Deflection of light by rotating regular black holes using the Gauss-Bonnet theorem

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In this paper, we study the weak gravitational lensing in the spacetime of rotating regular black hole geometries such as Ayon-Beato-García (ABG), Bardeen, and Hayward black holes. We calculate the deflection angle of light using the Gauss-Bonnet theorem (GBT) and show that the deflection of light can be viewed as a partially topological effect in which the deflection angle can be calculated by considering a domain outside of the light ray applied to the black hole optical geometries. Then, we demonstrate also the deflection angle via the geodesics formalism for these black holes to verify our results and explore the differences with the Kerr solution. These black holes have, in addition to the total mass and rotation parameter, different parameters of electric charge, magnetic charge, and deviation parameter. We find that the deflection of light has correction terms coming from these parameters, which generalizes the Kerr deflection angle.

I. INTRODUCTION

Since Einstein discovered the general theory of relativity in 1915 [1], Einstein's theory has been subjected to numerous experimental tests. It turns out that experimental results are quite well in agreement with theoretical predictions of this theory, starting from astrophysics observations, and also a number of other precise confirmed experiments [2–5]. Some of the predictions are exciting: gravitational waves which were recently detected by LIGO [3,4], and gravitational lensing and bending of light, black holes, wormholes and others. The gravitational lensing has been studied previously in the literature using different types of spacetimes with strong lensing or weak lensing.

In this paper, we focus on the weak gravitational lensing using the Gauss-Bonnet theorem (GBT), also known as the

Gibbons-Werner method (GWM). Because black holes cannot be observed directly, one way to ensure their existence is to study the geodesic equations of light rays in the curved spacetime geometry due to the presence of black holes. In this way, one can extract valuable information from black holes and detect their features. Weak gravitational lensing is an interesting method; however, in most of the cases, the strong lensing regime is needed; strong lensing provides more information from an experimental point of view to detect other exotic objects or ultra-compact objects such as boson stars [6]. In the near future, scientists expect to detect the horizon of a black hole using the Event Horizon Telescope (EHT) [7]; so, this topic has acquired a great interest, and many authors focus on it to obtain correct results [8–39].

Recently, Gibbons and Werner [40] changed the standard viewpoint related to the way we usually calculate the deflection angle. They showed that one can calculate the deflection angle in a very elegant way; namely, they have used the GBT in the context of optical geometry. The physical significance relies in the fact that one can view the bending of a light ray as a global effect, which is different

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from the standard viewpoint where the bending of light is usually associated within a region with a radius compared to the impact parameter. In this method, one shall only focus on a nonsingular domain outside of the light ray. For asymptotically flat spacetimes the deflection angle can be calculated by the following equation:

$$\hat{\alpha} = - \iint_{S_\infty} \mathcal{K} d\sigma,$$

where $\mathcal{K}$ is the Gaussian optical curvature and $d\sigma$ is the surface element of the optical geometry. Note that the above expression for the deflection angle holds in the case of asymptotically flat spacetimes, whereas in the case of a nonasymptotically flat metric, only finite distance corrections can be studied. Very recently, Werner has been able to extend this method to cover Kerr black holes using the Finsler-Randers-type metric. More specifically, he has applied Nazım's method to construct a Riemannian manifold osculating the Randers manifold [41]. In addition, this method has been extended to the wormhole geometries and nonasymptotically flat spacetimes with topological defects [42]. This method has been used in a number of papers [43–53]; among others, we note that the GBT has been used in interesting papers by Ishihara *et al.* [54–56], in which the deflection angle for finite distances for static (including the presence of the cosmological constant) and stationary metrics is studied in a rather different setup.

Classical singularities in general relativity break down the laws of physics. Singularities appear from the cosmological Big Bang theory to black holes where they are hidden behind the event horizon of a black hole. It is widely believed that quantum mechanics forbids the physics-ending singularities, but until today the problem of singularities remains an open problem in physics. To overcome the problem of singularities in black holes, many physicists have tried to construct regular black holes even in the context of classical general relativity. In this line of research we point out the Ayon-Beato-Garcia black hole (ABG) [57,58], Bardeen regular black holes [59] and Hayward regular black holes [60], which were obtained by finding a new mass function, and we obtain the deflection angles [61–66] in order to explore the difference with the Kerr black hole solution in the weak gravitational lensing. In doing so, we extend the GBT method to rotating black holes with electric and magnetic charge for the first time.

This paper is organized as follows. In Sec. II we start by reviewing some of the basic concepts related to Finsler geometry. By introducing the ABG-Randers optical metric and followed by the Gaussian optical curvature we

calculate the deflection angle. We study also the geodesic equation to verify our results. In Sec. III, we evaluate the deflection angle in the spacetime of a Bardeen black hole. In Sec. IV, we perform the same analysis for the Hayward regular black hole. Hence, we finalize our results in Sec. V.

II. DEFLECTION ANGLE OF ROTATING REGULAR ABG BLACK HOLE

In this section, first we use the rotating Ayon-Beato-García spacetime, which is a nonsingular exact black hole solution of Einstein field equations coupled to a nonlinear electrodynamics, and satisfy the weak energy condition. The metric of the ABG black hole is written in the form [61,62]:

$$ds^2 = -f(r,\theta)dt^2 + \frac{\Sigma}{\Delta}dr^2 - 2a\sin^2\theta(1 - f(r,\theta))d\phi dt$$
$$+ \Sigma d\theta^2 + \sin^2\theta[\Sigma - a^2(f(r,\theta) - 2)\sin^2\theta]d\phi^2, \quad (2.1)$$

with

$$f(r,\theta) = 1 - \frac{2mr\sqrt{\Sigma}}{(\Sigma + Q^2)^{3/2}} + \frac{Q^2\Sigma}{(\Sigma + Q^2)^2}, \quad (2.2)$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad (2.3)$$

$$\Delta = \Sigma f(r,\theta) + a^2\sin^2\theta. \quad (2.4)$$

The ABG black hole metric can be further simplified by setting $\theta = \pi/2$; in that case the function $f(r)$ takes the form

$$f(r) = 1 - \frac{2mr^2}{(r^2 + Q^2)^{3/2}} + \frac{Q^2r^2}{(r^2 + Q^2)^2}, \quad (2.5)$$

$$\Sigma = r^2, \quad (2.6)$$

$$\Delta = \Sigma f(r) + a^2. \quad (2.7)$$

We wish now to recast our ABG metric into the Finsler-Randers type metric of the general form [41],

$$\mathcal{F}(x,v) = \sqrt{\zeta_{ij}(x)v^iv^j} + \beta_i(x)v^i, \quad (2.8)$$

provided $\zeta^{ij}\beta_i\beta_j < 1$, in which ζ_{ij} gives the Riemannian metric to be calculated from the ABG metric, while β_i represents a one-form. If we solve Eq. (2.1) for the null geodesic case i.e., $ds^2 = 0$, and the problem is simplified to study the deflection of light in the equatorial plane by letting $\theta = \pi/2$. In that case, we find the following ABG-Randers optical metric:

$$\mathcal{F}\left(r, \phi, \frac{dr}{dt}, \frac{d\phi}{dt}\right) = \sqrt{\left[\frac{a^2(1 - f(r))^2}{f(r)^2} + \frac{\Sigma - a^2(f(r) - 2)}{f}\right]\left(\frac{d\phi}{dt}\right)^2 + \frac{\Sigma}{\Delta f(r)}\left(\frac{dr}{dt}\right)^2} - \frac{a(1 - f(r))}{f(r)}\frac{d\phi}{dt}. \quad (2.9)$$

