Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case

Sante Carloni and Daniele Vernieri

Centro de Astrofísica e Gravitação—CENTRA, Instituto Superior Tecnico—IST, Universidade de Lisboa—UL, Avenida Rovisco Pais 1, Lisboa 1049-001, Portugal

(Received 20 September 2017; published 21 June 2018)

We generalize the covariant Tolman-Oppenheimer-Volkoff equations proposed in Carloni and Vernieri [Phys. Rev. D **97**, 124056 (2018).]. to the case of static and spherically symmetric spacetimes with anisotropic sources. The extended equations allow a detailed analysis of the role of the anisotropic terms in the interior solution of relativistic stars and lead to the generalization of some well-known solutions of this type. We show that, like in the isotropic case, one can define generating theorems for the anisotropic Tolman-Oppenheimer-Volkoff equations. We also find that it is possible to define a reconstruction algorithm able to generate a double infinity of interior solutions. Among these, we derive a class of solutions that can represent "quasi-isotropic" stars.

DOI: 10.1103/PhysRevD.97.124057

I. INTRODUCTION

It has been clear for a long time that, different from the Newtonian case, in general relativity the structure of a spherical stellar object can be highly nontrivial. Indeed, in spite of the progress made so far, no completely satisfactory general solution of this problem has been found.

The isotropic case, in which the source of the gravitational field is a perfect fluid, was the focus of much research in the last few years. For example, in Ref. [1], an extensive selection of the known solutions was made using physical criteria like the positivity of the pressure and the negative gradient of the density distribution. In addition, a number of new methods to obtain analytic solutions representing relativistic stars were proposed (e.g., Ref. [2–5]), together with an analysis of the general properties of these solutions [6]. In Refs. [7,8], e.g., some general theorems were proven that connect different isotropic solutions to each other.

On the other hand, actual relativistic astrophysical stellar objects hardly resemble spheres of perfect fluids. This is due to a number of reasons. For example magnetic fields can be very intense and induce nontrivial deviations from isotropy. In addition, it is widely believed that in relativistic stellar objects matter is in a state which has exotic thermodynamical properties (see e.g., Refs. [9,10] and references therein). In fact, it has been suggested that even a simple mixture of real gas can behave as an anisotropic fluid (See Ref. [11]).

The study of solutions describing anisotropic stellar interiors presents a number of additional challenges, due to the extra degrees of freedom (d.o.f.) associated with the anisotropy. There are a number of interesting works in which solutions representing anisotropic stars are proposed and analyzed from different points of view (see e.g., Refs. [12–15] for some recent examples).

In Ref. [16], we developed a new covariant formalism to treat the Tolman-Oppenheimer-Volkoff (TOV) equations in the case of an isotropic fluid. In the new formalism, many aspects of the properties of these solutions, like the mathematical structure of the equations, become immediately clear. In addition, the generating theorems mentioned above, plus some new ones, can be easily formulated as deformations of the initial solutions. The formalism also allows the determination of a number of physically relevant solutions via the use, for example, of reconstruction algorithms.

The purpose of this paper is to extend this formalism to the case of anisotropic sources. As in the isotropic case, the new equations will clarify the mathematical structure of the problem and suggest in an intuitive way a number of purely analytical resolution strategies. The characterization of the anisotropy as a pressure term, will also allow the definition of a new class of generating theorems for this case. We will also prove, via the definition of a reconstruction algorithm, that, surprisingly, generating solutions in the anisotropic case is indeed easier than the isotropic one.

The paper is organized as follows. Section II introduces the basic formalism and a key set of variables which will be useful for our purposes. Section III is dedicated to the construction of the covariant TOV equations and to different resolution strategies of the new equations. In Sec. IV, we formulate some generating theorems of the anisotropic case. In Sec. V, instead, we will propose a reconstruction algorithm and we will use it to generate some physically interesting solutions. Section VI is dedicated to the conclusion.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper and Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $-,+,+,+$ signature, and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \quad (1)$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e., symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \quad (2)$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd}. \quad (3)$$

Finally, the Hilbert-Einstein action in the presence of matter is given by

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}[R + 2\mathcal{L}_m]. \quad (4)$$

II. 1+1+2 TOV EQUATIONS

We now derive the covariant TOV equations in the $1+1+2$ formalism in presence of anisotropic sources. A general and detailed presentation of the $1+1+2$ formalism can be found in Refs. [17–19] and in Ref. [16], the companion of this series of papers. Reference [16] also contains a general formulation of the junction conditions which will be used below.

In the case of the rotation-free, static and spherically symmetric spacetimes (LRSII) with anisotropic sources which will be considered in the following, all the $1+1+2$ vectors and tensors vanish as well as the variables Ω, ξ, $\mathcal{H}$, Θ, Σ and Q. Thus, one is left with the six scalars $\{\mathcal{A}, \phi, \mathcal{E}, \mu, p, \Pi\}$ which are related by the equations

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}, \quad (5)$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right), \quad (6)$$

$$\hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\mathcal{A}, \quad (7)$$

$$\hat{\mathcal{A}} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{2}(\mu + 3p), \quad (8)$$

$$\hat{K} = -\phi K, \quad (9)$$

with the constraints

$$0 = -\mathcal{A}\phi + \frac{1}{3}(\mu + 3p) - \mathcal{E} + \frac{1}{2}\Pi,$$
$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2. \quad (10)$$

We introduce at this point a parameter such that the Gaussian curvature K is given by [20]

$$K = K_0 e^{-\rho}. \quad (11)$$

The parameter ρ is connected to the standard area radius by

$$\rho = 2\ln(r/r_0), \quad (12)$$

where r_0 is an arbitrary constant. In the rest of this work, we will perform the calculations in ρ, but we will give the final results in terms of r to make the comparison with known results easier.

Using the parameter ρ and defining the following variables [20],

$$X = \frac{\phi_{,\rho}}{\phi}, \quad Y = \frac{\mathcal{A}}{\phi}, \quad \mathcal{K} = \frac{K}{\phi^2}, \quad E = \frac{\mathcal{E}}{\phi^2},$$
$$\mathbb{M} = \frac{\mu}{\phi^2}, \quad \mathbb{P} = \frac{\Pi}{\phi^2}, \quad P = \frac{p}{\phi^2}, \quad (13)$$

Eqs. (5)–(9) take the form

$$Y_{,\rho} = M + 3P - 2Y(X + Y + 1), \quad (14)$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X), \quad (15)$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = -2Y(M + \mathbb{P}) - 2P(2X + Y) - \mathbb{P}(4X + 3), \quad (16)$$

with the constraints

$$2\mathbb{M} + 2P + 2\mathbb{P} + 2X - 2Y + 1 = 0, \quad (17)$$

$$1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P} = 0, \quad (18)$$

$$2\mathbb{M} + 6P + 3\mathbb{P} - 6Y - 6E = 0. \quad (19)$$

These equations will be the starting point for the construction of the covariant TOV equations.

The form of $\mathcal{A}$, ϕ, K, Y and $\mathcal{K}$ in terms of the metric coefficients and their derivatives can be found in Ref. [16].

