Electromagnetic fields of slowly rotating magnetized compact stars in conformal gravity

Bobur Turimov,[1,*] Bobomurat Ahmedov,[1,2,†] Ahmadjon Abdujabbarov,[3,1,‡] and Cosimo Bambi[3,4,§]

[1]*Ulugh Beg Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan*
[2]*National University of Uzbekistan, Tashkent 100174, Uzbekistan*
[3]*Center for Field Theory and Particle Physics and Department of Physics, Fudan University, 2005 Songhu Road, Shanghai 200438, China*
[4]*Theoretical Astrophysics, Eberhard-Karls Universität Tübingen, Auf der Morgenstelle 10, Tübingen 72076, Germany*

(Received 26 December 2017; revised manuscript received 5 March 2018; published 6 June 2018)

In this paper we investigate the exterior vacuum electromagnetic fields of slow-rotating magnetized compact stars in conformal gravity. Assuming the dipolar magnetic field configuration, we obtain an analytical solution of the Maxwell equations for the magnetic and the electric fields outside a slowly rotating magnetized star in conformal gravity. Furthermore, we study the dipolar electromagnetic radiation and energy losses from a rotating magnetized star in conformal gravity. In order to get constraints on the L parameter of conformal gravity, the theoretical results for the magnetic field of a magnetized star in conformal gravity are combined with the precise observational data of radio pulsar period slowdown, and it is found that the maximum value of the parameter of conformal gravity is less than $L \lesssim 9.5 \times 10^5$ cm ($L/M \lesssim 5$).

DOI: 10.1103/PhysRevD.97.124005

I. INTRODUCTION

The end state of the life-cycle evolution of massive stars from several to $\sim$ hundred Solar masses through a supernova explosion may form either a neutron star or a black hole. Collapsed black hole according to the no hair theorem does not have intrinsic magnetic field (see e.g. [[1]–[5]]). On the contrary the formed neutron stars are highly magnetized objects and one of the main aims of the modern astrophysics of compact relativistic stars is to get a clear understanding of the configuration, structure and evolution of the stellar magnetic field. The precise measurements of electromagnetic signals from the radio-pulsars show that the magnetic fields of compact relativistic stars decrease in strength with the stellar age and the recycled old neutron stars have weaker magnetic fields. The strong electromagnetic field will affect observational data on high energetic processes in the vicinity of the compact star in all electromagnetic radiation spectra. Observational data of radio-pulsars and soft gamma ray repeaters (SGR) have shown that the surface magnetic field of a typical neutron star is about 10^{12} G, while for magnetars observed as SGRs and anomalous X-ray pulsars (AXP) it may reach the extreme values as 10^{15} G [6,7]. Therefore, the comparison of the evolution of magnetic fields and of the rotation observed in neutron stars with those modeled and theoretically predicted is a great challenge to get the constraints on the neutron star properties in the extreme physics regime. The continued analysis of the evolution of magnetic fields and the precise measurement of the spin of relativistic stars at various evolutionary stages is therefore necessary to get the constraints on alternate theories of gravity in the strong field regime. The electromagnetic signal detected from pulsars is mainly due the to the magneto-dipolar radiation of the rotating compact star, and energy loss due to electromagnetic radiation causes the spin-down of the relativistic star [8–16]. The structure of the pulsar magnetosphere and related astrophysical processes has been widely studied in the literature, see e.g., Refs. [17–25].

Thus the strong gravitational field regime near relativistic compact stars can play a role of laboratory to test general relativity versus other modified or alternative theories of gravity. Testing gravity theories using the strong field regime has been performed for X-ray sources from some black hole candidates [26–37]. The comparison of the electromagnetic field and radiation of the compact star with the pulsar spin down can also be used to constrain alternative theories of gravity [37,38].

The impact of strong electromagnetic fields can be observed by other astrophysical processes such as gravitational lensing, motion of test particles, and the

*bturimov@astrin.uz
†ahmedov@astrin.uz
‡ahmadjon@astrin.uz
§bambi@fudan.edu.cn

electromagnetic spectrum of accretion discs [39–44]. An analytical solution of the exterior electromagnetic field of a rotating magnetized star in the Newtonian limit has been found in [45]. Interior solutions for the electromagnetic fields of a constant magnetic density star are studied by many authors, see, for example, [46]. General relativistic corrections to the electric and magnetic field structure outside magnetized compact gravitational objects have been studied in [1] and have been further extended by a number of authors [2,10,11,24,25,47–74]. Magnetic fields of spherical compact stars in a braneworld have been studied in [75].

In this work we investigate the vacuum electromagnetic fields of slow-rotating magnetized compact stars in conformal gravity proposed in [76]. An example of Lagrangian in this large class of conformally invariant theories of gravity is

$$\mathcal{L} = \phi^2 R + 6 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{1}$$

where the scalar field ϕ (dilaton) is combined with the Ricci scalar. This Lagrangian is invariant under conformal transformations,

$$\begin{aligned} g_{\mu\nu} \to g^*_{\mu\nu} &= S g_{\mu\nu}, \\ \phi \to \phi^* &= S^{-1/2}\phi, \end{aligned} \tag{2}$$

where $S = S(x)$ is a function of the spacetime coordinates.

