# Revisiting hyperbolicity of relativistic fluids

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Motivated by the desire for highly accurate numerical computations of compact binary spacetimes in the era of gravitational wave astronomy, we reexamine hyperbolicity and well-posedness of the initial value problem for popular models of general relativistic fluids. Our analysis relies heavily on the dual-frame formalism, which allows us to work in the Lagrangian frame, where computation is relatively easy, before transforming to the desired Eulerian form. This general strategy allows for the construction of compact expressions for the characteristic variables in a highly economical manner. General relativistic hydrodynamics, ideal magnetohydrodynamics, and resistive magnetohydrodynamics are considered in turn. In the first case, we obtain a simplified form of earlier expressions. In the second, we show that the flux-balance law formulation used in typical numerical applications is only weakly hyperbolic and thus does not have a well-posed initial value problem. Newtonian ideal magnetohydrodynamics is found to suffer from the same problem when written in flux-balance law form. An alternative formulation, closely related to that of Anile and Pennisi, is instead shown to be strongly hyperbolic. In the final case, we find that the standard forms of resistive magnetohydrodynamics, relying upon a particular choice of "generalized Ohm's law," are only weakly hyperbolic. The latter problem may be rectified by adjusting the choice of Ohm's law, but we do not do so here. Along the way, weak hyperbolicity of the field equations for dust and charged dust is also observed. More sophisticated systems, such as multifluid and elastic models, are also expected to be amenable to our treatment.

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## I. INTRODUCTION

The first multimessenger observation of gravitational waves from a binary neutron star merger in Ref. [1] marks the beginning of a new era in astronomy. One of the main tasks of numerical relativity in the coming years will thus be in the *accurate* construction and modeling of gravitational waveforms from such spacetimes. This work is of course well underway, see, for example, Ref. [2], but from the point of view of accuracy suffers from a number of problems in practice and in principle. As a consequence, numerical relativity simulations of binary neutron star systems are less accurate than those of binary black holes. The principal cause of this difference is presumably the fact of shock formation in the fluid. For this, sophisticated methods can be employed, see, for example, Refs. [3,4] for introductions to shock-capturing methods in numerical relativity, but ultimately there is no avoiding the fact that a loss of differentiability means forfeiting accuracy. Since shocks are only expected to occur slightly before merger, we may expect that up until that point the quality of the neutron star data would be comparable to the vacuum case. This is also not the case, partially because the additional computational cost of the fluid forces the use of lower resolution but also because the singular nature of the fluid equations at the stellar surface, and the numerical hacks to treat this, serves as a constant source of error.

A mathematically pure approach to the problem would be to first give a proper analysis of the initial free boundary value problem for the full system consisting of the Einstein equations coupled to fluid matter. Unfortunately, such an analysis has not been undertaken for the standard form of the fluid equations in use in numerical relativity, although see Refs. [5,6] for interesting work in this direction. Our view is that this question deserves much more attention. After all, *no* numerical approximation can converge if the continuum partial differential equation (PDE) problem being approximated is ill posed. Such an analysis is, however, fiendishly difficult, not least because even the standard expressions for the characteristics of the relativistic Euler equations are complicated [7]. An alternative approach would be to switch completely to smoothed-particle hydrodynamics [8], although the mathematically inclined might ask similar questions also in that context.

Therefore, as a first step in this direction, we reexamine this basic question of hyperbolicity of general relativistic hydrodynamics (GRHD) and, relying heavily on insights from the dual-foliation and slightly more general dual-frame (DF) formalisms, as presented in Refs. [9–11], exploit structure in the field equations that simplify the resulting expressions. Consequently, we use the same methodology to give a characteristic analysis of the standard form of general relativistic magnetohydrodynamics (GRMHD) and resistive general relativistic magnetohydrodynamics (RGRMHD) as used in numerical relativity.

The paper is structured as follows. In Sec. II, for motivation, we explain the basic treatment of the stellar surface in numerical relativity and give examples of the consequent issues. We then give a brief review of the relevant PDEs theory and DF formalism as used in the paper. Section III contains our hyperbolicity analysis of GRHD, and Sec. IV contains that of GRMHD. In Sec. V, we investigate the hyperbolicity of RGRMHD. We conclude in Sec. VI. We work in $3+1$ dimensions; geometric units with $c = G = 1$ and the summation convention are used throughout. The calculations were performed primarily with xTensor for *Mathematica* [12]; our notebooks are available in Ref. [13].

## II. MOTIVATION AND THEORY OVERVIEW

### A. Stellar surfaces

In most numerical approaches for the treatment of relativistic hydrodynamics, the "Valencia" formulation [14] of the governing equations is employed [15–18]. This formulation is based on the use of two sets of variables: the primitive variables, such as the rest mass density $\rho$, the pressure $p$, and the fluid three-velocity $v^i$, and the corresponding conserved variables [3]. In practice, the flux-balance law PDE for the latter set is used for the time evolution. However, the primitive variables are also required for the flux calculation. The conserved variables can be expressed as simple functions of the primitives, whereas the inverse is usually done by a numerical root finding procedure [15,19]. A fundamental problem of this approach is that this mapping is singular for $\rho \to 0$. Therefore, a low density "atmosphere" is introduced as a threshold to avoid $\rho = 0$ in numerical schemes. Typically, this floor value is chosen to be around 8–12 orders of magnitude smaller than the maximum density of the star. Although an artificial atmosphere allows robust simulations of various neutron star setups, it does not constitute a satisfactory solution to the underlying problem. Furthermore, an artificial atmosphere poses a new problem for high-order schemes. In Fig. 1, the convergence results from the simulation of a single stationary, nonrotating neutron star [Tolman-Oppenheimer-Volkoff (TOV) solution] are shown.

In this simulation, a discontinuous Galerkin (DG) method of polynomial order $N = 3$ is employed. For the top panel result, only the star interior with analytical outer boundary conditions was evolved. Almost perfect pointwise fourth-order convergence can be observed, as expected. However, if stellar surface and artificial atmosphere are added as in a realistic simulation of the entire star, the convergence order rapidly decreases (middle panel), leaving behind no clear systematic behavior. The application of shock-capturing techniques, like the weighted essentially

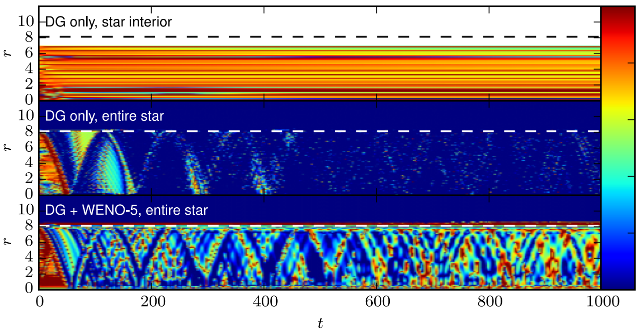


FIG. 1. Pointwise convergence order (color coding) for TOV star simulations with a DG scheme. Top: Only the interior of the star is simulated with a pure DG method and analytic outer boundary conditions. The stellar surface (dashed line) is not inside the numerical domain. Middle: Realistic setup of the entire star including its surface. It is surrounded by a low density atmosphere $\rho_{\rm atm} = 10^{-8}\rho_{\rm max}$. A pure DG method is used for the simulation. Bottom: Realistic setup of the entire star including its surface. The DG method is extended by a WENO-5 limiting procedure.

non-oscillatory (WENO) limiting methodology [20–22], partially cures this problem (bottom panel), and convergence in the $L_1$ norm does look somewhat better, although still not satisfactory. In any case, this strategy can only be seen as a workaround, which is clearly restricting the potential of high-order methods. It is possible that with a proper analysis it will turn out that full neutron star solutions have only a very limited level of regularity and that high-order schemes will never be of huge use in this context. In any case, it would be desirable to know so, since then the focus for developing numerical methods could be placed squarely on obtaining at least low-order convergence while maintaining perfect scalability.

As mentioned in the Introduction, such a "proper analysis" would require a treatment of the general relativistic initial free boundary value problem for the fluid models treated in numerical relativity. Presently, we are unable to do so, in part because of the algebraic complexity of the expressions involved in even the simplest hyperbolicity analysis of these models. This motivates us in what follows to revisit that question and look for structure in the equations that may not have been spotted or used in the past.

### B. PDE analysis

In this subsection, we introduce our notation and explain the key points in showing whether or not a system of PDEs is strongly hyperbolic. We are concerned purely with first-order PDE systems. The statements are taken primarily from Ref. [23], with only slight adjustment for our needs.

#### *1. Well-posedness of hyperbolic equations*

We start by considering a quasilinear system of evolution PDEs, with given time coordinate $t$, of the form

$$\partial_t \mathbf{U} = \mathbf{A}^p(x^\mu, \mathbf{U})\partial_p \mathbf{U} + \boldsymbol{\mathcal{S}}(x^\mu, \mathbf{U}), \tag{1}$$

where in this subsection $p$ stands for a spatial component index. We call $\mathbf{U}$ the state vector, and $\mathbf{A}^p(\mathbf{U}, x^\mu)$, the coefficient matrices of the spatial derivatives, are referred to as the principal matrix, although these are a number of matrices equal to the spatial dimensionality. The initial value problem (IVP) for (1) is called well posed if it admits a unique solution that depends continuously, in a suitable norm, on the initial data. The particular norm will not concern us in the present work. The source vector $\boldsymbol{\mathcal{S}}(x^\mu, \mathbf{U})$ contains all nonprincipal terms. These terms will not contribute to our PDE analysis whatsoever and, when they are included, are present only for completeness. From now on, the dependence of the principal matrix on both the solution and the coordinates $x^\mu$ will be suppressed in our notation. Let $s_i$ be a spatial 1-form normalized so that $(m^{-1})^{ij}s_i s_j = 1$, with $(m^{-1})^{ij}$ an arbitrary symmetric uniformly positive definite matrix which is permitted to depend upon the solution. Contracting the principal matrix with $s_i$, we call the resulting matrix

$$\mathbf{P}^s \equiv \mathbf{A}^s = \mathbf{A}^p s_p, \tag{2}$$

the principal symbol of the PDE system (1) (in the $s_i$-direction). At each point in spacetime, the system (1) is called:

(i) *weakly hyperbolic*, if for each such $s_i$ the eigenvalues of $\mathbf{P}^s$ are real;

(ii) *strongly hyperbolic*, if the system is weakly hyperbolic and for each such $s_i$ the principal symbol has a complete set of eigenvectors written as columns in a matrix $\mathbf{T}_s$ and there exists a constant $K > 0$, independent of $s_i$, such that,

$$|\mathbf{T}_s| + |\mathbf{T}_s^{-1}| \leq K; \tag{3}$$

(iii) *strictly hyperbolic*, if the system is weakly hyperbolic and if for each $s_i$ the eigenvalues are distinct;

(iv) *symmetric hyperbolic*, if there exists a symmetric positive definite *symmetrizer* $\mathbf{H}$, independent of $s_i$, such that $\mathbf{H}\mathbf{A}^p$ is symmetric for each $p$.

