Towards construction of ghost-free higher derivative gravity from bigravity

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In this paper, the ghost-freeness of the higher derivative theory proposed by Hassan *et al.* in [Universe **1**, 92 (2015)] is investigated. Hassan *et al.* believed the ghost-freeness of the higher derivative theory based on the analysis in the linear approximation. However, in order to obtain the complete correspondence, we have to analyze the model without any approximations. In this paper, we analyze the two-scalar model proposed in [Universe **1**, 92 (2015)] with arbitrary nonderivative interaction terms. In any order with respect to perturbative parameters, we prove that we can eliminate the ghost for the model with any nonderivative interaction terms.

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I. INTRODUCTION

The question whether the gravity could have the small mass or not has been argued from long time ago. In 1939, M. Fierz and W. Pauli derived the wave equations describing the second-order tensor corresponding to massive spin-two fields, which is called the Fierz-Pauli (FP) model [1]. Although their works were purely based on field theoretical motivations, some questions began with a negative observation by discovering the vDVZ discontinuity in 1970 [2]. The vDVZ discontinuity means that some observables calculated by the Fierz-Pauli theory do not coincide with those of the massless theory in the massless limit. From the fact, it seems that the possibility of the nonvanishing graviton mass had been excluded. On the other hand, in 1972, A. I. Vainshtein considered the gravitational model where the FP mass terms are added to the Einstein action [3]. He found that the spherical symmetric solution of the model does not have the discontinuity in the massless limit. Then, it had been obvious that, because the vDVZ discontinuity relies on the linear approximation, the discontinuity could be avoided by considering the nonlinear model. This mechanism is called the Vainshtein mechanism. In 1974, however, D. G. Boulware and S. Deser have pointed out that the large class of the massive spin-two models with nonlinear terms, which include the model considered by Vainshtein, has a scalar mode in addition to the massive spin-two modes [4]. This scalar mode has the kinetic term with the negative signature, so called the BD ghost. Then, it has become clear that the model is no longer unitary due to the BD ghost. The model satisfying both the Vainshtein mechanism and the BD ghost-freeness had not been constructed for a long time.

The situations changed in 2010. C. de Rham and G. Gabadadze considered the consistency of the nonlinear model in the high energy limit, so called the decoupling limit. By tuning the parameters of the interaction terms without any derivative, they have obtained the lower-order terms which make the theory ghost-free in the decoupling limit [5]. After that, they and A. J. Tolley have obtained the full nonlinear completion of the nonderivative interaction terms [6]. Now, this model is called the dRGT model. Although they had not completed the proof of the absence of the BD ghost in the full nonlinear level, S. F. Hassan and R. A. Rosen gave the complete proof by using the Hamiltonian analysis [7,8]. On the other hand, although the dRGT model includes a fixed metric $\eta_{\mu\nu}$ in addition to the dynamical metric $g_{\mu\nu}$ due to the violation of the diffeomorphism, the extension of the flat metric $\eta_{\mu\nu}$ to the general reference metric $f_{\mu\nu}$ was investigated. S. F. Hassan *et al.* have proved the BD ghost-freeness of the dRGT model with the general reference metric in [9]. In addition to the proof, S. F. Hassan and R. A. Rosen have considered a model where the reference metric $f_{\mu\nu}$ becomes dynamical by adding kinetic terms $\sqrt{-f}R(f)$ to the dRGT action, which is called bigravity model. Then, two metrics in this theory have already been interacted with each other. They have proved the BD ghost-freeness of the bigravity model, and they also showed that the bigravity includes one massless spin-two mode and one massive spin-two mode [8,10].

On the other hand, the theory which includes the modes identical with the modes in the bigravity has been also known in the context of the higher curvature theories. According to [11], the action where the general second-order terms, with respect to the curvature, are added to the Einstein-Hilbert action includes one scalar mode and one massive spin-two mode in addition to the massless spin-two

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mode. The scalar mode could be eliminated by tuning the parameters. We call the model where the scalar mode is eliminated as "the R-squared gravity", in this paper. Although the R-squared gravity has the modes similar to the bigravity, there is an essential difference between both of theories. Although the bigravity theory does not include any ghost, the signatures between the kinetic terms of the two modes in the R-squared gravity are opposite with each other. Therefore, the R-squared gravity violate the unitarity of the S matrix. The only exception is given in the three-dimensional spacetime. E. A. Bergshoeff, O. Hohm and P. K. Townsend proposed the R-squared gravity by tuning the parameters so that the massive spin-two mode has healthy propagation. The obtained model is called the "new massive gravity" (NMG) [12]. Although the signature of the kinetic term for the massless spin-two mode is negative, the massless mode does not propagate in the three-dimensional spacetime. Then, there is some parameter region which makes the theory perturbatively ghost-free. Furthermore, the NMG theory does not include the BD ghost, i.e., the NMG theory has 2 degrees of freedom in nonlinear level. For example, the proof using the Stükelberg trick was given in [13]. In this sense, the NMG theory can be regarded as a higher derivative gravity model conserving the unitarity of the S matrix. However, in the context of the AdS/CFT correspondence, it is well known that there is no parameter region which keep both the unitarity of the NMG theory, with negative cosmological constant, and the positivity of the central charge of its CFT dual [14]. The negativity of the central charge means the violation of the unitarity in the theory. Hence the bulk unitarity and the boundary unitarity are incompatible with each other.

In these backgrounds, the relationship between the bigravity and the R-squared gravity has been investigated after the discovery of the bigravity. In particular, M. F. Paulos and A. J. Tolley showed the equivalence between the bigravity in some limits of parameters and the R-squared gravity [13]. They also obtained some generalizations of the NMG theory without any BD ghosts. Moreover, S. F. Hassan, A. Schmidt-May and M. von Strauss tried to investigate the correspondence between the bigravity and the R-squared gravity without any limits of parameters. They proposed a higher derivative theory describing the same dynamics as bigravity under appropriate conditions. They have also shown that the higher derivative theory coincides with the R-squared gravity in small curvature approximation. In this way, they have concluded that the higher derivative theory is a ghost-free completion of the R-squared gravity. This analysis has been extended to a higher order in the context of the correspondence between the Weyl gravity and the partially massless gravity [15].

The reason why they believed the ghost-freeness of the higher derivative theory was based on the analysis in the linear approximation with respect to the fields, as given in the Appendix of [16]. However, in order to obtain the complete correspondence, we have to analyze the model without any approximations.

In this paper, we analyze the scalar model proposed in [16] with arbitrary nonderivative interaction terms, and investigate the possibility of the elimination of the ghost. As a result, we prove it for any nonderivative interaction terms, and any order with respect to the perturbative parameter.

II. PREVIOUS RESEARCH

In this section, we briefly review the analysis given in [16]. The action of the bigravity model [10] is given by

$$S[g,f] = M_g^{D-2}\int d^Dx\left[\sqrt{-g}R(g) + \alpha^{D-2}\sqrt{-f}R(f) - 2m^2\sqrt{-g}\sum_{n=0}^{D}\beta_n e_n(S)\right],$$
$$e_n(S) \equiv \frac{1}{n!}\delta^{\mu_1}{}_{\nu_1}{}^{\mu_2\ldots\mu_n}{}_{\nu_2\ldots\nu_n}S^{\nu_1}{}_{\mu_1}S^{\nu_2}{}_{\mu_2}\ldots S^{\nu_n}{}_{\mu_n},$$
$$S^\mu{}_\nu \equiv \sqrt{g^{-1}f}^{\mu}{}_{\nu}, \qquad S^\mu{}_\nu S^\nu{}_v = g^{\mu\nu}f_{\nu\rho}. \tag{1}$$

Here, D is the spacetime dimension, M_g is the Planck mass for the metric g, $\alpha \equiv M_f/M_g$ (M_f is the Planck mass for the metric f.) is the ratio of the Planck masses, β_n are free parameters without dimension, and m^2 is the mass parameter, which is introduced in order to make β_n dimensionless. The tensor $\delta^{\mu_1}{}_{\nu_1}{}^{\mu_2\ldots\mu_n}{}_{\nu_2\ldots\nu_n}$ is defined as follows,

$$\delta^{\mu_1}{}_{\nu_1}{}^{\mu_2\ldots\mu_n}{}_{\nu_2\ldots\nu_n} \equiv \frac{-1}{(D-n)!}\epsilon^{\mu_1\mu_2\ldots\mu_n\sigma_{n+1}\ldots\sigma_D}\epsilon_{\nu_1\nu_2\ldots\nu_n\sigma_{n+1}\ldots\sigma_D}. \tag{2}$$

Here the tensor $\epsilon^{\mu_1\ldots\mu_D}$ is the Levi-Civita anti-symmetric tensor, one of whose components is given by $\epsilon^{012\ldots D-1} = 1$. In the action (1), two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ interact with each other through the nonderivative interaction terms. Then the equation of motion given by the variation of $g_{\mu\nu}$ does not include any derivative of the metric $f_{\mu\nu}$,

$$\frac{\delta S[g,f]}{\delta g_{\mu\nu}} = 0. \tag{3}$$

Therefore, we can algebraically solve this equation with respect to $f_{\mu\nu}$.