Because the world around us is not conformally invariant, conformal symmetry must be broken, and one of the possibilities is that it is spontaneously broken. In such a case, Nature must select one of the vacua, namely a certain gauge corresponding to a specific choice of the conformal factor S. In the symmetric phase, the theory is invariant under conformal transformations, i.e., the physics is independent of the conformal factor S. In the broken phase, the choice of the conformal factor S does lead to observational effects. Such a choice may look arbitrary, but this is a fundamental feature of any spontaneously broken symmetry, not just of conformal gravity. In what follows, we will consider the infinite family of conformal factors found in [76] because they have the property to solve the singularity problem in the Kerr metric.

In the paper [77] the quasinormal modes of the scalar fields of a black hole in conformal gravity have been studied. The energy conditions of a black hole in conformal gravity have been studied in [78]. Conformal invariance preservation at the quantum level has been discussed in [79].

The present paper is organized as follows. Section II is devoted to the vacuum electromagnetic fields of a rotating magnetized compact star in conformal gravity, and we present an exact analytical solution of the general relativistic Maxwell equations for the magnetic and the electric fields of a slow-rotating neutron star in conformal gravity. In Sec. III, we calculate the energy losses from a slow-rotating neutron star in conformal gravity. In Sec. IV, we obtain astrophysical constraints on the value of the parameter of conformal gravity, L, from the comparison with current observational data. Finally, in Sec. V, we summarize our results and give a future outlook related to the present work. Throughout the paper, all physical quantities are denoted with "*". We use a space-like signature $(-,+,+,+)$, a system of units in which $G = c = 1$ and, we restore them when we need to compare our results with observational data. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

II. VACUUM ELECTROMAGNETIC FIELDS OF A ROTATING MAGNETIZED COMPACT STAR IN CONFORMAL GRAVITY

In this section we briefly discuss the electromagnetic fields in the spacetime of a magnetized compact star in conformal gravity. One of the most difficult mathematical problems is to solve the Einstein-Maxwell equations, which are coupled nonlinear differential equations, but one can solve them in some approximation when the electromagnetic field does not affect the spacetime around the compact star (see e.g., [10,11] for more details). Assuming that the electromagnetic field and its energy are too small to change the spacetime geometry around the compact star, we consider the electromagnetic field in the fixed spacetime geometry and investigate the effects of the background gravitational field on the electromagnetic field of the slow-rotating relativistic star in conformal gravity.

The spacetime of the most rapidly rotating compact (neutron) stars observed as millisecond pulsars can be described within the slow rotation limit [80]. In Boyer-Lindquist coordinates (t, r, θ, ϕ) the spacetime outside of the slowly rotating magnetized star in conformal gravity can be expressed by the following line element [76]:

$$\begin{aligned} ds^{*2} = S(r)\bigg[&-N^2 dt^2 + \frac{1}{N^2} dr^2 + r^2 d\theta^2 \\ &+ r^2 \sin^2\theta d\phi^2 - 2\omega(r) r^2 \sin^2\theta dt d\phi\bigg], \end{aligned} \tag{3}$$

with

$$N^2(r) = 1 - \frac{2M}{r}, \qquad r \geq R,$$

where M is the total mass and R is the radius of the compact star. $\omega(r) = 2aM/r^3$ is the angular velocity of the dragging of inertial frames, and a is the specific angular momentum, which is defined as $a = J/M$, and $J = I\Omega$ is the total angular momentum, with the moment of inertia I and the angular velocity Ω (or the period $P = 2\pi/\Omega$ of the star), which are very important and precisely measurable quantities/parameters in the observation of pulsars.

The function $S(r)$ in Eq. (3) is the scaling factor and, in the slow-rotating limit, has the form [76]

$$S(r) = \left(1 + \frac{L^2}{r^2}\right)^{2(n+1)}, \qquad n = 1, 2, 3, \ldots, \tag{4}$$

where L is a parameter with dimensions of a length and n is an integer positive number. The theory does not provide any prediction for the value of L, so we can expect that L is either on the order of the Planck length, $L \sim L_{\rm P}$, or on the order of the gravitational radius of the object, $L \sim M$, as these are the only two length scales of the system [76]. The first option is realized with the scale already present in the action, whereas the latter is with the scale that breaks conformal symmetry on-shell. *A priori*, both scenarios are possible and natural. In the present paper, we will consider the second option with $L \sim M$, as it is the only one with potential astrophysical implications in compact objects. If $L \sim L_{\rm P}$, modifications of Einstein's gravity would only show up in high-energy/high-curvature regimes. The choice of n is related to the symmetry breaking. As in any spontaneously broken symmetry, we cannot say why Nature selects a particular vacuum in the class of good vacua. In our work, we consider the simplest case, $n = 1$, and we briefly describe how our results change for larger values of n.

In the present paper we will investigate the electromagnetic properties of slow-rotating magnetized compact stars in conformal gravity. In order to study the electromagnetic fields of the compact star, one has to find the solutions of the general relativistic Maxwell equations which can be written as in [10,81].

Stellar Model:

Before doing any calculation, we list the stellar model assumptions.

(i) The magnetic moment of the star does not vary in time as a result of the high electrical conductivity of the stellar medium $\sigma \to \infty$; see e.g., [81].

(ii) In the case of the slow-rotating limit, one can consider only the linear approximation of the angular velocities as follows $\mathcal{O}(\omega)$ and $\mathcal{O}(\Omega)$, respectively.

(iii) The star has a spherical shape in the slow-rotating approximation. There is not a deformation due to rotation.

(iv) The medium outside of the star is vacuum.

