Dynamical system approach to Born-Infeld $f(R)$ gravity in Palatini formalism

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We investigate the dynamics of a number of $f(R)$ theories in the background of the spatially homogeneous as well as isotropic Friedmann-Leimaître-Robertson-Walker (FLRW) model and anisotropic Bianchi type I and V models in Palatini formalism using the dynamical systems approach. Considering theories of the type $f(R) = R + \gamma R^m - \beta/R^n$, it is examined whether this model is capable of allowing all four phases of cosmological evolution or not. Additionally, we have performed the analysis for $f(R) = R + \gamma R^2$, as these theories have attracted great attention in recent years. For the theories of the type $f(R) = R - \beta/R^n$, the sequence of the radiation-dominated, matter-dominated, and accelerated expanding epochs has been reproduced. In the case of anisotropic models, the evolution of the anisotropy parameter, i.e., the shear, is also investigated, where it is found that the initial anisotropic Universe evolves to a stable accelerated expanding Universe. Moreover, corresponding to the Bianchi type V model, the evolution of spatial curvature is also investigated.

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I. INTRODUCTION

Observational cosmology has strongly supported the notion of late-time accelerated expansion of the Universe. This has motivated several modifications in Einstein's general relativity (GR) to explain such expansion. $f(R)$ gravity theories [1–9] are widely used as one of the simplest modifications to GR. This modification corresponds to the replacement of the Ricci scalar (R) in Einstein-Hilbert action by any arbitrary function of R. Three different formalisms of $f(R)$ gravity are used to derive the field equations from the action. The first formalism is the metric $f(R)$ gravity, in which the affine connection depends on the metric, and the field equations are obtained by varying the action with respect to the metric only. The second formalism is the Palatini $f(R)$ gravity, where the field equations are obtained by varying the action with respect to the metric and the connection separately, as both are considered independent variables. In this formalism, an assumption is made regarding the matter action that it is independent of the affine connection. One can extend the Palatini $f(R)$ gravity by avoiding this particular assumption. This extension of Palatini $f(R)$ gravity is known as the metric-affine $f(R)$ gravity. In our analysis, we have considered the Palatini version of $f(R)$ gravity, as the field equations obtained are simple second order field equations, while at the same time it is found to be free from gravitational instability for the form $f(R) = R - \beta/R^n$ [5,8]. Metric $f(R)$ gravity does not show such stability in favor of this particular form of $f(R)$ [5,9]. In [9], it has been shown that gravitational instability occurs in $1/R$ theory. Such instability can be significantly improved [10,11] with the addition of the positive power (higher than 1) of the scalar curvature to the action. The author of [8] also studied this type of instability in the presence of matter considering a specific model of the form $f(R) = R - \mu^4/R$, where μ is a constant. Their analysis revealed that no such instability exists in the Palatini version of $f(R)$ gravity. They also investigated the reasons for the occurrence of the instability in metric formalism but not in the Palatini formalism of $f(R)$ gravity. The author of [12] has shown that the $1/R$ theory, which is equivalent to some scalar tensor theories of gravity, cannot pass the Solar System test. It has been proved in [10,11] that with the addition of the scalar curvature squared term to the action, the Solar System test for this section of modified gravity theory can be passed. The author of [13] studied the Palatini version of modified gravity models, including the higher order terms, in the scalar curvature in a nearly Newtonian regime and showed that as long as the coefficients corresponding to the higher order terms are reasonably small, such models can achieve a good Newtonian limit.

It is generally difficult to find the exact solution of the nonlinear field equations obtained by the variation of action. Hence, we are in need of a suitable approach to solve this problem. One such approach is the dynamical system approach (DSA) [14–18]. Chan in his Ph.D. thesis [14] gave a survey on the application of DSA and its role in

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cosmology. Leach [17] studied the extended theories of gravity using DSA in great detail. Fay *et al.* [5] presented a qualitative discussion on the application of DSA for the Friedmann-Leimaître-Robertson-Walker (FLRW) model corresponding to three different families of $f(R)$ in the case of Palatini $f(R)$ gravity.

The Einstein-Hilbert action has been the backbone of gravitational theory for almost a century. However, in 1924 Eddington proposed an intriguing, alternative theory of gravity based on a connection field without introducing a metric [19]. Eddington's original theory of gravity is incomplete because it does not include matter fields. A new Eddington-inspired Born-Infeld (EiBI) gravity theory [20] in the Palatini formulation including matter fields has been proposed by Bañados and Ferreira [21]. In vacuum, this theory reproduces Eddington's gravity and is completely equivalent to Einstein's general relativity (GR), but it leads to several attractive new features in the presence of matter [22]. We expect singularities such as in the beginning of the Universe or in the interior of black holes, but Eddington-inspired Born-Infeld gravity is often singularity free [21–23]. However, other singularities may appear in an astrophysical framework [24]. The appearance of such behavior makes Eddington-inspired Born-Infeld gravity interesting and exciting.

A purely metric proposal of Born-Infeld gravity can be found in [25]. Born-Infeld Lagrangians are viable modifications of the Einstein-Hilbert action provided the action is varied in accordance with the Palatini formalism, in contrast to the metric formalism [26]. A Palatini formulation of the gravitational action was first proposed in [27–29]. The Born-Infeld action is given by [21]

$$S = \frac{1}{\kappa^2 \epsilon} \int d^4x \left[\sqrt{-|g_{\mu\nu} + \epsilon R_{\mu\nu}(\Gamma)|} - \lambda \sqrt{-|g_{\mu\nu}|} \right] + S_m. \tag{1}$$

The theory introduces a coupling parameter ϵ (the Eddington parameter), a constant with inverse dimensions to that of a cosmological constant. λ is a dimensionless constant. For small values of ϵR, the EiBI action reduces to the Einstein-Hilbert action with cosmological constant $\Lambda = (\lambda - 1)/\epsilon$. From the observational perspective, it is an appealing property of Born-Infeld theory that it recovers GR with an effective cosmological constant. On the other hand, for large values of ϵR, the Eddington's action will be obtained approximately. Therefore, the coupling parameter ϵ interpolates between two different gravity theories. By suitably tuning this parameter the theory can be made to agree with all current observations [30].

The cosmological implications of the EiBI theory have also been investigated in the literature [30–35]. The evolution of the Universe filled with barotropic perfect fluid in the EiBI theory was considered in [31], for both isotropic and anisotropic universes. At the early stage of the Universe, when the energy density is high, the evolution is modified considerably as compared with that in GR. For the equation of state parameter $w > 0$, there is no initial singularity and for pressureless dust ($w = 0$), the initial state approaches a de Sitter evolution. This fact provides a new possibility of the singularity-free nature of the theory. The dynamics of a homogeneous as well as isotropic FLRW Universe considering the Eddington-inspired theory of gravity were explored in [32]. For a positive coupling parameter there is a singularity-avoiding behavior in the case of a perfect fluid with equation of state parameter $w > 0$. The range $-1/3 < w < 0$ leads to universes that experience an unbounded expansion rate, while still at a finite density. In the case of a negative coupling parameter the addition of spatial curvature leads to the possibility of oscillation between two finite densities. Domination by a scalar field with an exponential potential also leads to singularity-avoiding behavior when the coupling parameter is positive. The evolution of a spatially flat, homogeneous anisotropic Kasner universe filled with a scalar field, whose potential has various forms, was analyzed in Eddington-inspired Born-Infeld gravity in [33]. An exact solution for each scalar field potential, describing the initial state of the Universe, was found by imposing a maximal pressure condition. Makarenko *et al.* [30] worked out a gravity theory that combines the Born-Infeld gravity with an $f(R)$ gravity theory within the Palatini approach to address a number of important questions, such as the dynamics of the early Universe and the cosmic accelerated expansion of the late-time Universe. Harko *et al.* [34] considered a barotropic cosmological fluid in an anisotropic, Bianchi type I space-time in Eddington-inspired Born-Infeld gravity and obtained the general solution of the field equations. The behavior of the geometric and thermodynamic parameters of the Bianchi type I Universe has been studied, using both analytical and numerical methods, for some classes of high-density matter, described by the stiff causal, radiation, and pressureless fluid equations of state. For the dust-filled Universe, the cosmological evolution always ends in an isotropic phase, while for high-density, matter-filled universes, the isotropization of Bianchi type I universes is essentially determined by the initial conditions of the energy density. Bambi *et al.* [35] investigated the possibility of finding analytical solutions for axially symmetric magnetic fields (Melvin universe) in the Born-Infeld theory of gravity formulated in the Palatini approach. Their results set the basis for further extensions that could allow the embedding of pairs of black hole remnants in geometries with intense magnetic fields. Motivated by these studies, we perform an analysis that is aimed at investigating the Born-Infeld $f(R)$ gravity in Palatini formalism using DSA in the background of an isotropic as well as anisotropic model.

It is well established that the present state of the Universe is highly homogeneous and isotropic. The FLRW model

[5,21,26,30,36–38] mimics the best representation of this isotropic and homogeneous behavior of the present Universe. However, the existence of inhomogeneity and anisotropy in the early Universe has also been believed [39–48]. Anisotropy plays an important role in explaining the structure formation in the Universe existing in the early Universe that isotropizes with the evolution of time. The cosmic microwave background radiation (CMBR) described by Penzias and Wilson in 1965 fills the whole Universe with an almost isotropic thermal blackbody distribution at a temperature of about 2.725 K. However, in 1992, the Cosmic Background Explorer (COBE) team [49] found that the measure of the anisotropies available in the temperature of the CMBR is about one part in 10^5, and for this discovery J. C. Marther and G. F. Smoot were awarded the Nobel Prize in Physics in 2006. The present Universe is homogeneous as well as isotropic, but on large scales as recovered by the anisotropies present in the CMBR. Recent probes and experiments such as the Wilkinson Microwave Anisotropy Probe (WMAP) [50–53] and Planck's results [54,55] also confirmed the existence of small anisotropies of the CMBR. Therefore, anisotropic models such as Bianchi type models have gained importance in recent days to investigate large scale structures like galaxies, clusters of galaxies, etc., in the Universe. The Bianchi models are required not only to investigate the anisotropy of the early Universe, but also to study the accelerated expansion at late times. In our analysis, we have considered the FLRW model as well as the Bianchi type I and V models to investigate the behavior of the early and late Universe. In recent papers, the evolution of the anisotropic Bianchi model is explored using DSA in the case of metric [56,57] and Palatini [58–60] $f(R)$ gravity. Various physical and kinematical features of the spatially homogeneous and anisotropic Bianchi type I and V models have been discussed in [61–64].

There are excellent reviews [65,66] available in the literature on $f(R)$ gravity theory. The authors of [65] provided an overview of the latest developments in modified gravity, paying special attention to inflation, bouncing cosmology, and the late-time cosmic acceleration epoch. In their work, they assembled different types of modified gravity techniques and provided a virtual modified gravity toolbox containing all the necessary information on inflation, bouncing cosmology, and late-time acceleration in the context of modified gravity. The authors of [66] provided an excellent review on the unification of early-time inflation and late-time cosmic acceleration in the background of FLRW cosmology considering a number of modified gravity models, such as the $f(R)$ gravity model, modified Gauss-Bonnet and scalar-Gauss-Bonnet gravities, nonminimal models, nonlocal gravity, modified $f(R)$ Hořava-Lifshitz theory, and power-counting renormalizable covariant gravity. The authors of [10,11,66] unified the early- and late-time cosmic acceleration considering the theories of the form $f(R) = R + \alpha R^m - \beta/R^n (m, n > 0)$. At large curvature, the term R^m dominates, describing the inflationary phase of the early Universe. For $1 < m < 2$, one obtains the power law inflation, whereas for $m = 2$, anomaly-driven (Starobinsky) inflation appears at early times. At small curvature, the term $1/R^n$ dominates, which depicts the late-time cosmic acceleration. However, the authors of [5,67] have shown that theories of the form $f(R) = R + \alpha R^m - \beta/R^n$ have shown difficulties in producing the unified models of inflation and late-time acceleration with the radiation- and matter-dominated epochs. The authors of [5] also presented that it is possible to accomplish the sequences of a radiation-dominated, matter-dominated, and de Sitter period, corresponding to $f(R) = R + \alpha R^m - \beta/R^n$ when $0 < m < 1$. In addition, it has also been shown that the theories of the type $f(R) = R - \beta/R^n (n > -1)$ are capable of attaining the sequence of a radiation-dominated, matter-dominated, and late-time accelerated expanding period.

In this paper, we study a number of $f(R)$ theories which have attracted a great deal of attention, with a number of studies attempting to determine their viability as cosmological models. Despite the efforts carried out in this direction, the cosmological dynamics of models based on such theories are not fully understood. It is important to examine whether these models are capable of allowing all four phases of cosmological evolution or not. It should be noted that such theories should be able to produce all four phases as a maximal demand [5]. Thus the question of whether such theories could successfully account for a sequence of such phases or not provides a strong motivation for the analysis carried out here. In particular, the dynamical behavior of the chosen models has not been explored yet in the Born-Infeld $f(R)$ gravity.

The content of the paper is organized as follows. In Sec. II, the Born-Infeld $f(R)$ gravity in Palatini formalism is briefly discussed. In Sec. III, we proceed to study the cosmological dynamics for the FLRW model using DSA. The analysis is extended corresponding to anisotropic models, namely, Bianchi type I and V in Secs. IV and V, respectively. Ultimately, the conclusions are summarized in Sec. VI.

II. BORN-INFELD $f(R)$ THEORY IN PALATINI FORMALISM

Let us consider the action for Born-Infeld $f(R)$ theory of gravity as follows [30]:

$$S_{BI} = \frac{1}{\kappa^2 \epsilon} \int d^4x \left[\sqrt{-|g_{\mu\nu} + \epsilon R_{\mu\nu}(\Gamma)|} - \lambda \sqrt{-|g_{\mu\nu}|}\right] + \frac{\alpha}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m, \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor. $R_{\mu\nu}(\Gamma)$ represents a Ricci tensor which is constructed from the connection $\Gamma^{\alpha}_{\beta\gamma}$. $f(R)$

acts as a function of the Ricci scalar R, S_m is the matter action, λ represents a constant of first order, $\kappa^2 = 8\pi G$, and G is the gravitational constant.

For $\epsilon \to 0$, the above equation reproduces the action of $f(R)$ gravity theory and the Born-Infeld $f(R)$ action takes the form $(\int d^4x\sqrt{-g}\mathcal{L}_G + S_m)$, where the Lagrangian $\mathcal{L}_G = \frac{R-2\Lambda+\alpha f(R)}{2\kappa^2}$ and $\Lambda = \frac{\lambda-1}{\epsilon}$. On the other hand, for $\alpha \to 0$, we regenerate the Born-Infeld theory with action $(\frac{1}{\kappa^2\epsilon}\int d^4x[\sqrt{-|g_{\mu\nu}+\epsilon R_{\mu\nu}(\Gamma)|} - \lambda\sqrt{-|g_{\mu\nu}|}] + S_m)$. In addition, for $\alpha \to 0$ and $\epsilon \to 0$, we recover the GR term.

In the Palatini $f(R)$ gravity, the field equations are obtained by independent variations of the action, with respect to the metric and the affine connection. The variation of action (2) with respect to the metric $g_{\mu\nu}$ yields

$$\frac{\sqrt{-q}}{\sqrt{-g}}q^{\mu\nu} - \left[\left(\lambda - \frac{\alpha\epsilon}{2}f(R)\right)g^{\mu\nu} + \alpha\epsilon f'(R)g^{\mu\beta}g^{\nu\gamma}R_{\beta\gamma}\right] = -\kappa^2 T^{\mu\nu}, \tag{3}$$

where a prime stands for the derivatives with respect to R and $T_{\mu\nu}$ is the energy-momentum tensor.

The variation of action (2) with respect to the connection $\Gamma^\alpha_{\beta\gamma}$ yields

$$\nabla_\beta[\sqrt{-q}q^{\mu\nu} + \alpha f'(R)\sqrt{-g}g^{\mu\nu}] = 0. \tag{4}$$

In the above equations we have used the notation

$$q_{\mu\nu} = g_{\mu\nu} + \epsilon R_{\mu\nu}(\Gamma). \tag{5}$$

In Born-Infeld theory $q_{\mu\nu}$ can be written in terms of $g_{\mu\nu}$ as

$$q_{\mu\nu} = p(R)g_{\mu\nu}. \tag{6}$$

Using the above assumption in (5), the resulting equation reduces to

$$\nabla_\beta[(p(R) + \alpha f'(R))\sqrt{-g}g^{\mu\nu}] = 0. \tag{7}$$

Therefore we can define an auxiliary tensor of the form

$$u^{\mu\nu} = (p(R) + \alpha f'(R))g^{\mu\nu} \tag{8}$$

such that the corresponding equation becomes $\nabla_\beta[\sqrt{-u}u^{\mu\nu}] = 0$.