The obtained solution depends on the metric $g_{\mu\nu}$ and the curvature $R(g)$. Although there are generally $D-1$ solutions in Eq. (3), we choose one of them and denote as $f_{\mu\nu}[g]$.[1]

[1]In [16], this difference of the solutions is expressed by a parameter "a", which is a solution of $D-1$-dimensional polynomial equation. In later argument by using scalar fields, we adopt the specific solution. At present, however, we do not restrict the solution. The arguments after Eq. (5) are correct for any solution.

By substituting this solution, $f_{\mu\nu} = f_{\mu\nu}[g]$, to the equation of motion obtained by variation with respect to $f_{\mu\nu}$,

$$\left[\frac{\delta S[g,f]}{\delta f_{\mu\nu}}\right] = 0, \tag{4}$$

we obtain

$$\left[\frac{\delta S[g,f]}{\delta f_{\mu\nu}}\right]_{f=f[g]} = 0. \tag{5}$$

This equation of motion (5) has the following properties: The algebraic solution $f_{\mu\nu}[g]$ in (3) includes the second-order derivatives of g, and Eq. (4) is a second-order derivative equation. Then we find that the equation obtained by substituting the solution $f_{\mu\nu}[g]$ in (3) into (4) is the fourth-order Eq. (5). Moreover, the solutions of Eq. (5) are also the solutions of the original Eqs. (3) and (4), so that the dynamics described by Eq. (5) are stable despite being the fourth order.

In [16], Hassan *et al.* proposed the higher derivative model obtained by substituting the algebraic solution $f_{\mu\nu}[g]$ into the original action (1),

$$S'[g] \equiv S[g, f[g]]. \tag{6}$$

The dynamics described by this action $S'[g]$ do not completely coincide with the dynamics described by the original action $S[g,f]$. Indeed, by the variation of the action (6), we obtain

$$\begin{aligned}\frac{\delta S'[g]}{\delta g_{\mu\nu}(x)} &= \left[\left.\frac{\delta S[g,f]}{\delta g_{\mu\nu}(x)}\right|_f\right]_{f=f[g]} \\ &\quad + \int d^D y \frac{\delta f_{\rho\sigma}[g(y)]}{\delta g_{\mu\nu}(x)}\left[\left.\frac{\delta S[g,f]}{\delta f_{\rho\sigma}(y)}\right|_g\right]_{f=f[g]} \\ &= \int d^D y \frac{\delta f_{\rho\sigma}[g(y)]}{\delta g_{\mu\nu}(x)}\left[\left.\frac{\delta S[g,f]}{\delta f_{\rho\sigma}(y)}\right|_g\right]_{f=f[g]}.\end{aligned} \tag{7}$$

In the second line, we use the fact that the first term in the first line identically vanishes due to the fact that the algebraic solution $f[g]$ satisfies the equation of motion (3). Here, if we define

$$\frac{\delta f_{\rho\sigma}[g(y)]}{\delta g_{\mu\nu}(x)} \equiv \mathcal{O}^{\mu\nu}{}_{\rho\sigma}\delta(x-y), \tag{8}$$

the operator $\mathcal{O}$ becomes the second-order derivative operator because the function $f[g]$ contains $R(g)$. As a result, the equation of motion of the theory with the action $S'[g]$ in (6) is given by

$$\mathcal{O}^{\mu\nu}{}_{\rho\sigma}\left[\left.\frac{\delta S[g,f]}{\delta f_{\rho\sigma}(x)}\right|_g\right]_{f=f[g]} = 0. \tag{9}$$

Because the second-order differential operator $\mathcal{O}$ acts on the lhs of the original Eqs. (5), and (5) is the fourth order with respect to derivatives, we obtain the sixth-order differential equation (9). By introducing the auxiliary field $\lambda^{\rho\sigma}$, Eq. (9) could be decomposed as follows,

$$\mathcal{O}^{\mu\nu}{}_{\rho\sigma}\lambda^{\rho\sigma} = 0, \qquad \left[\left.\frac{\delta S[g,f]}{\delta f_{\rho\sigma}(x)}\right|_g\right]_{f=f[g]} = \lambda^{\rho\sigma}. \tag{10}$$

These equations express the system where the field $\lambda^{\rho\sigma}$ described by the second-order differential equation and the field $g_{\mu\nu}$ described by the fourth-order differential equation interact with each other. In order that the solution described by Eq. (10) is equivalent to the solution described by the original Eq. (5), it is necessary to be $\lambda^{\rho\sigma} = 0$ by choosing the initial conditions and/or the boundary conditions for $\lambda^{\rho\sigma}$.

When we obtain the action (6), we need to solve Eq. (3) for $f_{\mu\nu}$ explicitly. It is not, however, so easy to solve Eq. (3) because Eq. (3) is nonlinear matrix equation. Then, Hassan *et al.* have solved this equation perturbatively by expanding this equation with respect to $1/m^2$. As a result, by substituting the obtained solution into the original action (1), they have shown

$$\begin{aligned}S[g, f(g)] = M_g^{D-2}\int d^D x\sqrt{-g}\Big[\Lambda + c_R R(g) \\ - \frac{c_{RR}}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{D}{4(D-1)}R^2\right)\Big] + \mathcal{O}\left(\frac{1}{m^4}\right).\end{aligned} \tag{11}$$

Here, although the coefficients Λ, c_R, c_{RR} are defined by the parameters in the bigravity α, β_n in (1), because the explicit forms are a little bit complicated, we do not give these forms now. By neglecting the higher-order terms $\mathcal{O}(\frac{1}{m^4})$, the remaining terms are those in the R-squared gravity, which contains the healthy massless spin-two mode and the ghostlike massive spin-two mode [for example, see [11–13]]. Hassan *et al.* have conjectured that, although the truncated model, which contains the R-squared gravity, includes the ghost, but the complete form of this higher derivative theory could be ghost-free.

The reason why they believed that the higher derivative theory could be ghost-free is based on the analysis in the linear approximation with respect to the fields, as given in the Appendix of [16]. For avoiding the complication of our argument, we do not use their argument now. Their arguments are given in the Appendix, which we should read after the argument in Sec. IV A.

In order to obtain the complete correspondence, we have to analyze the theory without any approximations. Then, we consider the two-scalar model proposed in [16], keeping the interaction terms not equal to zero,

$$S_0[\phi,\psi] = \int d^D x \left[\frac{1}{2}\phi\Box\phi + \frac{1}{2}\psi\Box\psi - \frac{m^2}{2}(\phi+\psi)^2 - kV(\phi,\psi)\right], \quad (12)$$

and we investigate the possibility of eliminating the ghost. We define the function $\psi[\phi]$ as an algebraic solution with respect to ψ of the equation of motion obtained by variation with respect to ϕ. We consider the higher derivative model obtained by substituting the solution $\psi = \psi[\phi]$ to the original action $S_0[\phi,\psi]$ in (12),

$$S'[\phi] \equiv S_0[\phi,\psi[\phi]]. \quad (13)$$

We show that the amplitudes described by $S'[\phi]$, by choosing the appropriate physical space, coincide with the amplitudes of the original theory.

III. MODEL OF SCALAR FIELDS

In this section, we give the fundamental properties of the model proposed in this paper. Because most of the analysis is focused on the linear level, the obtained results are not so different from those obtained by Hassan *et al.* [16] but the spectrum is obtained by using the formulations different from those in [16].

A. Model of scalar fields

Although the model proposed in this paper has two modes with positive kinetic terms, the corresponding higher derivative model contains an additional mode. In this section, we explain this fact by focusing our analysis to the linear terms of fields. Let us consider the model of two scalar fields interacting with each other by a mass mixing,

$$S_0[\phi,\psi] = \int d^D x \left[\frac{1}{2}\phi\Box\phi + \frac{1}{2}\psi\Box\psi - \frac{m^2}{2}(\phi+\psi)^2 - kV(\phi,\psi)\right]. \quad (14)$$

In the analogy with the linearization of the bigravity action (1) which includes the mass mixing terms (see [10]), we add the mass mixing term. We assume that $V(\phi,\psi)$ is the interaction term including the third order or higher powers of fields without derivatives. This assumption is based on not only the analogy of the nonderivative interaction terms in bigravity, but also the necessity of expressing ψ as an algebraic function of ϕ, $\psi = \psi[\phi]$. More generally, although we should add some self-interaction terms with some derivatives of ϕ and ψ to the action (14), we do not include them just for simplicity.

Under the field redefinition,

$$\phi = \frac{1}{\sqrt{2}}(\xi+\eta), \qquad \psi = \frac{1}{\sqrt{2}}(\xi-\eta), \quad (15)$$

the linear terms of the action (14) are diagonalized as follows,

$$S_0[\phi(\xi,\eta),\psi(\xi,\eta)]|_{\text{linear}} = \int d^D x \left[\frac{1}{2}\xi(\Box - 2m^2)\xi + \frac{1}{2}\eta\Box\eta\right]. \quad (16)$$

From the above expression, we find this model includes two scalar fields with mass 0 and $2m^2$.