Israel's junction conditions [21,22] translate in the variables (13) as (see Ref. [16] for more details)

$$[\mathcal{K}] = 0, \quad [Y] = 0. \quad (20)$$

Using the constraint in Eq. (18) above, one gets

$$[P + \mathbb{P}] = 0. \tag{21}$$

As $[\mathcal{K}] = 0$, this is equivalent to saying that $[p_r] = 0$. In other words, in order to provide a smooth junction with the Schwarzschild metric, one has to seek a value of the radius in which the radial pressure is zero. In the following, we will impose the junction conditions requiring directly that p_r is zero on the boundary of the star. We will have, however, two cases in which this condition is incompatible with the structure of the solution and we will be required to introduce a shell with stress-energy tensor

$$T^{\mathcal{S}}_{ab} = u_a u_b [\phi + 2\mathcal{A}] + N_{ab}\left[\frac{\phi}{2} + \mathcal{A}\right]. \tag{22}$$

The condition Eq. (21) does not give any information on the energy density and the tangential pressure, implying that there is no constraint on these quantities. This can be verified breaking covariance. From the expressions connecting the $1+1+2$ scalars to the metric components in Ref. [16], we realize that the junction conditions (20) require the continuity of A, of its first derivative and also the continuity of B. From the Einstein equations, it is easy to see that the energy density and the tangential pressure depend on derivatives of the metric coefficients (like the first derivative of B) which have no constraint. As a consequence, both of these quantities can have a jump at the junction.

Considering Eq. (15) and eliminating X and Y from Eq. (16), one obtains

$$\begin{aligned} P_{,\rho} + \mathbb{P}_{,\rho} &= P\left(\mathbb{M} - 2\mathbb{P} - 3\mathcal{K} + \frac{7}{4}\right) + \mathbb{P}\left(\mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) \\ &\quad + \left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} - P^2 - \mathbb{P}^2, \\ \mathcal{K}_{,\rho} &= -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right). \end{aligned} \tag{23}$$

The structure of these equations is similar to the one of the isotropic case treated in Ref. [16], and therefore we have similar problems in determining a general analytical solution. The most important difference is that now two different pressure terms (isotropic and anisotropic) appear in the TOV equations. In the standard treatment, the equations above are written in a slightly different way. Indeed, the TOV equations are written for the combination $P + \mathbb{P}$ which would correspond to p_r/ϕ^2 (see e.g., Ref. [9]). We will make use of the latter form of the equations to give a special case in the following subsection.

In general, since anisotropic solutions possess additional d.o.f., it is natural to expect that this case is more complicated than the isotropic one. Surprisingly, we will find that this is not always the case.

The equations above are completely equivalent to the Einstein equations. However, not all the solutions of the Einstein equations with anisotropic fluids correspond to (the interior of) stellar objects. It is known [5,9] that a physical solution will have to fulfill the following conditions:

(1) μ, p_r and $p_\perp$ should be positive inside the object;
(2) the gradients of μ, p_r and $p_\perp$ should be negative;
(3) the speed of sound should be always less that the speed of light $0 < \frac{\partial p_r}{\partial \mu} < 1$, $0 < \frac{\partial p_\perp}{\partial \mu} < 1$;
(4) the energy conditions should be satisfied;
(5) the anisotropy Π should be zero in the center of the object, i.e., $p_r = p_\perp$ at the center.

In the following, we will present solutions that are compatible with these conditions at least for one set of their parameters.

III. SOME RESOLUTION STRATEGIES

There are several strategies which can be used to obtain solutions of the TOV equations in this case. The simplest one are based on making an ansatz on the behavior of the anisotropy Π and then solving for the corresponding pressure. This is one of the most common approaches in literature (see e.g., [9]).

Setting, for example,

$$\mathbb{P} = \frac{\mu_0^2 P_0 (1-h) e^{2\rho} (1 - \frac{\mu_0 e^\rho}{3K_0})^{h/2}}{18(3K_0 - \mu_0 e^\rho)^2 \left[\left(1 - \frac{\mu_0 e^\rho}{3K_0}\right)^{h/2} + 3P_0\right]^2}, \tag{24}$$

where h is an arbitrary constant, one obtains

$$\begin{aligned} P = \frac{\mu_0 e^\rho}{[(1 - \frac{\mu_0 e^\rho}{3K_0})^{h/2} + 3P_0]^2} &\left[\frac{3P_0^2 + (1 - \frac{\mu_0 e^\rho}{3K_0})^h}{4(\frac{\mu_0 e^\rho}{3K_0} - 1)}\right. \\ &\left. + \frac{2P_0[4(h+5)\mu_0 e^\rho - 72K_0](1 - \frac{\mu_0 e^\rho}{3K_0})^{h/2}}{(4\mu_0 e^\rho - 12K_0)^2}\right], \end{aligned} \tag{25a}$$

and

$$\mathcal{K} = -\frac{3K_0 e^{\rho/2}}{16\mu_0 e^{3\rho/2} - 12K_0 e^{\rho/2}}, \tag{26a}$$

$$Y = \frac{\mu_0 e^\rho (3K_0 - \mu_0 e^\rho)^{h/2}}{2(\mu_0 e^\rho - 3K_0)[(3K_0 - \mu_0 e^\rho)^{h/2} + 9K_0 P_0]}. \tag{26b}$$

This solution corresponds to the Bowers-Liang solution for a constant density object [23], which in radial coordinates reads

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (27a)$$

$$A = A_0\left[\left(1 - \frac{\mu_0}{3}r^2\right)^{h/2} + 3P_0\right]^{2/h}, \quad (27b)$$

$$B = \frac{3}{3 - \mu_0 r^2}, \quad (27c)$$

with

$$p_r = -\frac{\mu_0[(1 - \frac{\mu_0 r^2}{3})^{h/2} + P_0]}{(1 - \frac{\mu_0 r^2}{3})^{h/2} + 3P_0}, \quad (28a)$$

$$p_\perp = p_r - \frac{\mu_0^2 P_0 r^2 (1-h)(1 - \frac{\mu_0 r^2}{3})^{\frac{h}{2}-1}}{3[(1 - \frac{\mu_0 r^2}{3})^{h/2} + 3P_0]^2}, \quad (28b)$$

and $P_0 < 0$.

Another interesting strategy to obtain exact solutions of Eq. (23) is to find a convenient constraint for the anisotropic pressure term. For example, setting $\mathcal{P} = P + \mathbb{P}$ and imposing

$$\mathbb{P} = \frac{1}{6}\mathbb{M}(1 - 4\mathcal{K}), \quad (29)$$

the first of Eq. (23) reduces to

$$\mathcal{P}_{,\rho} + \mathcal{P}^2 + \mathcal{P}\left(3\mathcal{K} - \mathbb{M} - \frac{7}{4}\right) = 0, \quad (30)$$

which is analogous to the TOV equation of an isotropic system with pressure $\mathcal{P}$. It is easy to check that in the trivial case $\mathcal{P} = 0$, the above equations give the solution found by Florides [24]. As a consequence, Florides's solution can be considered the simplest element of an entire class of solutions for which Eq. (29) holds.