The physical significance of the ABG-Randers optical metric $\mathcal{F}$ relies in the remarkable feature of the Finsler geometry, namely it provides a way to actually compute the null geodesics. In other words, there is a link of finding null geodesics in our physical metric (2.1) with the problem of finding the null geodesics of a ABG-Randers optical metric which can be seen by recalling Fermat's principle. As $dt = \mathcal{F}(x, dx)$, Fermat's principle of least time in the context of general relativity suggests that the null geodesics can be found from the following condition:

$$\delta \int_{\gamma} dt = \delta \int_{\gamma_{\mathcal{F}}} \mathcal{F}(x, \dot{x}) dt = 0. \qquad (2.10)$$

Hence, it is clear that the Rander-Finsler metric $\mathcal{F}$ naturally appears in the problem of finding null geodesics and generalizes Fermat's principle. The Randers-Finsler metric is characterized by the Hessian,

$$g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2(x, v)}{\partial v^i \partial v^j}, \qquad (2.11)$$

where $x \in \mathcal{M}$, $v \in T_x M$. To this end we need to apply Nazım's method which allows us to construct a Riemannian manifold $(\mathcal{M}, \bar{g})$ that osculates the ABG-Randers manifold $(\mathcal{M}, \mathcal{F})$. For this purpose, we need to choose a vector field $\bar{v}$ tangent to the geodesic $\gamma_{\mathcal{F}}$, such that $\bar{v}(\gamma_{\mathcal{F}}) = \dot{x}$. In that case the Hessian reads

$$\bar{g}_{ij}(x) = g_{ij}(x, \bar{v}(x)). \qquad (2.12)$$

It is obvious that the choice of the vector field is not unique and affects the optical metric components, but the crucial result which should be noted is that a geodesic of the Randers manifold $\gamma_{\mathcal{F}}$ is also a geodesic $\gamma_{\bar{g}}$ of $(\mathcal{M}, \bar{g})$ (see [41] for details):

$$\ddot{x}^i + \Gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k = \ddot{x}^i + \bar{\Gamma}^i_{jk}(x) \dot{x}^j \dot{x}^k = 0, \qquad (2.13)$$

or $\gamma_F = \gamma_{\bar{g}}$. One can choose the nonsingular region $S_R \subset M$ to be bounded by the light ray $\gamma_{\mathcal{F}}$ and a curve γ_R in a radial distance R from the coordinate origin. Furthermore, these curves can be parameterized as follows [41]:

$$\gamma_{\mathcal{F}} : x^i(t) = \eta^i(t), \quad t \in [0, l], \qquad (2.14)$$

$$\gamma_R : x^i(t) = \zeta^i(t), \quad t \in [0, l^\star]. \qquad (2.15)$$

In particular one can introduce $\tau = t/l$ along the geodesic γ_F which belongs to the interval $\in (0, 1)$ and similarly $\tau^\star = 1 - t/l$ with the interval $\in (0, 1)$ along the curve γ_R. This means that one can pair each point $\eta^i(\tau)$ on $\gamma_{\mathcal{F}}$ with $\zeta^i(\tau^\star)$ on γ_R if we set $\tau = \tau^\star$. In other words, one can show that there exists a family of smooth curves $x^i(\sigma, \tau)$, such that for a pair of each point there is precisely one curve which touches the boundary curve (see Fig. 1).
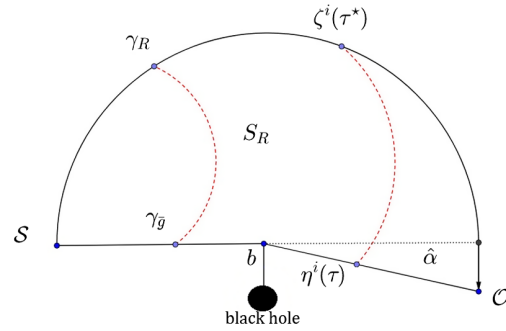

FIG. 1. Integration domain S_R, namely the equatorial plane (r, ϕ), in which $\hat{\alpha}$ is the total deflection angle and b is the impact parameter.

In addition we can say that $x^i(\sigma, \tau)$ touches the curve $\gamma_{\mathcal{F}}$ at the boundary when $\eta^i(\tau) = x^i(0, \tau)$, where σ is a new parameter. Then the following relation holds:

$$\dot{\eta}^i(\tau) = \frac{d\eta^i}{dt}(\tau) = \frac{dx^i}{d\sigma}(0, \tau). \qquad (2.16)$$

Likewise, we can say that $x^i(\sigma, \tau)$ touches the curve γ_R when $\zeta^i(\tau) = x^i(1, \tau)$. This suggests that

$$\dot{\zeta}^i(\tau) = \frac{d\zeta^i}{dt}(\tau) = \frac{dx^i}{d\sigma}(1, \tau). \qquad (2.17)$$

In general, one can construct a smooth and nonzero tangent vector field,

$$\bar{v}^i(x(\sigma, \tau)) = \frac{dx^i}{d\sigma}(\sigma, \tau), \qquad (2.18)$$

with a family of smooth curves which satisfy the following relation [41]:

$$x^i(\sigma, \tau) = \eta^i(\tau) + \dot{\eta}^i(\tau)\sigma + \mathcal{A}(\tau)\sigma^2 + \mathcal{B}(\tau)\sigma^3 \\ + y^i(\sigma, \tau)(1 - \sigma)^2 \sigma^2, \qquad (2.19)$$

with

$$\mathcal{A}(\tau) = 3\zeta^i(\tau) - 3\eta^i(\tau) - \dot{\zeta}^i(\tau) - 2\dot{\eta}^i(\tau)\eta^i(\tau),$$
$$\mathcal{B}(\tau) = 2\eta^i(\tau) - 2\zeta^i(\tau) + \dot{\zeta}^i(\tau) + \dot{\eta}^i(\tau).$$

That being said, and keeping in mind that our metric is asymptotically flat, we can choose an equation for the light rays as follows:

$$r(\phi) = \frac{b}{\sin \phi}, \qquad (2.20)$$

with b being the impact parameter. From the light ray equation one can deduce the following components for the vector field:

$$\bar{v}^r = \frac{dr}{dt} = -\cos\phi, \qquad \bar{v}^\phi = \frac{d\phi}{dt} = \frac{\sin^2\phi}{b}. \qquad (2.21)$$

It is worth noting that the choice of the vector field is dictated by the light ray equation r_γ. Note that our particular equation of the light ray represents a straight line approximation, and this will be important in the final result for the deflection angle. In other words, due to the straight line

approximation we expect our deflection angle to be correct in leading order terms.