Therefore we can consider

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}u^{\alpha\beta}(\partial_\mu u_{\nu\beta} + \partial_\nu u_{\mu\beta} - \partial_\beta u_{\mu\nu}). \tag{9}$$

This equation provides us with an exact and complete solution of the affine connection.

In the following, we will consider three different models to analyze the cosmological dynamics, using the above equations and considerations. We first study the spatially homogeneous and isotropic FLRW model and then continue for spatially homogeneous and anisotropic Bianchi type I and V models.

III. BORN-INFELD $f(R)$ THEORY IN FLRW COSMOLOGY

The line element of the spatially flat FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \tag{10}$$

where x, y, z represent the comoving coordinates, $a(t)$ act as the expansion scale factors, and t is the cosmological time.

In order to find the connection between the main and auxiliary tensor, let us consider $u(t) = (p(R) + \alpha f'(R))$ and $r(t) = (p(R) - 1)/\epsilon$. Using these relations, one can find that $R(u_{\alpha\beta})_{\mu\nu} = r(t)g_{\mu\nu}$. All these considerations lead to the following equations:

$$r(t) = \frac{3}{2}\left[2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{u}}{u} + \frac{\ddot{u}}{u} - \left(\frac{\dot{u}}{u}\right)^2\right] \tag{11}$$

$$r(t) = \left[\frac{\ddot{a}}{a} + \frac{5}{2}\frac{\dot{a}}{a}\frac{\dot{u}}{u} + \frac{\ddot{u}}{2u} + 2\left(\frac{\dot{a}}{a}\right)^2\right] \tag{12}$$

where the dots stand for the derivatives with respect to t and the Hubble parameter is given by $H = \dot{a}/a$.

Combining Eqs. (11) and (12), one can obtain

$$r(t) = 3\left(H + \frac{\dot{u}}{2u}\right)^2 \tag{13}$$

$$2\dot{H} = H\frac{\dot{u}}{u} + \frac{3}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{\ddot{u}}{u}, \tag{14}$$

and hence,

$$\frac{\dot{u}}{u} = \frac{\dot{r}}{r}. \tag{15}$$

The solution of Eq. (15) is

$$r(t) = cu(t), \tag{16}$$

where c is a constant. In the light of the equations above, one arrives at the following:

$$p(R)g^{\mu\nu} - \left[\left(\lambda - \frac{\alpha\epsilon}{2}f(R)\right)g^{\mu\nu} + \alpha f'(R)(p(R)-1)g^{\mu\nu}\right]$$
$$= -\kappa^2 T^{\mu\nu}. \tag{17}$$

The above equation in the background of the FLRW model can also be written as

$$\frac{2}{\epsilon}(\lambda-1) + \alpha(f'R - f) - 6(1+\alpha f')\left(H + \frac{\dot{u}}{2u}\right)^2$$
$$= \kappa^2(\rho_m + 2\rho_r), \tag{18}$$

where ρ_m and ρ_r represent the energy densities of matter and radiation, respectively. The conservation equations corresponding to both the energy densities are given by

$$\dot{\rho}_m + 3H\rho_m = 0 \tag{19}$$

$$\dot{\rho}_r + 4H\rho_r = 0. \tag{20}$$

Contracting Eq. (3) and considering that the trace of the radiative fluid vanishes, we obtain

$$4(1-\lambda) + \epsilon(1-\alpha f')R + 2\alpha\epsilon f = \kappa^2\epsilon\rho_m. \tag{21}$$

Using Eqs. (18)–(21), it can be shown that

$$\dot{R} = -3H\frac{\kappa^2\rho_m}{1-\alpha(f''R - f')}$$
$$= -3H\frac{4(1-\lambda) + \epsilon R - \alpha\epsilon(f'R - 2f)}{\epsilon - \alpha\epsilon(f''R - f')}. \tag{22}$$

Using Eqs. (18) and (22), one can find that

$$H^2 = \frac{2\kappa^2(\rho_m+\rho_r) + \alpha(f'R - f) + \frac{2}{\epsilon}(\lambda-1)}{6(1+\alpha f')\xi^2}, \tag{23}$$

where

$$\xi = 1 + \frac{\dot{u}}{2uH} = 1 - \frac{3}{2R}\frac{4(1-\lambda) + \epsilon R - \alpha\epsilon(f'R - 2f)}{\epsilon - \alpha\epsilon(f''R - f')}. \tag{24}$$

To study the complete and exact nature of cosmological solutions of the field equation using DSA, we define the following dimensionless variables:

$$\Omega_r = \frac{\kappa^2\rho_r}{3(1+\alpha f')\xi^2H^2}, \quad \Omega_m = \frac{\kappa\rho_m}{3(1+\alpha f')\xi^2H^2},$$
$$x = \frac{\alpha(f'R - f)}{6(1+\alpha f')\xi^2H^2}, \quad y = \frac{\lambda-1}{3\epsilon(1+\alpha f')\xi^2H^2}. \tag{25}$$

Therefore, the constraint equation reduces to

$$1 = \Omega_r + \Omega_m + x + y. \tag{26}$$

Differentiating Eq. (23) and using the variables (25) in the resulting equation, we get

$$2\frac{\dot{H}}{H^2} = -3 - \Omega_r + 3x + 3y + \frac{1}{6H^3}\frac{f''R\dot{R}}{(1+\alpha f')\xi^2}$$
$$-\frac{\alpha}{H}\frac{f''\dot{R}}{(1+\alpha f')} - \frac{\dot{\xi}}{H\xi}. \tag{27}$$

The first variable of (25) is

$$\Omega_r = \frac{\kappa^2\rho_r}{3(1+\alpha f')\xi^2H^2}.$$

Taking the log on both sides of the above equation, we have

$$\ln\Omega_r = \ln\frac{\kappa^2}{3} + \ln\rho_r - \ln(1+\alpha f') - 2\ln\xi - 2\ln H.$$

Differentiating with respect to time, we have

$$\frac{\dot{\Omega}_r}{\Omega_r} = \frac{\dot{\rho}_r}{\rho_r} - \frac{\alpha f''\dot{R}}{(1+\alpha f')} - 2\frac{\dot{\xi}}{\xi} - 2\frac{\dot{H}}{H}$$
$$= H[-1 + \Omega_r - 3x - 3y - C(R)x].$$

Now $\tau = \ln a$.

Therefore, the evolution equations corresponding to the variables Ω_r can be written as follows:

$$\frac{d\Omega_r}{d\tau} = \frac{1}{H}\frac{d\Omega_r}{dt} = \Omega_r[-1 + \Omega_r - 3x - 3y - C(R)x]. \tag{28}$$

The third variable of (25) is

$$x = \frac{\alpha(f'R - f)}{6(1+\alpha f')\xi^2H^2}.$$

Taking the log on both sides of the above equation, we have

$$\ln x = \ln\frac{\alpha}{6} + \ln(f'R - f) - \ln(1+\alpha f') - 2\ln\xi - 2\ln H.$$

Differentiating with respect to time, we have

$$\frac{\dot{x}}{x} = \frac{f''R\dot{R}}{(f'R - f)} - \frac{\alpha f''\dot{R}}{(1+\alpha f')} - 2\frac{\dot{\xi}}{\xi} - 2\frac{\dot{H}}{H}$$
$$= H[3 + \Omega_r - 3x - 3y + C(R)(1-x)].$$

Therefore, the evolution equations corresponding to the variables x can be written as follows:

$$\frac{dx}{d\tau} = \frac{1}{H}\frac{dx}{dt} = x[3 + \Omega_r - 3x - 3y + C(R)(1 - x)]. \qquad (29)$$

The fourth variable of (25) is

$$y = \frac{\lambda - 1}{3\epsilon(1 + \alpha f')\xi^2 H^2}.$$

Taking the log on both sides of the above equation, we have

$$\ln y = \ln\frac{\lambda - 1}{3\epsilon} - \ln(1 + \alpha f') - 2\ln\xi - 2\ln H.$$

Differentiating with respect to time, we have

$$\begin{aligned}\frac{\dot{y}}{y} &= -\frac{\alpha f''\dot{R}}{(1 + \alpha f')} - 2\frac{\dot{\xi}}{\xi} - 2\frac{\dot{H}}{H}\\ &= H[3 + \Omega_r - 3x - 3y - C(R)x].\end{aligned}$$

Therefore, the evolution equations corresponding to the variables y can be written as follows:

$$\frac{dy}{d\tau} = \frac{1}{H}\frac{dy}{dt} = y[3 + \Omega_r - 3x - 3y - C(R)x]. \qquad (30)$$

In the above equations $C(R)$ is defined as

$$\begin{aligned}C(R) &= \frac{1}{H}\frac{f''R\dot{R}}{(f'R - f)}\\ &= -3\frac{f''R(4(1 - \lambda) + \epsilon R - \alpha\epsilon(f'R - 2f))}{(f'R - f)(\epsilon - \alpha\epsilon(f''R - f'))}. \qquad (31)\end{aligned}$$

In the dynamical system (28)–(30) corresponding to FLRW cosmology, it is seen that only time dependence (or τ dependence) is contained in the parameter $C(R)$. However, in the following section it is seen that the value of this parameter is always constant. Therefore the above dynamical system does not depend on time (or τ) explicitly and hence the system is autonomous. The authors of [68] presented a detailed investigation on the autonomous dynamical system approach in $f(R)$.

Using Eqs. (22) and (25) in the constraint equation (26), one can have

$$\frac{4(1 - \lambda) + \epsilon R - \alpha\epsilon(f'R - 2f)}{\alpha\epsilon(f'R - f)} = \frac{1 - \Omega_r - x - y}{2x}. \qquad (32)$$

Equation (32) clearly shows that $C(R)$ can be expressed in terms of the dimensionless dynamical variables (25).

In order to study the dynamics of the system under consideration, we are interested in the equilibrium points of the system. Such points can be obtained by equating the evolution equations (28)–(30) to zero. The equilibrium points obtained in our case are as follows:

$$\begin{aligned}P_r&:\ (\Omega_r, x, y) = (1, 0, 0)\\ P_m&:\ (\Omega_r, x, y) = (0, 0, 0)\\ P_x&:\ (\Omega_r, x, y) = (0, 1, 0)\\ P_y&:\ (\Omega_r, x, y) = (0, 0, 1).\end{aligned}$$

To study the dynamics of a system, we also have to look at the stabilities of the equilibrium points. Such stabilities can be studied from the eigenvalues which are obtained in linearizing the system of equations (28)–(30). The eigenvalues associated with each of the equilibrium point are as follows:

$$\begin{aligned}P_r&:\ [\lambda_1, \lambda_2, \lambda_3] = [1, 4, 4 + C(R)]\\ P_m&:\ [\lambda_1, \lambda_2, \lambda_3] = [-1, 3, 3 + C(R)]\\ P_x&:\ [\lambda_1, \lambda_2, \lambda_3] = [-C(R), -3 - C(R), -4 - C(R)]\\ P_y&:\ [\lambda_1, \lambda_2, \lambda_3] = [-3, -4, C(R)].\end{aligned}$$

It is very useful to define the effective equation of state (EoS) parameter w_{eff}, in terms of the dynamical variables (25):

$$\begin{aligned}w_{\mathrm{eff}} &= \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}}\\ &= \frac{1}{3}\Omega_r - x - y - \frac{1}{3}C(R)x + \frac{\alpha f''\dot{R}}{3H(1 + \alpha f')} + \frac{\dot{\xi}}{3H\xi}. \qquad (33)\end{aligned}$$

We can also define the deceleration parameter q, which has a relation with the Hubble parameter H as

$$\frac{\dot{H}}{H^2} = -(1 + q), \qquad (34)$$

where q can be expressed in terms of the dynamical variables (25) as

$$\begin{aligned}q = -1 + \frac{3}{2}\Bigg[1 + \frac{1}{3}\Omega_r - x - y - \frac{1}{3}C(R)x\\ + \frac{\alpha f''\dot{R}}{3H(1 + \alpha f')} + \frac{\dot{\xi}}{3H\xi}\Bigg]. \qquad (35)\end{aligned}$$

Integrating Eq. (34) and setting the big bang time $t_0 = 0$, one can find the solution of the average scale factor a associated to each fixed point. For $q \neq -1$, the average scale factor is written as

$$a = a_0|t|^{\zeta}, \qquad (36)$$

where a_0 is a constant and $\zeta = (1 + q)^{-1}$.

For $q = -1$, the average scale factor is written as

$$a = a_0 e^{\frac{1}{3}\theta_0 t}. \quad (37)$$

Equations (36) and (37) indicate that a power law solution exists for $q \neq -1$, whereas a de Sitter solution exists for $q = -1$. One can use the average scale factor a and the deceleration parameter q to study the contracting as well as the expanding nature of solutions associated with all the fixed points [56–60].

A. Cosmological dynamics for model $f(R) = R + \gamma R^m - \beta/R^n$

To investigate the dynamics of the $f(R)$ model based on these theories, we will follow the approach used in [5]. In this case the terms $\gamma R^m (\gamma > 0)$ and $\beta/R^n (\beta > 0)$ are very important to describe the early and late times, respectively. This type of model was also discussed in [10,11,66]. The form of $f(R)$ considered in their analysis is given below:

$$f(R) = R - \frac{a}{(R - \Lambda_1)^n} + b(R - \Lambda_2)^m, \quad (38)$$

where the coefficients $n, m, a, b > 0$. In [11,66] it was shown that for $n = 1, m = 2, \Lambda_1 = \Lambda_2 = 0$, and small curvature, the scale factor is $a \propto t^2$, which is consistent with the result as discussed in [69,70].

Considering $\Lambda_1 = \Lambda_2 = 0, a = \beta$, and $b = \gamma$, we can attain the $f(R)$ model of the form

$$f(R) = R - \frac{\beta}{R^n} + \gamma R^m. \quad (39)$$

In this case Eq. (32) will become

$$\frac{(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} - (n+2)\alpha\beta R^{-(n+1)}}{(m-1)\alpha\gamma R^{(m-1)} + (n+1)\alpha\beta R^{-(n+1)}} = \frac{1 - \Omega_r - x + 3y}{2x}. \quad (40)$$

To study the dynamics of such an $f(R)$ model, we will start the analysis by investigating the early and late dynamics separately.

1. Cosmological dynamics for model $f(R) = R - \beta/R^n$

Considering $\gamma = 0$ in Eq. (39), we have

$$f(R) = R - \frac{\beta}{R^n}. \quad (41)$$

The term $\beta/R^n (\beta > 0, n > 0)$ [69–73] appearing in this model is very significant in studying the late-time accelerated expansion of the Universe.

For this particular case of $f(R)$, $C(R)$ and ξ can be expressed as

$$C(R) = \frac{3n}{((1+\alpha) + n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha) - (n+2)\alpha\beta R^{-(1+n)} + \frac{(1-c\epsilon)(1-\lambda)}{c\epsilon((1+\alpha) + n\alpha\beta R^{-(1+n)})}\right] \quad (42)$$

$$\xi = 1 - \frac{3}{2((1+\alpha) + n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha) - (n+2)\alpha\beta R^{-(1+n)} + \frac{(1-c\epsilon)(1-\lambda)}{c\epsilon((1+\alpha) + n\alpha\beta R^{-(1+n)})}\right], \quad (43)$$

where $R^{(n+1)}$ can be expressed in term of the dimensionless dynamical variables (25) as

$$R^{(1+n)} = \frac{\alpha\beta}{1+\alpha}\frac{1 - \Omega_r + 3x + 3y + n(1 - \Omega_r + x + 3y)}{2x}. \quad (44)$$

Let us assume

$$b = 1 + \frac{(1-c\epsilon)(1-\lambda)}{c\epsilon(1+\alpha)^2}. \quad (45)$$

Therefore, the equations of $C(R)$ and ξ reduce to

$$C(R) = \frac{3n}{((1+\alpha) + n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha) - (n+2)\alpha\beta R^{-(1+n)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha) + n\alpha\beta R^{-(1+n)}}\right] \quad (46)$$

$$\xi = 1 - \frac{3}{2((1+\alpha) + n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha) - (n+2)\alpha\beta R^{-(1+n)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha) + n\alpha\beta R^{-(1+n)}}\right]. \quad (47)$$

Fixed points and solutions.—For the fixed point P_r, both the numerator and denominator of Eq. (44) approach zero. Therefore we split the analysis into two different parts. The first part corresponds to $\beta/R^{(n+1)} \ll 1$ given by P_{r1}, and the second corresponds to $\beta/R^{(n+1)} \gg 1$ given by P_{r2} [5]. The fixed points and their associated solutions are listed in Table I.