Using the covariant TOV equations, one can easily find some other simple examples with a nontrivial $\mathcal{P}$. Indeed, working with a constant density, we have

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (31a)$$

$$A = \frac{A_0(e^{\mathcal{P}_0} r_0^2 + r^2)^2}{\sqrt{3 - \mu_0 r^2}}, \quad (31b)$$

$$B = \frac{3}{3 - \mu_0 r^2}, \quad (31c)$$

together with the radial and tangential pressure,

$$p_r = \frac{3 - \mu_0 r^2}{3(r^2 + e^{\mathcal{P}_0} r_0^2)}, \quad (32a)$$

$$p_\perp = \frac{\mu_0^2 r^2}{4(3 - \mu_0 r^2)} + p_r. \quad (32b)$$

The solution is well behaved in the origin. Unfortunately, in the value of r in which the radial pressure becomes zero, the orthogonal pressure diverges; thus, the solution can only be matched with Schwarzschild exterior before this singularity. Naturally, since $p_r \neq 0$ at the junction, the object represented by this solution will be enclosed in a shell, defined by the stress-energy tensor in Eq. (22).[1]

A more regular example is (we set $r_0 = 1$ for brevity)

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (33a)$$

$$A = A_0\psi(12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4)^{\frac{4b-5}{8b-4}}, \quad (33b)$$

$$\psi = \exp\left(\frac{\sqrt{3}(4b+1)\tan^{-1}\left(\frac{a(3-12b)+2b(4b-2)r^2}{a\sqrt{48b^2-24b-9}}\right)}{(2-4b)\sqrt{(4b-3)(4b+1)}}\right), \quad (33c)$$

$$B = \frac{12(br^2 - a)^2}{12a^2 - 3a(4b-1)r^2 + b(4b-2)r^4}, \quad (33d)$$

where a is an arbitrary constant and $b < -1/4$ or $b > 3/4$. The thermodynamical quantities are

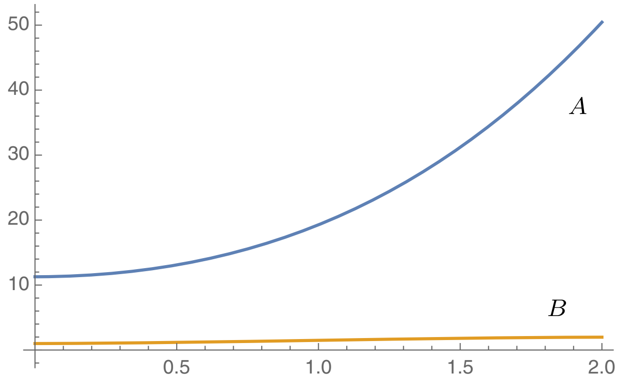
$$\mu = \frac{(4b+1)(9a^2 - 7abr^2 + 2b^2r^4)}{12(br^2 - a)^3}, \quad (34a)$$

$$p_r = \frac{1}{a - br^2}, \quad (34b)$$

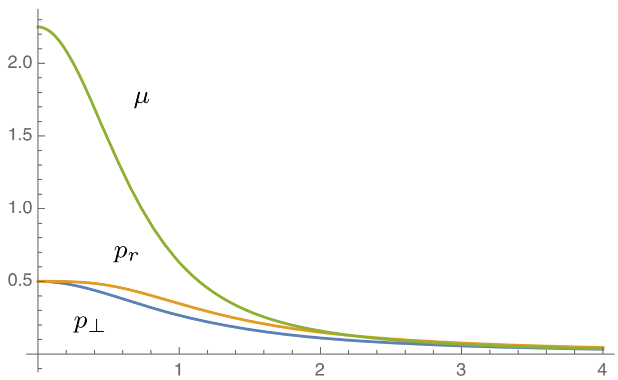
$$\begin{aligned} p_\perp = &\frac{[4b^2r^4 + 3a(4a + r^2) - b(12ar^2 + 2r^4)]^{-1}}{48(br^2 - a)^3} \\ &\times \{9a^3(64a + 19r^2) - b^3(624a^2r^4 + 800ar^6 + 100r^8) \\ &+ ab^2r^2(432a^2 + 1608ar^2 + 356r^4) \\ &- 9a^2br^2(168a + 47r^2) \\ &+ 80b^4r^6(4a + 2r^2) - 64b^5r^8\}. \end{aligned} \quad (34c)$$

Notice that the radial pressure in this solution is never zero. Hence, in principle, we have two different options. A first one is to consider the metric above as representing an object with a thin atmosphere that covers the entire spacetime. A second option would be to make a junction with the Schwarzschild solution by introduction a thin shell, like in the previous example. However, the component A of the metric and the speeds of sound are growing functions of the radial coordinate. This implies that the first option would lead to an unphysical situation and suggests a natural range of radii in which the solution can be soldered to the Schwarzschild solution.

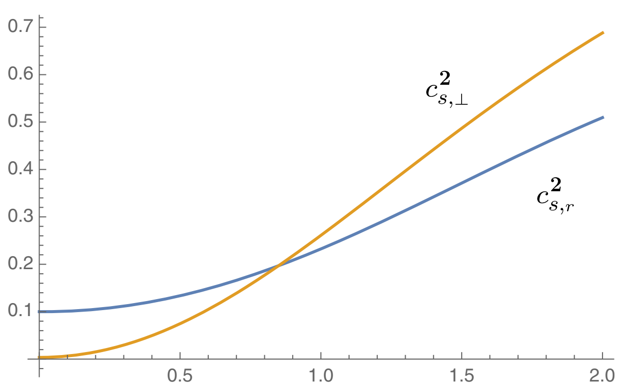
[1] Naturally, one could choose another type of exterior, like the Vaidya metric. In this case, since the exterior spacetime is nonempty no shell would be required. Although physically interesting, we will not consider this case here.



(a) The coefficients of the metric (33a). The blue line represents A and the orange B.



(b) The thermodynamic quantities (34) associated with (33a). The blue line represents p_r, the orange $p_\perp$ and the green μ.



(c) The speeds of sound associated with (33a).

FIG. 1. Graphs of the solution (33a) in the case $r_0 = 1$, $a = 2$, $\mu_1 = -1$, $a = -7/4$. The values of the parameters has been chosen in such a way to make the features of the solution as clear as possible.

In Fig. 1, we give the behavior of the solution for a convenient choice of the parameters.

Yet a different resolution strategy for Eq. (23) consists in separating the isotropic and anisotropic d.o.f. Shifting the position of the coupling term $2P\mathbb{P}$ and of the term $(\frac{1}{4} - \mathcal{K})\mathbb{M}$, one can decompose this equation in a system of equations which, given a choice for the behavior of the energy density, is composed by a Bernoulli and a Riccati equation.

A particularly interesting way to perform this separation is

$$\begin{aligned} &P_{,\rho} + P^2 - P\left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4}\right) \\ &\quad + \left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} = \mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right), \\ &\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right). \end{aligned} \tag{35}$$

This setting suggests that any isotropic ($\mathbb{P} = 0$) solution of the TOV equations can be associated to an anisotropic solution in which P, $\mathcal{K}$ and $\mathbb{M}$ are the same and the $\mathbb{P}$ is determined by the equation

$$\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) = 0. \tag{36}$$

Notice, however, that these new anisotropic solutions will have a different Y and therefore a different (0,0) component of the metric because the constraint (18) is changed by the presence of $\mathbb{P}$.