By the variation of the action (14) with respect to ϕ, we obtain

$$\frac{\delta S_0}{\delta\phi} = (\Box - m^2)\phi - m^2\psi - k\frac{\partial V(\phi,\psi)}{\partial\phi} = 0. \quad (17)$$

Because this equation does not include any derivatives of ψ, we can solve (17) with respect to ψ algebraically. Because Eq. (17) is the polynomial with respect to ψ, the solution is not unique and the number of the solutions depends on the exponent of the highest power terms of ψ. Nevertheless, the solution corresponding to the vacuum $\phi = 0 = \psi$ is uniquely determined, due to the existence of the mass mixing term. Then, just for simplicity, we adopt $\psi[\phi]$ which is the algebraic solution satisfying $\psi[\phi = 0] = 0$.[2] Through all of the later arguments in this paper, we adopt this class of solution. In this assumption, the linear part of $\psi[\phi]$ is expressed as follows,

$$m^2\psi[\phi] = (\Box - m^2)\phi + \mathcal{O}(\phi^2). \quad (18)$$

The higher derivative theory derived by Hassan *et al.* [16] corresponds to the new action $S_0[\phi,\psi[\phi]]$ obtained by substituting the algebraic solution (18) into the original action (14). Then the linear terms in the action are expressed as follows,

[2] In this paper, we analyze the scattering amplitudes in order to argue the ghost-freeness of the higher derivative theory (13). When we calculate the perturbative scattering amplitude, we have to choose the vacuum. As will be discussed later, the amplitudes of the higher derivative theory with the assumption $\psi[\phi = 0] = 0$ correspond to those of the original theory calculated around the vacuum $\phi = 0 = \psi$. On the other hand, another algebraic solution corresponds to the perturbative theories around another vacuum. We can extend our discussion to including another vacuum by replacing the mass terms $m^2(\phi+\psi)^2$ in the action (14) with more general terms $m_\phi^2\phi^2 + 2m_{\phi\psi}^2\phi\psi + m_\psi^2\psi^2$ (and regarding the replaced action as the action which have already been expanded around the interested vacuum). In the argument under the replaced action, the assumption $\psi[\phi = 0] = 0$ is no longer the specific case. But, just for simplicity, we do not extend the discussion in this paper.

$$S_0[\phi,\psi[\phi]] = \int d^Dx\left[\frac{1}{2m^4}\phi\Box(\Box-2m^2)(\Box-m^2)\phi + \mathcal{O}(\phi^3)\right]. \tag{19}$$

From the action (19), we find that there are the mass spectrum 0, $2m^2$, and an additional spectrum m^2. In the next part, we investigate whether each of the modes is ghost or not.

B. Spectrum

We have found the higher derivative model, whose linear parts are given in (19), contain an additional field with mass squared m^2. Now, in order to check whether the additional spectrum and original modes are ghost or not, we decompose the action (19) by introducing the Lagrange multiplier field λ,

$$S_0[\phi,\psi[\phi]]\to S[\phi,\psi,\lambda]\equiv S_0[\phi,\psi]+m^2\int d^Dx\lambda(\psi-\psi[\phi]). \tag{20}$$

Here, in order to simplify the later arguments, we put the coefficient m^2 in front of λ. Indeed, this coefficient does not affect the dynamics, no matter how we choose it. By using Eq. (18), the linear part of $S[\phi,\psi,\lambda]$ can be rewritten as follows,

$$\begin{aligned} S[\phi,\psi,\lambda]|_{\mathrm{linear}} &= \int d^Dx\left[\frac{1}{2}\phi\Box\phi+\frac{1}{2}\psi\Box\psi-\frac{m^2}{2}(\phi+\psi)^2 + \lambda(m^2\psi-(\Box-m^2)\phi)\right] \\ &= S_0[\phi,\psi]|_{\mathrm{linear}} - \int d^Dx\lambda(x)\frac{\delta S_0[\phi,\psi]|_{\mathrm{linear}}}{\delta\phi(x)}. \end{aligned} \tag{21}$$

From the last line of the above equations, it is obvious that the Lagrange multiplier terms could be eliminated by the field redefinition $\phi\to\phi+\lambda$,

$$S[\phi+\lambda,\psi,\lambda]|_{\mathrm{linear}} = \int d^Dx\left[\frac{1}{2}\phi\Box\phi+\frac{1}{2}\psi\Box\psi - \frac{m^2}{2}(\phi+\psi)^2-\frac{1}{2}\lambda(\Box-m^2)\lambda\right]. \tag{22}$$

Then we find that λ has the kinetic term with a negative signature; therefore, λ is a ghost with mass m^2. On the other hand, we also find that the terms of ϕ and ψ correspond to the linear terms of the original theory (14); therefore, ϕ and ψ are healthy fields. Indeed, under the field redefinitions,

$$\phi=\frac{1}{\sqrt{2}}(\xi+\eta), \qquad \psi=\frac{1}{\sqrt{2}}(\xi-\eta), \tag{23}$$

we obtain following diagonal expression,

$$\begin{aligned} \bar{S}[\xi,\eta,\lambda]|_{\mathrm{linear}} &= \int d^Dx\left[\frac{1}{2}\xi(\Box-2m^2)\xi+\frac{1}{2}\eta\Box\eta - \frac{1}{2}\lambda(\Box-m^2)\lambda\right], \\ \bar{S}[\xi,\eta,\lambda] &\equiv S[\phi(\xi,\eta)+\lambda,\psi(\xi,\eta),\lambda]. \end{aligned} \tag{24}$$

This action (24) has the healthy modes ξ and η, and the extra ghost mode λ. In other words, although the fields included in the original action $S_0[\phi,\psi]$ (14) are healthy, the new ghost field has appeared by the procedure of substitution. In the following sections, we would like to call (ξ,η) in the action (24) or (ϕ,ψ) in the action (22) "physical fields."

IV. CONJECTURE AND SAMPLE CALCULATION

The purpose of this section is to conjecture the correspondence between the original theory $S_0[\phi,\psi]$ (14) and the corresponding higher derivative theory $S_0[\phi,\psi[\phi]]$ (19). For this purpose, we consider specific interaction terms and calculate the tree-level amplitudes of the higher derivative theory described by the action $S_0[\phi,\psi[\phi]]$ (19).

A. The conjecture

As a result, "physical amplitudes" of the higher derivative theory coincide with the scattering amplitudes of the original theory. Here, "physical amplitudes" mean the scattering amplitudes where all external lines are taken to physical fields ξ and η in (23). In other words, the conjecture is expressed as the realization of the correspondence,

$$\begin{aligned} &\langle\xi(\vec{k}_1)\ldots\xi(\vec{k}_n)\eta(\vec{k}_{n+1})\ldots\eta(\vec{k}_{n+m});\mathrm{out}|\xi(\vec{p}_1)\ldots\xi(\vec{p}_N)\eta(\vec{p}_{N+1})\ldots\eta(\vec{p}_{N+M});\mathrm{in}\rangle_{\mathrm{Original}} \\ &= \langle\xi(\vec{k}_1)\ldots\xi(\vec{k}_n)\eta(\vec{k}_{n+1})\ldots\eta(\vec{k}_{n+m});\mathrm{out}|\xi(\vec{p}_1)\ldots\xi(\vec{p}_N)\eta(\vec{p}_{N+1})\ldots\eta(\vec{p}_{N+M});\mathrm{in}\rangle_{\mathrm{HD}}, \end{aligned} \tag{25}$$

in the tree level. As will be discussed later, because the Green functions of both theories are not identical with each other, we express the correspondence by using the S matrix elements. This means that the correspondence is only valid under the on-shell conditions. The reason why we find this conjecture and the proof for the specific case are given in the Appendix.

B. A sample calculation

In this part, in order to confirm the validity of the conjecture (25), we investigate the structure of the Feynman diagrams of the higher derivative theory for given interaction terms. As a result, we find the interesting structure of the diagrams. Now, we consider the third-order interaction term with respect to the massive field ξ,

$$S[\xi,\eta] = \int d^Dx\left[\frac{1}{2}\xi(\Box - 2m^2)\xi + \frac{1}{2}\eta\Box\eta - \mu\frac{\sqrt{2}}{3}\xi^3\right] = \int d^Dx\left[\frac{1}{2}\phi\Box\phi + \frac{1}{2}\psi\Box\psi - \frac{m^2}{2}(\phi+\psi)^2 - \frac{\mu}{3!}(\phi+\psi)^3\right]. \quad (26)$$

We start with deriving the corresponding higher derivative theory. The equation of motion obtained by the variation of the action (26) with respect to ϕ,

$$\frac{\delta S}{\delta\phi} = (\Box - m^2)\phi - m^2\psi - \frac{\mu}{2}(\phi+\psi)^2 = 0, \quad (27)$$

could be solved for ψ as follows:

$$\psi[\phi] = \phi - \frac{m^2}{\mu} \pm \sqrt{\frac{m^4}{\mu^2} + \frac{2}{\mu}\Box\phi}. \quad (28)$$

We find that there are two solutions because the equation of motion (27) is quadratic with respect to ψ. Now we restrict our arguments to the unique solution with the vacuum $\phi = 0 = \psi$. The signature satisfying this condition is "+" in (28). Under this selection, Eq. (28) can be expanded with respect to μ as follows,

$$m^2\psi[\phi] = (\Box - m^2)\phi - \frac{\mu}{2m^4}(\Box\phi)^2 + \frac{\mu^2}{2m^8}(\Box\phi)^3 + \mathcal{O}(\mu^3). \quad (29)$$