A. Gaussian optical curvature

We shall now continue to compute the metric components. To this end, we need to combine Eqs. (2.11) and (2.21), yielding the following nonzero components:

$$\bar{g}_{rr} = -\frac{2[-2(m+\frac{r}{4})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2} + amr^2\sin^6\phi]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}$$
$$-\frac{[18(m+\frac{r}{9})(\sin^4\phi r^2 + b^2\cos^2\phi)^{5/2} - 7(13(m+\frac{r}{13})r^2\sin^4\phi + b^2\cos^2\phi(m+\frac{r}{7}))r^2\sin^6\phi]Q^2}{r^3(\sin^4\phi r^2 + b^2\cos^2\phi)^{5/2}}, \qquad (2.22)$$

$$\bar{g}_{r\varphi} = \frac{2a\cos^3\phi m}{r(\frac{\sin^4\phi r^2 + b^2\cos^2\phi}{b^2})^{3/2}} - \frac{b^3 aQ^2\cos^3\phi(17\sin^4\phi mr^{19} + \sin^4\phi r^{20} + 11r^{17}\cos^2\phi b^2 m + \cos^2\phi b^2 r^{18})}{r^{20}(\sin^4\phi r^2 + b^2\cos^2\phi)^{5/2}}, \qquad (2.23)$$

$$\bar{g}_{\varphi\varphi} = \frac{2r[-2amr^2\sin^6\phi - 3ab^2 m\cos^2\phi\sin^2\phi + (m+\frac{r}{2})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}$$
$$-\frac{Q^2[7(m+\frac{r}{7})(\sin^4\phi r^2 + b^2\cos^2\phi)^{5/2} - 3a\sin^2\phi\Xi(r,a,m)]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{5/2}}, \qquad (2.24)$$

where

$$\Xi = 6\left(m+\frac{r}{9}\right)r^4\sin^8\phi + 15\cos^2\phi\left(m+\frac{r}{9}\right)b^2 r^2\sin^4\phi + 7b^4\cos^4\phi\left(m+\frac{r}{7}\right). \qquad (2.25)$$

The determinant is given as

$$\det\bar{g} = -\frac{6r^4 ma\sin^6\phi + 6b^2 amr^2\cos^2\phi\sin^2\phi + 33(-\frac{2}{11}mr^2 - \frac{1}{33}r^3)(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}$$
$$-\frac{Q^2[-51r^2(m+\frac{r}{11})a\sin^6\phi - 45b^2 a\cos^2\phi(m+\frac{r}{15})\sin^2\phi + 33(m+\frac{r}{11})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}. \qquad (2.26)$$

The Gaussian optical curvature then can be found by noticing the relation $\bar{R}_{r\phi r\phi} = \mathcal{K}\det\bar{g}$. In other words, we can compute $\mathcal{K}$ as follows:

$$\mathcal{K} = \frac{1}{\sqrt{\det\bar{g}}}\left[\frac{\partial}{\partial\phi}\left(\frac{\sqrt{\det\bar{g}}}{\bar{g}_{rr}}\bar{\Gamma}^\phi_{rr}\right) - \frac{\partial}{\partial r}\left(\frac{\sqrt{\det\bar{g}}}{\bar{g}_{rr}}\bar{\Gamma}^\phi_{r\phi}\right)\right]. \qquad (2.27)$$

Our computation reveals the following result:

$$\mathcal{K} = -2\frac{m}{r^3} + 3\frac{Q^2}{r^4} + 12\frac{Q^2 m}{r^5} + \frac{15a\mathcal{G}(r,\phi)}{r^5}. \qquad (2.28)$$

Note that the first term corresponds to the Schwarzschild black hole, whereas the second and third term give the charge contribution, and finally the last term is a consequence of the rotation. Note that the function $\mathcal{G}(r,\varphi)$ is rather complicated expression which is found to be

$$\mathcal{G}(r,\phi) = -\frac{\sin^2\phi}{(\sin^4\phi\, r^2 + b^2\cos^2\phi)^{9/2}}\left[\left(\frac{11Q^2mr^8}{10} + \frac{1}{2}Q^2r^9 - \frac{2}{5}mr^{10}\right)(\sin(\phi))^{16} - \frac{1}{2}r^6b^2\left(Q^2m + \frac{1}{5}Q^2r - \frac{2}{5}mr^2\right)(\sin(\phi))^{14}\right.$$

$$+ r^6b^2(\cos(\phi))^2\left(Q^2m + \frac{11Q^2r}{10} + \frac{2}{5}mr^2\right)(\sin(\phi))^{12} + \frac{42b^3r^5(\cos(\phi))^2(\sin(\phi))^{11}}{5}\left(Q^2m + \frac{5Q^2r}{42} - \frac{4mr^2}{21}\right)$$

$$+ \frac{9r^4b^2}{2}(\cos(\phi))^2(\sin(\phi))^{10}\left(\left(-\frac{10Q^2mr^2}{9} - \frac{2}{9}Q^2r^3 + \frac{4}{9}r^4m\right)(\cos(\phi))^2 + b^2\left(Q^2m + \frac{1}{15}Q^2r - \frac{2}{15}mr^2\right)\right)$$

$$+ \frac{84r^5b^3(\cos(\phi))^4(\sin(\phi))^9}{5}\left(Q^2m + \frac{5Q^2r}{42} - \frac{4mr^2}{21}\right) + \frac{96r^4b^4(\cos(\phi))^4(\sin(\phi))^8}{5}\left(Q^2m + \frac{1}{8}Q^2r - \frac{5mr^2}{96}\right)$$

$$- \frac{21r^3b^5(\cos(\phi))^4(\sin(\phi))^7}{5}\left(Q^2m - \frac{5Q^2r}{21} + \frac{2}{7}mr^2\right) - 2r^2b^4(\cos(\phi))^4\left(-9Q^2mr^2(\cos(\phi))^2\right.$$

$$\left. + b^2\left(Q^2m - \frac{1}{5}Q^2r + \frac{2}{5}mr^2\right)\right)(\sin(\phi))^6 - \frac{42r^3b^5(\cos(\phi))^6(\sin(\phi))^5}{5}\left(Q^2m - \frac{5Q^2r}{21} + \frac{2}{7}mr^2\right)$$

$$- \frac{7r^2b^6(\cos(\phi))^6(\sin(\phi))^4}{10}\left(Q^2m - \frac{20Q^2r}{7} + \frac{18mr^2}{7}\right) + \frac{1}{5}(7Q^2 + 2r^2)rb^7(\cos(\phi))^6m(\sin(\phi))^3$$

$$+ Q^2b^8(\cos(\phi))^8(m + r/5) - 5r^2b^6(\cos(\phi))^8\left(Q^2m - \frac{1}{5}Q^2r + \frac{2}{5}mr^2\right)(\sin(\phi))^2$$

$$\left. + \frac{1}{5}(14Q^2 + 4r^2)rb^7(\cos(\phi))^8m\sin(\phi)\right]. \tag{2.29}$$

Note that the Gaussian optical curvature depends on the black hole parameters, a, m, and Q^2. In the next section we are going to evaluate the deflection angle with the help of the above result.