One can indeed find a number of different ways to connect isotropic and anisotropic solutions. For example, setting

$$P = P_0 - \mathbb{P}, \qquad \mathbb{M} = \mathbb{M}_0 + \alpha\mathbb{M}_1, \tag{37}$$

where P_0 and $\mathbb{M}_0$ are part of a known solution of the isotropic equations. With these assumptions, the Eq. (35) return an algebraic equation that can be solved for $\mathbb{P}$:

$$\begin{aligned} \mathbb{P} = \frac{1}{6}[&P_0(4\alpha\mathbb{M}_1 + 4\mathbb{M}_0 - 12\mathcal{K} + 7) \\ &- 4P_{0,\rho} - 4P_0^2 - (4\mathcal{K} - 1)(\alpha\mathbb{M}_1 + \mathbb{M}_0)]. \end{aligned} \tag{38}$$

Now the TOV equations can be solved completely if one is able to integrate the second of Eq. (35). The latter equation is of the Bernoulli type and admits the following formal solution:

$$\begin{aligned} &\mathcal{K} = \frac{e^F}{\mathcal{K}_* - 2\int e^F d\rho}, \\ &F = \int \frac{1}{2}(4\alpha\mathbb{M}_1 + 4\mathbb{M}_0 + 1)d\rho. \end{aligned} \tag{39}$$

As an example, one can start with the classical isotropic constant density solution

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{40a}$$

$$A = A_0\left(\sqrt{3-\mu_0 r^2} + P_0\right)^2, \tag{40b}$$

$$B = \frac{3}{3-\mu_0 r^2}, \tag{40c}$$

with

$$p = -\frac{\mu_0(P_0 + 3\sqrt{3-\mu_0 r^2})}{3(P_0 + \sqrt{3-\mu_0 r^2})}. \tag{41}$$

Setting

$$\mathbb{M}_1 = \frac{\mu_1 e^\rho}{K_0}\mathcal{K}, \tag{42}$$

where μ_1 is a constant, the solution for $\mathcal{K}$ is

$$\mathcal{K} = \frac{3K_0}{4(3K_0 - 4e^\rho[\alpha\mu_1 + \mu_0])}. \tag{43}$$

This leads, after long but trivial calculations, to the following solution for the metric and the fluid thermodynamics,

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{44a}$$

$$A = \frac{A_0\sqrt{3-\mu_0 r^2}(3 - P_0^2 r_0^2 - \mu_0 r^2)}{\sqrt{3 - r^2(\alpha\mu_1 + \mu_0)}} \times \tag{44b}$$

$$\exp\left(2\tanh^{-1}\left(\frac{\sqrt{3-\mu_0 r^2}}{P_0 r_0}\right)\right), \tag{44c}$$

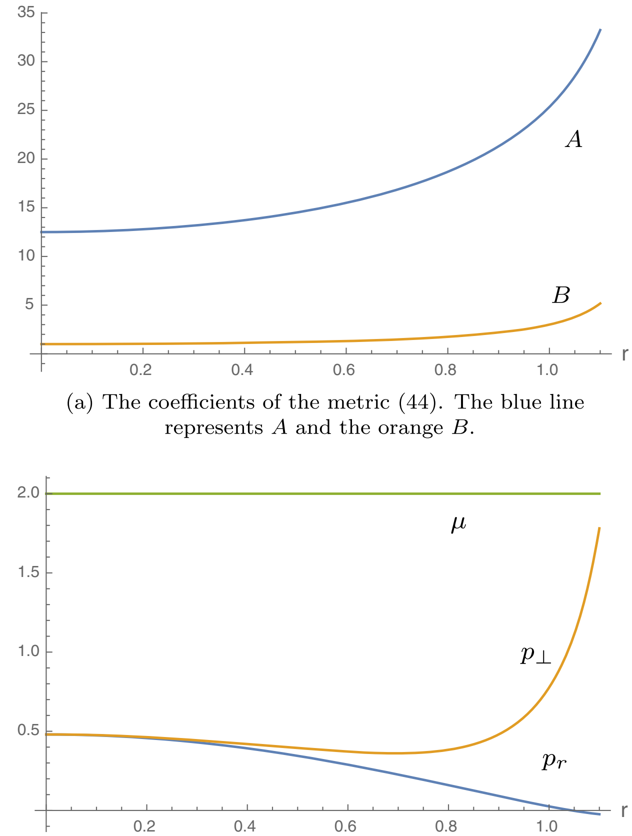
$$B = \frac{3}{3 - r^2(\alpha\mu_1 + \mu_0)}, \tag{44d}$$

where P_0 is an integration constant. The radial and tangential pressure read

$$p_r = \frac{\mu_0[3 - r^2(\alpha\mu_1 + \mu_0)](3\sqrt{3-\mu_0 r^2} + P_0 r_0)}{3(\mu_0 r^2 - 3)(\sqrt{3-\mu_0 r^2} + P_0 r_0)}, \tag{45a}$$

$$p_\perp = -192(\mu_0 r^2 - 3)^2(r^2(\beta\mu_1 + \mu_0) - 3)p_r - \frac{36\beta\mu_1 r^2(8\mu_0(r^2(\beta\mu_1 + \mu_0) - 3) - 12\beta\mu_1)}{48(\mu_0 r^2 - 3)^2(12 - 4r^2(\beta\mu_1 + \mu_0))}. \tag{45b}$$

In this solution, the pressures at the center are regular for any value of the parameters, and one can have the central value of these quantities to be positive. In addition, the parameters can be set in such a way to avoid any singularity. Figure 2 gives an example in which the radial pressure goes to zero at a finite radius.



(a) The coefficients of the metric (44). The blue line represents A and the orange B.

(b) The thermodynamic quantities (45) associated with (44). The blue line represents p_r, the orange $p_\perp$ and the green μ.

FIG. 2. Graphs of the solution (44) in the case $r_0 = 1$, $\mu_0 = 3/2$, $\mu_1 = -1/2$, $\beta = -1$, $P_* = -7$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

The reason behind the connection between isotropic and anisotropic solutions will become clear in the next section when we will look into the details of the generating theorems for anisotropic solutions. We will discover that ultimately these theorems are behind the methods presented above.

IV. GENERATING THEOREMS FOR ANISOTROPIC TOV SOLUTIONS

In Refs. [7,8,25], a number of interesting theorems were proved, dubbed *"generating theorems"*. These theorems allow us to connect different solutions of the TOV equations in the isotropic case, in the sense that given a solution of the equations, one can recover new solutions which differ only in a given set of quantities (e.g., the pressure and (0,0) component of the metric). In Ref. [16], using the $1+1+2$ covariant formalism, we were able to prove that these theorems can be related to linear deformations of the solutions of the isotropic TOV equations. We also showed that the constraints between the variables Y, $\mathcal{K}$, P, $\mathbb{M}$ can be used as a guide to deduce such generating theorems. In this section, we will extend these generating theorems to the case of anisotropic solutions. We will show that, in addition to the theorems already found for the isotropic case, new theorems can be formulated.

Let us start from a shift in the isotropic pressure and the variable Y,

$$P = P_0 + P_1, \qquad Y = Y_0 + Y_1, \tag{46}$$

which corresponds to the transformation

$$A \to A_0(\rho)\exp\left(\int Y_1 d\rho\right), \tag{47a}$$

$$B \to B_0(\rho), \tag{47b}$$

$$C \to C_0(\rho) \tag{47c}$$

of Theorem 2 in Ref. [7].

Using the Eq. (35), the constraint in Eq. (18) and Eq. (14), we obtain

$$\begin{aligned} &P_{1,\rho} + P_1^2 + P_1\left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4}\right) = 0,\\ &Y_1 = P_1. \end{aligned} \tag{48}$$

whose solution is

$$\begin{aligned} P_1 &= \frac{e^F}{P_* + \int e^F d\rho},\\ F &= \int\left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{7}{4}\right)d\rho. \end{aligned} \tag{49}$$

In other words, starting form the solution $(Y_0, \mathcal{K}_0, P_0, \mathbb{M}_0)$, we have obtained a new solution $(Y, \mathcal{K}, P, \mathbb{M})$ by solving two integrals.

In [16], we proved a similar theorem for the isotropic case. Comparing the two results, it is easy to see that the only difference consists in the presence of the additional term $2\mathbb{P}_0$ in the integral that defines F.