By replacing ψ in the original action $S_0[\phi,\psi]$ with $\psi[\phi]$, we obtain the higher derivative theory. However, for the simplicity of the analysis, we do not consider the higher derivative form. Instead of this, we analyze the action (24) expressed by ξ, η, and λ. Now, we expand the action with respect to μ,

$$S[\phi(\xi,\eta)+\lambda, \psi(\xi,\eta), \lambda] \equiv \sum_{n=0}^{\infty}\bar{S}^{(n)}[\xi,\eta,\lambda]. \quad (30)$$

Here $\bar{S}^{(n)}[\xi,\eta,\lambda]$ are the nth-order terms with respect to μ. By using this notation, the lower-order terms of $\bar{S}^{(n)}[\xi,\eta,\lambda]$ corresponding to the algebraic solution (29) are given by

$$\bar{S}^{(0)}[\xi,\eta,\lambda] = \int d^Dx\left[\frac{1}{2}\xi(\Box - 2m^2)\xi + \frac{1}{2}\eta\Box\eta - \frac{1}{2}\lambda(\Box - m^2)\lambda\right],$$

$$\bar{S}^{(1)}[\xi,\eta,\lambda] = \int d^Dx\left[-\frac{\sqrt{2}}{3}\mu\xi^3 + \frac{\mu}{4m^2}\lambda\{(\Box\xi)^2 - 4m^4\xi^2\} + \frac{\mu}{\sqrt{2}}\lambda\{\Box\xi\Box\lambda - m^4\xi\lambda\} + \frac{\mu}{4m^4}\lambda\{2\Box\xi + \Box\eta + 2\sqrt{2}\Box\lambda\}\Box\eta - \frac{\mu}{3!}\lambda^3 + \frac{\mu}{2m^4}\lambda(\Box\lambda)^2\right],$$

$$\bar{S}^{(2)}[\xi,\eta,\lambda] = \int d^Dx\left(-\frac{\mu^2}{2m^8}\right)\lambda\left[\frac{1}{2\sqrt{2}}(\Box\xi + \Box\eta)^3 + \frac{3}{2}(\Box\xi + \Box\eta)^2\Box\lambda + \frac{3}{\sqrt{2}}(\Box\xi + \Box\eta)(\Box\lambda)^2 + (\Box\lambda)^3\right]. \quad (31)$$

The lowest terms in $\bar{S}^{(0)}$ coincide with those in (24). The Feynman diagrams of the third-order terms are summarized in Fig. 1. Now let us investigate the sufficient condition for the realization of the conjecture given in (25). We should note again that the following arguments are only in the case of tree level.

In the first-order terms with respect to μ, $\bar{S}^{(1)}$, the first term, which is represented as ① in Fig. 1, coincides with the interaction term of the original action (26). For the realization of the conjecture (25), it is enough to show that the amplitudes including the vertexes except the vertex ① do not contribute to the physical amplitudes. Moreover, because the terms $\bar{S}^{(n)}, n \geq 2$, include the Lagrange multiplier field λ, we find that all terms except ① always include the ghost λ. Then, if any contribution from some vertexes except ① exists, there must be the ghost in the internal line. Therefore, for the realization of (25), it is enough to show that the diagrams including the ghost in internal line do not contribute to the physical amplitudes.

In order to confirm the sufficient condition above, it is enough to show there are no vertexes which could decrease the number of ghosts. The reason and the more exact

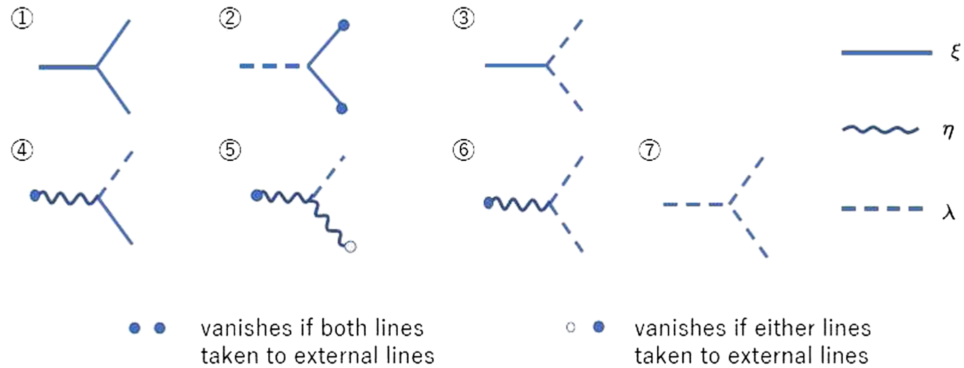

FIG. 1. Interaction terms in the third order.

meaning of this statement may be obvious by the following consideration: Let us consider any vertex which contains the some physical fields and some ghost fields, in the left-hand side of Fig. 2. Here, the solid lines represent the physical fields, and the broken lines represent the ghost. If we try to construct the physical amplitudes from this vertex, we must decrease the number of ghosts by acting some vertexes, as in the right-hand side of Fig. 2. The vertex which could decrease the number of the ghosts are only the first-order terms with respect to ghost. Then, if the action does not contain the first-order terms with respect to the ghost, we cannot construct the diagrams contributing to the physical amplitude from the vertex in the left-hand side of Fig. 2.

Although the action (31) seems to contain the first-order terms with respect to the ghost [that is, the vertexes ②, ④, and ⑤ in Fig. 1], these terms effectively do not contribute to the physical amplitudes. The vertex ② obtained from following terms,

$$\frac{\mu}{4m^2}\lambda\{(\Box\xi)^2 - 4m^4\xi^2\}, \tag{32}$$

vanishes if both of the physical fields ξ are taken to external line. Indeed, the on-shell condition is given by $\Box \to 2m^2$, then the terms (32) obviously vanish. Similarly, we find that the contributions from ④ and ⑤ vanish again under the on-shell condition of η. In order to emphasize this fact, we put some points on the diagrams represented in Fig. 1. The lines which connect the two points with the same color vanish by taking all the lines with the points to the external lines simultaneously. From this property, it is impossible to construct the nonvanishing physical amplitude by using the terms (32) at least lower than the six-points diagrams. The nontriviality appears in the six-points diagrams. Because we can construct the nonvanishing diagrams like (a) of Fig. 3 with an on-shell condition for the physical fields, without the on-shell condition of λ, the six-points diagram like (f) of Fig. 4 could survive under the on-shell condition. So that, we afraid if this diagram could contribute to the physical amplitudes. In this order, however, we cannot ignore the contributions from the higher-order terms,

$$\left(-\frac{\mu^2}{2m^8}\right)\lambda\frac{1}{2\sqrt{2}}(\Box\xi)^3, \tag{33}$$

in $\bar{S}^{(2)}$. Surprisingly, the summation of the diagrams (a) and (b) with the on-shell condition of the physical fields, without the on-shell condition of λ, becomes equal to zero. So we could regard the summation of these diagrams as the diagram (c) of Fig. 3. Therefore, under the on-shell condition, the summation of the nonvanishing amplitudes represented in Fig. 4 becomes equal to zero. In this way, we predict that the nonvanishing diagrams constructed from the lower-order terms could be eliminated by the diagrams constructed from the higher-order terms. In the following

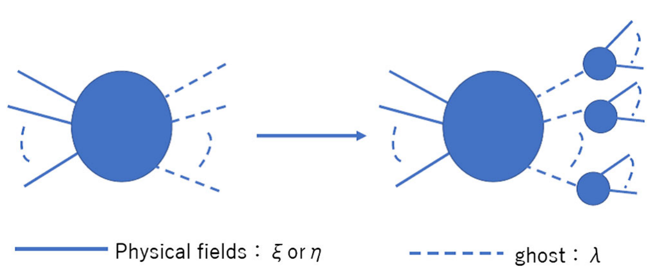

FIG. 2. The operation decreasing the number of ghost.

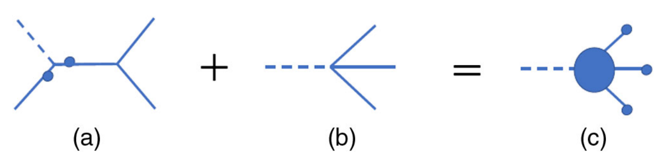

FIG. 3. The sums of the nonvanishing diagrams.

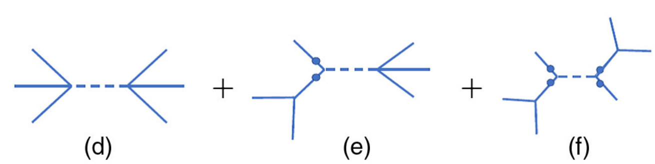

FIG. 4. The contribution of the nonvanishing diagrams to the physical amplitudes.

sections, we would like to prove this conjecture for general nonderivative interaction terms in any order.

V. GENERAL PROOF

In the previous sections, we specified the modes of the theory described by the action $S_0[\phi, \psi[\phi]]$, and obtained the conjecture for the correspondence between the higher derivative theory $S_0[\phi, \psi[\phi]]$ and the original theory $S_0[\phi, \psi]$. In this section, we would like to argue general nonderivative interaction terms $V(\phi, \psi)$ and prove the conjecture in any order with respect to the perturbative parameter k.

A. General algebraic solution

First, in this section, we derive the lower-order terms with respect to k in $\psi[\phi]$. The full-order solution will be derived in a later section.