B. Deflection angle

Theorem: Let $(S_R, \bar{g})$ be a non-singular and simply connected domain over the osculating Riemannian manifold $(\mathcal{M}, \bar{g})$ bounded by circular curve γ_R and the geodesic $\gamma_{\bar{g}}$. Let $\mathcal{K}$ be the Gaussian curvature of $(\mathcal{M}, \bar{g})$, and κ the geodesic curvature of $\partial S_R = \gamma_{\bar{g}} \cup \gamma_R$. Then, the GBT can be stated as follows [40,41]:

$$\iint_{S_R} \mathcal{K}d\sigma + \oint_{\partial S_R} \kappa dt + \sum_k \alpha_k = 2\pi\chi(S_R). \tag{2.30}$$

As we have already noted that $d\sigma$ gives the surface element, α_k represents the kth exterior angles, $\chi(S_R)$ is known as the Euler characteristic number. The geodesic curvature basically determines the deviation from the geodesic. By definition we have $\kappa(\gamma_{\bar{g}}) = 0$ because $\gamma_{\bar{g}}$ is a geodesic. Of particular importance is the geodesic curvature of γ_R in a radial coordinate R from the coordinate origin. It can be calculated via

$$\kappa(\gamma_R) = |\nabla_{\dot{\gamma}_R}\dot{\gamma}_R|. \tag{2.31}$$

We can choose $\gamma_R := r(\varphi) = R = \text{const}$, and in that case the radial part yields

$$(\nabla_{\dot{\gamma}_R}\dot{\gamma}_R)^r = \dot{\gamma}_R^\phi(\partial_\phi\dot{\gamma}_R^r) + \bar{\Gamma}^r_{\phi\phi}(\dot{\gamma}_R^\phi)^2. \tag{2.32}$$

It is noted that the first term vanishes, and the second term can be calculated by the unit speed condition i.e., $\bar{g}_{\phi\phi}\dot{\gamma}_R^\phi\dot{\gamma}_R^\phi = 1$. Because our optical geometry is asymptotically Euclidean we find that $\kappa(\gamma_R) \to R^{-1}$ as $R \to \infty$. The other point is the fact that as $R \to \infty$, the sum of jump angles ($\alpha_\mathcal{O}$), to the source $\mathcal{S}$, and observer $\mathcal{O}$, yields $\alpha_O + \alpha_S \to \pi$ [40]. For constant R, the optical metric gives

$$\lim_{R\to\infty} dt = \lim_{R\to\infty}\left[\sqrt{\frac{a^2(1-f(r))^2}{f(r)^2} + \frac{\Sigma - a^2(f(r)-2)}{f}}\right.$$
$$\left. - \frac{a(1-f(r))}{f(r)}\right]d\phi \to Rd\phi, \tag{2.33}$$

where we have used the fact that

$$\lim_{R\to\infty} f(R) \to 1. \tag{2.34}$$

Finally one can shows that

$$\lim_{R\to\infty} \kappa(\gamma_R)\frac{dt}{d\phi} \to 1. \tag{2.35}$$

Note that by construction, the source S and the observer O are assumed to be in the asymptotically Euclidean region, thus the last equation clearly reveals our assumptions that our optical metric is asymptotically Euclidean. Having computed the geodesic curvature from GBT it follows

$$\iint_{\mathcal{S}_R} \mathcal{K}dS + \oint_{\gamma_R} \kappa dt \stackrel{R\to\infty}{=} \iint_{\mathcal{S}_\infty} \mathcal{K}d\sigma + \int_0^{\pi+\hat{\alpha}} d\phi = \pi, \tag{2.36}$$

resulting with

$$\hat{\alpha} = -\iint_{\mathcal{S}_\infty} \mathcal{K}d\sigma. \tag{2.37}$$

After substituting the Gaussian optical curvature (2.28) into the last equation we find

$$\hat{\alpha} \simeq -\int_0^\pi \int_{\frac{b}{\sin\phi}}^\infty \left[-2\frac{m}{r^3} + 3\frac{Q^2}{r^4} + 12\frac{Q^2 m}{r^5} + \frac{15a\mathcal{G}(r,\phi)}{r^5} \right]$$
$$\times \sqrt{\det\bar{g}}\, dr\, d\phi. \tag{2.38}$$

Solving the nonrotating part in the above integral we find

$$\mathcal{I}_1 = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \left(-2\frac{m}{r^3} + 3\frac{Q^2}{r^4} + 12\frac{Q^2 m}{r^5} \right) \sqrt{\det\bar{g}}\, dr\, d\varphi$$
$$= \frac{4m}{b} - \frac{3\pi Q^2}{4b^2} - \frac{16Q^2 m}{3b^3}. \tag{2.39}$$

The rotating part gives

$$\mathcal{I}_2 = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \left(\frac{15a\mathcal{G}(r,\phi)}{r^5} \right) \sqrt{\det\bar{g}}\, dr\, d\varphi$$
$$= \pm\frac{4ma}{b^2} \pm \frac{9896maQ^2}{15b^4}. \tag{2.40}$$

The total deflection angle is found to be

$$\hat{\alpha} = \frac{4m}{b} - \frac{3\pi Q^2}{4b^2} \pm \frac{4ma}{b^2} + \mathcal{O}(Q^2, a, m), \tag{2.41}$$

where the signs of positive and negative stand for retrograde and prograde light rays.

C. Geodesics

The equations of the geodesics can be derived from the Lagrangian of a test particle [67]. For motion in the equatorial plane, that is, $\theta = \pi/2$ and $\dot{\theta} = 0$, the Lagrangian results to be

$$2\mathcal{L} = -f(r)\dot{t}^2 + \frac{r^2}{\Delta}\dot{r}^2 - 2a(1-f(r))\dot{\phi}\,\dot{t}$$
$$+ [r^2 - a^2(f(r) - 2)]\dot{\phi}^2, \tag{2.42}$$

where $\dot{q} = dq/d\tau$, and τ is an affine parameter along the geodesic. As the Lagrangian (2.42) is independent of the cyclic coordinates (t, ϕ), then their conjugate momenta (Π_t, Π_ϕ) are conserved. Then, the equations of motion are obtained from $\dot{\Pi}_q - \frac{\partial\mathcal{L}}{\partial q} = 0$ and yield

$$\dot{\Pi}_t = 0, \qquad \dot{\Pi}_\phi = 0, \tag{2.43}$$

where $\Pi_q = \partial\mathcal{L}/\partial\dot{q}$ are the conjugate momenta to the coordinate q and are given by

$$\Pi_t = -f(r)\dot{t} - a(1-f(r))\dot{\phi} \equiv -E,$$
$$\Pi_r = \frac{r^2}{\Delta}\dot{r} \text{ and}$$
$$\Pi_\phi = -a(1-f(r))\dot{t} + (r^2 - a^2(f(r) - 2))\dot{\phi} \equiv L, \tag{2.44}$$

where E and L are dimensionless integration constants associated with each of them. So, the Hamiltonian is given by

$$\mathcal{H} = \Pi_t \dot{t} + \Pi_\phi \dot{\phi} + \Pi_r \dot{r} - \mathcal{L}, \tag{2.45}$$

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \equiv -\bar{m}^2. \tag{2.46}$$

Now, by normalization, we shall consider $\bar{m}^2 = 0$ for photons. Therefore, we obtain

$$\dot{t} = \frac{2a^2 E - aL + Er^2 + a(L - aE)f(r)}{a^2 + r^2 f(r)}, \tag{2.47}$$

$$\dot{\phi} = \frac{aE + f(r)(L - aE)}{a^2 + r^2 f(r)}, \tag{2.48}$$

$$\dot{r}^2 = \frac{a(2aE^2 - 2EL) + E^2 r^2 - f(r)(L - aE)^2}{r^2}. \tag{2.49}$$

The distance of the closest approach r_0 for the metric (2.1) can be obtained from $\dot{r} = 0$, which yields

$$\frac{r_0}{b} = \sqrt{1 - \left(\frac{a}{b}\right)^2 + r_0^2\left(1 - \frac{a}{b}\right)^2\left(\frac{Q^2}{(Q^2 + r_0^2)^2} - \frac{2m}{(Q^2 + r_0^2)^{3/2}}\right)}, \tag{2.50}$$

where $b = L/E$ is the impact parameter.

