}{}_{r})_{,t}+f_{T_{G}}C^{q}{}_{pt}K^{bk}{}_{[q}K^{df}{}_{r]}\right]$$
$$+f_{T_{G}}[\epsilon^{cprt}(2\epsilon^{a}{}_{dkf}K^{bk}{}_{p}K^{d}{}_{qr}+\epsilon_{qdkf}K^{ak}{}_{p}K^{bd}{}_{r}+\epsilon^{ab}{}_{kf}K^{k}{}_{dp}K^{d}{}_{qr})K^{qf}{}_{t}$$
$$+\epsilon^{cprt}\epsilon^{ab}{}_{kd}K^{fd}{}_{p}\left(K^{k}{}_{fr,t}-\frac{1}{2}K^{k}{}_{fq}C^{q}{}_{tr}\right)+\epsilon^{cprt}\epsilon^{ak}{}_{df}K^{df}{}_{p}\left(K^{b}{}_{kr,t}-\frac{1}{2}K^{b}{}_{kq}C^{q}{}_{tr}\right)\Big], \tag{127}$$

$$C^{c}{}_{ab}=E_{a}{}^{\mu}E_{b}{}^{\nu}(e^{c}{}_{\mu,\nu}-e^{c}{}_{\nu,\mu}), \tag{128}$$

$$h^{ab}=f_{T}\epsilon^{a}{}_{kcd}\epsilon^{bpqd}K^{k}{}_{fp}K^{fc}{}_{q}. \tag{129}$$

Note that for the purpose of the gravitation wave investigation, the stress-energy contribution has been neglected.

In order to analyze the gravitational waves in this theory, we again consider the linear perturbations in the metric and in the tetrad around a Minkowski background, namely expressions (69) and (70), respectively. From the definition of T_G we deduce that it is at least a fourth-order quantity in the tetrad perturbation. Furthermore, for simplicity we assume that $f(T,T_G)$ is Taylor expandable around $T=T_G=0$. Therefore, the resulting zeroth- and first-order perturbation equations are

$$\eta^{ab}f(0,0)=0, \tag{130}$$

$$(H^{(1)[ac]b}+H^{(1)[ba]c}-H^{(1)[cb]a})_{,c}=0. \tag{131}$$

Similar to the previous discussion, the zeroth-order equation implies that no cosmological constant is present in the theory, which is consistent with the fact that the perturbations are performed around the Minkowski background.

From the definition of H^{abc} we remark that

$$H^{(1)abc}=f_{T}(0,0)(\eta^{ac}K^{(1)bd}{}_{d}-K^{(1)bca}). \tag{132}$$

Substituting in the first-order equation (131) yields

$$f_{T}(0,0)[\eta^{ab}K^{(1)cd}{}_{d}-\eta^{ac}K^{(1)bd}{}_{d}+K^{(1)bca}]_{,c}$$
$$=f_{T}(0,0)\partial_{c}S^{(1)abc}=0, \tag{133}$$

where the definition of the superpotential (6) has been used. Interestingly enough, for the nontrivial case $f_T(0,0)\neq 0$ (otherwise GR cannot be obtained at any limit), Eq. (133) coincides with Eq. (21) obtained in the case of simple $f(T)$ gravity (note that changing coordinate to tangent-space indices involves only the zeroth-order part of the tetrad which is just the Kronecker delta). Hence, we deduce that at the first-order perturbation level, the gravitational waves behave in the same way in $f(T)$ and $f(T,T_G)$ gravities, and thus the previously obtained result that no further polarization modes comparing to GR are obtained at this order holds for $f(T,T_G)$ gravity, too. This result was expected, since T_G is a higher-order torsion invariant, and therefore its effect switches on at higher perturbation orders.

VI. CONCLUSIONS

In this work we investigated the gravitational waves and their properties, in various modified teleparallel theories, such as $f(T)$, $f(T,B)$, and $f(T,T_G)$ gravities, by utilizing the perturbed equations. Furthermore, we performed the analysis in both the metric and the tetrad languages, in order to reveal the properties of the formalism. Additionally, we performed the perturbations around a Minkowski background, a case which is obtained in the absence of a cosmological constant, but also in the case where the presence of a cosmological constant changes the background around which the perturbations are realized. Finally, in the case of usual $f(T)$ gravity we performed the analysis both for the standard formulation of zero spin connection and for the most general and fully covariant case of a nonzero spin connection.

For the case of simple $f(T)$ gravity we verified the result that no further polarization modes comparing to GR are present at the first-order perturbation level, since the torsion scalar (which is quadratic in the torsion tensor) does not acquire any perturbative contribution at this level. Hence, as we showed, in order to see the effect of $f(T)$ gravity on the gravitational waves themselves one should look at third-order perturbations, in which a deviation from GR is obtained due to the contribution from the f_{TT} component. Nevertheless, we mention that this is the effect on the internal properties of the gravitational waves, such as their polarization modes, since in general the effect of $f(T)$ gravity on the cosmological gravitational wave propagation can be seen straightaway from the dispersion relation at first order, due to the effect of $f(T)$ gravity on the cosmological background on which the gravitational waves propagate [52].

For the case of $f(T,B)$ gravity with $f(T,B) \neq f(T)$, by examining the geodesic deviation equations, we showed that extra polarization modes, namely the longitudinal and breathing modes, do appear at first-order perturbation level. The reason for this behavior is the fact that although the first-order perturbation does not have any effect on T, it does affect the boundary term B. Additionally, in the case where a cosmological constant is present we have extracted the gravitational wave equations, obtaining the cosmological-constant corrections to the solutions, which reflect the fact that the background is not Minkowski anymore.

Finally, we investigated the gravitational waves in $f(T,T_G)$ gravity, which at the first-order perturbation level exhibit the same behavior as those of $f(T)$ gravity; that is they do not have extra polarization modes comparing to GR. This result was expected, since the teleparallel equivalent of the Gauss-Bonnet term T_G is a higher-order torsion invariant, and therefore its effect switches on at higher perturbation orders.

In summary, as we showed, apart from their difference from curvature-based gravity, different modified teleparallel gravities exhibit different gravitational wave properties amongst themselves, despite the fact that at first sight they might appear as similar theories. Hence, the advancing gravitational-wave astronomy would help to alleviate the degeneracy not only between curvature and torsional modified gravity but also between different subclasses of modified teleparallel gravities.

ACKNOWLEDGMENTS

The research work disclosed in this paper is partially funded by the ENDEAVOUR Scholarships Scheme. This article is based upon work from CANTATA COST (European Cooperation in Science and Technology) action CA15117, EU Framework Programme Horizon 2020.

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