The evolution of the dynamical variables Ω_r, x, and y, along with the EoS parameter $w_{\rm eff}$ for $\beta/R^{(n+1)} \ll 1$ ($n = 0.002, b = -1$, β arbitrary), are plotted in Fig. 1. The figure clearly depicts that the radiation-dominated era (P_{r1}) first approaches the matter-dominated era (P_m) and

TABLE I. Fixed points and their associated solutions for FLRW model corresponding to $f(R) = R - \beta/R^n$.

Points	Ω_m	EoS parameter $(w_{\rm eff})$	Deceleration parameter (q)	Average scale factor (a)	Physical behavior
P_{r1}	0	$\frac{1}{3}$	1	$a_0\vert t\vert^{\frac{1}{2}}$	Decelerated expansion
P_{r2}	0	$-\frac{2}{3} - \frac{1}{n}$	$-\frac{1}{2} - \frac{3}{2n}$	$a_0\vert t\vert^{\frac{2n}{n-3}}$	Decelerated expansion for $-3 < n < 0$ Accelerated expansion for $n \geq 3, n < -3$ Decelerated contraction for $0 < n < 3$ Milne evolution for $n = -3$
P_m	1	0	$\frac{1}{2}$	$a_0\vert t\vert^{\frac{2}{3}}$	Decelerated expansion
P_x	0	$-1 + \frac{n(n+2)(1-b)}{4(n+1)^2}$	$-1 + \frac{3n(n+2)(1-b)}{8(n+1)^2}$	$a_0\vert t\vert^{\frac{8(n+1)^2}{3n(n+2)(1-b)}}$	Accelerated expansion for $n > 0, b < 1,\ b < 1 - \frac{8(n+1)^2}{3n(n+2)}$ Decelerated expansion for $n > 0, b < 1,\ b > 1 - \frac{8(n+1)^2}{3n(n+2)}$
P_y	0	-1	-1	$a_0\vert t\vert^{\frac{1}{3}\theta_0 t}$	Accelerated expansion

then goes to either a power law (P_x) or de Sitter acceleration era (P_y).

Stability of fixed points.—It has already been mentioned in the previous section that the eigenvalues of the linearized equations (28)–(30) reveal information about the stability associated to each fixed point. The eigenvalues and the stability corresponding to each fixed point for the FLRW model are summarized in Table II.

Phase portrait analyses in the case of a FLRW model for $f(R) = R - \beta/R^n$ (β arbitrary, $n = 0.002, b = -1$) are plotted in Fig. 2. In this plot, two heteroclinic sequences of the form $P_r \to P_m \to P_x$ and $P_r \to P_m \to P_y$ are obtained. The first sequence mimics the evolution of the radiation-dominated (P_r) to the matter-dominated era (P_m) and then to the accelerated expanding (P_x) epoch. The second sequence depicts the evolution of the radiation-dominated (P_r) to the matter-dominated era (P_m) and then to the de Sitter expanding (P_y) epoch.

2. Cosmological dynamics for model $f(R) = R + \gamma R^m$

Considering $\beta = 0$ in Eq. (39), we have

$$f(R) = R + \gamma R^m \tag{48}$$

The term $\gamma R^m (\gamma > 0, m > 0)$ appearing in this model is very significant in studying the accelerated expansion of the early Universe.

For this particular case of $f(R)$, $C(R)$ and ξ can be expressed as

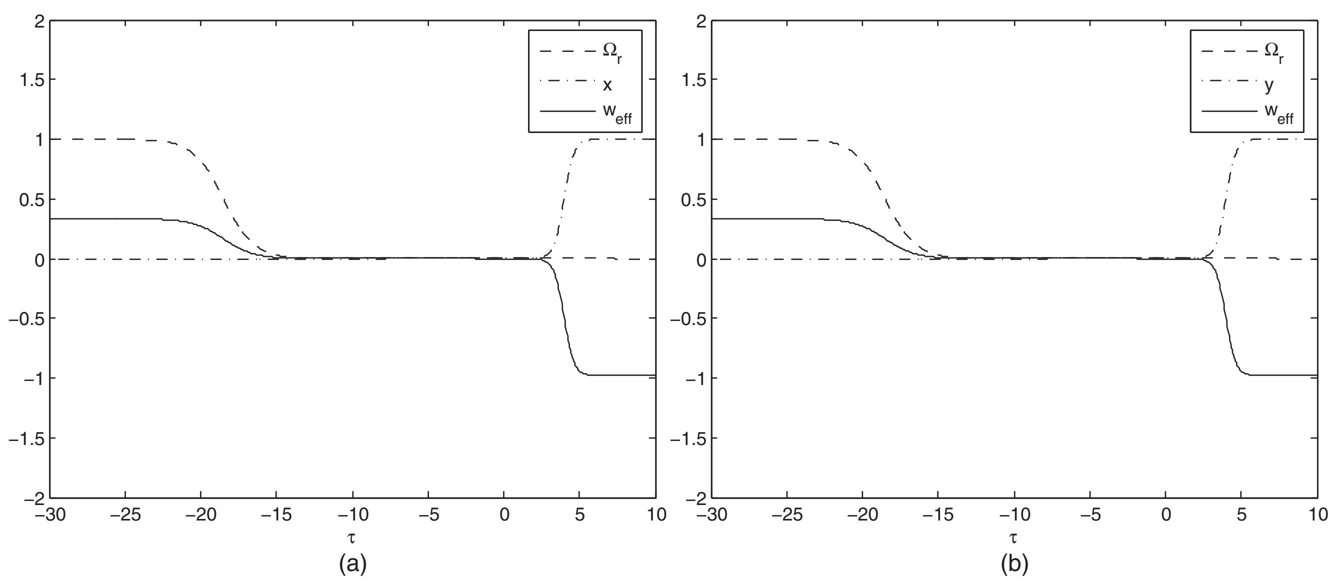

FIG. 1. Evolutions of the dynamical variables Ω_r, x and y along with the EoS parameter $w_{\rm eff}$, for FLRW model corresponding to $f(R) = R - \beta/R^n$ with $n = 0.002, b = -1$, and arbitrary β.

TABLE II. Eigenvalues and stability of the fixed points for FLRW model corresponding to $f(R) = R - \beta/R^n$.

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3]$	Stability
P_{r1}	$[1, 4, (4+3nb)]$	Unstable for $n > -\frac{4}{3b}$ Saddle otherwise
P_{r2}	$[1, 1, 4]$	Unstable
P_m	$[-1, 3, 3(1+nb)]$	Saddle
P_x	$\left[\frac{3n(n+2)(1-b)}{2(n+1)^2}, -3+\frac{3n(n+2)(1-b)}{2(n+1)^2}, -4+\frac{3n(n+2)(1-b)}{2(n+1)^2}\right]$	Stable for $n > 0, b > 1$ Saddle otherwise
P_y	$[-4, -3, 3nb]$	Stable for $n > 0, b < 0$

$$C(R) = \frac{-3m}{((1+\alpha) - m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right] \tag{49}$$

$$\xi = 1 - \frac{3}{2((1+\alpha) - m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right], \tag{50}$$

where $R^{(1-m)}$ can be expressed in terms of the dimensionless dynamical variables (25) as

$$R^{(1-m)} = \frac{\alpha\gamma}{1+\alpha}\frac{m(1-\Omega_r+x+3y) - (1-\Omega_r+3x+3y)}{2x}. \tag{51}$$

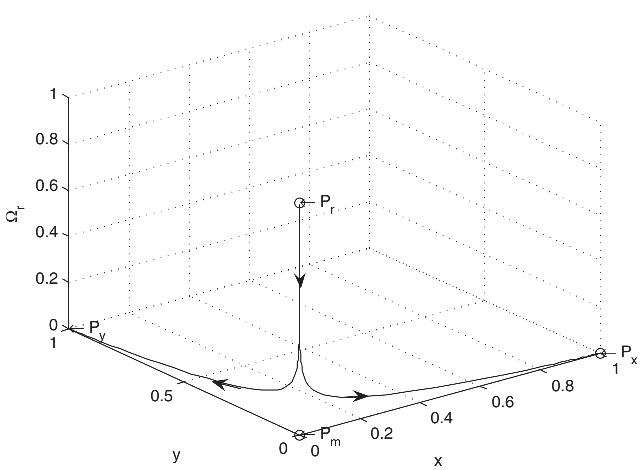

FIG. 2. Phase portraits of the FLRW model corresponding to $f(R) = R - \beta/R^n$, for $n = 0.002, b = -1$, and arbitrary β.

Fixed points and solutions.—For the fixed point P_r, both the numerator and denominator of Eq. (51) approach zero. Therefore we split the analysis into two different parts. The first part corresponds to the high-energy regime given by $\gamma R^{(m-1)} \gg 1 (P_{r2})$, and the second corresponds to the low-energy regime given by $\gamma R^{(m-1)} \ll 1 (P_{r1})$ [5]. The fixed points and their associated solutions are listed in Table III.

Stability of fixed points.—The eigenvalues and the stability corresponding to each fixed point for the FLRW model are summarized in Table IV.

3. Cosmological dynamics for model $f(R) = R + \gamma R^2$

Considering $m = 2$ in Eq. (48), we have

$$f(R) = R + \gamma R^2. \tag{52}$$

In this case Eq. (32) will become

$$\frac{\alpha\gamma R}{(1+\alpha)} = \frac{2x}{1-\Omega_r - x + 3y}. \tag{53}$$

In this case C can be written as

$$C = -6\left[1 + (b-1)\frac{1-\Omega_r - x + 3y}{1-\Omega_r + 3x + 3y}\right]. \tag{54}$$

For this particular form of $f(R)$, the expressions for the EoS parameter and deceleration parameter reduce to

$$w_{\rm eff} = \frac{1}{3}\Omega_r + x - y + (b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y} - \frac{4x}{1-\Omega_r+3x+3y}\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right] + \frac{8x\frac{1-\Omega_r-x+3y}{(1-\Omega_r+3x+3y)^2}\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right]}{2-3\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right]} \tag{55}$$

$$q = \frac{1}{2} + \frac{3}{2}\left[\frac{1}{3}\Omega_r + x - y + (b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y} - \frac{4x}{1-\Omega_r+3x+3y}\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right]\right] + \frac{12x\frac{1-\Omega_r-x+3y}{(1-\Omega_r+3x+3y)^2}\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right]}{2-3\left[1+(b-1)\frac{1-\Omega_r-x+3y}{1-\Omega_r+3x+3y}\right]}. \tag{56}$$

In this particular model, for the fixed point P_r, the EoS and deceleration parameters are undefined. Hence, this form of $f(R)$ does not possess a standard radiation-dominated era. However, the fixed points P_m, P_x, and P_y exist in this model. In this case point P_m exists only in the region $\gamma R \ll 1$, whereas P_x exists only in the region

TABLE III. Fixed points and their associated solutions for the FLRW model corresponding to $f(R) = R + \gamma R^m$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Physical behavior
P_{r1}	0	$\frac{1}{3}$	1	$a_0\|t\|^{\frac{1}{2}}$	Decelerated expansion
P_{r2}	0	$-\frac{2}{3}+\frac{1}{m}$	$-\frac{1}{2}+\frac{3}{2m}$	$a_0\|t\|^{\frac{2m}{m+3}}$	Decelerated expansion for $0 < m < 3$ Accelerated expansion for $m > 3, m \leq -3$ Decelerated contraction for $-3 < m < 0$ Milne evolution for $m = 3$
P_m	1	0	$\frac{1}{2}$	$a_0\|t\|^{\frac{2}{3}}$	Decelerated expansion
P_x	0	$-1+\frac{m(m-2)(1-b)}{4(m-1)^2}$	$-1+\frac{3m(m-2)(1-b)}{8(m-1)^2}$	$a_0\|t\|^{\frac{8(m-1)^2}{3m(m-2)(1-b)}}$	Accelerated expansion for $m > 2, b > 1, b < 1-\frac{8(m-1)^2}{3m(m-2)}$ Decelerated expansion for $m > 2, b > 1, b > 1-\frac{8(m-1)^2}{3m(m-2)}$
P_y	0	-1	-1	$a_0\|t\|^{\frac{1}{3}\theta_0 t}$	Accelerated expansion

TABLE IV. Eigenvalues and stability of the fixed points for the FLRW model corresponding to $f(R) = R + \gamma R^m$.

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3]$	Stability
P_{r1}	$[1, 4, (4-3mb)]$	Unstable for $m < \frac{4}{3b}$ Saddle otherwise
P_{r2}	$[1, 1, 4]$	Unstable
P_m	$[-1, 3, 3(1-mb)]$	Saddle
P_x	$\left[\frac{3m(m-2)(1-b)}{2(m-1)^2}, -3+\frac{3m(m-2)(1-b)}{2(m-1)^2}, -4+\frac{3m(m-2)(1-b)}{2(m-1)^2}\right]$	Stable for $m > 2, b > 1$ Saddle otherwise
P_y	$[-4, -3, -3mb]$	Stable for $m > 0, b > 0$; $m < 0, b < 0$ Saddle otherwise

$\gamma R \gg 1$ [5]. The fixed points and their associated solutions are listed in Table V.

An interesting feature of this form of $f(R)$ is that the fixed point (P_x) corresponding to the unstable node represents a dust Universe. Thus it implies that our Universe begins with the EoS of matter (P_x); enters the matter-dominated era (P_m), which corresponds to the saddle node; and finally ends in a stable de Sitter phase (P_y). This transition is plotted in Fig. 3, which shows the sequence of the form $P_x \to P_m \to P_y$.

4. Cosmological dynamics for model $f(R) = R + \gamma R^m - \beta/R^n$

The $f(R)$ of the form $f(R) = R + \gamma R^m$ describing the early Universe shows that it is possible to obtain a nonstandard high-energy radiation-dominated phase (P_{r2}), followed by an inflationary phase (P_x). The $f(R)$ of the form $f(R) = R - \beta/R^n$ describing the late-time Universe shows that a sequence originating from the radiation-dominated (P_{r1}) epoch approaches the matter-dominated era (P_m) and finally attains the accelerated expanding scenario (P_x, P_y). Combining the results of these two forms of $f(R)$, we can study the dynamics of $f(R) = R + \gamma R^m - \beta/R^n$ ($\gamma > 0, \beta > 0$). We plot the evolution of dynamical variables Ω_r, x, and y, along with the EoS parameter $w_{\rm eff}$ for $m = 1.9$, $b = -0.5$, and $n = 1$, in Fig. 4. The figure shows that the Universe begins with a nonstandard radiation-dominated phase, followed by inflation, and then followed by a standard radiation-dominated phase and a matter-dominated phase, until it finally attains the accelerated expanding Universe phase.

TABLE V. Fixed points and their associated solutions for the FLRW model corresponding to $f(R) = R + \gamma R^2$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3]$	Stability
P_r	0	...	...	...	...	...
P_m	1	0	$\frac{1}{2}$	$a_0\|t\|^{\frac{2}{3}}$	$[-1, 3, 3(1-2b)]$	Saddle
P_x	0	0	$\frac{1}{2}$	$a_0\|t\|^{\frac{2}{3}}$	$[2, 3, 6]$	Unstable
P_y	0	-1	-1	$a_0\|t\|^{\frac{1}{3}\theta_0 t}$	$[-6b, -4, -3]$	Stable for $b > 0$ Saddle otherwise

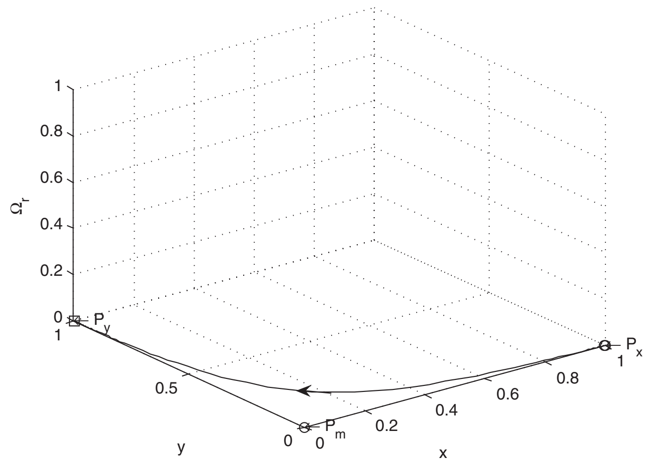

FIG. 3. Phase portraits of the FLRW model corresponding to $f(R) = R + \gamma R^2$, for $b = 1$ and arbitrary β.