(v) One can look for the stationary solutions of the Maxwell equations for the components of the magnetic field in the following form [10,81]

$$B^{\hat{r}}(r,\theta,\phi,t) = F^*(r) \times [\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda], \tag{5}$$

$$B^{\hat{\theta}}(r,\theta,\phi,t) = G^*(r) \times [\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda], \tag{6}$$

$$B^{\hat{\phi}}(r,\theta,\phi,t) = H^*(r) \times \sin\chi\sin\lambda, \quad \lambda = \phi - \Omega t, \tag{7}$$

where the unknown functions $F^*(r)$, $G^*(r)$, and $H^*(r)$ are corrections to the magnetic field due to general relativity and conformal gravity and χ is the inclination angle of the magnetic field relative to the stellar rotation axis.

In the paper [5], such a consideration has already been performed in the general relativistic case, and the expressions for the stationary vacuum electromagnetic fields of a slow-rotating relativistic star have been clearly shown. Following the techniques used in [5], we find the relations for the electromagnetic fields of a slow-rotating compact star in conformal gravity that are distinguished by the scaling factor $S(r)$ in comparison with the general relativistic ones. One can simply write them in the following form:

$$(B^{\hat{i}}, E^{\hat{i}})_{\rm CG} = \frac{1}{S}(B^{\hat{i}}, E^{\hat{i}})_{\rm GR} \qquad (i = 1, 2, 3), \tag{8}$$

or

$$(\mathbf{B}, \mathbf{E})_{\rm CG} = \frac{1}{S}(\mathbf{B}, \mathbf{E})_{\rm GR}, \tag{9}$$

where the vectors $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and the electric fields, respectively. Collecting all the statements which are introduced here, one can easily find the profile functions $F^*(r)$, $G^*(r)$, and $H^*(r)$ in the expressions (5)–(7) for the components of the magnetic field in the following form (see e.g., Ref. [5])

$$F^*(r) = -\frac{3\mu}{4M^3 S}\left[\ln N^2 + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right], \tag{10}$$

$$G^*(r) = H^*(r) = \frac{3\mu N}{4rM^2 S}\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right], \tag{11}$$

where μ is the magnetic dipole moment of the magnetized compact star. From the astrophysical point of view, the electric field of compact stars (pulsars and magnetars) is at least V/c times weaker than the stellar magnetic field, where V is the linear velocity of the neutron star surface. Analytical expressions for the electric field are given in Appendix.

Hereafter, introducing the normalized dimensionless radial coordinate $\eta = r/R$ and assuming zero inclination angle $\chi = 0$, one can write the exact solutions for the components of the magnetic field (5)–(7) in the following form

$$B^{\hat{r}}(\eta,\theta) = -\frac{3B_0}{\epsilon^3 S}\left[\ln N^2 + \frac{\epsilon}{\eta}\left(1 + \frac{\epsilon}{2\eta}\right)\right]\cos\theta, \tag{12}$$

$$B^{\hat{\theta}}(\eta,\theta) = \frac{3B_0 N}{\eta\epsilon^2 S}\left[\frac{2\eta}{\epsilon}\ln N^2 + \frac{1}{N^2} + 1\right]\sin\theta, \tag{13}$$

$$B^{\hat{\phi}}(\eta,\theta) = 0, \tag{14}$$

where $B_0 = 2\mu/R^3$ is the Newtonian value of the magnetic field at the polar cap of the star, $\epsilon = 2M/R$ is the compactness of the star, and $N^2(\eta) = 1 - \epsilon/\eta$ is the lapse function. The scaling factor can be rewritten in terms of the normalized dimensionless radial coordinate in the following form:

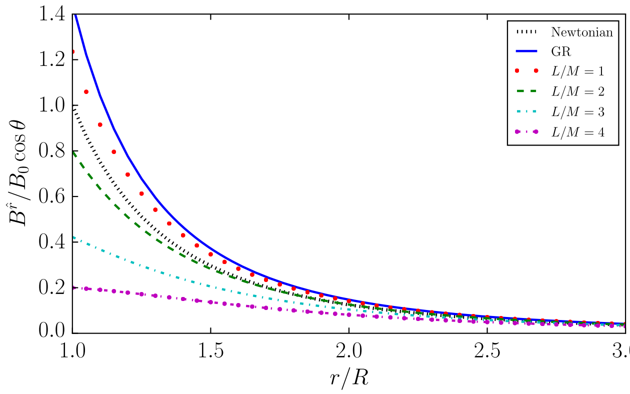

FIG. 1. Normalized dimensionless radial r/R dependence of the radial component of the magnetic field $B^{\hat{r}}/B_0 \cos\theta$ in conformal gravity for the compactness $\epsilon = 0.3$ with zero inclination angle $\chi = 0$.

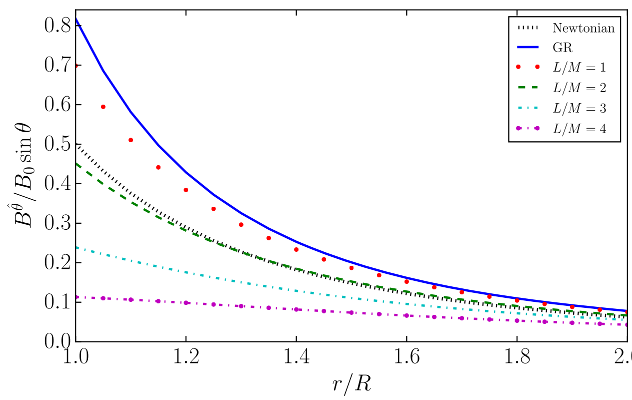

FIG. 2. Normalized dimensionless radial r/R dependence of the tangential component of the magnetic field $B^{\hat{\theta}}/B_0 \sin\theta$ in conformal gravity for the compactness $\epsilon = 0.3$ with zero inclination angle $\chi = 0$.