Note that if the eigenvectors depend continuously on $s^i$ then condition (3), which will typically be the case in physical systems, with the matrix norm $|\cdot|$ is automatically fulfilled. In that case, proving strong hyperbolicity at a point requires then showing that the principal symbol $\mathbf{P}^s$ has only real eigenvalues and a complete set of eigenvectors; i.e. $\mathbf{P}^s$ is diagonalizable. If a system is strictly and/or symmetric hyperbolic, it is also strongly hyperbolic [23,24]. Since the principal symbol is solution dependent, we note that the precise level of hyperbolicity is, too. For linear constant coefficient problems, strong hyperbolicity is equivalent to well-posedness of the IVP. In the more general case, strong hyperbolicity at each point is a necessary condition for well-posedness; additional smoothness conditions are needed to guarantee well-posedness. We are interested in the present study in establishing hyperbolicity of relativistic fluid models in an efficient manner.

### 2. Characteristic variables

Given a strongly hyperbolic system in the form of (1) with principal symbol $\mathbf{P}^s$ and matrix of right eigenvectors $\mathbf{T}_s$, the diagonalized form of $\mathbf{P}^s$ with its eigenvalues on the diagonal is given by

$$\boldsymbol{\Lambda}^s = \mathbf{T}_s^{-1}\mathbf{P}^s\mathbf{T}_s. \tag{4}$$

We introduce the orthogonal projector to $s_i$, that is, ${}^m\!\perp^j{}_i = \delta^j{}_i - (m^{-1})^{jk}s_k s_i$, and, in this subsection, use capital letters $A$, $B$, $C$ to denote projected component indices. We call the components of the transformed state vector $d_\mu \hat{\mathbf{U}} = \mathbf{T}_s^{-1}\partial_\mu \mathbf{U}$ the characteristic variables in direction $s_i$. The $d$ symbol here symbolizes the fact that the matrix $\mathbf{T}_s^{-1}$, which is generally both position and solution dependent, is *not* to be commuted with the partial derivative. In practice, we may think of the characteristic variables as being constructed from perturbations to the solution. When presenting them, we will employ a notation like $\delta\varphi$ to denote some derivative of a component $\varphi$ of the state vector. The characteristic variables have the property that they satisfy particularly simple equations of motion if we ignore derivatives transverse to $\hat{s}^i = (m^{-1})^{ij}s_j$ and the lower-order source terms,

$$d_t\hat{\mathbf{U}} = \boldsymbol{\Lambda}^s d_{\hat{s}}\hat{\mathbf{U}} + (\mathbf{T}_s^{-1}\mathbf{A}^A\mathbf{T}_s)d_A\hat{\mathbf{U}} + \mathbf{T}_s^{-1}\boldsymbol{\mathcal{S}}. \tag{5}$$

In the linear constant coefficient approximation, dropping the aforementioned terms leaves just decoupled advection equations propagating with speeds determined by the eigenvalues of $\mathbf{P}^s$.

### 3. General relativity with matter

In this paper, we will study different types of matter in full general relativity (GR). We are interested in solutions to the IVP for the Einstein equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{6}$$

which contain derivatives up to second order in space and time for the metric components $g_{\mu\nu}$ on the left-hand side with the energy-momentum tensor $T_{\mu\nu}$ as a source term on the right-hand side. These equations are supplemented with additional evolution equations for the matter variables. The latter may be fluid and/or electromagnetic variables, depending on the physical system under consideration. To treat the metric variables, we may proceed to use a first-order reduction and construct a suitably hyperbolic reformulation of the Einstein equations. In this way, one can write the principal symbol schematically as

$$\mathbf{P}^s = \begin{pmatrix} \mathbf{P}^s_{\mathrm{g}} & \mathbf{P}^s_{\mathrm{g}\times\mathrm{m}} \\ \mathbf{P}^s_{\mathrm{m}\times\mathrm{g}} & \mathbf{P}^s_{\mathrm{m}} \end{pmatrix}, \tag{7}$$

with the principal symbols for the metric $\mathbf{P}^s_{\mathrm{g}}$ and matter variables $\mathbf{P}^s_{\mathrm{m}}$. If the evolution equations for the matter variables contain no second-order derivatives of the metric, the matrix $\mathbf{P}^s_{\mathrm{m}\times\mathrm{g}}$ can be set to zero by replacing first derivatives with reduction variables. If furthermore the energy-momentum tensor contains no derivatives of the fluid variables, the equations of motion of which we assume to be first order, we have $\mathbf{P}^s_{\mathrm{g}\times\mathrm{m}} = 0$, and the statement above that $T_{\mu\nu}$ serves as a source term is justified from the PDEs point of view. In such a case, we may perform the characteristic analysis separately for $\mathbf{P}^s_{\mathrm{g}}$ and $\mathbf{P}^s_{\mathrm{m}}$. Thus, taking a strongly hyperbolic first-order formulation for the metric variables, one needs only to study the properties of $\mathbf{P}^s_{\mathrm{m}}$. In the following, the index m will be dropped. We assume such a minimal coupling throughout the work.

In the following sections, we write the equations of motion in various forms similar to (1), but for convenience instead of the partial derivative operator $\partial_\mu$, we use the spacetime covariant derivative $\nabla$, the Lie derivative $\mathcal{L}_n$, and various other operators to be introduced momentarily. The assumption of minimal coupling allows us to ignore first derivatives of the metric that appear in these expressions by implicitly assuming that they are replaced by the metric reduction variables. This approach is appropriate for any minimally coupled metric-based theory of gravity. Note that care is sometimes needed in avoiding violating the condition, which may render the analysis appropriate only in the Cowling approximation, in which the metric is simply given and only the matter variables must be evolved.

### C. Dual-frame formalism

In this subsection, we give a brief review of the DF approach of Refs. [9,10]. Since only some quantities and relations of the formalism will be given, Ref. [9] is required reading for deeper insights and a full understanding of the construction. Note that, despite the naming of the formalism, we will in fact here use two frames, only one of which defines a coordinate tensor basis.

#### *1. Index notation*

Throughout the paper, we use the latin letters $a-e$ as abstract indices. We also use $p$ as an abstract index, placing it always on the spatial derivative appearing on the right-hand side of our first-order PDE system. The inverse four-metric $g^{ab}$ is the only object permitted to raise and lower indices. Greek indices run from 0 to 3 and denote the components of tensors in the coordinate basis associated with our coordinates $x^\mu = (t, x^i)$. Latin indices $i-k$ run from 1 to 3 and stand for the spatial components in the same basis. The symbol $\partial_a$ stands for the flat covariant derivative naturally defined by $x^\mu$. Indices $n$, $N$, $u$, $V$, $S$, $Q_1$, $Q_2$, $s$, $q_1$, $q_2$, $\mathbb{s}$, $\mathbb{q}_1$, $\mathbb{q}_2$, and $z$ label contraction in that slot with $n^a$ or $n_a$ and so on, respectively. We take capital latin letters $A-C$ as abstract indices denoting the application of the projection operators ${}^{\mathrm{Q}}\!\perp$ or ${}^{\mathrm{q}}\!\perp$, to be defined later. Similarly, we use indices $\mathbb{A}-\mathbb{C}$ and $\hat{\mathbb{A}}-\hat{\mathbb{C}}$ to denote the application of the projection operator ${}^{\mathbb{q}}\!\perp$ over a vector or dual vector, respectively. This will become clear later. For products of different projectors, we write for instance ${}^{\mathbb{q}}\!\perp^a{}_{\hat{\mathbb{B}}}{}^{\mathrm{Q}}\!\perp^B{}_c \equiv {}^{\mathbb{q}}\!\perp^a{}_b{}^{\mathrm{Q}}\!\perp^b{}_c$. Please note that in our notebooks [13] the index notation convention differs somewhat from that used here (see README.txt accompanying the notebooks).

#### *2. Basic idea and objects*

The basic idea of the DF approach is to describe a region of spacetime in two different frames, called the lower- and the uppercase frames. In this paper, the lowercase frame is Eulerian, that is, a coordinate frame associated with coordinates $x^\mu$, as is standard in numerical relativity. It consists of the four vectors $\partial^a_\mu$. The associated coframe is $\nabla_a x^\mu$. Associated with the lowercase frame is also the usual future pointing timelike unit normal to spatial slices of constant $t$, which is, as usual, denoted by $n^a$. Tensors orthogonal to $n^a$ are called lowercase spatial, or just lowercase. The uppercase frame consists of a future pointing timelike unit vector $N^a$, which in our application will be identified with the fluid four-velocity $u^a$, plus any three linearly independent vector fields orthogonal to $N^a$. The latter vectors will be chosen for convenience. Tensors orthogonal to $N^a$ are called uppercase spatial, or just uppercase. We also employ a further frame, consisting of $n^a$ plus three linearly independent lowercase vectors which are to be fixed as and when required. The future pointing unit vectors of the lower- and uppercase frames can be mutually 3 + 1 decomposed as

$$n^a = W(N^a + V^a), \qquad N^a = W(n^a + v^a), \tag{8}$$

with the Lorentz factor $W = (1 - V^a V_a)^{-1/2} = (1 - v^a v_a)^{-1/2} = (1 + \hat{v}^a \hat{v}_a)^{1/2}$. The vectors $v^a = \hat{v}^a / W$ and $V^a$ are the boost vectors orthogonal to $n^a$ and $N^a$, respectively. We also define projection operators by

$$\gamma^b{}_a = \delta^b{}_a + n^b n_a, \qquad {}^{(\mathrm{N})}\gamma^b{}_a = \delta^b{}_a + N^b N_a, \tag{9}$$

which are obviously orthogonal to their associated normal vectors, $\gamma^b{}_a n_b = 0$, ${}^{(\mathrm{N})}\gamma^b{}_a N_b = 0$. The projection operator $\gamma^b{}_a$ becomes the natural induced metric $\gamma_{ab}$ on slices of constant $t$ when both indices are lowered. We call ${}^{(\mathrm{N})}\gamma_{ab}$ and $\gamma_{ab}$ the upper- and lowercase spatial metrics, respectively. Projecting the uppercase spatial metric with $\gamma^b{}_a$ yields