Now, following [29] the bending angle can be determined by the expression

$$\hat{\alpha} = 2\int_{r_0}^\infty \left|\frac{d\phi}{dr}\right| dr - \pi, \tag{2.51}$$

which yields

$$\hat{\alpha} = \frac{4m}{b} - \frac{3\pi Q^2}{4b^2} \pm \frac{4ma}{b^2} + \mathcal{O}(Q^2, a, m), \tag{2.52}$$

where we use the change of variables $u = r_0/r$; then, we substitute the impact parameter given by Eq. (2.50), and we expand in Taylor series around m, a, and Q. Finally, we consider $r_0 \approx b$. In Fig. 2 we plot the deflection of light in the background of a rotating regular ABG black hole by solving numerically Eqs. (2.48) and (2.49).
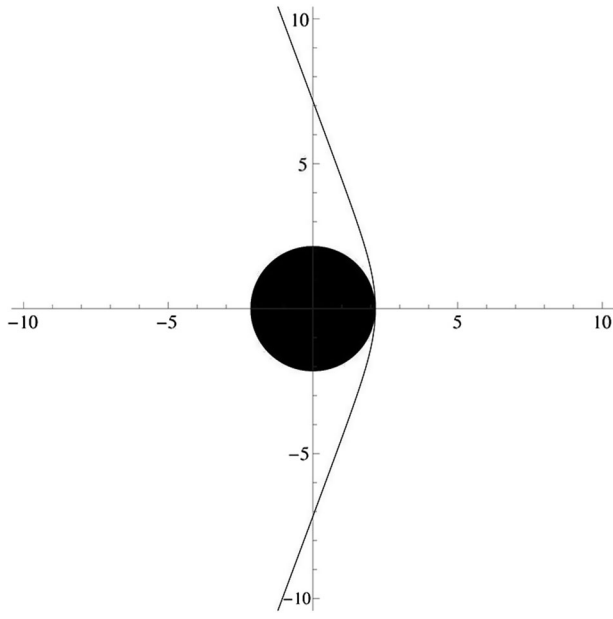
FIG. 2. Deflection of light in the background of a rotating regular ABG black hole with $E = 2$, $L = 5$, $m = 0.3$, $Q = 0.1$ and $a = 0.1$. The circle corresponds to the closest approach ($r_0 = 2.158$) to the black hole.

III. DEFLECTION ANGLE BY ROTATING REGULAR BARDEEN BLACK HOLE

In this section, we study the deflection angle by rotating Bardeen regular black hole. The spacetime metric of the rotating Bardeen regular black hole reads [62]:

$$ds^2 = -\left(1 - \frac{2\mathcal{M}_b r}{\Sigma}\right)dt^2 - \frac{4a\mathcal{M}_b r\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2$$
$$+ \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_b r\sin^2\theta}{\Sigma}\right)d\phi^2, \quad (3.1)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad (3.2)$$

$$\Delta = r^2 - 2\mathcal{M}_b r + a^2, \quad (3.3)$$

$$\mathcal{M}_b = m\left(\frac{r^2}{r^2 + g_\star^2}\right)^{3/2}, \quad (3.4)$$

in which $g_\star$ is the magnetic charge due to the nonlinear electromagnetic field. In this case, we find the following expression for the optical metric:

$$\mathcal{F}\left(r, \phi, \frac{dr}{dt}, \frac{d\phi}{dt}\right) = \sqrt{\frac{r^4}{\Delta(\Delta - a^2)}\left(\frac{dr}{dt}\right)^2 + \frac{r^4\Delta}{\Delta - a^2}\left(\frac{d\phi}{dt}\right)^2} - \frac{2mar}{\Delta - a^2}\left(\frac{r^2}{r^2 + g_\star^2}\right)^{3/2}\frac{d\phi}{dt}, \quad (3.5)$$

$$\bar{g}_{rr} = -\frac{2[-2(m + \frac{r}{4})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2} + amr^2\sin^6\phi]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}} - \frac{g_\star^2 m[3ar^2\sin^6\phi + 6(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{r^3(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}},$$

$$\bar{g}_{r\phi} = \frac{2ab^3\cos^3\phi m}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}} - \frac{3ab^3\cos^3\phi m g_\star^2}{r^3(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}, \quad (3.6)$$

$$\bar{g}_{\phi\phi} = \frac{2r[-2amr^2\sin^6\phi - 3ab^2 m\cos^\phi\sin^2\phi + (m + \frac{r}{2})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}$$
$$- \frac{3g_\star^2 m[-2ar^2\sin^6\phi - 3ab^2\cos^2\phi\sin^2\phi + (\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}. \quad (3.7)$$

With the determinant

$$\det\bar{g} = r^2 - \frac{6rm[a\sin^2\phi - \sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}]}{\sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}} + \frac{9g_\star^2 m[a\sin^2\phi - \sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}]}{r\sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}}. \quad (3.8)$$

Then the Gaussian optical curvature is

$$\mathcal{K} = -\frac{2m}{r^3} + \frac{18mg_\star^2}{r^5} + \frac{27am\mathcal{S}(r, \phi)}{r^5}, \quad (3.9)$$

with

$$\mathcal{S}(r,\phi) = -\frac{\sin^2\phi}{(\sin^4\phi r^2 + b^2\cos^2\phi)^{7/2}}\left[\left(\frac{14g_\star^2 r^6}{9} - 2/9r^8\right)(\sin(\phi))^{12} - 1/6b^2 r^4(g_\star^2 - 2/3r^2)(\sin(\phi))^{10}\right.$$

$$+ \frac{31b^2 r^4(\cos(\phi))^2(\sin(\phi))^8}{9}\left(g_\star^2 + \frac{4r^2}{31}\right) + \left(2b^3 g_\star^2 r^3 - \frac{8b^3 r^5}{9}\right)(\cos(\phi))^2(\sin(\phi))^7$$

$$+ 1/3(-5r^2(\cos(\phi))^2 + 2b^2)b^2 r^2(\cos(\phi))^2(g_\star^2 - 2/3r^2)(\sin(\phi))^6$$

$$+ \left(4b^3 g_\star^2 r^3 - \frac{16b^3 r^5}{9}\right)(\cos(\phi))^4(\sin(\phi))^5 + 1/18(97g_\star^2 - 18r^2)b^4 r^2(\cos(\phi))^4(\sin(\phi))^4$$

$$+ 1/3b^5 r(\cos(\phi))^4(g_\star^2 + 2/3r^2)(\sin(\phi))^3 + (\cos(\phi))^6 b^6 g^2$$

$$\left. + 5/3b^4 r^2(\cos(\phi))^6(g_\star^2 - 2/3r^2)(\sin(\phi))^2 + 2/3b^5 r(\cos(\phi))^6(g_\star^2 + 2/3r^2)\sin(\phi)\right]. \tag{3.10}$$

Substituting these relations from GBT it follows the integral

$$\hat{\alpha} \simeq -\int_0^\pi \int_{\frac{b}{\sin\phi}}^\infty \left(-\frac{2m}{r^3} + \frac{18mg_\star^2}{r^5} + \frac{27amS(r,\phi)}{r^5}\right)$$
$$\times \sqrt{\det\bar{g}}\,dr\,d\phi. \tag{3.11}$$