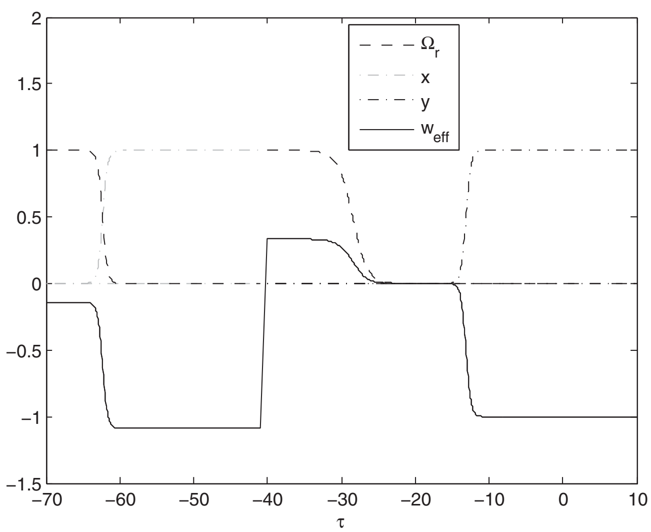

FIG. 4. Evolutions of the variables Ω_r, x, and y, as well as the EoS parameter w_{eff}, for the FLRW model corresponding to $f(R) = R + \gamma R^m - \beta/R^n$ with $m = 1.9; n = 1; b = -0.5;$ and β, γ arbitrary.

IV. BORN-INFELD $f(R)$ THEORY IN BIANCHI I COSMOLOGY

The line element of the spatially homogeneous and anisotropic Bianchi I metric is given by

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2], \quad (57)$$

where $A(t)$ and $B(t)$ are expansion scale factors.

As discussed in the previous model, it can be proved that, for this particular model also, one obtains $r(t) = cu(t)$.

Equation (17) in this case can be written as

$$\frac{2}{\epsilon}(\lambda - 1) + \alpha(f'R - f) - \frac{1}{2}(1 + \alpha f')\left(\left(\theta + \frac{\dot{u}}{2u}\right)^2 - \sigma^2\right) = \kappa^2(\rho_m + 2\rho_r), \quad (58)$$

where θ stands for the volume expansion scalar and σ for the shear scalar, which can be defined as

$$\theta = 3H = 3\frac{\dot{a}}{a} = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \quad (59)$$

$$\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2. \quad (60)$$

The conservation equations corresponding to the energy densities of matter and radiation are given by

$$\dot{\rho}_m + \theta\rho_m = 0 \quad (61)$$

$$\dot{\rho}_r + \frac{4}{3}\theta\rho_r = 0. \quad (62)$$

To study the evolution of the anisotropy parameter in the case of Bianchi I spacetime, we need to investigate the trace-free Gauss-Codazzi equation given by

$$\dot{\sigma} = -\left(\theta + \frac{\dot{u}}{u}\right)\sigma. \quad (63)$$

Using Eqs. (21) and (61), it can be shown that

$$\dot{R} = -\theta\frac{\kappa^2\rho_m}{1 - \alpha(f''R - f')} = -\theta\frac{4(1-\lambda) + \epsilon R(1 - \alpha\epsilon(f'R - 2f))}{\epsilon - \alpha\epsilon(f''R - f')}. \quad (64)$$

Using Eqs. (58) and (64), one can find that

$$\theta^2 = \frac{1}{2(1 + \alpha f')\xi^2}\left[6\sigma^2(1 + \alpha f') + 6\kappa^2(\rho_m + \rho_r) + 3\alpha(f'R - f) + \frac{6}{\epsilon}(\lambda - 1)\right], \quad (65)$$

where

$$\xi = 1 + \frac{3}{2}\frac{\dot{u}}{u\theta} = 1 - \frac{3}{2R}\frac{4(1-\lambda) + \epsilon R(1 - \alpha\epsilon(f'R - 2f))}{\epsilon - \alpha\epsilon(f''R - f')}. \quad (66)$$

The variables that have been considered in the case of a Bianchi I model are given below:

$$\Sigma=\frac{\sqrt{3}\sigma}{\xi\theta},\quad \Omega_r=\frac{3\kappa^2\rho_r}{(1+\alpha f')\xi^2\theta^2},\quad \Omega_m=\frac{3\kappa\rho_m}{(1+\alpha f')\xi^2\theta^2},$$
$$x=\frac{3\alpha(f'R-f)}{2(1+\alpha f')\xi^2\theta^2},\quad y=\frac{3(\lambda-1)}{\epsilon(1+\alpha f')\xi^2\theta^2}. \tag{67}$$

Therefore, the constraint equation reduces to

$$1=\Sigma^2+\Omega_r+\Omega_m+x+y. \tag{68}$$

The evolution equations corresponding to the variables (67) can be written as follows:

$$\frac{d\Sigma}{d\tau}=\frac{\Sigma}{2}[-3+3\Sigma^2+\Omega_r-3x-3y-9C(R)x-9D(R)(1-\Sigma^2) -18E(R)(1-\Sigma^2)] \tag{69}$$

$$\frac{d\Omega_r}{d\tau}=\Omega_r[-1+3\Sigma^2+\Omega_r-3x-3y-9C(R)x-9D(R)\Sigma^2 +18E(R)\Sigma^2] \tag{70}$$

$$\frac{dx}{d\tau}=x[3+3\Sigma^2+\Omega_r-3x-3y+9C(R)(1-x)-9D(R)\Sigma^2 +18E(R)\Sigma^2] \tag{71}$$

$$\frac{dy}{d\tau}=y[3+3\Sigma^2+\Omega_r-3x-3y-9C(R)x-9D(R)\Sigma^2 +18E(R)\Sigma^2], \tag{72}$$

where τ stands for $\ln a$, and $C(R)$, $D(R)$, and $E(R)$ are defined as

$$C(R)=\frac{1}{3\theta}\frac{f''R\dot{R}}{(f'R-f)} =-\frac{1}{3}\frac{f''R(4(1-\lambda)+\epsilon R-\alpha\epsilon(f'R-2f))}{(f'R-f)(\epsilon-\alpha\epsilon(f''R-f'))} \tag{73}$$

$$D(R)=\frac{\alpha}{3\theta}\frac{f''\dot{R}}{(1+\alpha f')} =-\frac{\alpha}{3}\frac{f''(4(1-\lambda)+\epsilon R-\alpha\epsilon(f'R-2f))}{(1+\alpha f')(\epsilon-\alpha\epsilon(f''R-f'))} \tag{74}$$

$$E(R)=\frac{1}{3\theta}\frac{\dot{u}}{u}=-\frac{1}{3R}\frac{(4(1-\lambda)+\epsilon R-\alpha\epsilon(f'R-2f))}{(\epsilon-\alpha\epsilon(f''R-f'))}. \tag{75}$$

In the dynamical system corresponding to the anisotropic Bianchi I model (69)–(72), it is observed that the time dependence (or τ dependence) is contained in the parameters $C(R)$, $D(R)$, and $E(R)$, and they are found to be constant in the investigation of the following sections. Therefore the dynamical system corresponding to an anisotropic Bianchi I model does not depend on time (or τ) explicitly and, hence, the system is autonomous.

Using Eqs. (64) and (67) in the constraint equation (68), one can have

$$\frac{4(1-\lambda)+\epsilon R-\alpha\epsilon(f'R-2f)}{\alpha\epsilon(f'R-f)}=\frac{1-\Sigma^2-\Omega_r-x-y}{2x}. \tag{76}$$

Equations (73)–(76) clearly show that $C(R)$, $D(R)$, and $E(R)$ can be expressed in terms of the dimensionless dynamical variables (67).

The fixed points in the case of a Bianchi I model can be obtained by equating the evolution equations (69)–(72) to zero. The fixed points obtained for this particular Bianchi model are as follows:

$$\begin{aligned} P_s^{\pm}&:\ (\Sigma,\Omega_r,x,y)=(\pm1,0,0,0)\\ P_r&:\ (\Sigma,\Omega_r,x,y)=(0,1,0,0)\\ P_m&:\ (\Sigma,\Omega_r,x,y)=(0,0,0,0)\\ P_x&:\ (\Sigma,\Omega_r,x,y)=(0,0,1,0)\\ P_y&:\ (\Sigma,\Omega_r,x,y)=(0,0,0,1). \end{aligned}$$

The eigenvalues associated with each fixed point are as follows:

$$\begin{aligned} P_s^{\pm}&:[\lambda_1,\lambda_2,\lambda_3,\lambda_4]=[2-9D(R)+18E(R),\\ &\quad -9D(R)+18E(R),\\ &\quad -6-9D(R)+18E(R),\\ &\quad 6+9C(R)-9D(R)+18E(R)]\\ P_r&:[\lambda_1,\lambda_2,\lambda_3,\lambda_4]=\left[1,4,4+9C(R),-1+\frac{9}{2}D(R)-9E(R)\right]\\ P_m&:[\lambda_1,\lambda_2,\lambda_3,\lambda_4]=\left[-1,3,3+9C(R),\right.\\ &\quad \left.-\frac{3}{2}+\frac{9}{2}D(R)-9E(R)\right]\\ P_x&:[\lambda_1,\lambda_2,\lambda_3,\lambda_4]=\left[-9C(R),-3-9C(R),-4-9C(R),\right.\\ &\quad \left.-3-\frac{9}{2}C(R)+\frac{9}{2}D(R)-9E(R)\right]\\ P_y&:[\lambda_1,\lambda_2,\lambda_3,\lambda_4]=\left[-3,-4,9C(R),-3+\frac{9}{2}D(R)-9E(R)\right]. \end{aligned}$$

The effective equation of state (EoS) parameter $w_{\rm eff}$ and the deceleration parameter q, in terms of the dynamical variables (67) for a Bianchi I model, can be defined as

$$w_{\mathrm{eff}}=\frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}}=\frac{1}{(1-\Sigma^2\xi^2)}\left[\Sigma^2+\frac{1}{3}\Omega_r-y-(1+3C(R))x+3D(R)(1-\Sigma^2)+6E(R)\Sigma^2-\Sigma^2\xi^2+2\frac{\dot{\xi}}{\xi\theta}\right] \tag{77}$$

$$q=-1+\frac{3}{2}\left[1+\Sigma^2+\frac{1}{3}\Omega_r-y-(1+3C(R))x+3D(R(1-\Sigma^2)+6E(R)\Sigma^2+2\frac{\dot{\xi}}{\xi\theta}\right], \tag{78}$$

where the relation between the deceleration parameter q and the volume expansion scalar θ is

$$\frac{\dot{\theta}}{\theta^2}=-\frac{1}{3}(1+q). \tag{79}$$

As we are interested in the evolution of the anisotropy parameter, it is very beneficial for us to express Eq. (63) in terms of the dynamical variables (67):

$$\dot{\sigma}=-\frac{1}{\sqrt{3}}[(1+3E(R))\xi\Sigma]\theta^2. \tag{80}$$

Integrating the above equation, we can find the evolution of shear for the anisotropic fixed points.

A. Cosmological dynamics for model $f(R)=R+\gamma R^m-\beta/R^n$

In this case Eq. (76) will become

$$\frac{(1+\alpha)-(m-2)\alpha\gamma R^{(m-1)}-(n+2)\alpha\beta R^{-(n+1)}}{(m-1)\alpha\gamma R^{(m-1)}+(n+1)\alpha\beta R^{-(n+1)}}=\frac{1-\Sigma^2-\Omega_r-x+3y}{2x}. \tag{81}$$

As discussed earlier in this case also, we start the analysis by investigating the early- and late-time dynamics separately.

1. Cosmological dynamics for model $f(R)=R-\beta/R^n$

For the particular case of $f(R)$, $C(R)$, $D(R)$, $E(R)$, and ξ can be expressed as

$$C(R)=\frac{n}{3((1+\alpha)+n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha)-(n+2)\alpha\beta R^{-(1+n)}+\frac{(b-1)(1+\alpha)^2}{(1+\alpha)+n\alpha\beta R^{-(1+n)}}\right] \tag{82}$$

$$D(R)=\frac{1}{3}\left[\frac{n(n+1)\alpha\beta R^{-(1+n)}}{(1+\alpha)+n\alpha\beta R^{-(1+n)}}\right]\times\frac{1}{((1+\alpha)+n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha)-(n+2)\alpha\beta R^{-(1+n)}+\frac{(b-1)(1+\alpha)^2}{(1+\alpha)+n\alpha\beta R^{-(1+n)}}\right] \tag{83}$$

$$E(R)=-\frac{1}{3((1+\alpha)+n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha)-(n+2)\alpha\beta R^{-(1+n)}+\frac{(b-1)(1+\alpha)^2}{(1+\alpha)+n\alpha\beta R^{-(1+n)}}\right] \tag{84}$$

$$\xi=1-\frac{3}{2((1+\alpha)+n(n+2)\alpha\beta R^{-(1+n)})}\left[(1+\alpha)-(n+2)\alpha\beta R^{-(1+n)}+\frac{(b-1)(1+\alpha)^2}{(1+\alpha)+n\alpha\beta R^{-(1+n)}}\right], \tag{85}$$

where $R^{(n+1)}$ can be expressed in terms of the dimensionless dynamical variables (67) as

$$R^{(n+1)}=\frac{\alpha\beta}{2x(1+\alpha)}[1-\Sigma^2-\Omega_r+3x+3y+n(1-\Sigma^2-\Omega_r+x+3y)]. \tag{86}$$

Fixed points and solutions.—For the fixed points $P_s^{\pm}$ and P_r, both the numerator and denominator of Eq. (86) approach zero. Therefore we split the analysis into two different parts. The first part corresponds to $\beta/R^{(n+1)}\ll 1$, and the second corresponds to $\beta/R^{(n+1)}\gg 1$ [5]. The fixed points and their associated solutions for a Bianchi I model are listed in Table VI.

The evolution of the dynamical variables Σ, Ω_r, x, and y, along with the EoS parameter w_{eff} for $\beta/R^{(n+1)}\ll 1$ ($n=0.002, b=-1$, β arbitrary), are plotted in Fig. 5. The figure clearly depicts that the anisotropic Universe ($P_s^{\pm}$) first approached the radiation-dominated (P_{r1}) and matter-dominated (P_m) era, respectively, and finally went to the accelerated expanding Universe (P_x, P_y).

Stability of fixed points.—The eigenvalues and the stability corresponding to each fixed point for a Bianchi I model are summarized in Table VII.