TABLE I. Overview of the relationship between the upper- and lowercase quantities.

| | Uppercase | Lowercase |
|---|---|---|
| Unit normal vector | $N^a = W(n^a + v^a)$ | $n^a = W(N^a + V^a)$ |
| Boost vector | $V^a$ | $v^a = \hat{v}^a/W$ |
| Lorentz factor | $W = (1 - V^a V_a)^{-1/2}$ | $W = (1 - v^a v_a)^{-1/2}$ |
| Projector | ${}^{(\mathrm{N})}\gamma^a{}_b = g^a{}_b + N^a N_b$ | $\gamma^a{}_b = g^a{}_b + n^a n_b$ |
| Boost metric | ${}^{(\mathrm{N})}\mathbb{g}_{ab} := {}^{(\mathrm{N})}\gamma_{ab} + W^2 V_a V_b$ | $\mathbb{g}_{ab} := \gamma_{ab} + \hat{v}_a \hat{v}_b$ |
| Inverse boost metric | ${}^{(\mathrm{N})}(\mathbb{g}^{-1})^{ab} = {}^{(\mathrm{N})}\gamma^{ab} - V^a V^b$ | $(\mathbb{g}^{-1})^{ab} = \gamma^{ab} - v^a v^b$ |

$$\mathbb{g}_{ab} := \gamma^c{}_a \gamma^d{}_b {}^{(\mathrm{N})}\gamma_{cd} = \gamma_{ab} + \hat{v}_a \hat{v}_b, \tag{10}$$

with inverse

$$(\mathbb{g}^{-1})^{ab} = \gamma^{ab} - v^a v^b, \tag{11}$$

which we call the boost metric and inverse boost metric, respectively. In the same way but projecting the lowercase projector $\gamma^b{}_a$ with ${}^{(\mathrm{N})}\gamma^b{}_a$, we define the uppercase boost metric and its inverse,

$$\begin{aligned} {}^{(\mathrm{N})}\mathbb{g}_{ab} &:= {}^{(\mathrm{N})}\gamma^c{}_a {}^{(\mathrm{N})}\gamma^d{}_b \gamma_{cd} = {}^{(\mathrm{N})}\gamma_{ab} + W^2 V_a V_b, \\ {}^{(\mathrm{N})}(\mathbb{g}^{-1})^{ab} &= {}^{(\mathrm{N})}\gamma^{ab} - V^a V^b. \end{aligned} \tag{12}$$

These various relations are collected in Table I.

The vector $n^a$ is by construction hypersurface orthogonal. The lapse function $\alpha$, shift vector $\beta^a$, and time vector $t^a \equiv \partial_t^a$ are defined and related via

$$\begin{aligned} \alpha &= (-\nabla_a t \nabla^a t)^{-\frac{1}{2}}, \qquad n^a = -\alpha \nabla^a t, \\ \beta^a &= \gamma^a{}_b t^b = t^a - \alpha n^a. \end{aligned} \tag{13}$$

The spacetime metric can be expanded in the lowercase frame as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}, \tag{14}$$

with inverse

$$g^{\mu\nu} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}. \tag{15}$$

The intrinsic covariant derivative operator, defined by projection of the spacetime covariant derivative acting on spatial tensors, is denoted by $D$ and has connection $\Gamma$. Finally, the extrinsic curvature $K_{ab}$ is defined using the standard numerical relativity sign convention, by

$$K_{ab} = -\gamma^c{}_a \nabla_c n_b. \tag{16}$$

In the present work, we need not define any such connection variables associated with the uppercase frame, since it will be used exclusively in an algebraic manner to simplify the various matrices that appear in our analysis. The key idea is that by using the DF formalism we may express the equations of motion in a Lagrangian frame that is, for fluid matter, in some sense preferred. This allows us to exploit structure in the field equations that is otherwise not obvious and consequently makes the computation necessary to analyze hyperbolicity relatively straightforward.

### *3. 2+1 decomposition*

In our analysis, we not only split the equations in a $3+1$ manner against the future pointing unit timelike vectors $n^a$ and $N^a$, but we furthermore decompose the two spatial projectors $\gamma^a{}_b$ and ${}^{(\mathrm{N})}\gamma^a{}_b$ against various arbitrary unit spatial vectors. The spatial vectors and associated orthogonal projectors are collected together in Table II. Please note that $g_{cb}{}^{\mathbb{q}}\!\perp^b{}_a$ is not symmetric. Therefore, we distinguish between the abstract indices $\mathbb{A}$ and $\hat{\mathbb{A}}$ of ${}^{\mathbb{q}}\!\perp$ when applied on a tensor.

### *4. PDE notation and characteristic analysis*

Starting from a four-dimensional formulation of a quasi-linear first-order system,

$$\boldsymbol{\mathcal{A}}^a \partial_a \mathbf{U} + \boldsymbol{\mathcal{S}} = 0, \tag{17}$$

we may $3+1$ split the equations against $N^a$ or $n^a$ by inserting $\delta^b{}_a = {}^{(\mathrm{N})}\gamma^b{}_a - N^b N_a = \gamma^b{}_a - n^b n_a$ between $\boldsymbol{\mathcal{A}}^a$ and the derivative operator $\partial_a$. We then obtain two

TABLE II. Summary of the various unit spatial vectors appearing in our $2+1$ decomposed equations, plus their associated projection operators.

| | Uppercase | Lowercase | Lowercase |
|---|---|---|---|
| Unit normal vector | $N^a$ | $n^a$ | $n^a$ |
| Spatial 1-form | $S_a$ | $\mathbb{s}_a$ | $s_a$ |
| Spatial vector | $S^a = {}^{(\mathrm{N})}\gamma^{ab} S_b$ | $\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab}\mathbb{s}_b$ | $s^a = \gamma^{ab} s_b$ |
| Norm | $S_a S^a = 1$ | $\mathbb{s}_a (\mathbb{g}^{-1})^{ab} \mathbb{s}_b = 1$ | $s_a s^a = 1$ |
| Orthogonal projector | ${}^{\mathrm{Q}}\!\perp^b{}_a = {}^{(\mathrm{N})}\gamma^b{}_a - S^b S_a$ | ${}^{\mathbb{q}}\!\perp^b{}_a = \gamma^b{}_a - \hat{\mathbb{s}}^b \mathbb{s}_a$ | ${}^{\mathrm{q}}\!\perp^b{}_a = \gamma^b{}_a - s^b s_a$ |
| Index notation | ${}^{\mathrm{Q}}\!\perp^B{}_A$ | ${}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}}$ | ${}^{\mathrm{q}}\!\perp^B{}_A$ |

potentially equivalent evolution systems for $\mathbf{U}$ in terms of $n^a$ and $N^a$. These are

$$\mathcal{A}^n\partial_n\mathbf{U} = \mathcal{A}^a\gamma^b{}_a\partial_b\mathbf{U} + \mathcal{S},$$
$$\mathcal{A}^N\partial_N\mathbf{U} = \mathcal{A}^{a(\mathrm{N})}\gamma^b{}_a\partial_b\mathbf{U} + \mathcal{S}. \tag{18}$$

To denote clearly the properties of the matrices, we make the following definitions:

$$\mathcal{A}^n \equiv \mathbf{A}^{\mathrm{n}}, \qquad \mathcal{A}^a\gamma^b{}_a \equiv \mathbf{A}^b, \qquad \mathbf{A}^b n_b = 0,$$
$$\mathcal{A}^N \equiv \mathbf{B}^{\mathrm{N}}, \qquad \mathcal{A}^{a(\mathrm{N})}\gamma^b{}_a \equiv \mathbf{B}^b, \qquad \mathbf{B}^b N_b = 0. \tag{19}$$

Let $S^a$ be an arbitrary unit uppercase spatial vector against $N^a$, so $S^aS_a = 1$, $S^aN_a = 0$, and let $\mathbb{s}_a$ be an arbitrary lowercase spatial 1-form against $n^a$, $\mathbb{s}_a n^a = 0$, normalized against the inverse boost metric $\mathbb{s}_a(\mathbb{g}^{-1})^{ab}\mathbb{s}_b = 1$. The eigenvalue problems of these systems in directions $\mathbb{s}_a$ and $S_a$ read

$$\mathbf{l}^{\mathrm{n}}_\lambda((\mathbf{A}^{\mathrm{n}})^{-1}\mathbf{A}^{\mathbb{s}} - \mathbb{1}\lambda) = 0,$$
$$\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}((\mathbf{B}^{\mathrm{N}})^{-1}\mathbf{B}^S - \mathbb{1}\lambda_{\mathrm{N}}) = 0, \tag{20}$$

with principal symbols $(\mathbf{A}^{\mathrm{n}})^{-1}\mathbf{A}^{\mathbb{s}}$ and $(\mathbf{B}^{\mathrm{N}})^{-1}\mathbf{B}^S$, left eigenvectors $\mathbf{l}^{\mathrm{n}}_\lambda$ and $\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}$, and eigenvalues $\lambda$ and $\lambda_{\mathrm{N}}$ for lowercase and uppercase, respectively. Please note that we place on the lowercase eigenvalues no index n. The eigenvalues will in general depend on the spatial vector chosen to obtain the principal symbol. Dependencies on spatial vectors will sometimes be explicitly indicated by square brackets.