Let us consider the action (14) with general nonderivative interaction terms,

$$S_0 = \int d^D x \left[\frac{1}{2}\phi \Box \phi + \frac{1}{2}\psi \Box \psi - \frac{m^2}{2}(\phi + \psi)^2 - kV(\phi, \psi) \right]. \tag{34}$$

Here, $V(\phi, \psi)$ consists of the general nonderivative interaction terms including third- or higher-order terms of fields. The equation of motion derived by the variation with respect to ϕ,

$$\frac{\delta S_0}{\delta \phi} = (\Box - m^2)\phi - m^2\psi - kV^{1,0}(\phi, \psi) = 0,$$
$$V^{n,m}(\phi, \psi) \equiv \frac{\partial^{n+m} V(\phi, \psi)}{\partial^n \phi \partial^m \psi}, \tag{35}$$

could be solved with respect to ψ around the vacuum $\phi = 0 = \psi$ for any $V(\phi, \psi)$. We assume a perturbative solution expanded with respect to k in the following form,

$$m^2\psi[\phi] = m^2\psi_0[\phi] + F[\phi], \qquad m^2\psi_0[\phi] \equiv (\Box - m^2)\phi,$$
$$F[\phi] \equiv \sum_{n=1}^{\infty} k^n F^{(n)}[\phi], \tag{36}$$

and determine the $F[\phi]$. By substituting (36) into (35), and expanding the obtained expression in powers of k, we find

$$\begin{aligned}
-\frac{1}{k}F[\phi] &= V^{1,0}\left(\phi, \psi_0[\phi] + \frac{1}{m^2}F[\phi]\right) \\
&= \sum_{n=0}^{\infty} \frac{V^{1,n}(\phi, \psi_0[\phi])}{n!} \left(\frac{F[\phi]}{m^2}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{m^{2n}} \frac{V^{1,n}(\phi, \psi_0[\phi])}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} F^{(k_1)}[\phi] \ldots F^{(k_n)}[\phi] k^{k_1+\cdots+k_n} \\
&= \sum_{n=0}^{\infty} \frac{1}{m^{2n}} \frac{V^{1,n}(\phi, \psi_0[\phi])}{n!} \sum_{N=1}^{\infty} k^N \sum_{1 \le k_1, \ldots, k_n \le N-n+1} F^{(k_1)}[\phi] \ldots F^{(k_n)}[\phi] \delta_{k_1+\ldots k_n, N} \\
&= V^{1,0}(\phi, \psi_0) + \sum_{N=1}^{\infty} k^N \sum_{n=1}^{N} \frac{1}{m^{2n}} \frac{V^{1,n}(\phi, \psi_0[\phi])}{n!} \sum_{s_1=1}^{N-n+1} F^{(s_1)}[\phi] \sum_{s_2=1}^{N-n+1-s_1} F^{(s_2)}[\phi] \ldots \\
&\qquad \sum_{s_{n-1}=1}^{N-n-s_1-s_2-\ldots-s_{n-2}} F^{(s_{n-1})}[\phi] F^{(N-n+1-s_1-s_2-\ldots-s_{n-1})}[\phi] \\
&= V^{1,0}(\phi, \psi_0[\phi]) + \sum_{N=1}^{\infty} k^N \sum_{n=1}^{N} \frac{1}{m^{2n}} \frac{V^{1,n}(\phi, \psi_0[\phi])}{n!} \\
&\quad \times \left[\Pi_{i=1}^{n-1} \sum_{s_i=1}^{N-n+1-\Sigma_{k=1}^{i-1} s_k} F^{(s_i)}[\phi] \right] F^{(N-n+1-\Sigma_{k=1}^{n-1} s_k)}[\phi].
\end{aligned} \tag{37}$$

By using the last expression for $F[\phi]$ in (36) and by comparing both sides in (37) order by order in k, we obtain the following recursion relations,

$$F^{(1)} = -V^{1,0}(\phi, \psi_0[\phi]),$$
$$F^{(N+1)} = -\sum_{n=1}^{N} \frac{1}{m^{2n}} \frac{V^{1,n}(\phi,\psi_0[\phi])}{n!} \times \left[\Pi_{i=1}^{n-1} \sum_{s_i=1}^{N-n+1-\Sigma_{k=1}^{i-1}s_k} F^{(s_i)}[\phi]\right] F^{(N-n+1-\Sigma_{k=1}^{n-1}s_k)}[\phi]. \tag{38}$$

Solving the recursion relations for lower-order terms, we find

$$F^{(1)}[\phi] = -V_0^{1,0},$$
$$F^{(2)}[\phi] = \frac{1}{m^2} V_0^{1,0} V_0^{1,1},$$
$$F^{(3)}[\phi] = -\frac{1}{m^4} V_0^{1,0}\left[(V_0^{1,1})^2 + \frac{1}{2} V_0^{1,2} V_0^{1,0}\right],$$
$$F^{(4)}[\phi] = \frac{1}{m^6} V_0^{1,0}\left[(V_0^{1,1})^3 + \frac{3}{2} V_0^{1,0} V_0^{1,1} V_0^{1,2} + \frac{1}{6}(V_0^{1,0})^2 V_0^{1,3}\right]. \tag{39}$$

Here, we express $V^{n,m}(\phi,\psi_0[\phi])$ as $V^{n,m}(\phi,\psi_0[\phi]) \equiv V_0^{n,m}$ for simplicity.

B. Validity of conjecture

In this section, we confirm the conjecture (25) for any nonderivative interaction terms, $V(\phi,\psi)$, in lower-order perturbations. As we have seen in Sec. IV, a sufficient condition realizing the conjecture (25) is that the action of the higher derivative theory does not include the first-order terms with respect to λ. Then we consider eliminating the first-order terms with respect to λ by some field redefinitions.

According to the Kamefuchi-O'Raifeartaigh-Salam's theorem [17], the S matrix elements are invariant, under the fields redefinitions expressed as follows,

$$\phi' = c\phi + u[\phi,\psi,\lambda], \tag{40}$$

where c must be a nonvanishing constant, and $u[\phi,\psi,\lambda]$ must be second- or higher-order terms with respect to fields or some derivatives of these fields. Then, it is enough to confirm that all field redefinitions [except for later (42)] satisfy the expression (40), in order to realize the conjecture (25).

Let us consider the action obtained by substituting the algebraic solution (36) into the action decomposed by the Lagrange multiplier field λ (20),

$$S_1[\phi_1,\psi,\lambda] \equiv \int d^Dx\left[\frac{1}{2}\phi_1\Box\phi_1 + \frac{1}{2}\psi\Box\psi - \frac{m^2}{2}(\phi_1+\psi)^2 - kV(\phi_1,\psi) + \lambda(m^2\psi - m^2\psi_0[\phi_1] - F[\phi_1])\right]$$
$$= S_0[\phi_1,\psi] + \int d^Dx[\lambda(m^2\psi - m^2\psi_0[\phi_1] - F[\phi_1])]. \tag{41}$$

Here, in order to regard the successive redefinitions of the action and ϕ in the following as some arithmetic progressions, we put the number "1" as a suffix on them. As we have seen in Sec. III B, the first-order terms with respect to λ, in the zero order of k, could be eliminated by the field redefinition,

$$\phi_1 = \phi_2 + \lambda. \tag{42}$$

Under the redefinition (42), the action (41) is transformed as follows,

$$S_2[\phi_2,\psi,\lambda] \equiv S_1[\phi_1,\psi,\lambda] = S_0[\phi_2,\psi] + \int d^Dx\lambda[-kV^{1,0}(\phi_2,\psi) + F[\phi_2]] + \mathcal{O}(\lambda^2)$$
$$= S_0[\phi_2,\psi] + \int d^Dx\lambda\left[-k(V^{1,0}(\phi_2,\psi) + F^{(1)}[\phi_2]) - \sum_{n=2}^{\infty} k^n F^{(n)}[\phi_2]\right] + \mathcal{O}(\lambda^2). \tag{43}$$

Because we are not interested in second- or higher-order terms with respect to λ, we could ignore these terms.

Now, the terms independent of k have vanished and new terms proportional to k have appeared. The new terms are expressed by the second term of the first line in (43), $-k\lambda V^{1,0}(\phi_2,\psi)$, that is, the contribution from S_0. Then, in the second line of (43), we pick up the first-order terms with respect to k. We could show that these terms vanish under the on-shell condition. By the result in (39), the first-order terms with respect to k are expressed as follows,

$$-k\lambda(V^{1,0}(\phi_2,\psi) - V^{1,0}(\phi_2,\psi_0[\phi_2])). \tag{44}$$

The only difference between the two terms is that the arguments are either ψ or $\psi_0[\phi_2]$. Now, because the linear terms have already been diagonalized, the condition,

$m^2\psi = m^2\psi_0[\phi_2] \equiv (\Box - m^2)\phi_2$, is the linear parts of the solution of the EoM obtained by the variation of the action respect to ϕ_2, i.e., the on-shell condition. Hence, the contributions from the vertexes (44) to the scattering amplitudes vanish when all the physical fields ϕ_2, ψ are taken to the external lines. These terms just correspond to the terms which vanish under the on-shell condition in the ξ^3-model,

$$\frac{\mu}{4m^2}\lambda\{(\Box\xi)^2 - 4m^4\xi^2\}. \tag{45}$$

Now, because we have assumed that $V^{1,0}(\phi,\psi)$ includes third- or higher-order terms of fields, we have verified that the diagrams, less than six points, including some internal lines of ghost, do not contribute to the physical amplitudes for general nonderivative interaction terms.