Phase portrait analyses in the case of a Bianchi I model for $f(R)=R-\beta/R^n$ ($n=0.002, b=-1, \beta$ arbitrary) are plotted in Fig. 6. In Fig. 6(a), we obtain heteroclinic

TABLE VI. Fixed points and their associated solutions for a Bianchi I model corresponding to $f(R) = R - \beta/R^n$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Shear (σ)	Physical behavior
$P_{s1}^{\pm}$	0	$\frac{1}{3} - \frac{2b}{4-3b}$	$2 - 3b$	$a_0\|t\|^{\frac{1}{3(1-b)}}$	$\sigma_0 a_0^{-3(1-b)}\|t\|^{-1}$	Decelerated expansion for $b < \frac{2}{3}$ Accelerated expansion for $\frac{2}{3} < b < 1$ Decelerated contraction for $b > 1$ Milne evolution for $b = \frac{2}{3}$
$P_{s2}^{\pm}$	0	$\frac{1}{3} + \frac{2}{4n+3}$	$2 + \frac{3}{n}$	$a_0\|t\|^{\frac{n}{3(n+1)}}$	$\sigma_0 a_0^{-3(1+\frac{1}{n})}\|t\|^{-1}$	Decelerated expansion for $n > 0, n < -\frac{3}{2}$ Accelerated expansion for $-\frac{3}{2} < n < -1$ Decelerated contraction for $-1 < n < 0$ Milne evolution for $n = -\frac{3}{2}$
P_{r1}	0	$\frac{1}{3}$	1	$a_0\|t\|^{\frac{1}{2}}$	0	Decelerated expansion
P_{r2}	0	$-\frac{2}{3} - \frac{1}{n}$	$-\frac{1}{2} - \frac{3}{2n}$	$a_0\|t\|^{\frac{2n}{n-3}}$	0	Decelerated expansion for $-3 < n < 0$ Accelerated expansion for $n \geq 3, n < -3$ Decelerated contraction for $0 < n < 3$ Milne evolution for $n = -3$
P_m	1	0	$\frac{1}{2}$	$a_0\|t\|^{\frac{2}{3}}$	0	Decelerated expansion
P_x	0	$-1 + \frac{n(n+2)(1-b)}{4(n+1)^2}$	$-1 + \frac{3n(n+2)(1-b)}{8(n+1)^2}$	$a_0\|t\|^{\frac{8(n+1)^2}{3n(n+2)(1-b)}}$	0	Accelerated expansion for $n > 0, b < 1,\ b < 1 - \frac{8(n+1)^2}{3n(n+2)}$ Decelerated expansion for $n > 0, b < 1,\ b > 1 - \frac{8(n+1)^2}{3n(n+2)}$
P_y	0	-1	-1	$a_0\|t\|^{\frac{1}{3}\theta_0 t}$	0	Accelerated expansion

sequences of the form $P_s^{\pm} \to P_r \to P_m \to P_x$ for $y = 0$. These sequences depict the evolution of an anisotropic Universe ($P_s^{\pm}$) to the radiation-dominated (P_r) and matter-dominated era (P_m), respectively, and then to the accelerated expanding (P_x) epoch. In Fig. 6(b) we have two heteroclinic sequences of the type $P_r \to P_m \to P_x$ and $P_r \to P_m \to P_y$ for $\Sigma = 0$. These sequences mimic the evolution of the radiation-dominated (P_r) to the matter-dominated era (P_m) and then to the accelerated expanding (P_x, P_y) epoch.

2. Cosmological dynamics for model $f(R) = R + \gamma R^m$

For the particular case of $f(R)$, $C(R)$, $D(R)$, $E(R)$, and ξ can be expressed as

$$C(R) = -\frac{m}{3((1+\alpha) - m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha) + m\alpha\gamma R^{(m-1)}}\right] \tag{87}$$

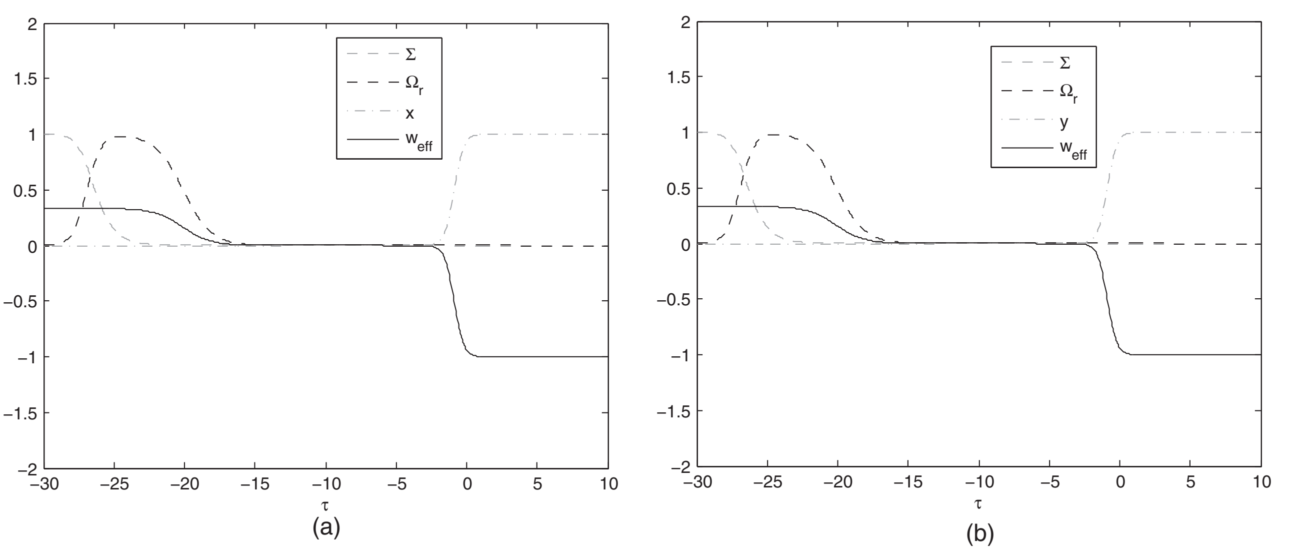

FIG. 5. Evolutions of the dynamical variables Σ, Ω_r, x, and y, along with the EoS parameter $w_{\rm eff}$, for a Bianchi I model corresponding to $f(R) = R - \beta/R^n$ with $n = 0.002, b = -1$, and arbitrary β.

TABLE VII. Eigenvalues and stability of the fixed points for a Bianchi I model corresponding to $f(R) = R - \beta/R^n$

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$	Stability
$P_{s1}^{\pm}$	$[2(1-3b), 3(1-2b), 6(1-b), 6+3(n-2)b]$	Stable for $b > 1, b > \frac{2}{2-n}$ Unstable for $b < \frac{1}{3}, b < \frac{2}{2-n}$ Saddle otherwise
$P_{s2}^{\pm}$	$[(5+\frac{9}{n}), (6+\frac{9}{n}), (6+\frac{9}{n}), (9+\frac{9}{n})]$	Stable for $-1 < n < 0$ Unstable for $n > 0, n < -\frac{9}{5}$ Saddle otherwise
P_{r1}	$[1, 4, (4+3nb), (-1+3b)]$	Unstable for $n > -\frac{4}{3b}, b > \frac{1}{3}$ Saddle otherwise
P_{r2}	$[1, 1, 4, -\frac{1}{2}(5+\frac{9}{n})]$	Unstable for $-\frac{9}{5} < n < 0$ Saddle otherwise
P_m	$[-1, 3, 3(1+nb), -\frac{3}{2}(1-6b)]$	Saddle
P_x	$\left[\frac{3n(n+2)(1-b)}{2(n+1)^2}, -3+\frac{3n(n+2)(1-b)}{2(n+1)^2}, -4+\frac{3n(n+2)(1-b)}{2(n+1)^2}, -3+\frac{3(n+2)(n-4)(1-b)}{8(n+1)^2}\right]$	Stable for $n > 0, b > 1$ Saddle otherwise
P_y	$[-4, -3, -3(1-b), 3nb]$	Stable for $n > 0, b < 0$

$$D(R) = \frac{1}{3}\left[\frac{m(m-1)\alpha\gamma R^{(m-1)}}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right] \times \frac{1}{((1+\alpha)-m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right] \tag{88}$$

$$E(R) = -\frac{1}{3((1+\alpha)-m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right] \tag{89}$$

$$\xi = 1 - \frac{3}{2((1+\alpha)-m(m-2)\alpha\gamma R^{(m-1)})}\left[(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} + \frac{(b-1)(1+\alpha)^2}{(1+\alpha)+m\alpha\gamma R^{(m-1)}}\right], \tag{90}$$

where $R^{(1-m)}$ can be expressed in terms of the dimensionless dynamical variables (67) as

$$R^{(1-m)} = \frac{\alpha\gamma}{2x(1+\alpha)}[m(1-\Sigma^2-\Omega_r+x+3y) - (1-\Sigma^2-\Omega_r+3x+3y)]. \tag{91}$$

Fixed points and solutions.—For the fixed points $P_s^{\pm}$ and P_r, both the numerator and denominator of Eq. (91) approach zero. Therefore, we split the analysis into two

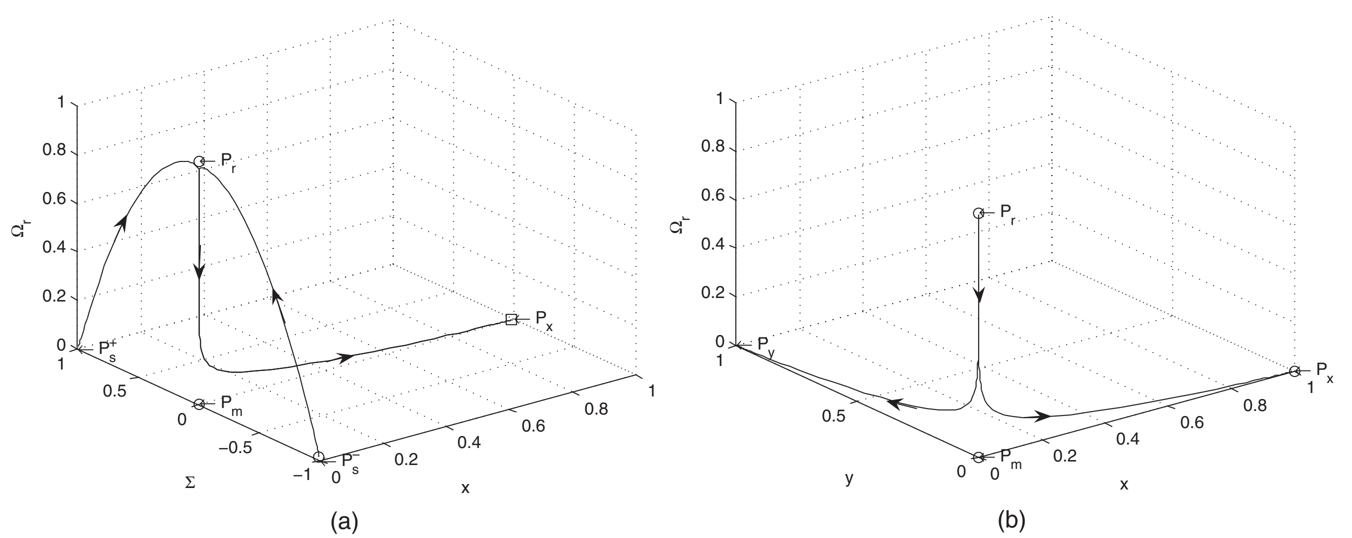

FIG. 6. Phase portraits of a Bianchi I model corresponding to $f(R) = R - \beta/R^n$ for $n = 0.002, b = -1$, and arbitrary β.

TABLE VIII. Fixed points and their associated solutions for a Bianchi I model corresponding to $f(R) = R + \gamma R^m$.

Points	Ω_m	EoS parameter (w_{eff})	Deceleration parameter (q)	Average scale factor (a)	Shear (σ)	Physical behavior
$P^{\pm}_{s1}$	0	$\frac{1}{3} - \frac{2b}{4-3b}$	$2-3b$	$a_0\|t\|^{\frac{1}{3(1-b)}}$	$\sigma_0 a_0^{-3(1-b)}\|t\|^{-1}$	Decelerated expansion for $b < \frac{2}{3}$ Accelerated expansion for $\frac{2}{3} < b < 1$ Decelerated contraction for $b > 1$ Milne evolution for $b = \frac{2}{3}$
$P^{\pm}_{s2}$	0	$\frac{1}{3} + \frac{2}{3-4m}$	$2 - \frac{3}{m}$	$a_0\|t\|^{\frac{m}{3(m-1)}}$	$\sigma_0 a_0^{-3(1-\frac{1}{m})}\|t\|^{-1}$	Decelerated expansion for $m < 0, m > \frac{3}{2}$ Accelerated expansion for $1 < m < \frac{3}{2}$ Decelerated contraction for $0 < m < 1$ Milne evolution for $m = \frac{3}{2}$
P_{r1}	0	$\frac{1}{3}$	1	$a_0\|t\|^{\frac{1}{2}}$	0	Decelerated expansion
P_{r2}	0	$-\frac{2}{3} + \frac{1}{m}$	$-\frac{1}{2} + \frac{3}{2m}$	$a_0\|t\|^{\frac{2m}{m+3}}$	0	Decelerated expansion for $0 < m < 3$ Accelerated expansion for $m > 3, m \leq -3$ Decelerated contraction for $-3 < m < 0$ Milne evolution for $m = 3$
P_m	1	0	$\frac{1}{2}$	$a_0\|t\|^{\frac{2}{3}}$	0	Decelerated expansion
P_x	0	$-1 + \frac{m(m-2)(1-b)}{4(m-1)^2}$	$-1 + \frac{3m(m-2)(1-b)}{8(m-1)^2}$	$a_0\|t\|^{\frac{8(m-1)^2}{3m(m-2)(1-b)}}$	0	Accelerated expansion for $m > 2, b > 1,\ b < 1 - \frac{8(m-1)^2}{3m(m-2)}$ Decelerated expansion for $m > 2, b > 1,\ b > 1 - \frac{8(m-1)^2}{3m(m-2)}$
P_y	0	-1	-1	$a_0\|t\|^{\frac{1}{3}\theta_0 t}$	0	Accelerated expansion

different parts. The first part corresponds to the high-energy regime given by $\gamma R^{(m-1)} \gg 1$ ($P^{\pm}_{s2}, P_{r2}$) and the second corresponds to the low-energy regime given by $\gamma R^{(m-1)} \ll 1$ ($P^{\pm}_{s1}, P_{r1}$) [5]. The fixed points and their associated solutions are listed in Table VIII.

Stability of fixed points.—The eigenvalues and the stability corresponding to each fixed point for an anisotropic Bianchi I model corresponding to $f(R) = R + \gamma R^m$ are summarized in Table IX.

3. Cosmological dynamics for model $f(R) = R + \gamma R^2$

In this case Eq. (76) will become

$$\frac{\alpha\gamma R}{(1+\alpha)} = \frac{2x}{1 - \Sigma^2 - \Omega_r - x + 3y}. \tag{92}$$

In this case C, D, and E can be written as

$$C = -\frac{2}{3}\left[1 + (b-1)\frac{1 - \Sigma^2 - \Omega_r - x + 3y}{1 - \Sigma^2 - \Omega_r + 3x + 3y}\right] \tag{93}$$

TABLE IX. Eigenvalues and stability of the fixed points for a Bianchi I model corresponding to $f(R) = R + \gamma R^m$.

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$	Stability
$P^{\pm}_{s1}$	$[2(1-3b), 3(1-2b), 6(1-b), 6-3(m+2)b]$	Stable for $b > 1, b > \frac{2}{2+m}$ Unstable for $b < \frac{1}{3}, b < \frac{2}{2+m}$ Saddle otherwise
$P^{\pm}_{s2}$	$[(5-\frac{9}{m}), (6-\frac{9}{m}), (6-\frac{9}{m}), (9-\frac{9}{m})]$	Stable for $0 < m < 1$ Unstable for $m < 0; m > \frac{9}{5}$ Saddle otherwise
P_{r1}	$[1, 4, (4-3mb), (-1+3b)]$	Unstable for $m < \frac{4}{3b}, b > \frac{1}{3}$ Saddle otherwise
P_{r2}	$[1, 1, 4, -\frac{1}{2}(5-\frac{9}{m})]$	Unstable for $0 < m < \frac{9}{5}$ Saddle otherwise
P_m	$[-1, 3, 3(1-mb), -\frac{3}{2}(1-6b)]$	Saddle
P_x	$\left[\frac{3m(m-2)(1-b)}{2(m-1)^2}, -3 + \frac{3m(m-2)(1-b)}{2(m-1)^2}, -4 + \frac{3m(m-2)(1-b)}{2(m-1)^2}, -3 + \frac{3m(m-2)(m+4)(1-b)}{2(m-1)^2}\right]$	Stable for $m > 2, b > 1$ Saddle otherwise
P_y	$[-4, -3, -3(1-b), -3mb]$	Stable for $m < 0, b < 0; m > 0, 0 < b < 1$

TABLE X. Fixed points and their associated solutions for a Bianchi I model corresponding to $f(R) = R + \gamma R^2$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3]$	Stability
$P_s^{\pm}$	0	⋯	⋯	⋯	⋯	⋯
P_r	0	⋯	⋯	⋯	⋯	⋯
P_m	1	0	$\frac{1}{2}$	$a_0\lvert t\rvert^{\frac{2}{3}}$	$[-1, 3, 3(1-2b), -\frac{3}{2}(1-6b)]$	Saddle
P_x	0	0	$\frac{1}{2}$	$a_0\lvert t\rvert^{\frac{2}{3}}$	$[2, 3, 3, 6]$	Unstable
P_y	0	-1	-1	$a_0\lvert t\rvert^{\frac{1}{3}\theta_0 t}$	$[-6b, -4, -3, -3(1+b)]$	Stable for $b > 0$ Saddle otherwise

$$D = -\frac{1}{3}\frac{4x}{1-\Sigma^2-\Omega_r+3x+3y}\left[1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right] \tag{94}$$

$$E = -\frac{1}{3}\left[1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right]. \tag{95}$$

For this particular form of $f(R)$, the expressions for the EoS parameter and deceleration parameter reduce to

$$\begin{aligned} w_{\rm eff} = {} & \frac{1}{1-\Sigma^2\left[1-\frac{3}{2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)\right]^2}\left[\frac{1}{3}\Omega_r+\Sigma^2+x-y+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right. \\ & -\frac{4x(1-\Sigma^2)}{1-\Sigma^2-\Omega_r+3x+3y}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)-2\Sigma^2\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right) \\ & \left.-\Sigma^2\left[1-\frac{3}{2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)\right]^2+\frac{8x\frac{1-\Sigma^2-\Omega_r-x+3y}{(1-\Sigma^2-\Omega_r+3x+3y)^2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)}{2-3\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)}\right] \end{aligned} \tag{96}$$

$$\begin{aligned} q = {} & \frac{1}{2}+\frac{3}{2}\left[\frac{1}{3}\Omega_r+\Sigma^2+x-y+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}-\frac{4x(1-\Sigma^2)}{1-\Sigma^2-\Omega_r+3x+3y}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)\right. \\ & \left.-2\Sigma^2\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)+\frac{8x\frac{1-\Sigma^2-\Omega_r-x+3y}{(1-\Sigma^2-\Omega_r+3x+3y)^2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)}{2-3\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)}\right]. \end{aligned} \tag{97}$$

In this case for the fixed points $P_s^{\pm}$ and P_r, the EoS and deceleration parameters are undefined. Hence this form of $f(R)$ does not possess a standard anisotropic and radiation-dominated era. However, the fixed points P_m, P_x, and P_y exist in this model. The fixed points and their associated solutions are listed in Table X.