Introducing the four-vectors $\phi^a$, $\tilde{\phi}^a$, we could also write the eigenvalue problems as

$$\mathbf{l}^{\mathrm{n}}_\lambda\mathcal{A}^a\phi_a = 0, \qquad \phi_a = -\lambda n_a + \mathbb{s}_a,$$
$$\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}\mathcal{A}^a\tilde{\phi}_a = 0, \qquad \tilde{\phi}_a = -\lambda_{\mathrm{N}}N_a + S_a. \tag{21}$$

### D. Frame independence of strong hyperbolicity

In Ref. [9], it is shown that strong hyperbolicity is unaffected by a switch of coordinates, provided that the boost vector is sufficiently small. Following that result, we will prove that strong hyperbolicity is independent of the choice of frame, provided that a specific estimate on the boost vector is satisfied. This estimate will depend on the maximum eigenvalue of the system. We start with the system of equations for the state vector $\mathbf{U}$ in the uppercase frame,

$$\partial_N\mathbf{U} = \mathbf{B}^p\partial_p\mathbf{U} + \mathcal{S}, \tag{22}$$

and suppose that it is strongly hyperbolic there, so that there is a complete set of (left) eigenvectors in all uppercase spatial directions. Expressing $N_a$ and ${}^{(\mathrm{N})}\gamma^b{}_a$ in terms of the lowercase quantities, the same system can be written as

$$W(\mathbb{1} + \mathbf{B}^V)\partial_n\mathbf{U}$$
$$= [B^a(\gamma^p{}_a + \hat{v}^pV_a) - (\mathbb{1} + \mathbf{B}^V)\hat{v}^p]\partial_p\mathbf{U} + \mathcal{S}, \tag{23}$$

where we have to first investigate the invertibility of $\mathbf{A}^{\mathrm{n}} = W(\mathbb{1} + \mathbf{B}^V)$. Let the uppercase boost vector be written as $V^a = |V|S^a_V$ with norm $|V| = (V^aV_a)^{1/2}$ and unit vector $S^a_V$ in the direction of $V^a$. Since $\mathbf{B}^{S_V}$ is diagonalizable with diagonal form $\mathbf{\Lambda}^{S_V}$, it has a complete set of right eigenvectors written as columns in the matrix $\mathbf{T}_{S_V}$ and $\mathbf{T}_{S_V}$ is invertible. Performing a similarity transformation, we obtain

$$(\mathbf{T}_{S_V})^{-1}(\mathbb{1} + \mathbf{B}^V)\mathbf{T}_{S_V} = \mathbb{1} + |V|\mathbf{\Lambda}^{S_V}, \tag{24}$$

and invertibility of $\mathbb{1} + \mathbf{B}^V$ is guaranteed if for each eigenvalue $\lambda_{\mathrm{N}}[S^a_V]$ the inequality

$$1 + |V|\lambda_{\mathrm{N}}[S^a_V] > 0 \tag{25}$$

for arbitrary unit $S^a_V$ holds. This condition will be guaranteed by assumption in the proof that follows.

Let $S^a$ be an arbitrary unit uppercase spatial vector. The eigenvalue problem in direction $S^a$ corresponding to the PDE system in (22) in the upper frame can be written as

$$\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}(\mathbf{B}^S - \mathbb{1}\lambda_{\mathrm{N}}[S^a]) = 0, \tag{26}$$

where $\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}$ is the uppercase left eigenvector for the principal symbol $\mathbf{B}^S$ with eigenvalue $\lambda_{\mathrm{N}}[S^a]$.

The eigenvalue problem for direction $\mathbb{s}_a$ in the lower frame for the PDE system (23) may be written as

$$\mathbf{l}^{\mathrm{n}}_\lambda(\mathbb{1} + \mathbf{B}^V)^{-1}[\mathbf{B}^S - (\mathbb{1} + \mathbf{B}^V)(\hat{v}^{\mathbb{s}} + W\lambda)] = 0 \tag{27}$$

for lowercase left eigenvector $\mathbf{l}^{\mathrm{n}}_\lambda$ with eigenvalue $\lambda$. The associated principal symbol is

$$\mathbf{P}^{\mathbb{s}} = \frac{1}{W}[(\mathbb{1} + \mathbf{B}^V)^{-1}\mathbf{B}^S - \mathbb{1}\hat{v}^{\mathbb{s}}], \tag{28}$$

and the lowercase spatial 1-form $\mathbb{s}_a$ is related to the uppercase one by $\mathbb{s}_a = S_a + W^2V^S(N_a + V_a)$; see also Table III. The projectors given in Table II satisfy ${}^{\mathbb{q}}\bot^a{}_b = (\mathbb{g}^{-1})^{ac\mathrm{Q}}\bot_{cd}\gamma^d{}_b$.

Introducing the modified lowercase left eigenvector $\mathbf{L}^{\mathrm{n}}_\lambda = \mathbf{l}^{\mathrm{n}}_\lambda(\mathbb{1} + \mathbf{B}^V)^{-1}$ and collecting terms of $\mathbf{B}$, we rewrite Eq. (27) as

$$\mathbf{L}^{\mathrm{n}}_\lambda[\mathbf{B}^{S - V(\hat{v}^{\mathbb{s}} + W\lambda)} - \mathbb{1}(\hat{v}^{\mathbb{s}} + W\lambda)] = 0. \tag{29}$$

By defining the new uppercase unit spatial vector

$$S^a_\lambda[S^b, \lambda] \coloneqq \frac{1}{N}(S^a - V^a(\hat{v}^{\mathbb{s}} + W\lambda)), \tag{30}$$

TABLE III. The relationship between upper- and lowercase unit spatial vectors.

| | Uppercase | Lowercase |
|---|---|---|
| Unit normal vector | $N^a$ | $n^a$ |
| Boost vector | $V^a$ | $v^a$ |
| Spatial vector | $S^a = {}^{(\mathrm{N})}\gamma^{ab}S_b$ | $\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab}\mathbb{s}_b$ |
| Spatial 1-form | $S_a = \mathbb{s}_a + v^{\mathbb{s}} n_a$ ${}^{(\mathrm{N})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\mathbb{s}_c$ | $\mathbb{s}_a = S_a + W^2 V^S(N_a + V_a)$ $\gamma^b{}_a S_b$ |

with normalization,

$$\begin{aligned} N &= [(S^a - V^a(\hat{v}^{\mathbb{s}} + W\lambda))(S_a - V_a(\hat{v}^{\mathbb{s}} + W\lambda))]^{1/2} \\ &= \sqrt{W^2(\lambda - WV^S)^2 + 1 + (V^S)^2W^2 - \lambda^2}, \\ &= \sqrt{W^2(\lambda + v^{\mathbb{s}})^2 + 1 + (v^{\mathbb{s}})^2 - \lambda^2}, \end{aligned} \tag{31}$$

we finally arrive at the eigenvalue problem

$$\mathbf{L}^{\mathrm{n}}_{\lambda}\left[\mathbf{B}^{S_\lambda} - \mathbb{1}\frac{1}{N}(\hat{v}^{\mathbb{s}} + W\lambda)\right] = 0, \tag{32}$$

for the redefined lowercase left eigenvector $\mathbf{L}^{\mathrm{n}}_{\lambda}$, principal symbol $\mathbf{B}^{S_\lambda}$, and eigenvalue $(\hat{v}^{\mathbb{s}} + W\lambda)/N$ in the direction of $S^a_\lambda$. The relation $WV^S = -v^{\mathbb{s}}$ follows by using relations given in Tables I and III. The lowercase eigenvalue problem (32) for fixed $\lambda$ is the same eigenvalue problem as for the uppercase system for eigenvalue $(\hat{v}^{\mathbb{s}} + W\lambda)/N$ in (26) where the spatial direction $S^a$ is replaced by $S^a_\lambda$. Therefore,

$$\frac{1}{N}(\hat{v}^{\mathbb{s}} + W\lambda) = \lambda_{\mathrm{N}}[S^a_\lambda] \tag{33}$$

must hold.

Equation (33) is a strong result, since we are now able to calculate the lowercase frame eigenvalues from knowledge of the uppercase results. Nevertheless, solving for $\lambda$ may be hard since both $N$ and $\lambda_{\mathrm{N}}$ contain polynomials in $\lambda$. The lowercase left eigenvector to eigenvalue $\lambda$ is then simply given by

$$\mathbf{l}^{\mathrm{n}}_{\lambda}[\mathbb{s}_b] = \mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}[S^a_\lambda](\mathbb{1} + \mathbf{B}^V), \tag{34}$$

and the right eigenvectors are given by

$$\mathbf{r}^{\mathrm{n}}_{\lambda}[\mathbb{s}_b] = \mathbf{r}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}[S^a_\lambda]. \tag{35}$$

The proof is as follows. We know that for arbitrary unit spatial $S^a$ the principal symbol $\mathbf{P}^S$ has:

(1) real eigenvalues $\lambda_{\mathrm{N}}$,
(2) a complete set of left and right eigenvectors obeying $|\mathbf{T}_S| + |\mathbf{T}_S^{-1}| \leq K$, where $\mathbf{T}_S$ is the matrix of right (or left) eigenvectors written as columns (or rows) and $K$ is independent of $S^a$.

We assume furthermore that:

(3) all uppercase eigenvalues fulfill the inequality $1 - |\lambda_{\mathrm{N}}||V| > 0$, for all uppercase unit spatial $S^a$.

This assumption automatically guarantees the condition (25) for the invertibility of $\mathbb{1} + \mathbf{B}^V$.

*The lowercase eigenvalues are real.*—We start by showing that the lowercase system is at least weakly hyperbolic. By use of (33), we obtain

$$\lambda = \frac{W^3V^S(1-\lambda_{\mathrm{N}}^2) + \lambda_{\mathrm{N}}W\sqrt{1 + \lambda_{\mathrm{N}}^2(1/W^2 - 1 + (V^S)^2)}}{W^2(1 - \lambda_{\mathrm{N}}^2(1 - 1/W^2))} \tag{36}$$

for given $\lambda_{\mathrm{N}}$. The only danger is that the terms within the square root are negative, but considering these, we have

$$\begin{aligned} 1 + \lambda_{\mathrm{N}}^2(1/W^2 - 1 + (V^S)^2) &= 1 - \lambda_{\mathrm{N}}^2(|V|^2 - (V^S)^2) \\ &\geq 1 - \lambda_{\mathrm{N}}^2|V|^2 > 0, \end{aligned} \tag{37}$$

where we have used assumptions (1) and (3). Therefore, all lowercase eigenvalues are real.

*The lowercase eigenvectors are linearly independent.*—Take a lowercase eigenvalue $\lambda$ with algebraic multiplicity $k$. Then, by Eq. (33), the corresponding uppercase eigenvalue $\lambda_{\mathrm{N}}[S^a_\lambda]$ has also algebraic multiplicity $k$. Thus, by assumptions (2), which ensures that we can find $k$ linearly independent eigenvectors to the associated eigenvalue problem (32), and (3), which guarantees the invertibility of $\mathbb{1} + \mathbf{B}^V$, and the use of Eq. (34), we know that we can find $k$ linearly independent lowercase left eigenvectors in the eigenspace of $\lambda$. This statement holds also for the right eigenvectors. Therefore, the lowercase principal symbol is diagonalizable.