$$S(\eta) = \left[1 + \frac{\epsilon^2}{4\eta^2}\left(\frac{L}{M}\right)^2\right]^{2(n+1)}, \qquad n = 1, 2, 3, \ldots, \quad (15)$$

and in what follows we will focus on the scenario in which L can be on the order of the gravitational radius of the system, hence also on the order of the stellar radius R. Note that the scaling factor is always greater than 1 ($S \geq 1$). This means, without doing any calculations, that one can conclude that the magnetic field of the compact star decreases in conformal gravity. More precisely, Figs. 1 and 2 show the normalized radial dependence of the radial and the tangential components of the magnetic field described by Eqs. (12) and (13) of a relativistic star in conformal gravity when $n = 1$. One can easily see that in both cases the components of the magnetic field strength are lowered by increasing the dimensional parameter L, which means the magnetic field of a relativistic star decreases in the spacetime of conformal gravity.

III. ASTROPHYSICAL APPLICATION

In this section, we will briefly study the electromagnetic dipole radiation from a rotating magnetized neutron star in conformal gravity. Note that such a phenomenon is at the basis of the observational evidence of radio pulsars identified with the rotating magnetized neutron stars. In the case of pure electromagnetic radiation, the luminosity of the magnetized star in conformal gravity can be calculated as [11]

$$L^*_{\rm em} = \frac{\Omega_R^{*4} R^6}{6c^3} B_R^{*2} \sin^2\chi, \quad (16)$$

where Ω^*_R is the angular velocity in the observer's frame and B^*_R is the value of the magnetic field strength at the surface of the star:

$$\Omega^*_R = \frac{\Omega}{\sqrt{S_R(1-\epsilon)}}, \quad (17)$$

and

$$B^*_R = B_0 \frac{f}{S_R}, \quad (18)$$

with

$$f = -\frac{3}{\epsilon^3}\left[\epsilon\left(1 + \frac{\epsilon}{2}\right) + \ln(1-\epsilon)\right], \quad (19)$$

where the subscript R indicates the value at $r = R$. From Eq. (16), one can easily see that the luminosity of a rotating magnetized neutron star in conformal gravity is decreased due to the decrease of the magnetic field strength and by the gravitational redshift of the effective rotational angular velocity Ω^*_R.

In the case of pure dipole electromagnetic radiation, the Newtonian value of the luminosity has the following form [82]

$$L_{0\rm em} = \frac{\Omega^4 R^6}{6c^3} B_0^2 \sin^2\chi. \quad (20)$$

In order to calculate the rate of the energy loss from the radio pulsar through dipolar electromagnetic radiation in conformal gravity, one has to consider the ratio of the luminosity in Newtonian and in conformal gravity [6]

$$\frac{L^*_{\rm em}}{L_{0\rm em}} = \left(\frac{f}{1-\epsilon}\right)^2 \left[1 + \frac{\epsilon^2}{4}\left(\frac{L}{M}\right)^2\right]^{-8(n+1)}. \quad (21)$$

The dependence of the rate of energy loss from the compactness of the magnetized neutron star in conformal gravity for different values of the parameter L/M is illustrated in Fig. 3. The plot shows the increase of the rate of energy loss with the increase of the compactness of the star.

Figure 4 shows the dependence of the rate of energy loss of a magnetized neutron star in conformal gravity from the module of the parameter L/M for different values of the compactness ϵ of the star.

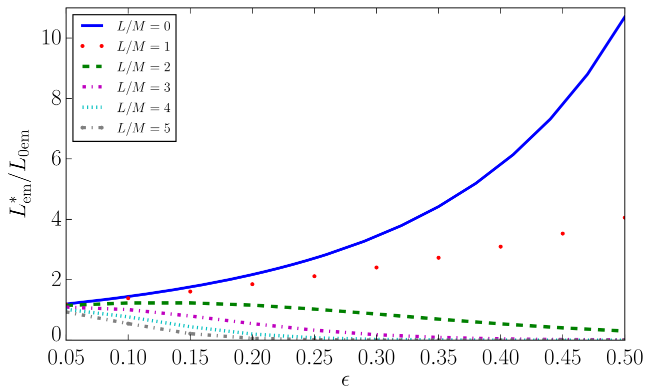

FIG. 3. Dependence of the energy losses $L^*_{\rm em}/L_{0\rm em}$ from the compactness ϵ of the star for different values of the parameter L/M.

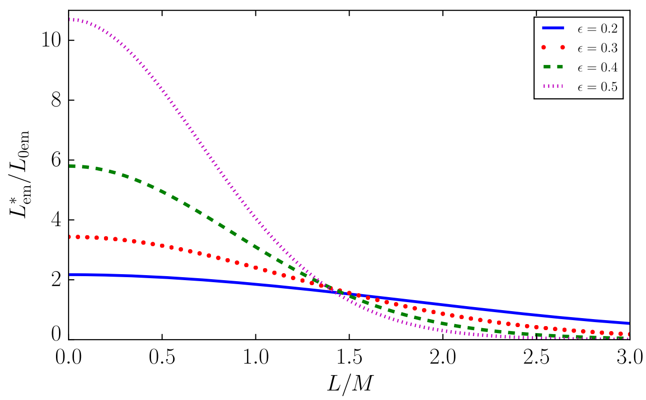

FIG. 4. Dependence of the energy losses $L^*_{\rm em}/L_{0\rm em}$ from the parameter L/M for different values of the compactness ϵ of the star.