As discussed in the previous case of FLRW geometry in this geometry also, the fixed point (P_x) corresponding to the unstable node represents a dust Universe. Thus it implies that our Universe begins with the EoS of matter (P_x), approaches the matter-dominated era (P_m), and finally reaches the stable de Sitter phase (P_y). This sequence is plotted in Fig. 3.

4. Cosmological dynamics for model $f(R) = R + \gamma R^m - \beta/R^n$

The $f(R)$ of the form $f(R) = R + \gamma R^m$ describing the early Universe shows that it is possible to obtain a nonstandard high-energy anisotropic phase ($P_{s2}^{\pm}$) and radiation-dominated phase (P_{r2}), followed by an inflationary phase (P_x). The $f(R)$ of the form $f(R) = R - \beta/R^n$ describing the late-time Universe shows that a sequence originating from the anisotropic phase ($P_{s1}^{\pm}$) approaches the radiation-dominated era (P_{r1}), then the matter-dominated era (P_m), and finally reaches the accelerated expanding scenario (P_x, P_y). Combining the results of these two forms of $f(R)$, we can study the dynamics of $f(R) = R + \gamma R^m - \beta/R^n$ ($\gamma > 0, \beta > 0$). We plot the evolution of dynamical variables Σ, Ω_r, x, and y along with the EoS parameter $w_{\rm eff}$ for $m = 1.9$ and $n = 1$ in Fig. 7. The figure shows that the Universe begins with a nonstandard anisotropic phase, then it goes to a nonstandard radiation-dominated era, followed by inflation, a standard anisotropic, standard radiation-dominated as well as matter-dominated phase, and finally attains the accelerated expanding epoch.

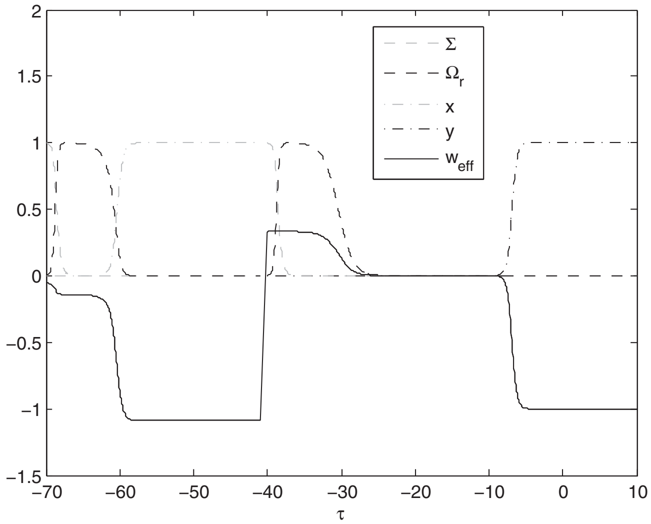

FIG. 7. Evolutions of the variables Σ, Ω_r, x, and y, as well as the EoS parameter $w_{\rm eff}$, for a Bianchi I model corresponding to $f(R) = R + \gamma R^m - \beta/R^n$ with $m = 1.9$; $n = 1$; $b = -0.5$; and β, γ arbitrary.

V. BORN-INFELD $f(R)$ THEORY IN BIANCHI V COSMOLOGY

The line element of the spatially homogeneous and anisotropic Bianchi V metric is given by

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{-2px}[B^2(t)dy^2 + C^2(t)dz^2], \tag{98}$$

where $A(t)$, $B(t)$, and $C(t)$ are expansion scale factors and p acts as a constant.

For this particular model putting $\mu = 1$ and $\nu = 0$ in Eq. (17), one can obtain

$$2\frac{\dot{A}}{A} = \frac{\dot{B}}{B} + \frac{\dot{C}}{C}. \tag{99}$$

Solving the above equation, one obtains

$$A = c_1 BC, \tag{100}$$

where c_1 is a constant.

As discussed in the previous geometries, in the case of Bianchi V geometry also, it can be proved that $r(t) = cu(t)$, and Eq. (17) can be written as

$$\frac{2}{\epsilon}(\lambda - 1) + \alpha(f'R - f) - \frac{1}{2}(1 + \alpha f')\left[\left(\theta + \frac{\dot{u}}{2u}\right)^2 - \sigma^2 - 3p^2({}^2K)\right] = \kappa^2(\rho_m + 2\rho_r), \tag{101}$$

where ${}^2K$ stands for the spatial curvature given by

$${}^2K = \frac{1}{A^2}. \tag{102}$$

The volume expansion scalar θ and shear scalar σ in case of a Bianchi V metric can be defined as

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = 3\frac{\dot{A}}{A} = \frac{3}{2}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{103}$$

$$\sigma^2 = \frac{1}{3}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2 - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{C}\dot{A}}{CA}\right] = \left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right)^2. \tag{104}$$

To study the evolution of the anisotropy parameter and spatial curvature corresponding to Bianchi V spacetime, we need to investigate the following equations:

$$\dot{\sigma} = -\left(\theta + \frac{\dot{u}}{u}\right)\sigma \tag{105}$$

$${}^2\dot{K} = -\frac{2}{3}\theta({}^2K). \tag{106}$$

Using Eq. (64) in (101), one can find that

$$\theta^2 = \frac{1}{2(1+\alpha f')\xi^2}\left[6\sigma^2(1+\alpha f') + 18p^2({}^2K)(1+\alpha f') + 6\kappa^2(\rho_m + \rho_r) + 3\alpha(f'R - f) + \frac{6}{\epsilon}(\lambda - 1)\right]. \tag{107}$$

The variables that have been considered in case of a Bianchi V model are given below:

$$\Sigma = \frac{\sqrt{3}\sigma}{\xi\theta}, \quad \Omega_r = \frac{3\kappa^2\rho_r}{(1+\alpha f')\xi^2\theta^2}, \quad \Omega_m = \frac{3\kappa\rho_m}{(1+\alpha f')\xi^2\theta^2},$$
$$K = -\frac{3p^2({}^2K)}{\xi^2\theta^2}, \quad x = \frac{3\alpha(f'R - f)}{2(1+\alpha f')\xi^2\theta^2},$$
$$y = \frac{3(\lambda - 1)}{\epsilon(1+\alpha f')\xi^2\theta^2}. \tag{108}$$

Therefore, the constraint equation reduces to

$$1 = \Sigma^2 + \Omega_r + \Omega_m - 3K + x + y. \tag{109}$$

The evolution equations corresponding to the variables (108) can be written as follows:

$$\frac{d\Sigma}{d\tau} = \frac{\Sigma}{2}[-3 + 3\Sigma^2 + \Omega_r + 3K - 3x - 3y - 9C(R)x - 9D(R)(1 - \Sigma^2 + 3K) - 18E(R)(1 - \Sigma^2)] \tag{110}$$

$$\frac{d\Omega_r}{d\tau} = \Omega_r[-1 + 3\Sigma^2 + \Omega_r + 3K - 3x - 3y - 9C(R)x - 9D(R)(\Sigma^2 - 3K) + 18E(R)\Sigma^2] \tag{111}$$

$$\frac{dK}{d\tau} = K[1 + 3\Sigma^2 + \Omega_r + 3K - 3x - 3y - 9C(R)x + 9D(R)(1 - \Sigma^2 + 3K) + 18E(R)\Sigma^2] \tag{112}$$

$$\frac{dx}{d\tau} = x[3 + 3\Sigma^2 + \Omega_r + 3K - 3x - 3y + 9C(R)(1 - x) - 9D(R)(\Sigma^2 - 3K) + 18E(R)\Sigma^2] \tag{113}$$

$$\frac{dy}{d\tau} = y[3 + 3\Sigma^2 + \Omega_r + 3K - 3x - 3y - 9C(R)x - 9D(R)(\Sigma^2 - 3K) + 18E(R)\Sigma^2]. \tag{114}$$

The expressions for $C(R)$, $D(R)$, and $E(R)$ are given in (73)–(75).

In the dynamical system corresponding to the anisotropic Bianchi V model (110)–(114), it is observed that the time dependence (or τ dependence) is contained in the parameters $C(R)$, $D(R)$, and $E(R)$, and they are found to be constant in the investigation of the following sections. Therefore the dynamical system corresponding to an anisotropic Bianchi V model does not depend on time (or τ) explicitly and, hence, the system is autonomous.

Using Eqs. (64) and (108) in the constraint equation (109), one can have

$$\frac{4(1-\lambda) + \epsilon R - \alpha\epsilon(f'R - 2f)}{\alpha\epsilon(f'R - f)} = \frac{1 - \Sigma^2 - \Omega_r + 3K - x - y}{2x}. \tag{115}$$

The fixed points in the case of a Bianchi V model can be obtained by equating the evolution equations (110)–(114) to zero. The fixed points obtained for this particular Bianchi model are as follows:

$$P_s^{\pm}\colon\ (\Sigma, \Omega_r, K, x, y) = (\pm 1, 0, 0, 0, 0)$$
$$P_r\colon\ (\Sigma, \Omega_r, K, x, y) = (0, 1, 0, 0, 0)$$
$$P_m\colon\ (\Sigma, \Omega_r, K, x, y) = (0, 0, 0, 0, 0)$$
$$P_k\colon\ (\Sigma, \Omega_r, K, x, y) = \left(0, -\frac{1}{3}, 0, 0, 0\right)$$
$$P_x\colon\ (\Sigma, \Omega_r, K, x, y)) = (0, 0, 0, 1, 0)$$
$$P_y\colon\ (\Sigma, \Omega_r, K, x, y) = (0, 0, 0, 0, 1).$$

The eigenvalues associated with each fixed point are as follows:

$$P_s^{\pm}\colon\ [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = [2 - 9D(R) + 18E(R), 3 - 9D(R) + 18E(R), -6 - 9D(R) + 18E(R), 4 + 18E(R)6 + 9C(R) - 9D(R) + 18E(R)]$$
$$P_r\colon\ [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = \left[1, 4, 2 + 9D(R), 4 + 9C(R), -1 + \frac{9}{2}D(R) - 9E(R)\right]$$
$$P_m\colon [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = \left[-1, 3, 3 + 9C(R), 1 + 9D(R), -\frac{3}{2} + \frac{9}{2}D(R) - 9E(R)\right]$$
$$P_k\colon\ [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = [-2 - 9D(R), -1 - 9D(R), 2 - 9D(R), -2 - 9E(R), 2 + 9C(R) - 9D(R)]$$
$$P_x\colon\ [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = \left[-9C(R), -3 - 9C(R), -4 - 9C(R), -2 - 9C(R) + 9D(R), -3 - \frac{9}{2}C(R) + \frac{9}{2}D(R) - 9E(R)\right]$$
$$P_y\colon\ [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = \left[-3, -4, 9C(R), -2 - 9D(R), -3 + \frac{9}{2}D(R) - 9E(R)\right].$$

The effective equation of state (EoS) parameter $w_{\rm eff}$ and the deceleration parameter q, in terms of the dynamical variables (108) for Bianchi I model, can be defined as

$$w_{\rm eff} = \frac{1}{(1 - (\Sigma^2 - K)\xi^2)}\left[\Sigma^2 + \frac{\Omega_r}{3} + K - y - (1 + 3C(R))x + 3D(R)(1 - \Sigma^2 + 3K) + 6E(R)\Sigma^2 - (\Sigma^2 + K)\xi^2 + 2\frac{\dot{\xi}}{\xi\theta}\right] \tag{116}$$

$$q = -1 + \frac{3}{2}\left[1 + \Sigma^2 + \frac{1}{3}\Omega_r + K - y - (1 + 3C(R))x + 3D(R(1 - \Sigma^2 + 3K) + 6E(R)\Sigma^2 + 2\frac{\dot{\xi}}{\xi\theta}\right]. \tag{117}$$

As we are interested in the evolution of the anisotropy parameter and spatial curvature, it is useful to express Eqs. (105) and (106) in terms of the dynamical variables (108):

TABLE XI. Fixed points and their associated solutions for a Bianchi V model corresponding to $f(R) = R - \beta/R^n$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Shear (σ)	Spatial curvature (^2K)	Physical behavior
$P_{s1}^{\pm}$	0	$\frac{1}{3} - \frac{2b}{4-3b}$	$2 - 3b$	$a_0\lvert t\rvert^{\frac{1}{3(1-b)}}$	$\sigma_0 a_0^{-3(1-b)}\lvert t\rvert^{-1}$	0	Decelerated expansion for $b < \frac{2}{3}$ Accelerated expansion for $\frac{2}{3} < b < 1$ Decelerated contraction for $b > 1$ Milne evolution for $b = \frac{2}{3}$
$P_{s2}^{\pm}$	0	$\frac{1}{3} + \frac{2}{4n+3}$	$2 + \frac{3}{n}$	$a_0\lvert t\rvert^{\frac{n}{3(n+1)}}$	$\sigma_0 a_0^{-3(1+\frac{1}{n})}\lvert t\rvert^{-1}$	0	Decelerated expansion for $n > 0, n < -\frac{3}{2}$ Accelerated expansion for $-\frac{3}{2} < n < -1$ Decelerated contraction for $-1 < n < 0$ Milne evolution for $n = -\frac{3}{2}$
P_{r1}	0	$\frac{1}{3}$	1	$a_0\lvert t\rvert^{\frac{1}{2}}$	0	0	Decelerated expansion
P_{r2}	0	$-\frac{2}{3} - \frac{1}{n}$	$-\frac{1}{2} - \frac{3}{2n}$	$a_0\lvert t\rvert^{\frac{2n}{n-3}}$	0	0	Decelerated expansion for $-3 < n < 0$ Accelerated expansion for $n \geq 3, n < -3$ Decelerated contraction for $0 < n < 3$ Milne evolution for $n = -3$
P_m	1	0	$\frac{1}{2}$	$a_0\lvert t\rvert^{\frac{2}{3}}$	0	0	Decelerated expansion
P_k	0	$-\frac{1}{3}$	0	$a_0\lvert t\rvert$	0	$^2K_0 a_0^{-2}\lvert t\rvert^{-2}$	Milne evolution
P_x	0	$-1 + \frac{n(n+2)(1-b)}{4(n+1)^2}$	$-1 + \frac{3n(n+2)(1-b)}{8(n+1)^2}$	$a_0\lvert t\rvert^{\frac{8(n+1)^2}{3n(n+2)(1-b)}}$	0	0	Accelerated expansion for $n > 0, b < 1,\ b < 1 - \frac{8(n+1)^2}{3n(n+2)}$ Decelerated expansion for $n > 0, b < 1,\ b > 1 - \frac{8(n+1)^2}{3n(n+2)}$
P_y	0	-1	-1	$a_0\lvert t\rvert^{\frac{1}{3}\theta_0 t}$	0	0	Accelerated expansion

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}[(1 + 3E(R))\xi\Sigma]\theta^2 \tag{118}$$

$$^2\dot{K} = -\frac{2}{3}(^2K)\theta. \tag{119}$$

Integrating the above equations, we can find the evolution of the shear as well as the spatial curvature for the equilibrium points.