*Show necessary regularity conditions.*—Let us label the left and right eigenvectors and eigenvalues, making duplicates to account for their multiplicity if necessary, with an index, writing $\mathbf{l}_{\lambda_{(i)}}$, $\mathbf{r}_{\lambda_{(i)}}$, and $\lambda_{(i)}$, respectively. Please note that only in this proof indices $i$, $j$ label characteristic quantities and do not stand for spatial tensor basis components. We denote $\mathbf{T}_{\mathbb{s}}$ as the matrix of lowercase right vectors, where the $i$th column of $\mathbf{T}_{\mathbb{s}}$ is $\mathbf{r}_{\lambda_{(i)}}$. We order so that the $i$th row of $\mathbf{T}_{\mathbb{s}}^{-1}$ is $\mathbf{l}_{\lambda_{(i)}}$. Thus, $\mathbf{l}_{\lambda_{(i)}}\mathbf{r}_{\lambda_{(j)}} = \delta_{ij}$. By Eqs. (34) and (35), we can express for each $i$ the lowercase eigenvectors $\mathbf{l}_{\lambda_{(i)}}, \mathbf{r}_{\lambda_{(i)}}$ as $\mathbf{l}^{\mathrm{N}}_{\lambda^{\mathrm{N}}_{(i)}}[S^a_{\lambda_{(i)}}](\mathbb{1} + \mathbf{B}^V)$ and $\mathbf{r}^{\mathrm{N}}_{\lambda^{\mathrm{N}}_{(i)}}[S^a_{\lambda_{(i)}}]$, respectively. The uppercase principal symbol is diagonalizable by assumption (2), so for each $i$, we may extend each such left or right eigenvector with the remaining linearly independent eigenvectors of the uppercase principal symbol for spatial vector $S^a_{\lambda_{(i)}}$. We denote by $\mathbf{T}_{S_{\lambda_{(i)}}}$ the matrices of those completed sets of eigenvectors expanding the chosen $\mathbf{r}^{\mathrm{N}}_{\lambda^{\mathrm{N}}_{(i)}}[S^a_{\lambda_{(i)}}]$ (and $\mathbf{l}^{\mathrm{N}}_{\lambda^{\mathrm{N}}_{(i)}}[S^a_{\lambda_{(i)}}]$) written as columns (rows). The chosen $i$th right (left) eigenvector is placed in the $i$th column (row). By assumption (2), we then have

$$|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}| + |\mathbf{T}_{S_{\lambda_{(i)}}}| \leq K_{(i)} \tag{38}$$

for each $i$.

Define now the square diagonal quadratic matrix $\mathbf{D}_{(i)}$, which has in the $i$th entry of its diagonal 1 and otherwise zeros,

$$\mathbf{D}_{(i)} \coloneqq \mathrm{diag}(\underbrace{0, \ldots, 0}_{i-1\ \text{times}}, 1, 0, \ldots, 0); \qquad \sum_i \mathbf{D}_{(i)} = \mathbb{1}.$$

Their norm is $|\mathbf{D}_{(i)}| = \max_{|\mathbf{y}|=1} |\mathbf{y}_{(i)}| = 1$, where $\mathbf{y}_{(i)}$ is the $i$th component of $\mathbf{y}$. Then, with the above definitions,

$$\mathbf{T}_{\mathbb{S}} = \sum_i \mathbf{T}_{S_{\lambda_{(i)}}} \mathbf{D}_{(i)},$$
$$\mathbf{T}^{-1}_{\mathbb{S}} = \sum_i \mathbf{D}_{(i)} \mathbf{T}^{-1}_{S_{\lambda_{(i)}}} (\mathbb{1} + \mathbf{B}^V), \tag{39}$$

and we can give the estimate

$$\begin{aligned} |\mathbf{T}^{-1}_{\mathbb{S}}| + |\mathbf{T}_{\mathbb{S}}| &\leq \sum_i (|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}| |\mathbb{1} + \mathbf{B}^V| + |\mathbf{T}_{S_{\lambda_{(i)}}}|) \\ &\leq \sum_i (|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}| + |\mathbf{T}_{S_{\lambda_{(i)}}}|) \max\{1, |\mathbb{1} + \mathbf{B}^V|\} \\ &\leq \sum_i K_{(i)} \max\{1, |\mathbb{1} + \mathbf{B}^V|\} \equiv K. \end{aligned} \tag{40}$$

In the first step, we inserted (39) for the matrices and used the submultiplicity of the norm. In the second, we estimated the prefactors, and finally, in the last step, we used assumption (2) given by (38) for each $i$. We thus arrive at the inequality (3), which together with the above properties gives strong hyperbolicity in the lowercase frame.

*Multiplicity and degeneracies.*—The definition of strong hyperbolicity does not require that the multiplicity of the eigenvalues be constant as the spatial direction is varied. In the literature on relativistic fluids, special cases in which the algebraic multiplicity of a particular eigenvalue increases when looking in particular special directions are called degeneracies of the system. All such possible degeneracies must be taken into account in the demonstration of strong hyperbolicity since diagonalizability of the principal symbol is required in all directions. Note that the relationship between the occurrence of degeneracies in the uppercase and lowercase systems is, however, not trivial. The key point is that when transforming from the lowercase system to the associated uppercase eigenvalue problem (32) we consider the latter only for a fixed eigenvalue. For different eigenvalues, we naturally assign *different* uppercase eigenvalue problems. Therefore, it may be that, for example, uppercase degeneracies always occur in pairs, while the same is not true in the lowercase frame. Indeed, we will see that this is the case for a particular formulation of GRMHD. The relationship between the degeneracies plays no role in the foregoing proof of the equivalence of strong hyperbolicity across the two frames.

*Discussion.*—All systems we study in relativistic physics will satisfy, by construction, that the boost velocities $v_a$ are always smaller than the speed of light. We will furthermore immediately reject any equation of state that results in wave speeds, that is, eigenvalues of the principal symbol, that are greater than the speed of light. This is reasonable in the current study since we are concerned exclusively with relativistic fluid models. On the other hand, however, one should not get the false impression that this must always be the case in relativistic physics. Theories with gauge freedom, such as the electromagnetism and GR, do admit hyperbolic formulations with superluminal speeds. In GR, the obvious example of such a gauge is the popular moving-puncture family. In that case, when the boost vector becomes too large, uppercase strong hyperbolicity will not be sufficient to guarantee the same in the lowercase frame, since the crucial inequality $|\lambda_{\mathrm{N}}||V| < 1$ can be violated. In fact, since GRMHD also inherits some gauge freedom from the Maxwell equations, the same could be said for that model. Such subtleties will not affect us in practice.

### E. Variable independence of strong hyperbolicity

Let $\mathbf{U}$ be a state vector for which the principal symbol $\mathbf{P}^s_{\mathbf{U}}$ is diagonalizable for each unit spatial 1-form $s_a$. Let $\mathbf{V}$ be another state vector of the same dimension, the components of which depend smoothly on the components of $\mathbf{U}$. Derivatives of the two state vectors are then related by the Jacobian $\mathbf{J}$,

$$\partial_a \mathbf{V} = \mathbf{J} \partial_a \mathbf{U}. \tag{41}$$

The principal symbol for $\mathbf{V}$ is then

$$\mathbf{P}^s_{\mathbf{V}} = \mathbf{J} \mathbf{P}^s_{\mathbf{U}} \mathbf{J}^{-1}. \tag{42}$$

Since this transformation is nothing more than a similarity transformation, the eigenvalues remain the same, and the (left) right eigenvectors for $\mathbf{V}$ are just modified by a matrix multiplication with the (inverse) Jacobian. Thus, as is well known, strong hyperbolicity is independent of the choice of evolved variables. Note that, during the hyperbolicity analysis, one choice of variables may make the practical computations very much easier than another.

### F. Recovering the eigenvalues and eigenvectors of the lowercase frame

In this subsection, we explain how we use the results of Sec. II D. As mentioned before, the upper frame will be chosen as the frame of a comoving observer with the fluid, so we take

$$N^a \equiv u^a, \qquad {}^{(\mathrm{u})}\gamma^a{}_b = g^a{}_b + u^a u_b, \tag{43}$$

with the four-velocity of the fluid $u^a$. Despite that we are in the fluid frame or so-called Lagrangian frame, we never set one of the boost vectors to zero.

Since we obtain all our results using computer algebra, it is convenient to introduce a basis to obtain scalar quantities as entries in the matrices. The various basis vectors are given in Table IV. Given a spatial vector $s^a$ of unit magnitude with respect to some metric, we consider a set $\{s^a, q_1{}^a, q_2{}^a\}$ forming a right-handed orthonormal basis with respect to the same metric.

Let $S^a$ be an arbitrary unit uppercase spatial vector. Given a strongly hyperbolic system of PDEs in the form of Eq. (22) with $N^a \equiv u^a$, we write the principal symbol as $\mathbf{P}^S$. We denote the known eigenvalues of $\mathbf{P}^S$ by $\lambda_{\mathrm{u}}[S^a]$ and the known complete set of left eigenvectors, obtained by Eq. (26), by $\mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]$. Then, the lowercase eigenvalues are given by Eq. (33), and the lowercase left eigenvectors $\mathbf{l}^{\mathrm{n}}_\lambda$ for eigenvalue $\lambda$ are given by Eq. (34), that is, for a specific choice of a basis,

$$\begin{aligned}\mathbf{l}^{\mathrm{n}}_\lambda|_{\mathbb{s}} &= \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}}(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}})\\ &= \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}_\lambda}\mathbf{T}_\lambda(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}})\\ &= \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}_\lambda}(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}_\lambda})\mathbf{T}_\lambda,\end{aligned} \tag{44}$$

and the lowercase right eigenvectors $\mathbf{r}^{\mathrm{n}}_\lambda$ are obtainable by

$$\begin{aligned}\mathbf{r}^{\mathrm{n}}_\lambda|_{\mathbb{s}} &= \mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}}\\ &= (\mathbf{T}_\lambda)^{-1}\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}_\lambda}\end{aligned} \tag{45}$$

for given uppercase right eigenvector $\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]$. We denote by $\mathbf{T}_\lambda$ the transformation matrix between bases associated to $S^a$ and $S^a_\lambda$ on the level of eigenvectors and matrices.