Let us consider now the case of a combined shift of the pressure and the energy density by setting

$$P = P_0 + P_1, \qquad \mathbb{M} = \mathbb{M}_0 + \mathbb{M}_1, \qquad \mathcal{K} = \frac{1}{\mathcal{K}_0 + \mathcal{K}_1}, \tag{50}$$

which leads to

$$A \to A_0(\rho), \tag{51a}$$

$$B^{-1} \to B_0(\rho) + \frac{e^\rho}{K_0}\mathcal{K}_1, \tag{51b}$$

$$C \to C_0(\rho), \tag{51c}$$

and corresponds to Theorem 1 in Ref. [7] (see also [25]). Equation (35), the constraint in Eq. (18) and (14) return

$$\begin{aligned} \mathcal{K}_{1,\rho} &= -\mathcal{K}_1\Phi + \Gamma,\\ \Phi &= \frac{12\mathcal{K}_0 - 4\mathbb{M}_0 - 1 + Y_0(8\mathcal{K}_0 - 8\mathbb{M}_0 - 2)}{2(1 + 2Y_0)},\\ \Gamma &= \frac{8\mathcal{K}_0^2(Y_0 + 1) + \mathcal{K}_0(4\mathbb{M}_0 + 1)(2Y_0 + 1) - 4}{1 + 2Y_0},\\ P_1 &= \frac{\mathcal{K}_0^2 + \mathcal{K}_1\mathcal{K}_0 - 1}{\mathcal{K}_0 + \mathcal{K}_1},\\ \mathbb{M}_1 &= -\frac{[\mathcal{K}_0(\mathcal{K}_0 + \mathcal{K}_1) - 1](2Y_0 + 3)}{(\mathcal{K}_0 + \mathcal{K}_1)(2Y_0 + 1)}. \end{aligned} \tag{52}$$

The first equation above is a linear differential equation which can always be solved exactly as

$$\begin{aligned} \mathcal{K}_1 &= e^{-F}\left(\mathcal{K}_* - \int e^F\Gamma d\rho\right),\\ F &= \int \Phi d\rho. \end{aligned} \tag{53}$$

Different from the previous case, the theorem that we have obtained matches exactly the corresponding theorem in the isotropic case given in [16].

Exploiting the similarity of the role of the isotropic and anisotropic pressures in Eq. (23), we can, moreover, give additional theorems. For example, keeping the isotropic pressure unchanged, one can set

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, \qquad Y = Y_0 + Y_1, \tag{54}$$

and we obtain

$$\begin{aligned} Y_1 &= \mathbb{P}_1,\\ \mathbb{P}_1 &= \frac{e^F}{\mathbb{P}_* + \int e^F d\rho},\\ F &= \int\left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 + 2\mathbb{P}_0 - \frac{1}{4}\right)d\rho, \end{aligned} \tag{55}$$

which mirrors the theorem of Eq. (46).

Also the theorem in Eq. (50) has a similar analogue. Indeed, setting

$$\begin{aligned} \mathbb{P} &= \mathbb{P}_0 + \mathbb{P}_1,\\ \mathbb{M} &= \mathbb{M}_0 + \mathbb{M}_1,\\ \mathcal{K} &= \frac{1}{\mathcal{K}_0 + \mathcal{K}_1}, \end{aligned} \tag{56}$$

we obtain

$$\begin{aligned}
\mathcal{K}_{1,\rho} &= \mathcal{K}_1\Phi + \Gamma, \\
\Phi &= \frac{Y_0(8\mathcal{K}_0 - 8\mathbb{M}_0 - 2) - 4\mathbb{M}_0 - 1}{2(1+2Y_0)}, \\
\Gamma &= \frac{\mathcal{K}_0^2(8Y_0+2) - \mathcal{K}_0(4\mathbb{M}_0+1)(2Y_0+1)+2}{1+2Y_0}, \\
\mathbb{P}_1 &= \frac{\mathcal{K}_0^2 + \mathcal{K}_1\mathcal{K}_0 - 1}{\mathcal{K}_0+\mathcal{K}_1}, \\
\mathbb{M}_1 &= -\frac{2[\mathcal{K}_0(\mathcal{K}_0+\mathcal{K}_1)-1]Y_0}{(\mathcal{K}_0+\mathcal{K}_1)(2Y_0+1)}.
\end{aligned} \tag{57}$$

Again, the first equation above is a linear differential equation which can be solved exactly as

$$\mathcal{K}_1 = e^F\left(\mathcal{K}_* - \int e^{-F}\Gamma d\rho\right), \qquad F = \int \Phi d\rho. \tag{58}$$

A third generating theorem allows us to shift the isotropic pressure, maintaining the energy density fixed. Setting

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, \qquad P = P_0 + P_1, \qquad \mathcal{K} = \frac{1}{\mathcal{K}_0+\mathcal{K}_1}, \tag{59}$$

we obtain

$$\begin{aligned}
\mathcal{K}_{1,\rho} &= -\mathcal{K}_1\Phi + \Gamma, \\
\Phi &= \frac{1}{2}(4\mathbb{M}_0+1), \\
\Gamma &= \frac{1}{2}[4\mathcal{K}_0^2 - 2\mathcal{K}_0(4\mathbb{M}_0+1)+4], \\
\mathbb{P}_1 &= \frac{(\mathcal{K}_0(\mathcal{K}_0+\mathcal{K}_1)-1)(2Y_0+3)}{3(\mathcal{K}_0+\mathcal{K}_1)}, \\
P_1 &= \frac{2}{3}\left(\frac{1}{\mathcal{K}_0+\mathcal{K}_1} - \mathcal{K}_0\right)Y_0,
\end{aligned} \tag{60}$$

which can be solved by

$$\begin{aligned}
\mathcal{K}_1 &= e^{-F}\left(\mathcal{K}_* - \int e^F\Gamma d\rho\right), \\
F &= \int \Phi d\rho.
\end{aligned} \tag{61}$$

Like in the isotropic case, one can further consider non-linear deformations of a known solution. For example, in the case of a linear shift of the isotropic and anisotropic pressure with a generic change of $\mathcal{K}$, that is

$$\mathbb{P} = \mathbb{P}_0 + \mathbb{P}_1, \qquad P = P_0 + P_1, \qquad \mathcal{K} = \mathcal{K}_1, \tag{62}$$

we obtain

$$\begin{aligned}
\mathcal{K}_{1,\rho} &= -\mathcal{K}_1\left(-2\mathbb{M}_0 - \frac{1}{2}\right) - 2\mathcal{K}_1^2, \\
\mathbb{P}_1 &= (2\mathcal{K}_0-1)P_0 - \mathbb{P}_0 + Y_0 + \frac{1}{4} \\
&\quad \times \frac{1}{6}\mathcal{K}_1(-4P_0 - 4\mathbb{P}_0 - 5) + \frac{2}{3}\mathcal{K}_0(\mathbb{M}_0 + 3\mathbb{P}_0), \\
P_1 &= \frac{1}{6}\mathcal{K}_1(4P_0 + 4\mathbb{P}_0 - 1) \quad - \frac{2}{3}\mathcal{K}_0(\mathbb{M}_0 + 3P_0 + 3\mathbb{P}_0).
\end{aligned} \tag{63}$$

Similar results can also be found if one considers variations of P and $\mathbb{M}$ or $\mathbb{P}$ and $\mathbb{M}$ together with a change of $\mathcal{K}$. As in the isotropic case, one can obtain hints about the existence of new theorems by looking at the constraints in Eqs. (17)–(19).