IV. RESULTS AND DISCUSSION

Now one can get constraints on the conformal parameter L by comparing the obtained theoretical results on electromagnetic radiation from the rotating magnetized star in conformal gravity with the observational data on spin down for the well known rotating magnetized compact stars and magnetars observed as radio pulsars and SGRs/AXPs. In order to get the upper limit for the parameter L, one can consider the $P-\dot{P}$ diagram for typical pulsars [9,83–87]. From the observation data [62] shown in Fig. 5, one can see that the average value of the magnetic field strength for a typical radio pulsar is about $B_{\rm Av}=B_0\simeq 10^{12}$ G, its period is $P\simeq 1$ s, the period derivative is about $\dot{P}\simeq 10^{-15}\ {\rm s\,s^{-1}}$, and the lowest value of the magnetic field strength in observation is around $B^*_R=B_{\rm Low}\simeq 10^{11}$ G (with $P\simeq 1$ s and $\dot{P}\ \simeq 10^{-17}\,{\rm s s^{-1}}$). Using these observational values and the magnetodipolar formula (18) one can find the upper limit for the value of the parameter as $L\lesssim 9.5\times 10^5$ cm ($L/M\lesssim 5$) for $n=1$. This statement is in agreement with Figs. 6 and 7 on the dependence of the magnetic field at the surface of the NS from the parameter L/M for different values of the compactness of the star.

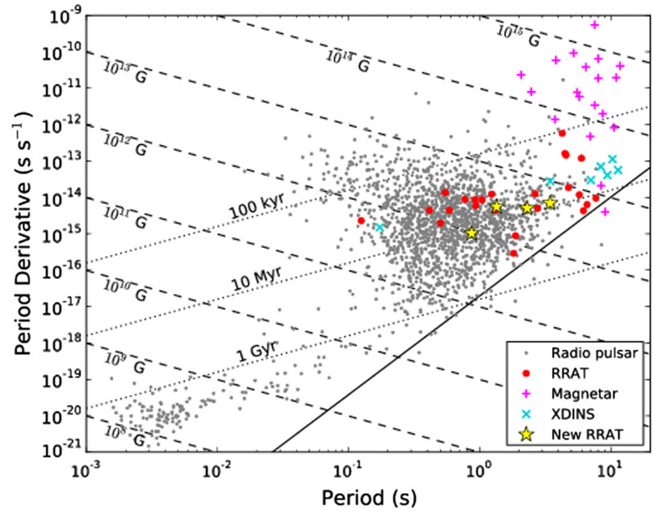

FIG. 5. $P-\dot{P}$ diagram for the observable pulsars and magnetars from the paper [88].

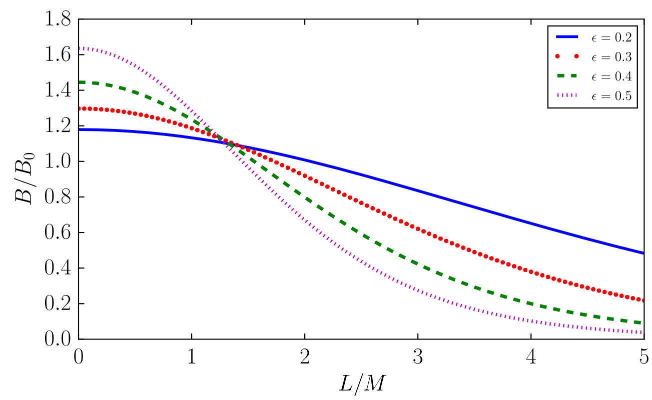

FIG. 6. Dependence of the ratio of the magnetic field from the parameter L/M for different values of the compactness ϵ.

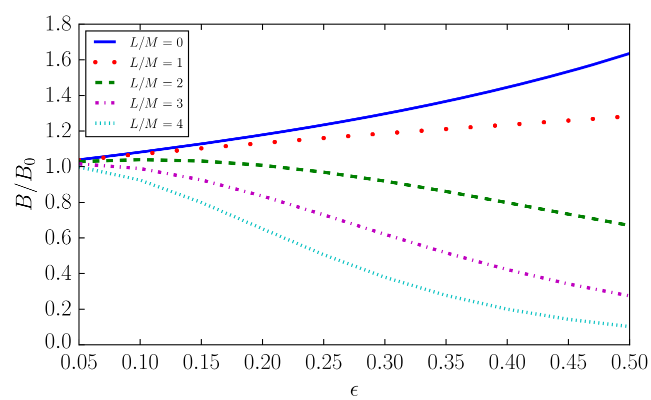

FIG. 7. Dependence of the ratio of the magnetic field from the compactness of the star ϵ for different values of the parameter L/M.

TABLE I. Dependence of the parameters n and L/M after comparison of magnetodipolar formula (18) with observation data for the value of the compactness $\epsilon \simeq 0.4$.

n	1	2	3	5	10	20	100
$L/$M	4.87	3.74	3.15	2.49	1.79	1.28	0.58

In the Table I, dependence of the model parameters n and L/M is obtained on comparison of the the magnetodipolar formula (18) with the observational data on spin down of the radio pulsars.

V. SUMMARY

In the present work we have investigated the modifications of the electromagnetic fields of a rotating magnetized compact star arising from the parameters of the conformal gravity and their astrophysical implications to the neutron stars observed as pulsars. We have studied the general relativistic Maxwell equations for the dipolar electromagnetic fields of a slowly rotating magnetized compact star in terms of the parameter of conformal gravity. and then obtained the analytical solution for the dipolar magnetic field in terms of the parameter L. Along with the magnetic field, we have obtained the analytical expressions for the electric field of a rotating magnetized star in conformal gravity.