TABLE IV. Overview of the upper- and lowercase basis vectors.

| | Uppercase | | Lowercase | |
|---|---|---|---|---|
| Unit normal vector | $N^a$ | $N^a$ | $n^a$ | $n^a$ |
| Spatial 1-form | $S^\lambda_a$ | $S_a$ | $\mathbb{s}_a$ | $s_a$ |
| Spatial vector | $S^a_\lambda$ | $S^a$ | $\hat{\mathbb{s}}^a$ | $s^a$ |
| Orthogonal basis 1-forms | $Q_1{}^\lambda_a, Q_2{}^\lambda_a$ | $Q_{1a}, Q_{2a}$ | $\mathbb{q}_{1a}, \mathbb{q}_{2a}$ | $q_{1a}, q_{2a}$ |
| Orthogonal basis vectors | $Q_1{}^a_\lambda, Q_2{}^a_\lambda$ | $Q_1{}^a, Q_2{}^a$ | $\hat{\mathbb{q}}^a_1, \hat{\mathbb{q}}^a_2$ | $q_1{}^a, q_2{}^a$ |
| Normalized/orthogonal via | ${}^{(\mathrm{u})}\gamma^{ab}$ | ${}^{(\mathrm{u})}\gamma^{ab}$ | $(\mathbb{g}^{-1})^{ab}$ | $\gamma^{ab}$ |

Two opportunities to obtain the lower eigenvectors are possible: Either we take the uppercase principal symbol $\mathbf{B}^{S_\lambda}|_{\mathbf{S}}$ in a basis associated to $S^a$ and calculate for given $\lambda_{\mathrm{u}}[S^a_\lambda]$ the new uppercase eigenvectors or we take the uppercase eigenvectors to $\mathbf{B}^S|_{\mathbf{S}}$ in a basis associated to $S^a$ and make the replacement $\boldsymbol{S} \to \boldsymbol{S}_\lambda = (S^a_\lambda, Q_1{}^a_\lambda, Q_2{}^a_\lambda)^T$ which naturally defines a SO(3)-transformation $\mathbf{R}$. Using the first way, the left and right eigenvectors are given by the formulas in the first line of Eqs. (44) and (45). However, the principal symbol might lose its easy form, which could be especially crucial for a high number of evolved variables. Therefore, we chose the second procedure in our notebooks [13], where the second (and/or third) lines of Eqs. (44) and (45) are used to obtain the lower eigenvectors.

The recovery will be explicitly shown for the system of GRMHD in Sec. IV. For the analysis of GRHD, the procedure is given in the corresponding notebook [13] but not in the paper.

For the sake of clarity, we finally want to relate all our explanations with the covariant form of characteristic analysis using the vector $\phi_a$ and the eigenvalue problem as in (21). Taking the four-vector of the form $\phi_a = -\lambda n_a + \mathbb{s}_a$ with $\lambda = \lambda[\mathbb{s}_b]$ and writing the lowercase vectors in terms of $u^a$, $V^a$, and $S^a$, we obtain

$$\begin{aligned}\phi_a &= -\lambda n_a + \mathbb{s}_a\\ &= -\lambda(W u_a + W V_a) + S_a + W^2 V^S(u_a + V_a)\\ &= (W^2 V^S - W\lambda)u_a + S^a + (W^2 V^S - W\lambda)V^a\\ &= N(-\lambda_{\mathrm{u}}[S^b_\lambda]u_a + S^\lambda_a)\\ &\propto -\lambda_{\mathrm{u}}[S^b_\lambda]u_a + S^\lambda_a.\end{aligned} \tag{46}$$

The last step is done since $\phi_a$ is defined up to an arbitrary scalar factor and we always consider unit spatial vectors for the characteristic analysis.

## III. HYPERBOLICITY OF GRHD

We now start applying the formalism of the last section to concrete examples of fluid matter models. We begin with the simple case of an ideal fluid. Because a full characteristic analysis has been nicely given in Ref. [7], the calculations here serve first as a sanity check in a nontrivial example but second as a proof of principle that the DF approach to the analysis results in an economic treatment. Thus, we consider the energy-momentum tensor of an ideal fluid,

$$T^{ab} = \rho_0 h u^a u^b + p g^{ab}, \tag{47}$$

with the four-velocity of the fluid elements $u^a$, rest mass density $\rho_0$, specific enthalpy $h$, and pressure $p$. The specific enthalpy $h$ can be expressed in terms of $\rho_0$, $p$ and the specific internal energy $\varepsilon$ as

$$h = 1 + \varepsilon + \frac{p}{\rho_0}. \tag{48}$$

The evolution equations of the system are the conservation of energy momentum

$$\nabla_a(T^{ab}) = 0 \tag{49}$$

and the conservation of particle number

$$\nabla_a(\rho_0 u^a) = 0. \tag{50}$$

Projecting Eq. (49) along and perpendicular to the fluid four-velocity $u^a$, we get the equations

$$\rho_0 h \nabla_a u^a + u^a \nabla_a(\rho_0 + \varepsilon\rho_0) = 0 \tag{51}$$

and

$$\rho_0 h {}^{(\mathrm{u})}\gamma^c{}_b u^a \nabla_a u^b + {}^{(\mathrm{u})}\gamma^{ca}\nabla_a p = 0, \tag{52}$$

respectively. We choose an arbitrary equation of state (EOS) of the form

$$p = p(\rho_0, \varepsilon). \tag{53}$$

Equations (50)–(53) provide us with six equations for the six unknown quantities $(\rho_0, \varepsilon, p, \hat{v}_a)$. By (53), we only need to evolve the state vector $\mathbf{U} = (p, \hat{v}_a, \varepsilon)^T$. The components of $\mathbf{U}$, expanded in our lowercase (Eulerian) tensor basis, may be viewed as a slightly modified version of the *primitive variables* $\rho_0$, $\varepsilon$, $v_i$ commonly used in the literature. The characteristic analysis will be performed on the system of equations (50)–(52), the state vector $\mathbf{U}$, in particular in a non-flux-balance law form. Since there is no gauge freedom in the system, the analysis applies unambiguously even after a change of variables, for example, to the conservative variables $D, \tau, S_i$ defined in, for example, Ref. [15]. This is assured by the proof in Sec. II E.

### A. Lowercase formulation

We split Eqs. (50)–(52) now against $n^a$ and $\gamma^a{}_b$ to get a system of first-order partial differential equations for the variables $(p, \hat{v}_a, \varepsilon)$. Doing so, it is easy to show that the system of equations can be rewritten as

$$\mathcal{L}_n p = (c_s^2 - 1)W_{c_s}^2 v^a D_a p - c_s^2 \rho_0 h \frac{W_{c_s}^2}{W}(\mathbb{g}^{-1})^{ab} D_a \hat{v}_b + c_s^2 \rho_0 h W_{c_s}^2 (\mathbb{g}^{-1})^{ab} K_{ab}, \tag{54}$$

$$\gamma^b{}_a \mathcal{L}_n \hat{v}_b = -\frac{1}{W\rho_0 h}(\gamma^c{}_a + c_s^2 W_{c_s}^2 v^c v_a) D_c p - v^c D_c \hat{v}_a + c_s^2 W_{c_s}^2 v_a (\mathbb{g}^{-1})^{bc} D_b \hat{v}_c - c_s^2 W_{c_s}^2 (\mathbb{g}^{-1})^{bc} K_{bc} \hat{v}_a - W D_a \ln\alpha, \tag{55}$$

$$\mathcal{L}_n \varepsilon = \frac{p}{\rho_0^2 h}\frac{W_{c_s}^2}{W^2} v^a D_a p - \frac{p}{\rho_0}\frac{W_{c_s}^2}{W}(\mathbb{g}^{-1})^{ab} D_a \hat{v}_b - v^a D_a \varepsilon + \frac{p}{\rho_0} W_{c_s}^2 (\mathbb{g}^{-1})^{ab} K_{ab}, \tag{56}$$

with $W_{c_s} = 1/\sqrt{1 - c_s^2 v^2}$, where $c_s$ is the local speed of sound and

$$c_s^2 = \frac{1}{h}\left(\chi + \frac{p}{\rho_0^2}\kappa\right), \quad \chi = \left(\frac{\partial p}{\partial \rho_0}\right)_\varepsilon, \quad \kappa = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho_0}. \tag{57}$$

Unless otherwise stated, we consider only matter or EOS with speed of sound $0 < c_s \leq 1$. As one can see, we have used the Lie derivative $\mathcal{L}_n$ along the timelike unit normal vector $n^a$ instead of $\partial_t$ and have written the covariant derivative $D_a$ associated to the intrinsic metric $\gamma_{ab}$ instead of $\partial_i$, but as discussed in Sec. II, this makes no difference to our analysis. Writing the system (54)–(56) as a vectorial equation of the form

$$\mathbf{A}^n \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}}, \tag{58}$$

we can identify

$$\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^b{}_a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{A}^p = \begin{pmatrix} (c_s^2 - 1)W_{c_s}^2 v^p & -c_s^2 \rho_0 h \frac{W_{c_s}^2}{W}(\mathbb{g}^{-1})^{pc} & 0 \\ -\frac{1}{W\rho_0 h} f^p{}_a & c_s^2 W_{c_s}^2 (\mathbb{g}^{-1})^{pc} v_a - v^p \gamma^c{}_a & 0 \\ \frac{p}{\rho_0^2 h}\frac{W_{c_s}^2}{W^2} v^p & -\frac{p}{\rho_0}\frac{W_{c_s}^2}{W}(\mathbb{g}^{-1})^{pc} & -v^p \end{pmatrix}, \tag{59}$$

with shorthand $f^p{}_a = \gamma^p{}_a + c_s^2 W_{c_s}^2 v^p v_a$ and can write the source vector here as

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} c_s^2 \rho_0 h W_{c_s}^2 (\mathbb{g}^{-1})^{ab} K_{ab} \\ -c_s^2 W_{c_s}^2 (\mathbb{g}^{-1})^{bc} K_{bc} \hat{v}_a - W D_a \ln\alpha \\ \frac{p}{\rho_0} W_{c_s}^2 (\mathbb{g}^{-1})^{ab} K_{ab} \end{pmatrix}. \tag{60}$$