Moreover, the terms (44) could be eliminated by additional field redefinition. The fact that the terms (44) become equal to zero under the on-shell condition $\psi = \psi_0[\phi_2]$, means that the terms (44) could be factored by $(\psi - \psi_0[\phi_2])$. Indeed, by expanding $V^{1,0}(\phi_2,\psi)$ around $\psi = \psi_0[\phi_2]$, because the leading terms are canceled each other out, the terms (44) could obviously be factored by $(\psi - \psi_0[\phi_2])$. Because the terms $(\psi - \psi_0[\phi_2])$ are the linear part of EoM, we could eliminate these terms by some field redefinition. Indeed, under the field redefinition,

$$\phi_2 = \phi_3 + \lambda u[\phi_3,\psi], \tag{46}$$

the contributions from S_0 are given by

$$\begin{aligned} &S_0[\phi_2 = \phi_3 + \lambda u[\phi_3,\psi],\psi] - S_0[\phi_3,\psi] \\ &\quad = -\int d^Dx[m^2(\psi - \psi_0[\phi_3]) + kV^{1,0}(\phi_3,\psi)]\lambda u(\phi_3,\psi) \\ &\qquad + \mathcal{O}(\lambda^2). \end{aligned} \tag{47}$$

The contributions from the other terms could be included in $\mathcal{O}(\lambda^2)$. Then (44) could be eliminated, if we choose the term $u[\phi_3,\psi]$ as follows,

$$\begin{aligned} u[\phi_3,\psi] &= -\frac{k}{m^2}\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}, \\ \Delta V^{1,0}(\phi_3,\psi) &\equiv V^{1,0}(\phi_3,\psi) - V^{1,0}(\phi_3,\psi_0[\phi_3]), \\ \Delta\psi &\equiv \psi - \psi_0[\phi_3]. \end{aligned} \tag{48}$$

We should note that, although the representation (48) seems to be defined as a division of $\Delta\psi$, because ΔV proportional to $\Delta\psi$, it is in fact some polynomial of the fields. So that, the field redefinition (46) satisfies the expression (40), and the S matrix elements are invariant under this field redefinition.

For the convenience of later arguments, we now define the operator Δ for any function $f(\psi)$ of ψ as follows,

$$\Delta f(\psi) \equiv f(\psi) - \lim_{\psi\to\psi_0} f(\psi). \tag{49}$$

We should note that $\Delta V^{1,0}(\phi,\psi)$ and $\Delta\psi$ in (48) surely satisfy this definition. Here, for the field redefinitions later, the argument ϕ of the limiting value $\psi_0[\phi]$ is defined as the adopted variable ϕ_n for each frame. In other words, more correctly, we should define the operator Δ_n as follows,

$$\Delta_n f(\psi) \equiv f(\psi) - \lim_{\psi\to\psi_0[\phi_n]} f(\psi), \tag{50}$$

but we ignore the index n just for simplicity. Indeed, the difference of each index n could be included into the ignored term $\mathcal{O}(\lambda^2)$ in all of the following relevant equations.

After the field redefinition (46), the action could be expressed as follows,

$$\begin{aligned} &S_3[\phi_3,\psi,\lambda] \\ &\quad \equiv S_2[\phi_2 = \phi_3 + \lambda u[\phi_3,\psi],\psi,\lambda] \\ &\quad = S_0[\phi_3,\psi] + \int d^Dx\left[\frac{\lambda k^2}{m^2}\left[V^{1,0}(\phi_3,\psi)\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}\right.\right. \\ &\qquad \left.\left. - m^2F^{(2)}[\phi_3]\right] - \lambda\sum_{n=3}^{\infty} k^n F^{(n)}[\phi_3] + \mathcal{O}(\lambda^2)\right]. \end{aligned} \tag{51}$$

From the result of the previous part, $m^2F^{(2)}[\phi_3] = V^{1,0}(\phi_3,\psi_0[\phi_3])V^{1,1}(\phi_3,\psi_0[\phi_3])$, we find the terms proportional to second powers of k are given by

$$\begin{aligned} &\frac{\lambda k^2}{m^2}\left[V^{1,0}(\phi_3,\psi)\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}\right. \\ &\qquad \left. - V^{1,0}(\phi_3,\psi_0[\phi_3])V^{1,1}(\phi_3,\psi_0[\phi_3])\right]. \end{aligned} \tag{52}$$

We should note the following important fact: By taking the on-shell limit, $\psi \to \psi_0[\phi_3]$, because the component $\Delta V^{1,0}(\phi_3,\psi)/\Delta\psi$ in the first term goes to the differential coefficient $V^{1,1}(\phi_3,\psi_0[\phi_3])$, the terms (52) cancel each other out. So that, now, we have verified that the diagrams, less than eight points, including some internal lines of ghost, are not contributing to the physical amplitudes.

Moreover, because the second term in (52) is the on-shell limit of the first term, these terms could be expressed by using Δ defined in (49), i.e.,

$$\begin{aligned} &\left[V^{1,0}(\phi_3,\psi)\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}\right. \\ &\qquad \left. - V^{1,0}(\phi_3,\psi_0[\phi_3])V^{1,1}(\phi_3,\psi_0[\phi_3])\right] \\ &\quad = \Delta\left[V^{1,0}(\phi_3,\psi)\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}\right]. \end{aligned} \tag{53}$$

We should note that the operation of the overall Δ also acts on $\Delta\psi$. On the other hand, Δ in the numerator of $\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}$ should only be operated to $V^{1,0}$. Hence, more correctly, $\frac{\Delta V^{1,0}(\phi_3,\psi)}{\Delta\psi}$ is the term divided by $\Delta\psi$ after operating the Δ to $V^{1,0}$; i.e., we should represent it as follows,

$$\mathrm{rhs} = \Delta\left[V^{1,0}(\phi_3,\psi)\frac{1}{\Delta\psi}\Delta V^{1,0}(\phi_3,\psi)\right]. \tag{54}$$

Now, we find that the terms proportional to k^2 are also equal to zero under the on-shell condition, and find also that these terms are proportional to $\Delta\psi$. So that, these terms could also be eliminated by new field redefinition satisfying the expression (40). We could easily predict the appearance of the terms proportional to k^3 which are also equal to zero under the on-shell condition after the field redefinition. Therefore, we could predict the realization of this mechanism in any order of k. In the next part, we prove the realization of the mechanism.

C. General proof

In the previous part, in lower orders of k, we saw that the first-order terms with respect to λ could be eliminated by the field redefinitions conserving the S matrix elements invariant. In this part, we prove the possibility of the eliminations of the first-order terms with respect to λ in any order of k. Then, by mathematical induction, the proof is performed for some general terms of the actions, the field redefinitions, and the $F^{(n)}$, given by the analogy of the previous part. The general terms given by the analogy in the previous part are expressed as

$$S_n[\phi_n,\psi,\lambda] = S_0[\phi_n,\psi] + \int d^Dx\left[(-1)^{n+1}\frac{k^{n-1}}{m^{2n-4}}\lambda\Delta\left(V^{1,0}(\phi_n,\psi)\frac{1}{\Delta\psi}\Delta\right)^{n-2}V^{1,0}(\phi_n,\psi) - \lambda\sum_{m=n}^{\infty}k^mF^{(m)}[\phi_n]\right] + \mathcal{O}(\lambda^2),\quad (n\geq 2), \tag{55}$$

$$\phi_n = \phi_{n+1} + \frac{(-1)^{n+1}k^{n-1}}{m^{2n-2}}\lambda\frac{1}{\Delta\psi}\Delta\left(V^{1,0}(\phi_{n+1},\psi)\frac{1}{\Delta\psi}\Delta\right)^{n-2}V^{1,0}(\phi_{n+1},\psi),\quad (n\geq 2), \tag{56}$$

$$F^{(n)}[\phi] = \lim_{\psi\to\psi_0}\frac{(-1)^n}{m^{2n-2}}\left(V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\Delta\right)^{n-1}V^{1,0}(\phi,\psi).(n\geq 1) \tag{57}$$

The operator Δ is defined as acting on all the functions in the right-hand side of Δ. An example is given by

$$\begin{aligned}\left(V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\Delta\right)^2V^{1,0}(\phi,\psi) &= V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\Delta V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\Delta V^{1,0}(\phi,\psi)\\ &= V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\Delta V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}(V^{1,0}(\phi,\psi) - V^{1,0}(\phi,\psi_0[\phi]))\\ &= V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}\left[V^{1,0}(\phi,\psi)\frac{1}{\Delta\psi}(V^{1,0}(\phi,\psi) - V^{1,0}(\phi,\psi_0[\phi]))\right.\\ &\quad\left. - V^{1,0}(\phi,\psi_0[\phi])V^{1,1}(\phi,\psi_0[\phi])\right].\end{aligned} \tag{58}$$

We should also note that the second- or higher-order terms with respect to the operator $(1/\Delta\psi)\Delta$ are NOT asymptotic to the one just replaced the operator $(1/\Delta\psi)\Delta$ to normal derivative $d/d\psi$ in the limit of the on-shell condition. The nth-order derivative of the function $f(\psi)$ is asymptotic to the following term,

$$\lim_{\psi\to\psi_0}\left(\frac{1}{\Delta\psi}\Delta\right)^nf(\psi) = \frac{1}{n!}\left[\frac{d^nf(\psi)}{d\psi^n}\right]_{\psi=\psi_0}. \tag{59}$$

There is a difference by the factor $1/n!$. Furthermore, the Leibniz rule is normally given by

$$\frac{1}{\Delta\psi}\Delta(f(\psi)g(\psi)) = \left(\frac{1}{\Delta\psi}\Delta f(\psi)\right)(\lim_{\psi\to\psi_0}g(\psi)) + (\lim_{\psi\to\psi_0}f(\psi))\left(\frac{1}{\Delta\psi}\Delta g(\psi)\right). \tag{60}$$

By using (59) and (60), we can check Eqs. (55)–(57) straightforwardly.