As an important application of the obtained results, we have calculated energy losses of slow rotating magnetized neutron stars in conformal gravity through magneto-dipolar radiation and found that the rotating star with nonzero L parameter will lose less energy when compared to a rotating neutron star in general relativity. This permits us to check the effects of the scaling factor arising from the conformal gravity in the vicinity of a rotating magnetized star, especially, when one calculates the electromagnetic luminosity from the star. The latter is a very important measurable quantity in pulsar astrophysics. The obtained dependence has been combined with the astrophysical data on precise measurement of pulsar period slowdown in order to constrain the L parameter. We have found the upper limit for the parameter of conformal gravity as $L \lesssim 9.5 \times 10^5$ cm.

ACKNOWLEDGMENTS

This research is supported by Grants No. VA-FA-F-2-008 and No. YFA-Ftech-2018-8 of the Uzbekistan Ministry for Innovation Development, and by the Abdus Salam International Centre for Theoretical Physics through Grant No. OEA-NT-01. This research is partially supported by an Erasmus+exchange grant between SU and NUUz. C. B. acknowledges support from the National Natural Science Foundation of China (Grant No. U1531117), Fudan University (Grant No. IDH1512060), and the Alexander von Humboldt Foundation. B. A. would like to acknowledge Nazarbayev University, Astana, Kazakhstan for the warm hospitality through support from ORAU Grant No. SST 2015021. B. A. would like to acknowledge the support of the German Academic Exchange Service DAAD for supporting his stay at Frankfurt University.

APPENDIX: ELECTRIC FIELD OF A COMPACT STAR IN CONFORMAL GRAVITY

The components of the electric field can be chosen in the following form [5]

$$E^{\hat{r}}(r,\theta,\phi,t) = [f^*(r) + f_3^*(r)]\cos\chi(3\cos^2\theta - 1) + 3[g_1^*(r) + g_3^*(r)]\sin\chi\sin\theta\cos\theta\cos\lambda, \tag{A1}$$

$$E^{\hat{\theta}}(r,\theta,\phi,t) = [f_2^*(r) + f_4^*(r)]\cos\chi\sin\theta\cos\theta + [g_2^*(r) + g_4^*(r)]\sin\chi\cos\lambda - [g_5^*(r) + g_6^*(r)]\cos 2\theta\sin\chi\cos\lambda, \tag{A2}$$

$$E^{\hat{\phi}}(r,\theta,\phi,t) = [g_5^*(r) + g_6^*(r)]\sin\chi\cos\theta\sin\lambda - [g_2^*(r) + g_4^*(r)]\sin\chi\cos\theta\sin\lambda, \tag{A3}$$

where $\{f_i^*(r)\}$ and $\{g_i^*(r)\}$ are functions of the radial coordinate r. The explicit form of the profile functions in the spacetime of conformal gravity is given by [5]

$$f_1^*(r) = \frac{\mu\Omega C^* C_1^*}{6cR^2S}\left[\frac{2M^2}{3r^2} + \frac{2M}{r} - 4 + \left(3 - \frac{2r}{M}\right)\ln N^2\right], \tag{A4}$$

$$f_2^*(r) = -\frac{\mu\Omega C^* C_1^*}{cR^2S} N\left[\left(1 - \frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right], \tag{A5}$$

$$f_3^*(r) = \frac{15\mu\omega r^3}{16cM^5S}\left\{C_3^*\left[\frac{2M^2}{3r^2} + \frac{2M}{r} - 4 + \left(3 - \frac{2r}{M}\right)\ln N^2\right] + \frac{2M^2}{5r^2}\ln N^2 + \frac{2M^3}{5r^3}\right\}, \tag{A6}$$

$$f_4^*(r) = -\frac{45\mu\omega r^3}{8cM^5S} N\left\{C_3^*\left[\left(1 - \frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] + \frac{M^4}{15r^4N^2}\right\}, \tag{A7}$$

$$g_2^*(r) = \frac{3\mu\Omega r}{8cM^3NS}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right], \tag{A8}$$

$$g_4^*(r) = -\frac{3\mu\omega r}{8cM^3NS}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]. \tag{A9}$$

where the constants of integration C^*, C_1^*, and C_3^* can be found from the boundary conditions. The other functions are related to those above as follows [5]:

$$g_1^* = f_1^*, \qquad g_3^* = f_3^*, \qquad g_5^* = \frac{1}{2}f_2^*, \qquad g_6^* = \frac{1}{2}f_4^*.$$