Note that written in this form the principal parts of special and general relativistic hydrodynamics take an almost identical form. Let $\mathbb{s}_a$ be an arbitrary lowercase spatial 1-form, normalized against the inverse boost metric so that $(\mathbb{g}^{-1})^{ab}\mathbb{s}_a\mathbb{s}_b = 1$, and let ${}^{\mathbb{q}}\!\perp^b{}_a := \gamma^b{}_a - (\mathbb{g}^{-1})^{bc}\mathbb{s}_c\mathbb{s}_a$ be the orthogonal projector. Recalling the definition of $\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab}\mathbb{s}_b$ given in Table II, we write $\gamma^a{}_b = \hat{\mathbb{s}}^a\mathbb{s}_b + {}^{\mathbb{q}}\!\perp^a{}_b$. Inserting this relation into (58) and expanding leads to

$$(\mathcal{L}_n \mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} \simeq \mathbf{P}^{\mathbb{s}} (D_{\hat{\mathbb{s}}} \mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{B}}}, \tag{61}$$

with the principal symbol $\mathbf{P}^{\mathbb{s}} = \mathbf{A}^{\mathbb{s}} =$

$$\begin{pmatrix} W_{c_s}^2(c_s^2-1)v^{\mathbb{s}} & -\frac{W_{c_s}^2}{W}c_s^2\rho_0 h & 0^{\mathbb{B}} & 0 \\ -\frac{W^2+c_s^2(v^{\mathbb{s}})^2W_{c_s}^2}{W^3\rho_0 h} & -\frac{W^2-c_s^2W_{c_s}^2}{W^2}v^{\mathbb{s}} & 0^{\mathbb{B}} & 0 \\ -\frac{c_s^2W_{c_s}^2}{W\rho_0 h}v_{\hat{\mathbb{A}}}v^{\mathbb{s}} & c_s^2W_{c_s}^2 v_{\hat{\mathbb{A}}} & -v^{\mathbb{s}}{}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}} & 0_{\hat{\mathbb{A}}} \\ \frac{pW_{c_s}^2}{W^2\rho_0^2 h}v^{\mathbb{s}} & -\frac{pW_{c_s}^2}{W\rho_0} & 0^{\mathbb{B}} & -v^{\mathbb{s}} \end{pmatrix}. \quad (62)$$

The symbol "$\simeq$" denotes equality up to transverse principal and source terms. For any derivative operator $\delta$ and vector $z^a$, we write $(\delta\hat{v})_z \equiv z^a\delta\hat{v}_a$, and for the state vector, $(\delta\mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} = (\delta p, (\delta\hat{v})_{\hat{\mathbb{s}}}, (\delta\hat{v})_{\hat{\mathbb{A}}}, \delta\varepsilon)^T$. As explained earlier in Sec. II C, we introduce here furthermore the indices $\mathbb{A}$ and $\hat{\mathbb{A}}$, which are abstract but which indicate application of the orthogonal projector ${}^{\mathbb{q}}\!\perp^b{}_a$, meaning $z_{\hat{\mathbb{A}}} = {}^{\mathbb{q}}\!\perp^a{}_{\hat{\mathbb{A}}}z_a$ and $z^{\mathbb{A}} = {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_b z^b$ for any object $z$. Then, for example, we get $\gamma^a{}_b(\delta\hat{v})_a = \mathbb{s}_b(\delta\hat{v})_{\hat{\mathbb{s}}} + {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_b(\delta\hat{v})_{\hat{\mathbb{A}}}$.

Before we proceed with the characteristic analysis, a comment should be made. By the use of $\hat{v}_a$ in the state vector, the inverse boost metric arose in the principal part (59). By taking $\mathbb{s}_a$ normalized by $(\mathbb{g}^{-1})^{ab}$, we were able to get rid of this complication in the principal symbol, which became "easy," in the sense that it is highly structured. The principal symbol as well as the eigenvalues and eigenvectors for a state vector $(p, v_a, \varepsilon)$ can be found in the Appendix. Since we normalize the spatial 1-form $\mathbb{s}_a$ against the inverse boost metric, the eigenvalues and vectors take a form that is slightly modified in comparison with the literature, but these differences are purely superficial.

Solving the characteristic polynomial, one gets the five real eigenvalues

$$\lambda_{(0,1,2)} = -v^{\mathbb{s}},$$
$$\lambda_{(\pm)} = -\frac{1}{1-c_s^2v^2}\left((1-c_s^2)v^{\mathbb{s}} \pm \frac{c_s}{W}\sqrt{1-c_s^2v_\perp^2}\right), \quad (63)$$

with the shorthand $v_\perp^2 \coloneqq v^{\mathbb{A}}v_{\hat{\mathbb{A}}}$.

Please note that all eigenvalues in this paper have the opposite sign in comparison to the literature by our definition of the principal symbol. In the one-dimensional limit, i.e., $v_\perp = 0$, the eigenvalues $\lambda_{(\pm)}$ reduce to

$$\lambda_{(\pm)} = -\frac{v^{\mathbb{s}} \pm Wc_s}{1 \pm \frac{c_s v^{\mathbb{s}}}{W}},$$

which, as noted elsewhere [25], is just the special relativistic addition of two velocities multiplied with $W$. Due to our choice of a three-basis normalized by the inverse boost metric, the eigenvalues are slightly different as compared to the results in the Appendix.

The left eigenvectors of the principal symbol with our variable choice for the respective eigenvalues $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are

$$\left(-\frac{p}{c_s^2\rho_0^2h} \quad 0 \quad 0^{\mathbb{A}} \quad 1\right), \qquad \left(\frac{1}{\rho_0 h}\hat{v}_{\hat{\mathbb{C}}} \quad 0 \quad {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_{\hat{\mathbb{C}}} \quad 0\right),$$
$$\left(\pm\frac{\sqrt{1-c_s^2v_\perp^2}}{c_s\rho_0 h} \quad 1 \quad 0^{\mathbb{A}} \quad 0\right), \quad (64)$$

respectively. The associated right eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \\ 0_{\hat{\mathbb{B}}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ {}^{\mathbb{q}}\!\perp^{\mathbb{C}}{}_{\hat{\mathbb{B}}} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{c_s^2\rho_0^2 h}{p} \\ \pm\frac{c_s\rho_0}{p}\sqrt{1-c_s^2v_\perp^2} \\ -\frac{c_s^2\rho_0}{p}\hat{v}_{\hat{\mathbb{B}}} \\ 1 \end{pmatrix}, \quad (65)$$

respectively. Since there is a complete set of eigenvectors for each $\mathbb{s}_a$ which depend furthermore continuously on $\mathbb{s}_a$, the system is strongly hyperbolic. The characteristic variables corresponding to the speeds $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are given by

$$\hat{\mathrm{U}}_0 = \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p, \qquad \hat{\mathrm{U}}_{\hat{\mathbb{A}}} = (\delta\hat{v})_{\hat{\mathbb{A}}} + \frac{1}{\rho_0 h}\hat{v}_{\hat{\mathbb{A}}}\delta p,$$
$$\hat{\mathrm{U}}_\pm = (\delta\hat{v})_{\hat{\mathbb{s}}} \pm \frac{\sqrt{1-c_s^2v_\perp^2}}{c_s\rho_0 h}\delta p. \quad (66)$$

### B. Dust

A special case for the equation of state (53) is that of dust, in which the pressure is identically zero everywhere, $p \equiv 0$, and the energy density coincides with the rest mass density, $\varepsilon = 0$. It follows that the fluid elements then follow timelike geodesics and that the conservation of the number of particles (50) is automatically fulfilled by the conservation of energy momentum in equation (49). For the analysis of hyperbolicity, we use in this subsection $\mathbf{U} = (\rho_0, \hat{v}_a)$ as the state vector.

Using Eqs. (51) and (52) with $\varepsilon = p = 0$ and splitting the equations against $n^a$ and $\gamma^a{}_b$, the PDE system can be written as

$$\mathcal{L}_n\rho_0 = -v^aD_a\rho_0 - \frac{\rho_0}{W}(\mathbb{g}^{-1})^{ab}D_a\hat{v}_b + \rho_0(\mathbb{g}^{-1})^{ab}K_{ab},$$
$$\gamma^b{}_a\mathcal{L}_n\hat{v}_b = -v^bD_b\hat{v}_a - WD_a\ln\alpha. \quad (67)$$

Using again an arbitrary spatial 1-form $\mathbb{s}_a$ as in Sec. III A, one ends up with the principal symbol $\mathbf{P}^{\mathbb{s}}$ for $(\delta\mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}}$ as

$$\mathbf{P}^{\mathbb{s}} = \begin{pmatrix} -v^{\mathbb{s}} & -\frac{\rho_0}{W} & 0^{\mathbb{B}} \\ 0 & -v^{\mathbb{s}} & 0^{\mathbb{B}} \\ 0_{\hat{\mathbb{A}}} & 0_{\hat{\mathbb{A}}} & -v^{\mathbb{s}}{}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}} \end{pmatrix}, \quad (68)$$

which evidently contains a Jordan block. The principal symbol is thus missing an eigenvector. The system is only weakly hyperbolic, and hence the IVP is ill posed.