Now, let us prove Eqs. (55)–(57). First of all, we should note that the recursion equations (56) for ϕ_n are not the proposition that should be proved, but the given redefinitions which define the functional forms of the actions S_n. In other words, the functional $S_n[\phi_n, \psi, \lambda]$ is defined by the recursions $S_{n-1}[\phi_{n-1}[\phi_n], \psi, \lambda] = S_n[\phi_n, \psi, \lambda]$ with the initial term (43). In order to verify the correctness of the actions (55), it is necessary to verify the correctness of Eq. (57) in advance.

Let us verify the correctness of Eq. (57). For this purpose, we again solve Eq. (36), which $F[\phi] \equiv \sum_{n=1}^{\infty} k^n F^{(n)}[\phi]$ should satisfy, by using the operator Δ. The form of Eq. (36) is now given by

$$F[\phi] + kV^{1,0}\left(\phi, \psi_0[\phi] + \frac{F[\phi]}{m^2}\right) = 0. \tag{61}$$

By decomposing the term $V^{1,0}(\phi, \psi_0[\phi] + \frac{1}{m^2}F)$ in the terms which do not include any k and the other terms, we obtain

$$\begin{aligned} F[\phi] &= -k\left[V^{1,0}(\phi, \psi_0[\phi]) + \Delta V^{1,0}\left(\phi, \psi_0[\phi] + \frac{F[\phi]}{m^2}\right)\right], \\ \Delta V^{1,0}\left(\phi, \psi_0[\phi] + \frac{F[\phi]}{m^2}\right) &\equiv V^{1,0}\left(\phi, \psi_0[\phi] + \frac{F[\phi]}{m^2}\right) - \lim_{F\longrightarrow 0} V^{1,0}\left(\phi, \psi_0[\phi] + \frac{F[\phi]}{m^2}\right). \end{aligned} \tag{62}$$

Here, the definition of operator Δ is identical with the one used in (49) or (50) in the previous section. Now, the first term $V^{1,0}(\phi, \psi_0[\phi])$ becomes independent of k, and the second term $\Delta V^{1,0}$ includes the first- or higher-order terms with respect to k. Then, the first-order term $F^{(1)}[\phi]$ has been decided. In this way, we can pick up the lowest-order terms with respect to k by using the operator Δ.

Moreover, the terms $\Delta V^{1,0}$ could be decomposed as follows,

$$\begin{aligned} \Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) &= \frac{F}{m^2}\frac{1}{F/m^2}\Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) \\ &= -\frac{k}{m^2}V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) \\ &= -\frac{k}{m^2}\left[\lim_{F\to 0} V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\right. \\ &\quad \left. + \Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\right]. \end{aligned} \tag{63}$$

Here, just for simplicity, we omit the dependence $[\phi]$ in the expressions of ψ_0 and F. In the second line, we substitute Eq. (61) into the component F in the numerator. In the third line, the first term becomes the lowest-order term with respect to k, and the second term includes the higher terms. Then the second-order term $F^{(2)}$ has been decided.

In this way, the term $F^{(n)}[\phi]$ could be decided order by order, by using Eq. (61) and Δ. In the same way as the above procedure, we obtain the general recursion equations,

$$\begin{aligned} &\Delta\left\{V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta\right\}^n V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) \\ &= -\frac{k}{m^2}V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta\left\{V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta\right\}^n V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) \\ &= -\frac{k}{m^2}\left[\lim_{F\to 0}\left\{V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta\right\}^{n+1} V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\right. \\ &\quad \left. + \Delta\left\{V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\frac{1}{F/m^2}\Delta\right\}^{n+1} V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)\right]. \end{aligned} \tag{64}$$

By substituting the recursion equations into Eq. (62) order by order, we obviously obtain the complete form of F as follows,

$$F = \sum_{n=1}^{\infty} \lim_{F\to 0} \frac{(-k)^n}{m^{2n-2}} \left(V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right) \frac{1}{F/m^2}\Delta \right)^{n-1} V^{1,0}\left(\phi, \psi_0 + \frac{F}{m^2}\right)$$
$$= \sum_{n=1}^{\infty} \lim_{\psi\to \psi_0} \frac{(-k)^n}{m^{2n-2}} \left(V^{1,0}(\phi, \psi) \frac{1}{\Delta\psi}\Delta \right)^{n-1} V^{1,0}(\phi, \psi). \tag{65}$$

Therefore, we have verified Eq. (57).

Finally, let us complete our proof by confirming the correctness of general terms of the action (55) by the mathematical induction. In the case of $n = 2$, it is obvious that the action (55) coincide with (43). Now, for the fixed n, we assume that Eq. (55) is correct. Under the field redefinition (56), it is obvious from Eq. (47) that the original action $S_0[\phi_n, \psi]$ of Eq. (55) transforms as follows,

$$S_0[\phi_n, \psi] - S_0[\phi_{n+1}, \psi] = -\int d^D x [m^2 \Delta\psi + k V^{1,0}(\phi_{n+1}, \psi)]$$
$$\times \frac{(-1)^{n+1} k^{n-1}}{m^{2n-2}} \lambda \frac{1}{\Delta\psi} \Delta \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-2} V^{1,0}(\phi_{n+1}, \psi) + \mathcal{O}(\lambda^2)$$
$$= \int d^D x \left[-\frac{(-1)^{n+1} k^{n-1}}{m^{2n-4}} \lambda \Delta \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-2} V^{1,0}(\phi_{n+1}, \psi) \right.$$
$$\left. + \frac{(-1)^{n+2} k^n}{m^{2n-2}} \lambda \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi) \right] + \mathcal{O}(\lambda^2). \tag{66}$$

Then the $(n-1)$th-order terms with respect to k have been eliminated as follows,

$$S_{n+1}[\phi_{n+1}, \psi] \equiv S_n[\phi_n[\phi_{n+1}], \psi]$$
$$= S_0[\phi_{n+1}, \psi] + \int d^D x \left[\frac{(-1)^{n+2} k^n}{m^{2n-2}} \lambda \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi) \right.$$
$$\left. - \lambda k^n F^{(n)}[\phi_{n+1}] - \lambda \sum_{m=n+1}^{\infty} k^m F^{(m)}[\phi_{n+1}] \right] + \mathcal{O}(\lambda^2). \tag{67}$$

By using Eq. (57) which has already been proved, the nth-order terms with respect to k are expressed as follows,

$$\frac{(-1)^{n+2} k^n}{m^{2n-2}} \lambda \left[\left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi) - \lim_{\psi\to\psi_0} \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi) \right]$$
$$= \frac{(-1)^{n+2} k^n}{m^{2n-2}} \lambda \Delta \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi). \tag{68}$$

Finally, we obtain the $(n+1)$th action,

$$S_{n+1}[\phi_{n+1}, \psi] = S_0[\phi_{n+1}, \psi] + \int d^D x \left[\frac{(-1)^{n+2} k^n}{m^{2n-2}} \lambda \Delta \left(V^{1,0}(\phi_{n+1}, \psi) \frac{1}{\Delta\psi} \Delta \right)^{n-1} V^{1,0}(\phi_{n+1}, \psi) \right.$$
$$\left. - \lambda \sum_{m=n+1}^{\infty} k^m F^{(m)}[\phi_{n+1}] \right] + \mathcal{O}(\lambda^2). \tag{69}$$

Therefore, we have completed the proof by using the mathematical induction.

Because the above proof has been a little bit complicated, we now summarize how the conjecture have been proved. By using Eq. (55), in the limit $n \to \infty$, we obtain the following expression,

$$S_\infty[\phi_\infty, \psi, \lambda] = S_0[\phi_\infty, \psi] + \mathcal{O}(\lambda^2). \tag{70}$$

From the argument of Sec. IV B, the terms $\mathcal{O}(\lambda^2)$ could not contribute to any physical amplitude. The interaction terms which contribute to the physical amplitudes are only given by the original interaction term $V(\phi, \psi)$. Therefore, in this frame, it is obvious that the physical amplitudes of the higher derivative theory coincide with the corresponding amplitudes of the original theory.

Moreover, this frame is related with the initial frame obtained by (43) through the field redefinitions (56). Because the field redefinitions (56) satisfy the expression 40)), any scattering amplitudes calculated in the initial frame (43) coincide with the amplitudes of the asymptotic frame (70).

So that, through the asymptotic frame, the physical amplitudes calculated in the initial frame coincide with the corresponding amplitudes in original theory. Because the conjecture (25) is given in the initial frame (43), [that is, (24) in Sec. III B], the conjecture has been proved.