We can split this integral in two parts. The nonrotating contribution yields

$$\mathcal{I}_1 = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \left(-\frac{2m}{r^3} + \frac{18mg_\star^2}{r^5}\right)\sqrt{\det\bar{g}}\,dr\,d\phi$$
$$= \frac{4m}{b} - \frac{8g_\star^2 m}{b^3}. \tag{3.12}$$

For the second integral we find

$$\mathcal{I}_2 = -\int_0^\pi \int_{\frac{b}{\sin\phi}}^\infty \left(\frac{\mathcal{S}(r,\phi)}{4r^{11}}\right)\sqrt{\det\bar{g}}\,dr\,d\phi$$
$$= \pm\frac{4ma}{b^2} \mp \frac{24mag_\star^2}{b^4}. \tag{3.13}$$

Finally the total deflection angle of rotating Bardeen regular black hole is

$$\hat{\alpha} = \frac{4m}{b} - \frac{8g_\star^2 m}{b^3} \pm \frac{4ma}{b^2} + \mathcal{O}(m,a,g_\star^2). \tag{3.14}$$

A. Geodesics equations

In this case, the Lagrangian associated with the motion of particles in the equatorial plane ($\theta = \pi/2$ and $\dot{\theta} = 0$) results to be

$$2\mathcal{L} = -\left(1 - \frac{2\mathcal{M}_b}{r}\right)\dot{t}^2 - \frac{4a\mathcal{M}_b}{r}\dot{t}\,\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2$$
$$+ \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_b}{r}\right)\dot{\phi}^2, \tag{3.15}$$

where $\dot{q} = dq/d\tau$, and τ is an affine parameter along the geodesic. Because the Lagrangian (3.15) is independent of the cyclic coordinates (t,ϕ), then their conjugate momenta (Π_t, Π_ϕ) are conserved. Then, the equations of motion can be obtained from $\dot{\Pi}_q - \frac{\partial\mathcal{L}}{\partial q} = 0$, and we obtain

$$\dot{\Pi}_t = 0, \qquad \dot{\Pi}_\phi = 0, \tag{3.16}$$

where $\Pi_q = \partial\mathcal{L}/\partial\dot{q}$ are the conjugate momenta to the coordinate q and are given by

$$\Pi_t = -\left(1 - \frac{2\mathcal{M}_b}{r}\right)\dot{t} - \frac{2a\mathcal{M}_b}{r}\dot{\phi} \equiv -E, \qquad \Pi_r = \frac{r^2}{\Delta}\dot{r}$$

$$\text{and} \quad \Pi_\phi = -\frac{2a\mathcal{M}_b}{r}\dot{t} + \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_b}{r}\right)\dot{\phi} \equiv L, \tag{3.17}$$

where E and L are dimensionless integration constants associated with each of them. Therefore, the Hamiltonian is given by

$$\mathcal{H} = \Pi_t\dot{t} + \Pi_\phi\dot{\phi} + \Pi_r\dot{r} - \mathcal{L}, \tag{3.18}$$

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \equiv -\bar{m}^2. \tag{3.19}$$

Now, by normalization, we consider $\bar{m}^2 = 0$ for photons. Thus, for photons we obtain

$$\dot{t} = \frac{-2a\mathcal{M}_b L + Er^3 + a^2 E(2\mathcal{M}_b + r)}{r(r^2 - 2\mathcal{M}_b r + a^2)}, \tag{3.20}$$

$$\dot{\phi} = \frac{2a\mathcal{M}_b E - 2\mathcal{M}_b L + rL}{r(r^2 - 2\mathcal{M}_b r + a^2)}, \tag{3.21}$$
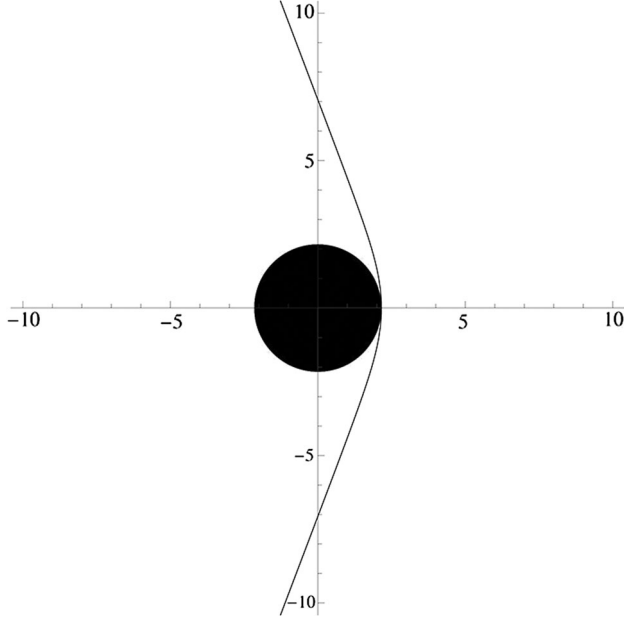
FIG. 3. Deflection of light in the background of a rotating regular Bardeen black hole with $E = 2$, $L = 5$, $m = 0.3$, $g_\star = 0.1$ and $a = 0.1$. The circle corresponds to the closest approach ($r_0 = 2.154$) to the black hole.

$$\dot{r}^2 = \frac{2(L - aE)^2 \mathcal{M}_b + (a^2 E^2 - L^2) r + E^2 r^3}{r^3}. \quad (3.22)$$

The distance of the closest approach r_0 for the metric (3.1) can be obtained from $\dot{r} = 0$, which yields

$$\frac{r_0}{b} = \sqrt{1 - \left(\frac{a}{b}\right)^2 - \frac{2m}{r_0}\left(1 - \frac{a}{b}\right)^2 \left(\frac{r_0^2}{g_\star^2 + r_0^2}\right)^{3/2}}, \quad (3.23)$$

where $b = L/E$ is the impact parameter.

Therefore, the bending angle Eq. (2.51) is given by

$$\hat{\alpha} = \frac{4m}{b} - \frac{8g_\star^2 m}{b^3} \pm \frac{4ma}{b^2} + \mathcal{O}(a, m, g_\star^2), \quad (3.24)$$

where, similar to the previous case, we use the change of variables $u = r_0/r$, and next, we substitute the impact parameter given by Eq. (3.23), then, we expand in Taylor series around m, a, and $g_\star$, and finally, we consider $r_0 \approx b$. In Fig. 3 we plot the deflection of light in the background of a rotating Bardeen black hole by solving numerically Eqs. (3.21) and (3.22).