V. RECONSTRUCTING ANISOTROPIC SOLUTIONS

To conclude, we give a reconstruction algorithm for anisotropic stellar interior solutions. In Ref. [16], we proposed a similar algorithm for isotropic stellar interior solutions. In that case, it turned out that the metric coefficients have to satisfy a differential constraint which can be difficult to solve. We will make a similar construction here, showing that, surprisingly, the algorithm for anisotropic solutions does not present any such constraints.

Solving Eqs. (14)–(19) for the matter variables, one obtains

$$\mathbb{M} = \frac{\mathcal{K}_\rho}{2\mathcal{K}} + \mathcal{K} - \frac{1}{4}, \tag{64}$$

$$P = \frac{1}{3}(2Y_\rho + 2Y^2 + Y) - \frac{2Y+1}{6}\frac{\mathcal{K}_\rho}{\mathcal{K}} - \frac{1}{3}\mathcal{K} + \frac{1}{12}, \tag{65}$$

$$\mathbb{P} = (2Y+1)\mathcal{K}_\rho - 4\mathcal{K}^2 - \mathcal{K}[4Y_\rho + 4(Y-1)Y - 1]. \tag{66}$$

In the isotropic case, one should set $\mathbb{P} = 0$ in (66) and this relation corresponds to the differential constraint we encountered in Ref. [16]. This result implies that the differential constraint found in the isotropic case corresponds to the very condition of isotropy.

The structure of Eqs. (64)–(66) shows that reconstructing anisotropic solutions is considerably simpler than reconstructing isotropic ones. Indeed, in the anisotropic case the equation above leads to a double infinity of solutions.

It is useful at this stage to give the expression of the speeds of sound in terms of the variables $\mathcal{K}$ and Y:

$$\begin{aligned}
c_{s,r}^2 &= \frac{dp_r}{d\mu} \\
&= \frac{\mathcal{K}^2(-4\mathcal{K} - 4Y_{,\rho} + 4Y + 1) + (4Y+1)\mathcal{K}\mathcal{K}_{,\rho}}{4\mathcal{K}^3 - \mathcal{K}^2 + \mathcal{K}(\mathcal{K}_{,\rho} - 2\mathcal{K}_{,\rho\rho}) + 4\mathcal{K}_{,\rho}^2},
\end{aligned} \tag{67}$$

$$\begin{aligned} c_{s,\perp}^2 &= \frac{dp_\perp}{d\mu} \\ &= -\frac{1+16\mathcal{K}Y^2(\mathcal{K}_{,\rho}+\mathcal{K})+\mathcal{K}\mathcal{K}_{,\rho}(1-12Y_{,\rho})}{[4\mathcal{K}^3-\mathcal{K}^2+\mathcal{K}(\mathcal{K}_{,\rho}-2\mathcal{K}_{,\rho\rho})+4\mathcal{K}_{,\rho}^2]} \\ &\quad+\frac{2\mathcal{K}^2(-8Y_{,\rho}+8Y_{,\rho\rho}+1)-8\mathcal{K}^3}{[4\mathcal{K}^3-\mathcal{K}^2+\mathcal{K}(\mathcal{K}_{,\rho}-2\mathcal{K}_{,\rho\rho})+4\mathcal{K}_{,\rho}^2]} \\ &\quad+\frac{Y(4\mathcal{K}(\mathcal{K}_{,\rho}+\mathcal{K}(-4\mathcal{K}+8Y_{,\rho}+1))-2)}{[4\mathcal{K}^3-\mathcal{K}^2+\mathcal{K}(\mathcal{K}_{,\rho}-2\mathcal{K}_{,\rho\rho})+4\mathcal{K}_{,\rho}^2]}. \end{aligned} \tag{68}$$

These expressions can be used to select the suitable forms of the metric variables Y and $\mathcal{K}$.

We will now use the above algorithm to generate some physically relevant solutions in the sense of Sec. II. We shall limit ourselves to give all the results directly in the parameter r.

Let us start form the metric coefficients,

$$\begin{aligned} A &= (a+br^{2\rho})^\delta, \\ B &= \frac{3\mu_0}{3\mu_0-r^2(\mu_0-\mu_1 r^{2\alpha})^{\beta+1}\mathcal{F}}, \end{aligned} \tag{69}$$

where $\mathcal{F}$ is the Gaussian hypergeometric function

$$\mathcal{F} = {}_2F_1\left(1,\beta+\frac{3}{2\alpha}+1;1+\frac{3}{2\alpha};\frac{r^{2\alpha\rho}\mu_1}{\mu_0}\right). \tag{70}$$

In this way, setting for brevity $r_0=1$, we get

$$\begin{aligned} \mu &= (\mu_0-\mu_1 r^{2\alpha})^\beta, \\ p_r &= \frac{2b\delta\mu_0}{\mu_0(a+br^2)} - \frac{[a+b(2\delta+1)r^2](\mu_0-\mu_1 r^{2\alpha})^{\beta+1}}{3\mu_0(a+br^2)}\mathcal{F}, \\ p_\perp &= \frac{2[a^2+ab(2-3\delta)r^2+b^2(-2\delta^2+\delta+1)r^4]\mathcal{F}}{\mu_0 12(a+br^2)^2(\mu_0-\mu_1 r^{2\alpha})^{-(\beta+1)}} \\ &\quad+\frac{12b\delta(2a+b\delta r^2)-6(a+br^2)(a+b(\delta+1)r^2)}{12(a+br^2)^2(\mu_0-\mu_1 r^{2\alpha})^{-\beta}}. \end{aligned} \tag{71}$$

Notice that the radial and tangential pressure converge to the same value in the center of the star. The radial and tangential speed of sound read

$$\begin{aligned} c_{s,r}^2 &= \frac{(\mu_0-\mu_1 r^{2\alpha})}{3\alpha\beta\mu_0\mu_1 r^{2\alpha}(a+br^2)^2}\left\{\frac{3}{2}\mu_0(a+br^2)[a+b(2\delta+1)r^2]-\frac{\mathcal{F}}{2}(\mu_0-\mu_1 r^{2\alpha})[2b\delta r^2(a+3br^2)+3(a+br^2)^2]\right. \\ &\quad\left.+6b^2\delta\mu_0 r^2(\mu_0-\mu_1 r^{2\alpha})^{-\beta}\right\}, \end{aligned} \tag{72}$$

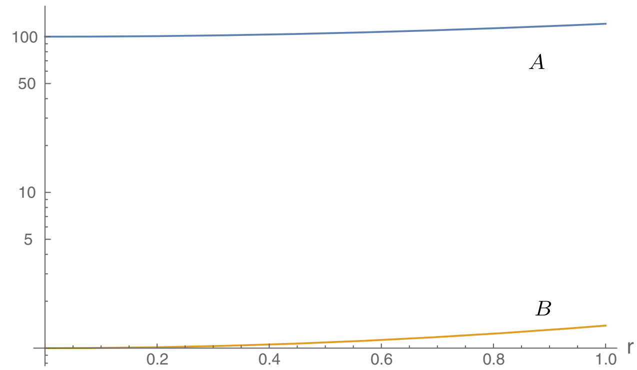
$$\begin{aligned} c_{s,\perp}^2 &= \frac{r^{-2\alpha}}{12\alpha\beta\mu_1(a+br^2)^3}\left\{\frac{\mathcal{F}}{\mu_0}(\mu_0-\mu_1 r^{2\alpha})^2[3a^3-3a^2b(\delta-3)r^2+ab^2(2(\delta-8)\delta+9)r^4+3b^3(-2\delta^2+\delta+1)r^6]\right. \\ &\quad+12b^2\delta r^2[b\delta r^2-a(\delta-4)](\mu_0-\mu_1 r^{2\alpha})^{1-\beta}-3\mu_0(a+br^2)[a^2+ab(2-5\delta)r^2+b^2(-2\delta^2+\delta+1)r^4] \\ &\quad\left.-3\mu_1 r^{2\alpha}(a+br^2)[ab\delta r^2(2\alpha\beta+5)+2b^2\delta^2 r^4+b^2\delta r^4(2\alpha\beta-1)+(2\alpha\beta-1)(a+br^2)^2]\right\}. \end{aligned} \tag{73}$$

In Figs. 3 and 4, we give a plot of the above solution for a set of values of the parameters which return physically relevant results.