VI. SUMMARY

In this paper, we have investigated the possibility of the elimination of the ghost in the higher derivative theory proposed in [16]. Although the possibility of the elimination of the ghost in the linear level had been argued by Hassan *et al.* [16], it is not so trivial to check whether the ghost could be eliminated in the nonlinear level or not. We have considered the model with two scalar fields interacting by a mass mixing (14), which was proposed in [16], but in the analysis of [16], the nonlinear interaction terms were neglected. We have analyzed the model without neglecting the interaction terms. In Sec. III, although there are many algebraic solutions, we have adopted the unique solution $\psi[\phi]$ which satisfy the condition $\psi[\phi = 0] = 0$. Under this assumption, we have found that in the higher derivative theory, there appears a ghost mode in addition to two healthy modes corresponding to the modes in the original theory. We have called these healthy modes as "physical fields." In Sec. IV, we have defined "physical amplitudes" as the amplitudes where all the external lines are taken to "physical fields." We have also proposed the conjecture (25), where "physical amplitudes" of the higher derivative theory coincide with the amplitudes of the original theory. In this setup, we have proved the conjecture (25) without any additional assumption besides the ones given in Sec. V.

It could be straightforward to extend the analyses given in this paper to the bigravity theory. There is, however, one concern. In order to apply the arguments in this paper to the bigravity theory, we need to pay more attentions to the commutativity between the substitution of the algebraic solution and the procedure of the gauge fixing. Because, as argued in the Appendix, the procedure of the derivation of the higher derivative theory could be regarded as the equivalent rewriting of the path integral. In the path integral formulation, we integrate out the field ϕ first. In the case of the gauge theory, we cannot perform the integral without the gauge fixing. Then we should fix the gauge first of all. Therefore, it could be necessary to investigate the commutativity between the substitution of the algebraic solution and the procedure of the gauge fixing.

In order to investigate this problem if the commutativity, it could be also better to investigate some toy model first. There is a candidate of the toy model: The pseudolinear model [18] is the massive spin-two model which has the nonderivative interaction terms in addition to the linear terms in the Fierz-Pauli model. Moreover, by Hinterbicher in [18], it has been proved that the BD ghost does not appear in this model. The curved space extension of the proof has been given in [19–21]. By using the pseudolinear model and the linearized Einstein-Hilbert action, we easily construct the model with mass mixing such as the bigravity. This theory has the structure very similar to that in the ξ^3-model which have used in Sec. IV B. By this future work, we may investigate the commutativity between the substitution of the algebraic solution and the procedure of the gauge fixing.

APPENDIX: PATH INTEGRAL

In order to show the correspondence between the higher derivative theory and the original theory, we consider how the higher derivative theory is obtained by the equivalent transformations of the generating function of the original theory. Let us consider the path integral with the external sources J_ϕ, J_ψ,

$$Z[J_\phi, J_\psi] = \int D\phi D\psi \exp\left[iS_0[\phi,\psi] + i\int d^Dx(\phi J_\phi + \psi J_\psi)\right]. \quad (A1)$$

By integrating out ϕ, we obtain

$$Z[J_\phi, J_\psi] = \int D\psi \exp\left[iS_0[\phi[\psi, J_\phi],\psi] + i\int d^Dx(\phi[\psi, J_\phi]J_\phi + \psi J_\psi)\right]. \quad (A2)$$

Here, in the tree level, $\phi[\psi, J_\phi]$ is defined as the perturbative solution of the equations of motion,

$$\frac{\delta S_0[\phi,\psi]}{\delta\phi} + J_\phi = 0, \quad (A3)$$

with respect to ϕ around the vacuum $\phi = 0 = \psi$. Around this vacuum, the unique inverse function $\psi[\phi, J_\phi]$, which is the algebraic solution of Eq. (A3) with respect to ψ with the condition $\psi[\phi = 0, J_\phi = 0] = 0$, exists. Under the field redefinition $\psi = \psi[\phi, J_\phi]$, because the inverse function $\psi[\phi, J_\phi]$ satisfies the identity $\phi[\psi[\phi, J_\phi], J_\phi] = \phi$, we obtain

$$Z[J_\phi, J_\psi] = \int D\phi \mathrm{Det}\left[\frac{\delta\psi[\phi, J_\phi]}{\delta\phi}\right] \exp\left[iS_0[\phi, \psi[\phi, J_\phi]] + i\int d^D x(\phi J_\phi + \psi[\phi, J_\phi]J_\psi)\right]. \tag{A4}$$

By introducing the Lagrange multiplier field λ and the FP-ghosts C, $\bar{C}$, we obtain

$$Z[J_\phi, J_\psi] = \int D\phi D\psi D\lambda DC D\bar{C} \exp\left[iS_0[\phi, \psi] + i\int d^D x\left(m^2\lambda(\psi - \psi[\phi, J_\phi]) + i\bar{C}\frac{\delta\psi[\phi, J_\phi]}{\delta\phi}C + \phi J_\phi + \psi J_\psi\right)\right]. \tag{A5}$$

Here, we omit an integral for the FP-ghosts. The exponent of this expression is similar to that in (20) with source terms for ϕ and ψ. The different parts are the terms $\psi[\phi, J_\phi]$ in the Lagrange multiplier terms and the FP-ghosts terms.

Now, we consider the specific case where two fields interacted with each other only through the mass mixing term, i.e., $V(\phi, \psi) = V_\phi(\phi) + V_\psi(\psi)$. In this case, the solution of Eq. (A3) is expressed as follows:

$$m^2\psi[\phi, J_\phi] = m^2\psi[\phi] + J_\phi, \qquad \psi[\phi] \equiv \psi[\phi, 0]. \tag{A6}$$

By substituting this expression into the path integral (A5), we obtain

$$Z[J_\phi, J_\psi] = \int D\phi D\psi D\lambda DC D\bar{C} \exp\left[iS_0[\phi, \psi] + i\int d^D x\left(m^2\lambda(\psi - \psi[\phi]) + i\bar{C}\frac{\delta\psi[\phi]}{\delta\phi}C + (\phi - \lambda)J_\phi + \psi J_\psi\right)\right]. \tag{A7}$$

In the tree level, we could ignore the FP-ghost terms. Under the field redefinition $\phi \to \phi + \lambda$, we obtain

$$\begin{aligned} Z[J_\phi, J_\psi] &\approx \int D\phi D\psi D\lambda \exp\left[iS_0[\phi + \lambda, \psi] + i\int d^D x(m^2\lambda(\psi - \psi[\phi + \lambda]) + \phi J_\phi + \psi J_\psi)\right] \\ &= \int D\phi D\psi D\lambda \exp\left[iS[\phi + \lambda, \psi, \lambda] + i\int d^D x(\phi J_\phi + \psi J_\psi)\right]. \end{aligned} \tag{A8}$$

Here, the equal "≈" means the equivalence up to the FP-ghost terms, the action $S[\phi + \lambda, \psi, \lambda]$ is defined in (20), and the linear part of $S[\phi + \lambda, \psi, \lambda]$ is given in (22). By the fields redefinition (15), we obtain the correspondence between the Green functions,

$$\int D\xi D\eta \exp\left[iS_0[\phi(\xi, \eta), \psi(\xi, \eta)] + i\int d^D x(\xi J_\xi + \eta J_\eta)\right] \approx \int D\xi D\eta D\lambda \exp\left[i\bar{S}[\xi, \eta, \lambda] + i\int d^D x(\xi J_\xi + \eta J_\eta)\right],$$
$$J_\xi \equiv \frac{1}{\sqrt{2}}(J_\phi + J_\psi), \qquad J_\eta \equiv \frac{1}{\sqrt{2}}(J_\phi - J_\psi). \tag{A9}$$

Here, the $\bar{S}[\xi, \eta, \lambda]$ is given in Eq. (24). Therefore, in the case of $V(\phi, \psi) = V_\phi(\phi) + V_\psi(\psi)$, the conjecture (25) can be trivially shown.

In the case of $V(\phi, \psi) \neq V_\phi(\phi) + V_\psi(\psi)$, however, the correspondence is not so trivial, due to the nonlinear dependence of J_ϕ in Eq. (A5). These terms contribute to the Green functions as some composite fields. In order to confirm the conjecture (25), we should show that the diagrams with such composite fields vanish under the on-shell condition. We do not, however, continue the further analysis by using the path integral but we prove the conjecture in another way in this paper.

Now, let us compare the above argument with the argument by Hassan *et al.* in [16]. They also considered the model with the source terms (A1), but they did not argue by using the integration as given above. Their arguments were more straightforward. First, they straightforwardly calculated the algebraic solution of Eq. (A3) in the case of $V(\phi, \psi) = 0$, and substituted the solution to the original action (A1), which coincides with Eq. (A4) without using the Jacobian. After that, by integrating out the obtained higher derivative theory, they obtained the generating function identical with the original theory. They also commented on the extension to the case of $V(\phi, \psi) = V_\phi(\phi) + V_\psi(\psi)$. They claimed the equivalence of both theories based on the above arguments, so they have not shown the correspondence in the case of $V(\phi, \psi) \neq V_\phi(\phi) + V_\psi(\psi)$. This is our motivation for considering the nonlinear case.

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