IV. DEFLECTION ANGLE BY ROTATING REGULAR HAYWARD BLACK HOLE

In this section, we investigate the deflection angle by rotating Hayward regular black hole. The spacetime metric of the rotating Hayward regular black hole reads [62]:

$$ds^2 = -\left(1 - \frac{2\mathcal{M}_h r}{\Sigma}\right)dt^2 - \frac{4a\mathcal{M}_h r\sin^2\theta}{\Sigma}dt d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_h r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \quad (4.1)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad (4.2)$$

$$\Delta = r^2 - 2\mathcal{M}_h r + a^2, \quad (4.3)$$

and in the equatorial plane the mass function is given by

$$\mathcal{M}_h = m\frac{r^3}{r^3 + g^3}, \quad (4.4)$$

where g is the rotation parameter. We find the following expression for the optical metric:

$$\mathcal{F}\left(r, \phi, \frac{dr}{dt}, \frac{d\phi}{dt}\right) = \sqrt{\frac{r^4}{\Delta(\Delta - a^2)}\left(\frac{dr}{dt}\right)^2 + \frac{r^4\Delta}{\Delta - a^2}\left(\frac{d\phi}{dt}\right)^2} - \frac{2mar}{\Delta - a^2}\left(\frac{r^3}{r^3 + g^3}\right)\frac{d\phi}{dt}, \quad (4.5)$$

with metric components,

$$\bar{g}_{rr} = -\frac{2[-2(m + \frac{r}{4})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2} + amr^2\sin^6\phi]}{r(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}} + \frac{2mg^3(ar^2\sin^6\phi - 2(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2})}{r^4(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}},$$

$$\bar{g}_{r\phi} = \frac{2ma\cos^3\phi}{r\left(\frac{\sin^4\phi r^2 + b^2\cos^2\phi}{b^2}\right)^{3/2}} - \frac{2ma\cos^3\phi g^3}{r^4\left(\frac{\sin^4\phi r^2 + b^2\cos^2\phi}{b^2}\right)^{3/2}}, \quad (4.6)$$

$$\bar{g}_{\phi\phi} = \frac{2r[-2amr^2\sin^6\phi - 3ab^2m\cos^2\phi\sin^2\phi + (m + \frac{r}{2})(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}} \tag{4.7}$$

$$- \frac{2mg^3[-2ar^2\sin^6\phi - 3ab^2\cos^2\phi\sin^2\phi + (\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}]}{r^2(\sin^4\phi r^2 + b^2\cos^2\phi)^{3/2}}. \tag{4.8}$$

The determinant is given by

$$\det\bar{g} = r^2 - \frac{6rm[a\sin^2\phi - \sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}]}{\sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}} + \frac{6g^3m[a\sin^2\phi - \sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}]}{r^2\sqrt{\cos^4\phi r^2 + (b^2 - 2r^2)\cos^2\phi + r^2}}. \tag{4.9}$$

Our computation reveals the following relation:

$$\mathcal{K} = -2\frac{m}{r^3} + 20\frac{mg^3}{r^6} + \frac{36am\mathcal{H}(r,\phi)}{r^6}, \tag{4.10}$$

in which

$$\mathcal{H}(r,\phi) = \frac{\sin^2\phi}{(\sin^4\phi r^2 + b^2\cos^2\phi)^{7/2}}\bigg[(5/4g^3r^6 - 1/6r^9)(\sin(\phi))^{12} + (-1/12g^3r^4 + 1/12r^7)b^2(\sin(\phi))^{10} + (\cos(\phi))^6b^6g^3$$

$$+ 10/3r^4b^2(\cos(\phi))^2(g^3 + 1/10r^3)(\sin(\phi))^8 + 7/6r^3(g^3 - 4/7r^3)b^3(\cos(\phi))^2(\sin(\phi))^7$$

$$+ 1/3r^2b^2(\cos(\phi))^2(g^2 + gr + r^2)(-5/2r^2(\cos(\phi))^2 + b^2)(g - r)(\sin(\phi))^6$$

$$+ 7/3r^3(g^3 - 4/7r^3)b^3(\cos(\phi))^4(\sin(\phi))^5 + 13/3r^2\left(g^3 - \frac{9r^3}{52}\right)b^4(\cos(\phi))^4(\sin(\phi))^4$$

$$+ 1/3rb^5(\cos(\phi))^4(g^3 + 1/2r^3)(\sin(\phi))^3 + (5/6g^3r^2 - 5/6r^5)b^4(\cos(\phi))^6(\sin(\phi))^2$$

$$+ 2/3rb^5(\cos(\phi))^6(g^3 + 1/2r^3)\sin(\phi)\bigg]. \tag{4.11}$$

Going through the same procedure the deflection angle can be calculated by the following integral:

$$\hat{\alpha} \simeq -\int_0^\pi \int_{\frac{b}{\sin\phi}}^\infty \left(-2\frac{m}{r^3} + 20\frac{mg^3}{r^6} + \frac{36am\mathcal{H}(r,\phi)}{r^6}\right)\sqrt{\det\bar{g}}\,dr\,d\phi. \tag{4.12}$$

After we evaluate the first integral we find

$$\mathcal{I}_1 = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \left(-2\frac{m}{r^3} + 20\frac{mg^3}{r^6}\right)\sqrt{\det\bar{g}}\,dr\,d\phi = \frac{4m}{b} - \frac{15m\pi g^3}{8b^4}. \tag{4.13}$$

The second integral gives

$$\mathcal{I}_2 = -\int_0^\pi \int_{\frac{b}{\sin\phi}}^\infty \left(\frac{36am\mathcal{H}(r,\phi)}{r^6}\right)\sqrt{\det\bar{g}}\,dr\,d\phi = \pm\frac{4ma}{b^2}. \tag{4.14}$$

Consequently the total deflection angle of rotating Hayward regular black hole results in

$$\hat{\alpha} = \frac{4m}{b} - \frac{15m\pi g^3}{8b^4} \pm \frac{4ma}{b^2}. \tag{4.15}$$

A. Geodesics equations

The Lagrangian associated with the motion of particles in the equatorial plane ($\theta = \pi/2$ and $\dot{\theta} = 0$) of a rotating regular Hayward black hole is given by

$$2\mathcal{L} = -\left(1 - \frac{2\mathcal{M}_h}{r}\right)\dot{t}^2 - \frac{4a\mathcal{M}_h}{r}\dot{t}\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 + \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_h}{r}\right)\dot{\phi}^2, \tag{4.16}$$

where $\dot{q} = dq/d\tau$, and τ is an affine parameter along the geodesic. Similar to the previous analysis the Lagrangian (4.16) is independent of the cyclic coordinates (t, ϕ). So, their conjugate momenta (Π_t, Π_ϕ) are conserved. Then, the equations of motion are obtained from $\dot{\Pi}_q - \frac{\partial \mathcal{L}}{\partial q} = 0$ and yield

$$\dot{\Pi}_t = 0, \qquad \dot{\Pi}_\phi = 0, \qquad (4.17)$$

where $\Pi_q = \partial \mathcal{L}/\partial \dot{q}$ are the conjugate momenta to the coordinate q, which is given by

$$\Pi_t = -\left(1 - \frac{2\mathcal{M}_h}{r}\right)\dot{t} - \frac{2a\mathcal{M}_h}{r}\dot{\phi} \equiv -E,$$

$$\Pi_r = \frac{r^2}{\Delta}\dot{r}, \quad \text{and}$$

$$\Pi_\phi = -\frac{2a\mathcal{M}_h}{r}\dot{t} + \left(r^2 + a^2 + \frac{2a^2\mathcal{M}_h}{r}\right)\dot{\phi} \equiv L, \quad (4.18)$$

where E and L are dimensionless integration constants associated with each of them. Thus, the Hamiltonian is given by

$$\mathcal{H} = \Pi_t \dot{t} + \Pi_\phi \dot{\phi} + \Pi_r \dot{r} - \mathcal{L}, \qquad (4.19)$$

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \equiv -\bar{m}^2. \qquad (4.20)$$

Now, by normalization, we consider $\bar{m}^2 = 0$ for photons. So, we obtain

$$\dot{t} = \frac{rE(r^2 + a^2) - 2a(L - aE)\mathcal{M}_h}{r(r^2 - 2r\mathcal{M}_h + a^2)}, \qquad (4.21)$$

$$\dot{\phi} = \frac{2aE\mathcal{M}_h + rL - 2L\mathcal{M}_h}{r(r^2 - 2r\mathcal{M}_h + a^2)}, \qquad (4.22)$$

$$\dot{r}^2 = \frac{r(E^2(a^2 + r^2) - L^2) + 2\mathcal{M}_h(L - aE)^2}{r^3}. \qquad (4.23)$$

In this case, the distance of the closest approach r_0 for the metric (3.1) obtained from $\dot{r} = 0$ yields

$$\frac{r_0}{b} = \sqrt{1 - \left(\frac{a}{b}\right)^2 - \frac{2m}{r_0}\left(1 - \frac{a}{b}\right)^2\left(\frac{mr_0^3}{g^3 + r_0^3}\right)}, \quad (4.24)$$

where $b = L/E$ is the impact parameter.