A. Cosmological dynamics for model $f(R) = R + \gamma R^m - \beta/R^n$

In this case Eq. (76) will become

$$\frac{(1+\alpha) - (m-2)\alpha\gamma R^{(m-1)} - (n+2)\alpha\beta R^{-(n+1)}}{(m-1)\alpha\gamma R^{(m-1)} + (n+1)\alpha\beta R^{-(n+1)}} = \frac{1 - \Sigma^2 - \Omega_r + 3K - x + 3y}{2x}. \tag{120}$$

As discussed earlier, we start the analysis by investigating the early- and late-time dynamics separately.

1. Cosmological dynamics for model $f(R) = R - \beta/R^n$

For the particular case of $f(R)$, $R^{(n+1)}$ can be expressed in terms of the dimensionless dynamical variables (108) as

$$R^{(1+n)} = \frac{\alpha\beta}{2x(1+\alpha)}[1 - \Sigma^2 - \Omega_r + 3K + 3x + 3y + n(1 - \Sigma^2 - \Omega_r + 3K + x + 3y)]. \tag{121}$$

Fixed points and solutions.—For the fixed points $P_s^{\pm}$, P_r, and P_k, both the numerator and denominator of Eq. (121) approach zero. Therefore we split the analysis into two different parts. The first part corresponds to $\beta/R^{(n+1)} \ll 1$, and the second corresponds to $\beta/R^{(n+1)} \gg 1$ [5]. The fixed points and their associated solutions are listed in Table XI.

The evolution of the dynamical variables Σ, Ω_r, K, x, and y, along with the EoS parameter $w_{\rm eff}$ for $\beta/R^{(n+1)} \ll 1$ ($n = 0.002, b = -1, \beta$ arbitrary), are plotted in Fig. 8. The figure clearly depicts that the anisotropic Universe ($P_s^{\pm}$) first approaches the radiation-dominated (P_{r1}) and matter-dominated (P_m) era, respectively; then tends to a spatially nonflat isotropic Universe (P_{k1}); and finally approaches the phase of an accelerated expanding Universe (P_x, P_y).

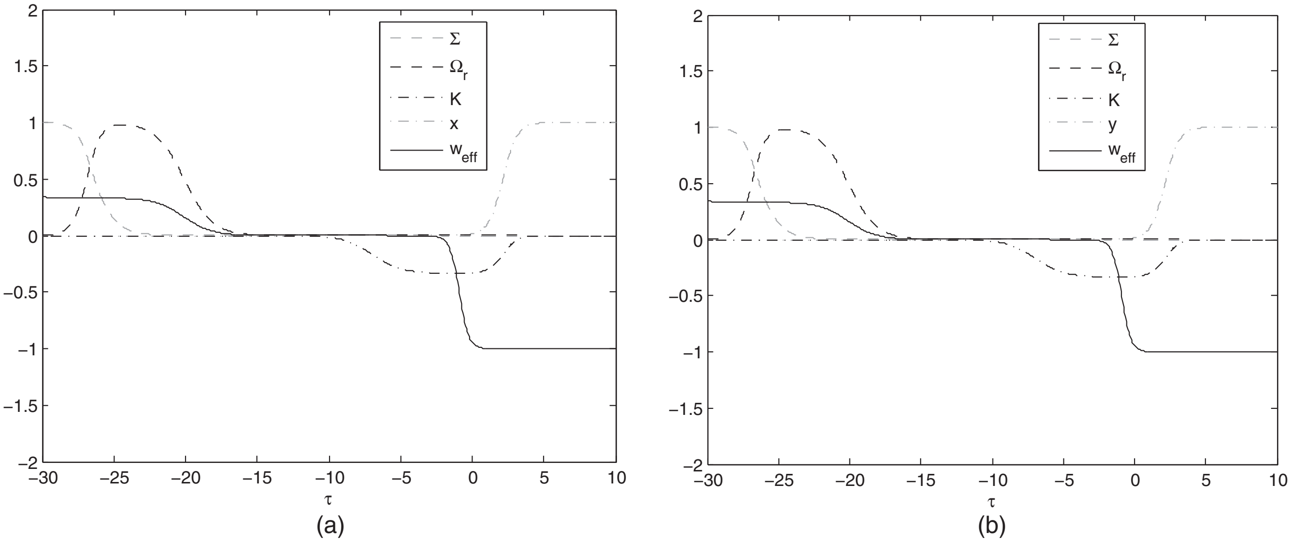

FIG. 8. Evolutions of the dynamical variables Σ, Ω_r, K, x, and y along with the EoS parameter $w_{\rm eff}$, for a Bianchi V model corresponding to $f(R) = R - \beta/R^n$ with $n = 0.002, b = -1$, and arbitrary β.

Stability of fixed points.—The eigenvalues and the stability corresponding to each fixed point for a Bianchi V model are summarized in Table XII.

Phase portrait analyses in the case of a Bianchi V model $f(R) = R - \beta/R^n$ ($n = 0.002, b = -1$, β arbitrary) are plotted in Fig. 9. In Fig. 9(a) we have heteroclinic sequences of the type $P_s^\pm \to P_m \to P_k \to P_x$ for $\Omega_r = y = 0$. These sequences mimic the evolution of an anisotropic Universe ($P_s^\pm$) to the matter-dominated era (P_m), followed by a spatially nonflat isotropic (P_k) epoch, and ultimately to the accelerated expanding (P_x) epoch.

In Fig. 9(b) we have heteroclinic sequences of the form $P_s^\pm \to P_r \to P_m \to P_x$ for $K = y = 0$. These sequences depict the evolution of an anisotropic Universe ($P_s^\pm$) to the radiation-dominated (P_r) and matter-dominated era (P_m), respectively, and then to the accelerated expanding (P_x) epoch.

In Fig. 9(c) we have a heteroclinic sequence of the form $P_r \to P_m \to P_k \to P_x$ for $\Sigma = y = 0$. This sequence shows the evolution of a radiation-dominated (P_r) and matter-dominated era (P_m), followed by a spatially nonflat isotropic (P_k) epoch, and then the stable de Sitter expanding (P_x) Universe.

TABLE XII. Eigenvalues and stability of the fixed points for a Bianchi V model corresponding to $f(R) = R - \beta/R^n$.

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$	Stability
$P_{s1}^\pm$	$[2(1-3b), 3(1-2b), 2(2-3b), 6(1-b), 6+3(n-2)b]$	Stable for $b > 1, b > \frac{2}{2-n}$ Unstable for $b < \frac{1}{3}, b < \frac{2}{2-n}$ Saddle otherwise
$P_{s2}^\pm$	$[(4+\frac{6}{n}), (5+\frac{9}{n}), (6+\frac{9}{n}), (6+\frac{9}{n}), (9+\frac{9}{n})]$	Stable for $-1 < n < 0$ Unstable for $n > 0, n < -\frac{9}{5}$ Saddle otherwise
P_{r1}	$[1, 2, 4, (4+3nb), (-1+3b)]$	Unstable for $n > -\frac{4}{3b}, b > \frac{1}{3}$ Saddle otherwise
P_{r2}	$[1, 1, 4, -(1+\frac{3}{n}), -\frac{1}{2}(5+\frac{9}{n})]$	Unstable for $-\frac{9}{5} < n < 0$ Saddle otherwise
P_m	$[-1, 1, 3, 3(1+nb), -\frac{3}{2}(1-6b)]$	Saddle
P_{k1}	$[-2, -1, 2, -(2-3b), (2+3nb)]$	Saddle
P_{k2}	$[-(2+\frac{3}{n}), (1+\frac{3}{n}), (2+\frac{3}{n}), (2+\frac{3}{n}), (5+\frac{3}{n})]$	Saddle
P_x	$\left[\frac{3n(n+2)(1-b)}{2(n+1)^2}, -3+\frac{3n(n+2)(1-b)}{2(n+1)^2}, -4+\frac{3n(n+2)(1-b)}{4(n+1)^2}, -2+\frac{3n(n+2)(1-b)}{2(n+1)^2}, -3+\frac{3(n+2)(n-4)(1-b)}{8(n+1)^2}\right]$	Stable for $n > 0, b > 1$ Saddle otherwise
P_y	$[-4, -3, -2, -3(1-b), 3nb]$	Stable for $n > 0, b < 0$

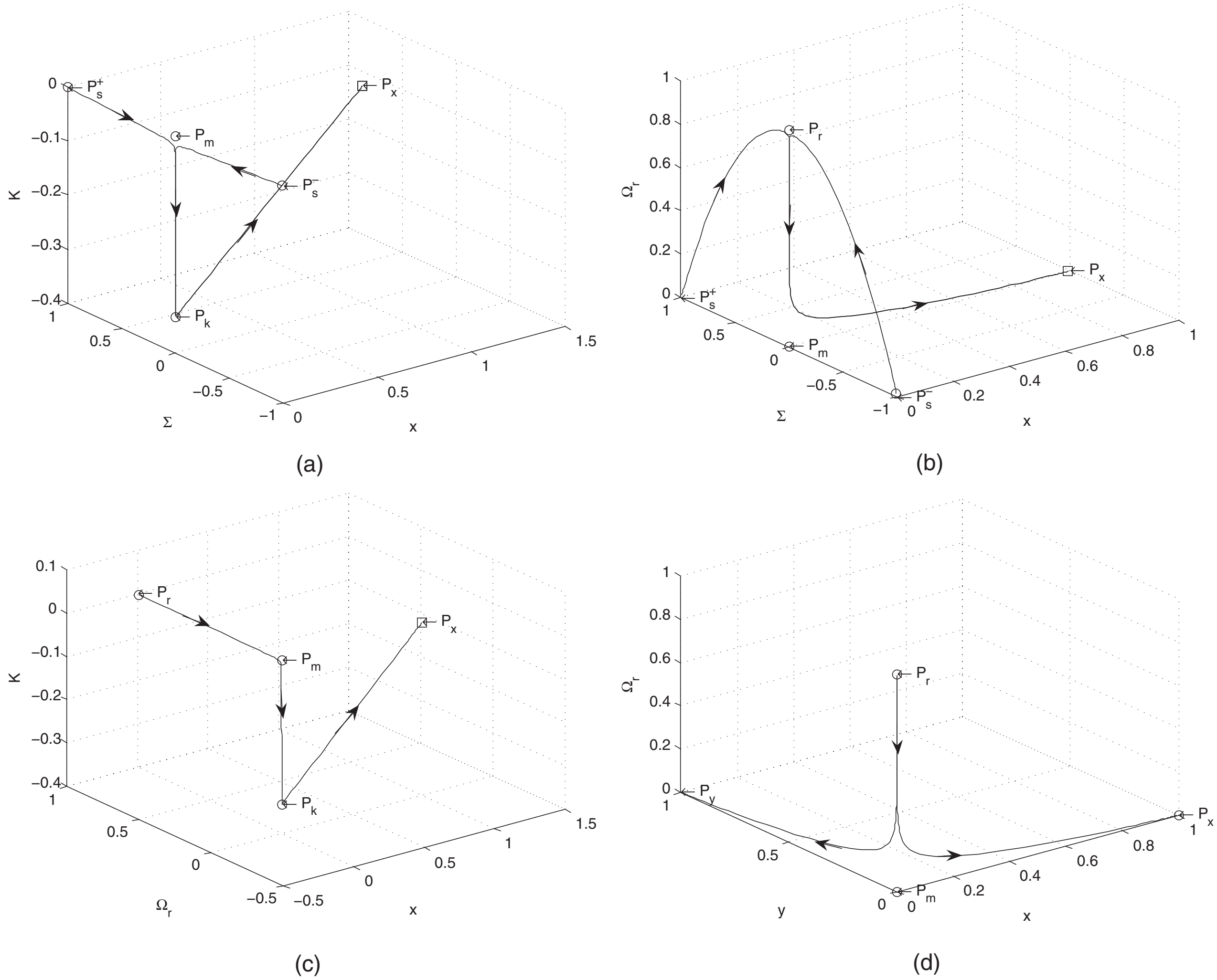

FIG. 9. Phase portraits of a Bianchi V model corresponding to $f(R) = R - \beta/R^n$, for $n = 0.002, b = -1$, and arbitrary β.

In Fig. 9(d) we have heteroclinic sequences of the type $P_r \to P_m \to P_x$ and $P_r \to P_m \to P_y$ for $\Sigma = K = 0$. These sequences mimic the evolution of the radiation-dominated (P_r) to matter-dominated era (P_m) and then to the accelerated expanding (P_x, P_y) epoch.

2. Cosmological dynamics for model $f(R) = R + \gamma R^m$

For the particular case of $f(R)$, $R^{(1-m)}$ can be expressed in term of the dimensionless dynamical variables (108) as

$$R^{(1-m)} = \frac{\alpha\gamma}{2x(1+\alpha)}[m(1 - \Sigma^2 - \Omega_r + 3K + x + 3y) - (1 - \Sigma^2 - \Omega_r + 3K + 3x + 3y)]. \quad (122)$$

Fixed points and solutions.—For the fixed points $P_s^{\pm}, P_r$, and P_k, both the numerator and denominator of Eq. (122) approach zero. Therefore, we split the analysis into two different parts. The first part corresponds to the high-energy regime given by $\gamma R^{(m-1)} \gg 1$ ($P_{s2}^{\pm}, P_{r2}, P_{k2}$), and the second corresponds to the low-energy regime given by $\gamma R^{(m-1)} \ll 1$ ($P_{s1}^{\pm}, P_{r1}, P_{k1}$) [5]. The fixed points and their associated solutions are listed in Table XIII.

Stability of fixed points.—The eigenvalues and the stability corresponding to each fixed point for a Bianchi V model corresponding to $f(R) = R + \gamma R^m$ are summarized in Table XIV.

3. Cosmological dynamics for model $f(R) = R + \gamma R^2$

In this case Eq. (115) will become

$$\frac{\alpha\gamma R}{(1+\alpha)} = \frac{2x}{1 - \Sigma^2 - \Omega_r + 3K - x + 3y}. \quad (123)$$

For this particular form of $f(R)$, the expressions for the EoS parameter and deceleration parameter reduce to

TABLE XIII. Fixed points and their associated solutions for a Bianchi V model corresponding to $f(R) = R + \gamma R^m$.

Points	Ω_m	EoS parameter ($w_{\rm eff}$)	Deceleration parameter (q)	Average scale factor (a)	Shear (σ)	Spatial curvature (^2K)	Physical behavior
$P^{\pm}_{s1}$	0	$\frac{1}{3} - \frac{2b}{4-3b}$	$2-3b$	$a_0 \lvert t\rvert^{\frac{1}{3(1-b)}}$	$\sigma_0 a_0^{-3(1-b)} \lvert t\rvert^{-1}$	0	Decelerated expansion for $b < \frac{2}{3}$ Accelerated expansion for $\frac{2}{3} < b < 1$ Decelerated contraction for $b > 1$ Milne evolution for $b = \frac{2}{3}$
$P^{\pm}_{s2}$	0	$\frac{1}{3} + \frac{2}{3-4m}$	$2 - \frac{3}{m}$	$a_0 \lvert t\rvert^{\frac{m}{3(m-1)}}$	$\sigma_0 a_0^{-3(1-\frac{1}{m})} \lvert t\rvert^{-1}$	0	Decelerated expansion for $m < 0, m > \frac{3}{2}$ Accelerated expansion for $1 < m < \frac{3}{2}$ Decelerated contraction for $0 < m < 1$ Milne evolution for $m = \frac{3}{2}$
P_{r1}	0	$\frac{1}{3}$	1	$a_0 \lvert t\rvert^{\frac{1}{2}}$	0	0	Decelerated expansion
P_{r2}	0	$-\frac{2}{3} + \frac{1}{m}$	$-\frac{1}{2} + \frac{3}{2m}$	$a_0 \lvert t\rvert^{\frac{2m}{m+3}}$	0	0	Decelerated expansion for $0 < m < 3$ Accelerated expansion for $m > 3, m \le -3$ Decelerated contraction for $-3 < m < 0$ Milne evolution for $m = 3$
P_m	1	0	$\frac{1}{2}$	$a_0 \lvert t\rvert^{\frac{2}{3}}$	0	0	Decelerated expansion
P_k	0	$-\frac{1}{3}$	0	$a_0 \lvert t\rvert$	0	$^2K_0 a_0^{-2} \lvert t\rvert^{-2}$	Milne evolution
P_x	0	$-1 + \frac{m(m-2)(1-b)}{4(m-1)^2}$	$-1 + \frac{3m(m-2)(1-b)}{8(m-1)^2}$	$a_0 \lvert t\rvert^{\frac{8(m-1)^2}{3m(m-2)(1-b)}}$	0	0	Accelerated expansion for $m > 2, b > 1,\ b < 1 - \frac{8(m-1)^2}{3m(m-2)}$ Decelerated expansion for $m > 2, b > 1,\ b > 1 - \frac{8(m-1)^2}{3m(m-2)}$
P_y	0	-1	-1	$a_0 \lvert t\rvert^{\frac{1}{3}\theta_0 t}$	0	0	Accelerated expansion

TABLE XIV. Eigenvalues and stability of the fixed points for a Bianchi V model corresponding to $f(R) = R + \gamma R^m$.