[1] L. M. Ozernoi and V. L. Ginzburg, Zh. Eksp. Teor. Fiz. **47**, 1030 (1964).
[2] J. L. Anderson and J. M. Cohen, Astrophys. Space Sci. **9**, 146 (1970).
[3] Ya. B. Zeldovich and I. D. Novikov, *Relativistic astrophysics. Vol. 1: Stars and relativity* (University of Chicaho Press, Chicago, 1971).
[4] R. Price, Phys. Rev. D **5**, 2419 (1972)
[5] K. Thorne, in *Magic Without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972), p. 231.
[6] R. C. Duncan and C. Thompson, Astrophys. J. Lett. **392**, L9 (1992).
[7] C. Thompson and R. C. Duncan, Mon. Not. R. Astron. Soc. **275**, 255 (1995).
[8] C. Thompson, M. Lyutikov, and S. R. Kulkarni, Astrophys. J. **574**, 332 (2002).
[9] T. A. Thompson, P. Chang, and E. Quataert, Astrophys. J. **611**, 380 (2004).
[10] L. Rezzolla, B. J. Ahmedov, and J. C. Miller, Mon. Not. R. Astron. Soc. **322**, 723 (2001).
[11] L. Rezzolla and B. J. Ahmedov, Mon. Not. R. Astron. Soc. **352**, 1161 (2004).
[12] V. S. Morozova, B. J. Ahmedov, A. A. Abdujabbarov, and A. I. Mamadjanov, Astrophys. Space Sci. **330**, 257 (2010).
[13] V. S. Morozova and B. J. Ahmedov, Astrophys. Space Sci. **333**, 133 (2011).
[14] B. J. Ahmedov, B. B. Ahmedov, and A. A. Abdujabbarov, Astrophys. Space Sci. **338**, 157 (2012).
[15] B. J. Ahmedov, A. A. Abdujabbarov, and D. B. Fayzullaev, Astrophys. Space Sci. **346**, 507 (2013).
[16] J. R. Rayimbaev, B. J. Ahmedov, N. B. Juraeva, and A. S. Rakhmatov, Astrophys. Space Sci. **356**, 301 (2015).
[17] P. Goldreich and W. H. Julian, Astrophys. J. **157**, 869 (1969).
[18] P. A. Sturrock, Astrophys. J. **164**, 529 (1971).
[19] L. Mestel, Nature **233**, 149 (1971).
[20] M. A. Ruderman and P. G. Sutherland, Astrophys. J. **196**, 51 (1975).
[21] J. Arons and E. T. Scharlemann, Astrophys. J. **231**, 854 (1979).
[22] A. G. Muslimov and A. I. Tsygan, Sov. Astron. **34**, 133 (1990).
[23] A. G. Muslimov and A. I. Tsygan, *The Magnetospheric Structure and Emission Mechanisms of Radio Pulsars*, edited by T. Hankins, J. Rankin, and J. Gil (Kluwer, Dordrecht, 1991), IAU Colloq. No. 128, p. 340.
[24] A. G. Muslimov and A. I. Tsygan, Mon. Not. R. Astron. Soc. **255**, 61 (1992).
[25] A. Muslimov and A. K. Harding, Astrophys. J. **485**, 735 (1997).
[26] Z. Cao, S. Nampalliwar, C. Bambi, T. Dauser, and J. A. Garcia, Phys. Rev. Lett. **120**, 051101 (2018).
[27] C. Bambi, J. Phys. Conf. Ser. **942**, 012004 (2017).
[28] C. Bambi, A. Cárdenas-Avendaño, T. Dauser, J. A. García, and S. Nampalliwar, Astrophys. J. **842**, 76 (2017).
[29] H. Zhang, M. Zhou, C. Bambi, B. Kleihaus, J. Kunz, and E. Radu, Phys. Rev. D **95**, 104043 (2017).
[30] C. Bambi, Rev. Mod. Phys. **89**, 025001 (2017).
[31] C. Bambi, Z. Cao, and L. Modesto, Phys. Rev. D **95**, 064006 (2017).
[32] M. Ghasemi-Nodehi and C. Bambi, Phys. Rev. D **94**, 104062 (2016).
[33] C. Bambi and S. Nampalliwar, Europhys. Lett. **116**, 30006 (2016).
[34] A. Abdujabbarov and B. Ahmedov, Phys. Rev. D **81**, 044022 (2010).
[35] A. Abdujabbarov, B. Ahmedov, and A. Hakimov, Phys. Rev. D **83**, 044053 (2011).
[36] A. Hakimov, A. Abdujabbarov, and B. Narzilloev, Int. J. Mod. Phys. A **32**, 1750116 (2017).
[37] A. Hakimov, A. Abdujabbarov, and B. Ahmedov, Phys. Rev. D **88**, 024008 (2013).
[38] B. V. Turimov, B. J. Ahmedov, and A. A. Hakimov, Phys. Rev. D **96**, 104001 (2017).
[39] I. D. Novikov and K. S. Thorne, in *Black Holesv*, edited by C. Dewitt and B. S. Dewitt (Gordon and Breach, New York, 1973), pp. 343–450.
[40] N. I. Shakura and R. A. Sunyaev, Astron. Astrophys. **24**, 337 (1973).
[41] D. N. Page and K. S. Thorne, Astrophys. J. **191**, 499 (1974).
[42] K. S. Thorne, Astrophys. J. **191**, 507 (1974).