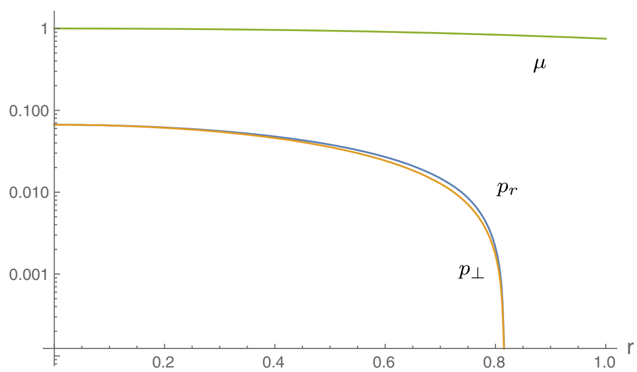
As a second example, let us consider the case

$$\begin{aligned} A &= [a+(c-br^2)^\gamma]^\delta, \\ B &= \frac{3\mu_0}{3\mu_0-r^2(\mu_0-\mu_1 r^{2\alpha})^{\beta+1}\mathcal{F}}, \\ \mathcal{F} &= {}_2F_1\left(1,\beta+\frac{3}{2\alpha}+1;1+\frac{3}{2\alpha};\frac{r^{2\alpha\rho}\mu_1}{\mu_0}\right), \end{aligned} \tag{74}$$

which can be seen as a generalization of the Bowers-Liang solution for nonconstant densities.

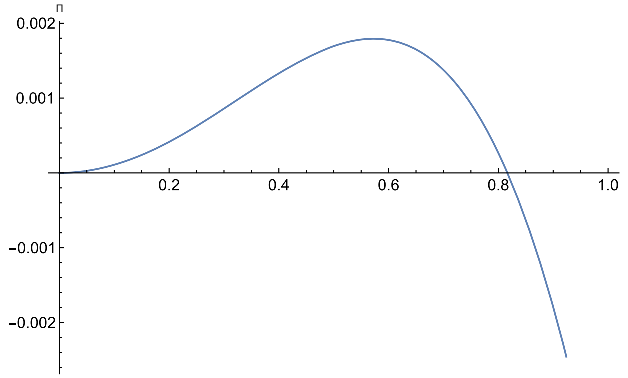


(a) A semilogarithmic plot of the coefficients of the metric (69). The blue line represents A and the orange B.

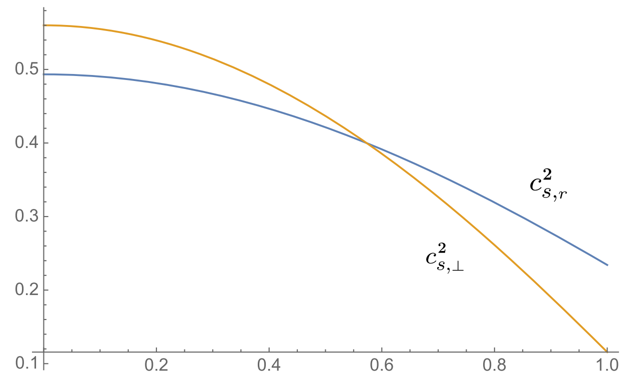


(b) A semilogarithmic plot of the thermodynamic quantities (71) associated with (69). The blue line represents p_r, the orange $p_\perp$ and the green μ.

FIG. 3. Graphs of the solution (69) in the case $r_0 = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 2$, $a = 10$, $b = 1$, $\mu_0 = 1$, $\mu_1 = 1/4$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.



(a) The behaviour of the anisotropic pressure Π associated with (69).



(b) The radial and orthogonal speeds of sound (72) and (73) (blue and orange line, respectively) associated with (69).

FIG. 4. Graphs of the solution (69) in the case $r_0 = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 2$, $a = 10$, $b = 1$, $\mu_0 = 1$, $\mu_1 = 1/4$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

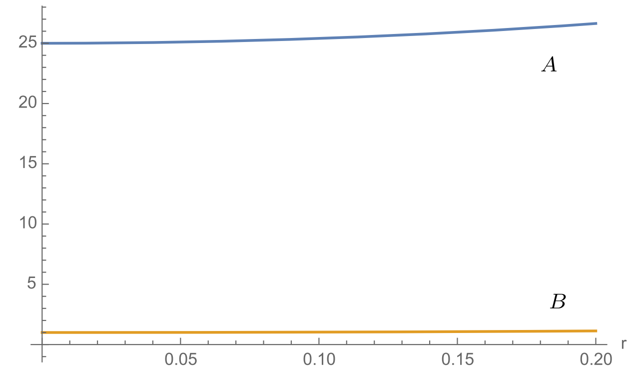
Setting for brevity $r_0 = 1$, we have

$$\mu = (\mu_0 - \mu_1 r^{2\alpha})^\beta, \tag{75}$$

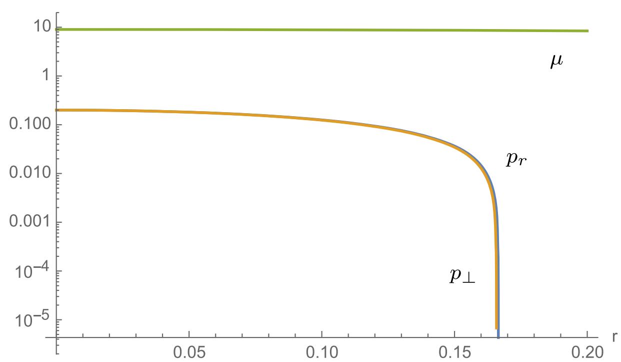
and

$$p_r = \frac{1}{6(br^2 - c)(a + (c - br^2)^\gamma)} \left\{ 12b\gamma\delta(c - br^2)^\gamma - 2\frac{\mathcal{F}}{\mu_0}[a(br^2 - c) + (c - br^2)^\gamma(br^2(2\gamma\delta + 1) - c)] \right\}, \tag{76}$$