Therefore, the bending angle Eq. (2.51) yields

$$\hat{\alpha} = \frac{4m}{b} - \frac{15m\pi g^3}{8b^4} \pm \frac{4ma}{b^2}. \qquad (4.25)$$

Here we use the change of variables $u = r_0/r$; we substitute the impact parameter given by Eq. (4.24) and expand in Taylor series around m, a, and g; finally, we consider
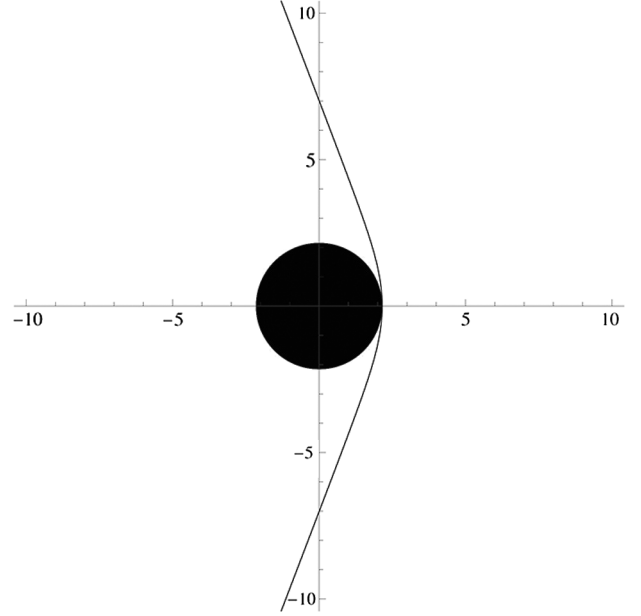

FIG. 4. Deflection of light in the background of a rotating Hayward black hole with $E = 2$, $L = 5$, $m = 0.3$, $g = 0.1$ and $a = 0.1$. The circle corresponds to the closest approach ($r_0 = 2.153$) to the black hole.

$r_0 \approx b$. In Fig. 4 we plot the deflection of light in the background of a rotating Hayward black hole by solving numerically Eqs. (4.22) and (4.23).

V. CONCLUSION

In this paper, we have investigated the deflection angle of light by rotating regular black holes such as Ayon-Beato-García, Bardeen and Hayward black hole. Starting from the physical metrics we have found the corresponding Rander-Finsler-type metric which basically provides a way to compute the deflection angle in terms of GBT. We have extended Werner's geometric method by including the electric charge Q, magnetic charge $g_\star$, and deviation parameter g which generalizes the expression for the Gaussian optical curvature, optical metric components, and finally the deflection angle.

In particular we have found that for the rotating regular black holes the total deflection angles are

$$\hat{\alpha}_{ABG} = \frac{4m}{b} - \frac{3\pi Q^2}{4b^2} \pm \frac{4ma}{b^2} + \mathcal{O}(Q^2, a, m), \qquad (5.1)$$

$$\hat{\alpha}_B = \frac{4m}{b} - \frac{8g_\star^2 m}{b^3} \pm \frac{4ma}{b^2} + \mathcal{O}(a, m, g_\star^2), \qquad (5.2)$$

$$\hat{\alpha}_H = \frac{4m}{b} - \frac{15m\pi g^3}{8b^4} \pm \frac{4ma}{b^2} + \mathcal{O}(a, m, g^3), \quad (5.3)$$

for the ABG, Bardeen and Hayward regular black holes, respectively. Thus, as these black holes have in addition to the total mass and rotation parameter, different parameters
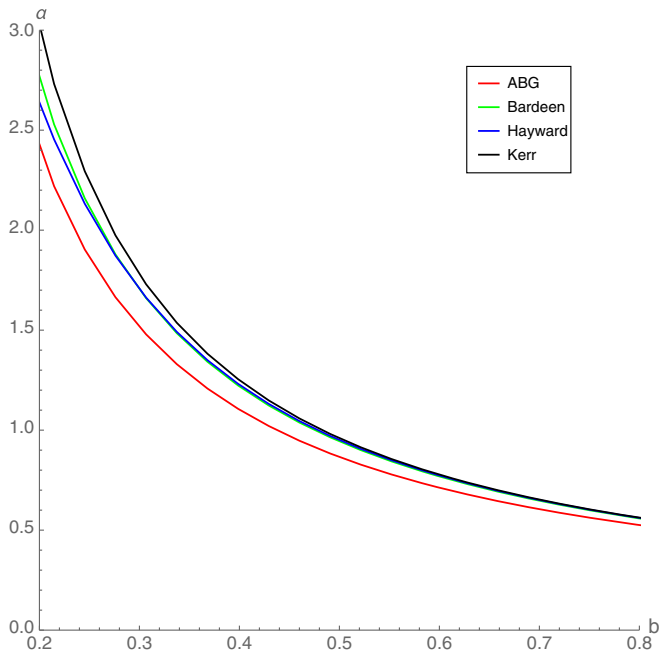
FIG. 5. Deflection of light by ABH, Bardeen, Hayward and Kerr black holes for the values of $m = Q = a = g = g_{\star} = 0.1$.

such as electric charge, magnetic charge, and deviation parameter, the deflection of light has correction terms coming from these parameters which generalizes the Kerr deflection angle,

$$\hat{\alpha}_K = \frac{4m}{b} \pm \frac{4ma}{b^2}. \qquad (5.4)$$

It is worth noting that our results show that the deflection angle is smaller than the Kerr deflection angle (see Fig. 5). In other words the contribution coming from the black hole parameters such as Q, $g_{\star}$ and g, is different in magnitude; however, in all three cases the light rays always bend

outward from the black holes which is indicated by the minus sign. In Fig. 5, we show the behavior of the deflection angle of the light for the regular black hole geometries as a function of the impact parameter. We observe that discrepancies could exist between predictions for the value of the deflection angle of the light for regular black holes for small values of the impact parameter, in which the deflection angle is smaller than the Kerr deflection angle. However, such discrepancy decreases when the impact parameter increases. In addition, we have checked our results of the deflection angle using the geodesics formalism, and we have shown it to be exact in leading order terms.

It is important to realize that the agreement between these two methods breaks down for second order terms; for example, in the case of a rotating Bardeen regular black hole the geodesic approach gives $\delta\hat{\alpha}_B = \mp 12mag_{\star}^2/b^4$, whereas with the Gauss-Bonnet theorem one finds $\delta\hat{\alpha}_B = \mp 24mag_{\star}^2/b^4$. Such inconsistency is to be expected considering the fact that one must choose a different equation for the light ray r_{γ} which incorporates the black hole parameters. We plan in the near future to extend our analytical analysis in the Gauss-Bonnet theorem to the second order terms to remove such an inconsistency.

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