[43] C. Bambi and L. Modesto, Phys. Lett. B **721**, 329 (2013).
[44] D. Pugliese and Z. Stuchlík, Astrophys. J. Suppl. Ser. **223**, 27 (2016).
[45] A. J. Deutsch, Ann. Astrophys. **18**, 1 (1955).
[46] R. Ruffini and A. Treves, Astrophys. Lett. **13**, 109 (1973).
[47] J. A. Petterson, Phys. Rev. D **10**, 3166 (1974).
[48] I. Wasserman and S. L. Shapiro, Astrophys. J. **265**, 1036 (1983).
[49] A. R. Prasanna and A. Gupta, Nuovo Cimento B **112**, 1089 (1997).
[50] U. Geppert, D. Page, and T. Zannias, Phys. Rev. D **61**, 123004 (2000).
[51] D. Page, U. Geppert, and T. Zannias, Astron. Astrophys. **360**, 1052 (2000).
[52] Y. Kojima, N. Matsunaga, and T. Okita, Mon. Not. R. Astron. Soc. **348**, 1388 (2004).
[53] L. Rezzolla and B. J. Ahmedov, Mon. Not. R. Astron. Soc. **459**, 4144 (2016).
[54] V. S. Morozova, L. Rezzolla, and B. J. Ahmedov, Phys. Rev. D **89**, 104030 (2014).
[55] L. Rezzolla, B. J. Ahmedov, and J. C. Miller, Found. Phys. **31**, 1051 (2001).
[56] A. Abdujabbarov, B. Ahmedov, O. Rahimov, and U. Salikhbaev, Phys. Scr. **89**, 084008 (2014).
[57] A. A. Abdujabbarov, B. J. Ahmedov, and N. B. Jurayeva, Phys. Rev. D **87**, 064042 (2013).
[58] A. A. Abdujabbarov, A. A. Tursunov, B. J. Ahmedov, and A. Kuvatov, Astrophys Space Sci **343**, 173 (2013).
[59] B. B. Ahmedov, Astrophys. Space Sci. **331**, 565 (2011).
[60] V. S. Morozova, B. J. Ahmedov, and O. Zanotti, Mon. Not. R. Astron. Soc. **444**, 1144 (2014).
[61] O. Zanotti, V. Morozova, and B. Ahmedov, Astron. Astrophys. Rev. **540**, A126 (2012).
[62] V. S. Morozova, B. J. Ahmedov, and O. Zanotti, Mon. Not. R. Astron. Soc. **419**, 2147 (2012).
[63] V. S. Morozova, B. J. Ahmedov, and O. Zanotti, Mon. Not. R. Astron. Soc. **408**, 490 (2010).
[64] B. J. Ahmedov and V. S. Morozova, Astrophys. Space Sci. **319**, 115 (2009).
[65] V. S. Morozova, B. J. Ahmedov, and V. G. Kagramanova, Astrophys. J. **684**, 1359 (2008).
[66] E. B. Abdikamalov, B. J. Ahmedov, and J. C. Miller , Mon. Not. R. Astron. Soc. **395**, 443 (2009).
[67] S. E. Gralla, A. Lupsasca, and A. Philippov , Astrophys. J. **833**, 258 (2016).
[68] Y. Kojima and Y. E. Kato , Prog. Theor. Exp. Phys. **2014**, 23E01 (2014).
[69] Y. Kojima, Mon. Not. R. Astron. Soc. **477**, 3530 (2018).
[70] V. Karas, J. Kovar, O. Kopacek, Y. Kojima, P. Slany, and Z. Stuchlik, in *Regular and Chaotic Motion in General Relativity. Case of Magnetized Black Hole and a Massive Magnetic Dipole* (American Astronomical Society, 2012), Vol. 220, p. 430.07.
[71] B. V. Turimov, B. J. Ahmedov, and A. A. Abdujabbarov, Modern Physic Letters A **24**, 733 (2009).
[72] B. V. Turimov, Int. J. Mod. Phys. D, DOI: 10.1142/S021827181850092X (2018).
[73] J. Kovar, Z. Stuchlik, and V. Karas, Classical Quantum Gravity **25**, 095011 (2008).
[74] J. Kovar, O. Kopacek, V. Karas, and Z. Stuchlik, Classical Quantum Gravity **27**, 135006 (2010).
[75] B. J. Ahmedov and F. J. Fattoyev, Phys. Rev. D **78**, 047501 (2008).
[76] C. Bambi, L. Modesto, and L. Rachwał, J. Cosmol. Astropart. Phys. 05 (2017) 003.
[77] B. Toshmatov, C. Bambi, B. Ahmedov, Z. Stuchlík, and J. Schee, Phys. Rev. D **96**, 064028 (2017).
[78] B. Toshmatov, C. Bambi, B. Ahmedov, A. Abdujabbarov, and Z. Stuchlík, Eur. Phys. J. C **77**, 542 (2017).
[79] L. Modesto and L. Rachwał, Nucl. Phys. **B889**, 228 (2014).
[80] J. B. Hartle and K. S. Thorne, Astrophys. J. **153**, 807 (1968).
[81] L. Rezzolla, F. K. Lamb, D. Marković, and S. L. Shapiro, Phys. Rev. D **64**, 104013 (2001).
[82] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields, Course of Theoretical Physics, Volume 2* (Elsevier Butterworth-Heinemann, Oxford, 2004).
[83] O. H. Guseinov, A. Ankay, and S. O. Tagieva, Int. J. Mod. Phys. D **13**, 1805 (2004).
[84] K. Chen and M. Ruderman, Astrophys. J. **408**, 179 (1993).
[85] S. Johnston and A. Karastergiou, Mon. Not. R. Astron. Soc. **467**, 3493 (2017).
[86] J. P. Ridley and D. R. Lorimer, Mon. Not. R. Astron. Soc. **404**, 1081 (2010).
[87] M. C. Miller and J. M. Miller, Phys. Rep. **548**, 1 (2015).
[88] C. Karako-Argaman, V. M. Kaspi, R. S. Lynch, J. W. T. Hessels, V. I. Kondratiev, M. A. McLaughlin, S. M. Ransom, A. M. Archibald, J. Boyles, F. A. Jenet, D. L. Kaplan, L. Levin, D. R. Lorimer, E. C. Madsen, M. S. E. Roberts, X. Siemens, I. H. Stairs, K. Stovall, J. K. Swiggum, and J. van Leeuwen, Astrophys. J. **809**, 67 (2015).