Points	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$	Stability
$P^{\pm}_{s1}$	$[2(1-3b), 3(1-2b), 2(1-3b), 6(1-b), 6-3(m+2)b]$	Stable for $b > 1, b > \frac{2}{2+m}$ Unstable for $b < \frac{1}{3}, b < \frac{2}{2+m}$ Saddle otherwise
$P^{\pm}_{s2}$	$[(4-\frac{6}{m}), (5-\frac{9}{m}), (6-\frac{9}{m}), (6-\frac{9}{m}), (9-\frac{9}{m})]$	Stable for $0 < m < 1$ Unstable for $m < 0; m > \frac{9}{5}$ Saddle otherwise
P_{r1}	$[1, 2, 4, (4-3mb), (-1+3b)]$	Unstable for $m < \frac{4}{3b}, b > \frac{1}{3}$ Saddle otherwise
P_{r2}	$[1, 1, 4, -(1-\frac{3}{m}), -\frac{1}{2}(5-\frac{9}{m})]$	Unstable for $0 < m < \frac{9}{5}$ Saddle otherwise
P_m	$[-1, 1, 3, 3(1-mb), -\frac{3}{2}(1-6b)]$	Saddle
P_{k1}	$[-2, -1, 2, -(2-3b), (2-3mb)]$	Saddle
P_{k2}	$[-(2-\frac{3}{m}), (1-\frac{3}{m}), (2-\frac{3}{m}), (2-\frac{3}{m}), (5-\frac{3}{m})]$	Saddle
P_x	$\left[\frac{3m(m-2)(1-b)}{2(m-1)^2}, -2+\frac{3m(m-2)(1-b)}{2(m-1)^2}, -3+\frac{3m(m-2)(1-b)}{2(m-1)^2}, -4+\frac{3m(m-2)(1-b)}{2(m-1)^2}, -3+\frac{3m(m-2)(m+4)(1-b)}{2(m-1)^2}\right]$	Stable for $m > 2, b > 1$ Saddle otherwise
P_y	$[-4, -3, -2, -3(1-b), -3mb]$	Stable for $m < 0, b < 0;$ $m > 0, 0 < b < 1$

$$w_{\mathrm{eff}} = \frac{1}{1-(\Sigma^2-3K)\left[2-\frac{3}{2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r-x+3y}{1-\Sigma^2-\Omega_r+3x+3y}\right)\right]^2}\left[\frac{1}{3}\Omega_r+\Sigma^2+K+x-y+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right.$$
$$-\frac{4x(1-\Sigma^2+3K)(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y})}{1-\Sigma^2-\Omega_r+3K+3x+3y}-2\Sigma^2\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)$$
$$-(\Sigma^2-K)\left[1-\frac{3}{2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)\right]^2$$
$$\left.+\frac{8x\frac{1-\Omega_r+3K-x+3y}{(1-\Sigma^2-\Omega_r+3K+3x+3y)^2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)}{2-3\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)}\right] \tag{124}$$

$$q = \frac{1}{2}+\frac{3}{2}\left[\frac{1}{3}\Omega_r+\Sigma^2+K+x-y+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right.$$
$$-\frac{4x(1-\Sigma^2+3K)\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)}{1-\Sigma^2-\Omega_r+3K+3x+3y}-2\Sigma^2\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)$$
$$\left.+\frac{8x\frac{1-\Omega_r+3K-x+3y}{(1-\Sigma^2-\Omega_r+3K+3x+3y)^2}\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)}{2-3\left(1+(b-1)\frac{1-\Sigma^2-\Omega_r+3K-x+3y}{1-\Sigma^2-\Omega_r+3K+3x+3y}\right)}\right]. \tag{125}$$

In this case, for the fixed points $P_s^{\pm}$, P_r, and P_k, the EoS and deceleration parameters are undefined. Hence this form of $f(R)$ does not possess a standard anisotropic, radiation-dominated, and spatially nonflat Universe. However, the fixed points P_m, P_x, and P_y exist in this model. The fixed points and their associated solutions are listed in Table XV.

As discussed in the previous two models in this form of $f(R)$ also, the fixed point (P_x) corresponding to the unstable node mimics a dust Universe. Thus it implies that our Universe begins with the EoS of matter (P_x), approaches the matter-dominated era (P_m), and ultimately accomplishes the stable de Sitter phase (P_y). This transition is plotted in Fig. 3.

4. Cosmological dynamics for model $f(R) = R + \gamma R^m - \beta/R^n$

The $f(R)$ of the form $f(R) = R + \gamma R^m$ describing the early Universe shows that it is possible to obtain a nonstandard high-energy anisotropic phase ($P_{s2}^{\pm}$), radiation-dominated phase (P_{r2}), and spatially nonflat phase (P_{k2}), followed by an inflationary phase (P_x). The $f(R)$ of the form $f(R) = R - \beta/R^n$ describing the late-time Universe shows that a sequence originating from the anisotropic phase ($P_{s1}^{\pm}$) approaches the radiation-dominated era (P_{r1}), matter-dominated era (P_m), and spatially nonflat phase (P_{k1}), and finally reaches the accelerated expanding scenario (P_x, P_y). Combining the results of these two forms of $f(R)$, we can study the dynamics of $f(R) = R + \gamma R^m - \beta/R^n$ ($\gamma > 0, \beta > 0$). We plot the evolution of dynamical variables Σ, Ω_r, K, x, and y, along with the EoS parameter w_{eff} for $m = 1.9$ and $n = 1$ in Fig. 10. The figure

TABLE XV. Fixed points and their associated solutions for a Bianchi V model corresponding to $f(R) = R + \gamma R^2$.

Points	Ω_m	EoS parameter (w_{eff})	Deceleration parameter (q)	Average scale factor (a)	Eigenvalues $[\lambda_1, \lambda_2, \lambda_3]$	Stability
$P_s^{\pm}$	0	...	...	...	...	...
P_r	0	...	...	...	...	...
P_m	1	0	$\frac{1}{2}$	$a_0\lvert t\rvert^{\frac{2}{3}}$	$[-1, 1, 3, 3(1-2b), -\frac{3}{2}(1-6b)]$	Saddle
P_x	0	0	$\frac{1}{2}$	$a_0\lvert t\rvert^{\frac{2}{3}}$	$[2, 3, 3, 4, 6]$	Unstable
P_k	0	...	...	...	...	...
P_y	0	-1	-1	$a_0\lvert t\rvert^{\frac{1}{3}\theta_0 t}$	$[-6b, -4, -3, -2, -3(1+b)]$	Stable for $b > 0$ Saddle otherwise

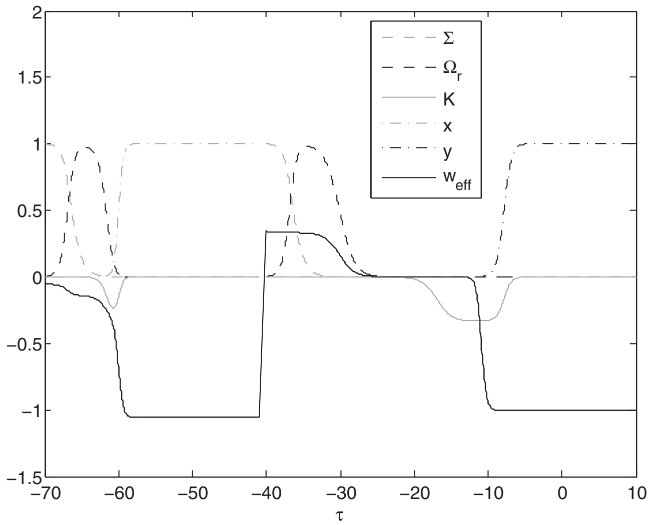

FIG. 10. Evolutions of the variables Σ, Ω_r, K, x, and y, as well as the EoS parameter w_{eff}, for a Bianchi V model corresponding to $f(R) = R + \gamma R^m - \beta/R^n$ with $m = 1.9; n = 1; b = -0.5$; and β, γ arbitrary.

shows that the Universe begins with a nonstandard anisotropic phase, then it goes to a nonstandard radiation-dominated era and spatially nonflat phase, followed by inflation, a standard anisotropic, standard radiation-dominated, matter-dominated, and spatially nonflat phase, and finally attains the accelerated expanding epoch.

VI. CONCLUSION

In this paper we have investigated the Born-Infeld theory in Palatini $f(R)$ gravity and explored the field equations in the background of FLRW, Bianchi type I, and Bianchi type V models. We have adopted a very useful approach known as the dynamical system approach. We tried to find the equilibrium points and to study the physical behavior for the models under consideration. We then extended our analysis for a number of families of $f(R)$ of the forms $f(R) = R + \gamma R^m - \beta/R^n$, $f(R) = R - \beta/R^n$, and $f(R) = R + \gamma R^m$.

In the case of a FLRW model for $f(R) = R - \beta/R^n$, it is shown that points describing the accelerated expanding Universe are found to be stable. In addition, the sequence of the radiation-dominated, matter-dominated, and accelerated expanding epochs is also realized. For $f(R) = R + \gamma R^m - \beta/R^n$, a sequence is obtained which shows that the Universe begins with a nonstandard radiation-dominated phase, followed by inflation, and then followed by a standard radiation-dominated phase and a matter-dominated phase, and finally attains the accelerated expanding Universe. We also perform an analysis for $f(R) = R + \gamma R^2$, which shows that this model does not possess a standard radiation-dominated phase and also shows that the Universe begins with the EoS of matter and ends in a stable de Sitter phase following a matter-dominated era.

The Bianchi I model is spatially homogeneous as well as anisotropic; therefore the corresponding model imparts information about the anisotropy parameter, i.e., shear. In this background for $f(R) = R - \beta/R^n$, we observed that the anisotropic Universe first approaches the radiation-dominated and matter-dominated era, respectively, and eventually attains the accelerated expanding Universe. The theories of the form $f(R) = R + \gamma R^m - \beta/R^n$ show that the Universe begins with a nonstandard anisotropic phase, then it goes to a nonstandard radiation-dominated era, followed by inflation, a standard anisotropic, standard radiation-dominated, as well as matter-dominated phase, and ultimately ends in the late-time accelerating phase. For $f(R) = R + \gamma R^2$, it is realized that a standard anisotropic and radiation-dominated phase is absent. We have also examined the equations depicting the evolution of shear and it is realized that anisotropy disappears as time evolves.

A Bianchi type V model is a simple generalization of a FLRW model comprised of negative curvature. Being spatially homogeneous and anisotropic, this model not only provides information about the anisotropy parameter but also about the spatial curvature. In this geometry for $f(R) = R - \beta/R^n$, the system initiates from an anisotropic Universe, followed by a radiation-dominated, matter-dominated, and spatially nonflat isotropic epoch, respectively, and ultimately reaches the accelerated expanding Universe. Theories of the type $f(R) = R + \gamma R^m - \beta/R^n$ display a sequence starting with a nonstandard anisotropic phase, which then goes to a nonstandard radiation-dominated era and spatially nonflat phase, followed by inflation, a standard anisotropic, standard radiation-dominated, matter-dominated, as well as spatially nonflat phase, and then enters a phase of accelerated expansion. Corresponding to the theories of the form $f(R) = R + \gamma R^2$, it is found that a standard anisotropic, radiation-dominated, and spatially nonflat phase is not present. Equations characterizing the evolution of shear as well as spatial curvature have also been investigated and it is fascinating to witness that the anisotropy and spatial curvature die out as time evolves, so that the anisotropic and nonflat Universe proceeds to the flat and isotropic Universe at late times.

In our analysis we have been able to reproduce the sequence of a radiation-dominated epoch followed by a matter-dominated and accelerating expanding epoch for the isotropic FLRW model as well as for anisotropic Bianchi I and V models in the case of $f(R) = R - \beta/R^n$. As the FLRW model is an isotropic model, it does not provide any information about the anisotropy parameter, i.e., the shear parameter. Bianchi I and V not only reveal information about the shear parameter but also show that the anisotropy vanishes with the evolution of time. In addition to the shear parameter, the spatial curvature disappears as time evolves in the case of a Bianchi V model.

The phase space portrait of the inflationary $f(R)$ theories containing a Starobinsky model and powers higher than two in the Ricci scalar were already examined by Capozziello *et al.* [74]. They found interesting trajectories that undergo an inflationary expansion and reach a Friedmannian asymptotic stable phase. Makarenko *et al.* [30] considered a Born-Infeld gravity Lagrangian plus an $f(R)$ term in a homogeneous and isotropic FLRW background, and then focused on a particular form of $f(R) \propto R^2$. Using such a form of $f(R)$, they studied the inflationary stage of the early Universe and realized that inflationary behavior may exist in a radiation-dominated era. In our analysis, we have considered $f(R) = R + \gamma R^m - \beta/R^n$, which can be used to study the early-time as well as late-time accelerating phase of the Universe. This particular form of $f(R)$ has also been considered for an isotropic FLRW model considering an action with $f(R)$ gravity in [5]. They have shown the evolution from a radiation-dominated epoch to an accelerated expanding Universe followed by a matter-dominated era, which is consistent with our investigation. Harko *et al.* [34] considered a barotropic cosmological fluid in an Eddington-inspired Born-Infeld gravity within an anisotropic Bianchi type I cosmology and studied the mean anisotropy parameter in detail. In their analysis, they realized that for the Universe filled with high-density matter, the isotropization of the anisotropic Universe depends on the initial conditions of energy density. However, corresponding to the dust-filled Universe, the anisotropic phase always accomplished the isotropic scenario. In our analysis, considering Bianchi type I and V models in $f(R)$ gravity, we show that the evolution of the anisotropy parameter decreases with time, and also we have realized the sequence of an anisotropic Universe, radiation-dominated epoch, matter-dominated epoch, and finally the accelerated expanding epoch. In case of a Bianchi V model, we also performed an investigation for the spatial curvature and have shown that the curvature disappears as time evolves. In this model, we obtain a sequence that originates from the anisotropic stage and enters the stable phase of accelerated expansion, followed by a radiation-dominated, matter-dominated, and spatially nonflat isotropic Universe.

The exact cosmological solutions of the dynamical system in $f(R)$ gravity within FLRW cosmology were discussed in [75]. The dynamics of a system was investigated considering the Noether symmetry approach, which allows us to procure the conserved quantities that lead to the reduction of dynamics and thus minimizes the difficulty when solving a dynamical system. In our analysis, we consider the dynamical system approach to achieve the exact cosmological solution of the dynamical system. Capozziello *et al.* [76] introduced the Noether symmetry approach in a Dirac-Born-Infeld Lagrangian considering a tachyonic potential $V(T)$, where T is a a tachyon scalar field. They also assumed a canonical scalar potential ϕ that is coupled to the tachyonic potential through an interacting potential $B(T,\phi)$. In their analysis, it has been shown that the scale factor (a) shows an exponential solution which is consistent with the accelerated behavior of the fixed point (P_y) in our analysis.

The authors of [77] studied a specific nonlinear gravity-scalar system in the Palatini formalism of $f(R)$ gravity, which leads to a FLRW cosmology different from the standard metric one. In their work, it was shown that the Palatini formalism in the case of nonlinear gravity-scalar systems allows an acceptable realization of the dark energy dominance. It is also shown that Palatini $f(R)$ gravity produces the effective quintessence or effective phantom phase at late times in the same qualitative way as discussed in the standard metric formalism. They have also established that the dynamical mechanism used to resolve the cosmological constant problem suggested in metric formalism also works in Palatini formalism. Thus one can observe that, even though both formalisms of $f(R)$ gravity apparently deliver different gravitational physics, the same cosmological phenomena may qualitatively occur in both formalisms.

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