$$\begin{aligned} p_\perp = {} & \frac{\mu_0^\beta(\mu_0 - \mu_1 r^{2\alpha})^{-\beta}}{12(br^2 - c)^2(a + (c - br^2)^\gamma)^2} \{2\mathcal{F}(\mu_0 - \mu_1 r^{2\alpha})^\beta[a^2(br^2 - c)^2 - a(c - br^2)^\gamma[b^2r^4(4\gamma^2\delta - \gamma\delta - 2) \\ & + bcr^2(4 - 3\gamma\delta) - 2c^2] + (c - br^2)^{2\gamma}(b^2r^4(-2\gamma^2\delta^2 + \gamma\delta + 1) + bcr^2(3\gamma\delta - 2) + c^2)] \\ & - 3\mu_0^{-\beta}(\mu_0 - \mu_1 r^{2\alpha})^\beta[2a^2(br^2 - c)^2(\mu_0 - \mu_1 r^{2\alpha})^\beta + 2a(c - br^2)^\gamma(b^2r^4(\gamma\delta + 2)(\mu_0 - \mu_1 r^{2\alpha})^\beta \\ & - 4b^2\gamma^2\delta r^2 + bc(4\gamma\delta - r^2(\gamma\delta + 4)(\mu_0 - \mu_1 r^{2\alpha})^\beta) + 2c^2(\mu_0 - \mu_1 r^{2\alpha})^\beta) \\ & + (c - br^2)^{2\gamma}(b^2r^2(2r^2(\gamma\delta + 1)(\mu_0 - \mu_1 r^{2\alpha})^\beta - 4\gamma^2\delta^2) \\ & + 2bc(4\gamma\delta - r^2(\gamma\delta + 2)(\mu_0 - \mu_1 r^{2\alpha})^\beta) + 2c^2(\mu_0 - \mu_1 r^{2\alpha})^\beta)]\}. \end{aligned} \tag{77}$$

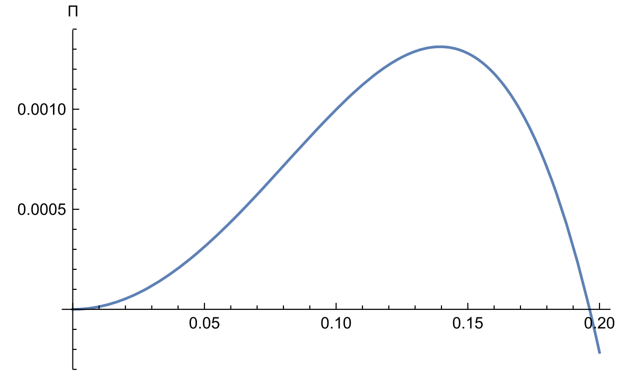


(a) A semilogarithmic plot of the coefficients of the metric (74).The blue line represents A and the orange B.

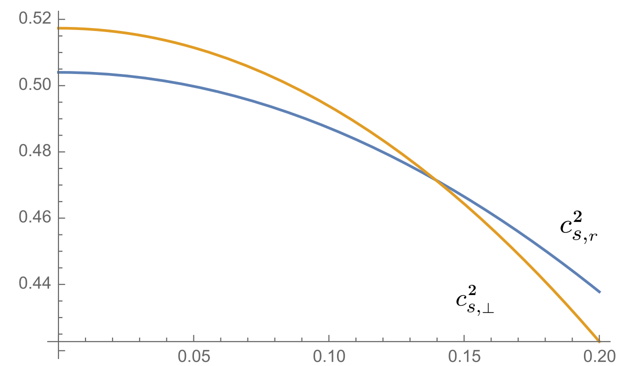


(b) A semilogarithmic plot of the thermodynamic quantities (75) and (77) associated with (74). The blue line represents p_r, the orange $p_\perp$ and the green μ.

FIG. 5. Graphs of the solution (74) in the case $r_0 = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 2$, $\delta = 2$, $a = 10$, $b = -1$, $c = 2$, $\mu_0 = 9$, $\mu_1 = 15$, $A_0 = 1$. The values of the parameters have been chosen in order to make the features of the solution as clear as possible.



(a) The behaviour of the anisotropic pressure Π associated with (74).



(b) The radial and orthogonal speeds of sound (blue and orange line, respectively) associated with (74).

FIG. 6. Graphs of the solution (74) in the case $r_0 = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 2$, $\delta = 2$, $a = 10$, $b = -1$, $c = 2$, $\mu_0 = 9$, $\mu_1 = 15$, $A_0 = 1$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

Notice that the radial and tangential pressure converge to the same value in the center of the star. The radial and orthogonal speeds of sound are too long to be reported here, but their calculation is trivial. In Figs. 5 and 6, we give a plot of the above solution for a set of values of the parameters fulfilling the physical criteria of Sec. II. In light of the theorems that we have presented in the previous section, this solution is somehow expected. It corresponds to the application of the theorem on the shift of the energy density, the isotropic pressure and the radial component of the metric.

We end this section pointing out an interesting aspect of the above solutions. At least in the cases that we have explored in Figs. 3 and 5, the radial and tangential pressure are relatively close to each other. For an object of this type, therefore, the degree of anisotropy even if present is rather small. This fact points to the conclusion that a class of regular objects can exist which are *quasi-isotropic*. Quasi-isotropic stars would behave like an isotropic star upon isolated observation, but they would differ dynamically because of the different properties of the anisotropic pressure. Such differences might appear evident and be studied, for example, at the perturbative level.

VI. CONCLUSIONS

In this paper, we have used the $1 + 1 + 2$ covariant formalism and a tailored variable choice to develop a covariant version of the Tolman-Oppenheimer-Volkoff equations for objects that present the maximum degree of anisotropy compatible with spherical symmetry. Within this framework, it becomes clear that the anisotropy generates additional pressure terms which modify one of the isotropic TOV equations. This anisotropic pressure corresponds to the anisotropy term Δ which generally appears in literature.

The covariant equations clarify the role of the anisotropic pressure in the structure of an object interior and immediately suggest a number of analytical resolution strategies. Indeed, some of these strategies have already been employed in literature in specific cases like the Bowers-Liang one.

The structure of the covariant TOV also suggests that there exists a number of algorithms that allow us to map isotropic solutions in anisotropic ones. Indeed, one of these methods has been recently proposed in Ref. [26]. These procedures can be useful to appreciate the physical role of the anisotropy in these systems as well as to explore further the solution space for anisotropic stars. We proposed some new algorithms of this type. In some cases, these methods allow us to recognize general properties of known solutions, like in the case of Florides's one.

In the isotropic case, a number of generating theorems were discovered which allow us to connect different isotropic solutions to each other. The formalism that we have used allows us to extend these theorems to the anisotropic case in a straightforward way. Indeed, we were able to find a number of new theorems that involve directly the anisotropic pressure. As in the isotropic case we can therefore talk about seed metrics and organize the known solutions in terms of their relations via these theorems.

Finally, the new equations were used to derive a reconstruction algorithm able to generate anisotropic solutions. Unexpectedly this algorithm is much easier than its isotropic counterpart. In fact, it allows us to straightforwardly generate a double infinity of solutions. We should however bear in mind that only few of these solutions correspond to sources with the necessary physical characteristics (see Sec. III). Hence, we can expect that most of these solutions will not have relevance for compact objects.

We used this algorithm to derive some new exact solutions. One of them is in a generalization of the Bowers-Liang solution for which the density is not constant. Its existence is expected by the generating theorems we have proven, but a closer analysis revealed an unexpected feature: for some values of the parameters, they show a radial and tangential pressure very close to each other. Since the other solution we reconstructed has the same properties one is lead to think that these "quasi-isotropic relativistic stars" might be a new class of objects never considered before. It would be interesting to explore further and in more detail the properties of this class of objects. A future work will be focused specifically on this task.

ACKNOWLEDGMENTS

S. C. and D. V. were supported by the Fundação para a Ciência e Tecnologia through Project No. IF/00250/2013 and acknowledge financial support provided under the European Union's H2020 ERC Consolidator Grant "Matter and strong-field gravity: New frontiers in Einstein's theory" Grant Agreement No. MaGRaTh646597.

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