### C. Uppercase formulation

We start again with Eqs. (50), (51), and (52) but split them against $u^a$ and ${}^{(\mathrm{u})}\gamma^b{}_a$. Using the definition of the local speed of sound (57), we derive after some algebra the following PDEs for the components of the state vector:

$$\nabla_u p = -c_s^2\rho_0 h^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b\hat{v}_c - c_s^2 W\rho_0 h^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b n_c, \tag{69}$$

$${}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u\hat{v}_c = -\frac{1}{\rho_0 h}{}^{(\mathrm{u})}\gamma^b{}_a\nabla_b p - W^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u n_c, \tag{70}$$

$$\nabla_u\varepsilon = -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b\hat{v}_c - \frac{Wp}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b n_c. \tag{71}$$

Here, we have used the relationship ${}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc} = {}^{(\mathrm{u})}\gamma(\mathbb{g}^{-1})_{ab}\gamma^{bc}$. Proceeding as when splitting against the lowercase frame, we write the system (69)–(71) as an equation for the state vector $\mathbf{U}$,

$$\mathbf{B}^{\mathrm{u}}\nabla_u\mathbf{U} = \mathbf{B}^p\nabla_p\mathbf{U} + \boldsymbol{\mathcal{S}}, \tag{72}$$

and identify

$$\mathbf{B}^{\mathrm{u}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{73}$$

and

$$\mathbf{B}^p = \begin{pmatrix} 0 & -c_s^2\rho_0 h^{(\mathrm{u})}\gamma^p{}_d(\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h}{}^{(\mathrm{u})}\gamma^p{}_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^p{}_d(\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix}. \tag{74}$$

The source vector is written as

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} -c_s^2 W\rho_0 h^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b n_c \\ -W^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u n_c \\ -\frac{Wp}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathbb{g}^{-1})^{dc}\nabla_b n_c \end{pmatrix}. \tag{75}$$

It is straightforward to verify that $\mathbb{1} + \mathbf{B}^V$ is invertible for all $v_a v^a < 1$. Therefore, as long as the various speeds in the system are not superluminal, that is, $|\lambda| \le 1$, expected since we are considering here a fluid model with no gauge freedom, by the argument of Sec. II D, we may analyze strong hyperbolicity equivalently in the upper- or lowercase frames.

Let $S_a$ be an arbitrary uppercase spatial vector, normalized against ${}^{(\mathrm{u})}\gamma_{ab}$ so that $S_a S^a = 1$, and let ${}^{\mathrm{Q}}\!\perp^b{}_a = {}^{(\mathrm{u})}\gamma^b{}_a - S^b S_a$ be the orthogonal projector. Decomposing ${}^{(\mathrm{u})}\gamma^a{}_b$ against $S^a$ and using relations in Table III to $\mathbb{s}_a$, we write Eq. (72) as

$$(\nabla_u\mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} \simeq \mathbf{P}^S(\nabla_S\mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{B}}}, \tag{76}$$

with principal symbol

$$\mathbf{P}^S = \mathbf{B}^S = \begin{pmatrix} 0 & -c_s^2\rho_0 h & 0^B & 0 \\ -\frac{1}{\rho_0 h} & 0 & 0^B & 0 \\ 0_A & 0_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & 0 \end{pmatrix}. \tag{77}$$

Since the uppercase projector is pushed through the lowercase inverse boost metric, we have $S_a S_b(\mathbb{g}^{-1})^{bc}(\delta\hat{v})_c = S_a\hat{\mathbb{s}}^c(\delta\hat{v})_c = S_a(\delta\hat{v})_{\hat{\mathbb{s}}}$, and for the orthogonal projector, ${}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_b(\delta\hat{v})_{\hat{\mathbb{A}}} = {}^{\mathrm{Q}}\!\perp^A{}_b{}^{(\mathrm{u})}\gamma_{Ac}(\mathbb{g}^{-1})^{cd}(\delta\hat{v})_d$.

By employing the uppercase frame, the principal symbol has become much simpler than before, see (62), exhibiting now essentially the same shape as that of a simple wave equation. In the present example, the extra structure is not required to complete the analysis, because in practice, computer algebra tools can already manage the more complicated form. In more sophisticated models, however, additional structure may become crucial if we wish to perform such an analysis. An obvious question to ask is: why is the uppercase form of the principal symbol so much cleaner? The reason, which in hindsight is obvious, is that the four-dimensional form of the fluid equations of motion contains the fluid four-velocity, and so any frame adapted to that fact naturally annihilates many terms in the principal symbol, uncovering the beautiful structure of (77). The five eigenvalues of $\mathbf{P}^S$ are

$$\lambda_{(0,1,2)} = 0, \qquad \lambda_{(\pm)} = \pm c_s, \tag{78}$$

with the corresponding left eigenvectors

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^A & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & {}^{\mathrm{Q}}\!\perp^A{}_C & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mp\frac{1}{c_s\rho_0 h} & 1 & 0^A & 0 \end{pmatrix}; \tag{79}$$

right eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ {}^{\mathrm{Q}}\!\perp^C{}_B \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \frac{c_s^2\rho_0^2 h}{p} \\ \mp\frac{c_s\rho_0}{p} \\ 0_B \\ 1 \end{pmatrix}; \tag{80}$$

and characteristic variables

$$\hat{\mathrm{U}}_0 = \delta\epsilon - \frac{p}{c_s^2 \rho_0^2 h}\delta p, \qquad \hat{\mathrm{U}}_A = (\delta\hat{v})_{\hat{\mathbb{A}}},$$

$$\hat{\mathrm{U}}_\pm = (\delta\hat{v})_{\hat{\mathbb{s}}} \mp \frac{\delta p}{c_s \rho_0 h}. \tag{81}$$

Using the recovery procedure described in Sec. II F gives the same results for eigenvalues and eigenvectors and characteristic variables as in our lowercase analysis. For details, see the notebooks [13] that accompany the paper.

## IV. HYPERBOLICITY OF GRMHD

In this section, we investigate whether or not two different formulations of GRMHD are strongly hyperbolic. The field equations will be expressed for a set of eight variables corresponding to those evolved numerically. The first characteristic analysis for RMHD was done by Ref. [26]. They worked covariantly and considered an augmented system of *ten* evolved variables, assuming implicitly a "free-evolution" style [24] to treat the two additional algebraic constraints, $u^a u_a = -1, u^a b_a = 0$, thus introduced, as well as the Gauss constraint besides. The analysis was then reviewed and expanded in Ref. [27]. The conclusion was that the augmented formulation of RMHD is strongly hyperbolic. Another augmented system for RMHD using ten variables was later derived in Ref. [28]. On the basis of Ref. [26], several authors, e.g., Refs. [29,30], reexamined the characteristic analysis and treated degeneracies. In particular, a very detailed discussion is given in Ref. [30].

For numerical implementation, a flux-balance law form of the equations was needed, as shocks can arise, and used in slightly different forms by, for example, Refs. [29–35]. A detailed overview is given in the review of Ref. [3]. In the flux-balance law form considered here, a total of eight variables including the magnetic field are evolved. It is important to stress that changing the number of variables can cause a breakdown of hyperbolicity, so in general, it is not enough to know that there is *some* good form of the system being treated. Rather, it is required that the *particular* formulation being employed should itself be at least strongly hyperbolic. The analysis of Ref. [27] therefore does not necessarily apply to the system in use in applications.

Our analysis begins with two observations that motivate a careful reconsideration of GRMHD. First, when numerical schemes to treat GRMHD are constructed, one sometimes sees that the longitudinal component of the magnetic field is ignored in evaluating the fluxes. This is ultimately because the approximation works by repeated application of a one-dimensional scheme, which is of course a perfectly legitimate approach. It is, however, easy to overlook the fact that when performing hyperbolicity analysis we are not free to discard any variable and must find a complete set of eigenvectors of the principal symbol, including that associated with the Gauss constraint. We must therefore be careful not to be misled by tricks that apply only to the method, rather than the system of equations itself.

Second, even if we can show strong hyperbolicity for a formulation of GRMHD that requires the evolution of only eight variables, we still may not claim that the flux-balance law formulation used in applications satisfies the same property. Like the field equations of GR and electrodynamics, those of GRMHD have a gauge freedom, which, from the free-evolution point of view is just the freedom to add combinations of the constraint to the evolution equations. Different choices of this addition affect the level of hyperbolicity of the formulation.

Neither of these subtleties has been completely taken care of by the earlier analyses, and indeed a first indication that the system of GRMHD used, for example, in Refs. [33,36] differs from that used in the analysis of Ref. [26] is the fact that the eigenvalues associated with the Gauss constraint differ between the two systems. In Ref. [26], the "entropy eigenvalue" is found with multiplicity 2. Of these, one corresponds to the Gauss constraint. In Ref. [36], for the system of eight variables, the entropy eigenvalue has only multiplicity 1, and the constraint eigenvalue is zero. We suppose that the reason these points have not been carefully unraveled before is chiefly that the lowercase principal symbol of GRMHD is a complicated matrix of which the structure is very difficult to spot. Remarkably, there is enough structure in the symbol so that the calculation of the eigenvalues and eigenvectors is possible in closed form, but the expressions are *very* long. For example, before developing the DF approach to the problem, which we will see simplifies matters greatly, we attempted a brute force treatment; the magnetosonic eigenvalues arrived with more than $10^4$ terms.

This section is structured as follows. In Sec. IV A, we recapitulate the basic definitions and equations for GRMHD following Refs. [27,30]. Afterward, we $3+1$ decompose the PDEs and derive the evolution equations, where in each multiples of the Gauss constraint are manually added (see Sec. IV B). We then analyze the characteristic structure of the principal symbol, taking all constraint addition coefficients to zero, which forms a set of PDEs that is in some sense analogous to the set of equations in Ref. [26], but with their algebraic constraints explicitly imposed; see Sec. IV C. In Secs. IV D and IV E, we do the analysis in the upper- and lowercase frames and give some comments about how the eigenvectors have to be rescaled to take account of degeneracies. Finally, in Sec. IV F, we take a different choice of constraint addition coefficients to obtain a set of equations equal to the flux-balance law system, comparing explicitly with Ref. [36], and show that this formulation of GRMHD which is used in numerical relativity is only weakly hyperbolic.

## A. Basics of GRMHD

In this subsection, we give a brief review about the basic definitions and equations of GRMHD following Refs. [27,30]. However, this will be done in a primarily mathematical fashion, suppressing some important physical insights and statements. We use Lorentz-Heaviside units for electromagnetic quantities with $\varepsilon_0 = \mu_0 = 1$ throughout, where $\varepsilon_0$ is the vacuum permittivity (or electric constant) and $\mu_0$ is the vacuum permeability (or magnetic constant).

### 1. Faraday tensor and Ohm's law

We start by introducing the Faraday electromagnetic tensor field (or for short field strength tensor) $F^{ab}$. For a generic observer with four-velocity $\mathtt{N}^a$, the field strength tensor and its dual can be expressed via the electric and magnetic four-vectors, $\mathtt{E}^a$, $\mathtt{B}^a$, as

$$F^{ab} = \mathtt{N}^a\mathtt{E}^b - \mathtt{N}^b\mathtt{E}^a + \epsilon^{abcd}\mathtt{N}_c\mathtt{B}_d,$$
$$^*F^{ab} = \mathtt{N}^a\mathtt{B}^b - \mathtt{N}^b\mathtt{B}^a - \epsilon^{abcd}\mathtt{N}_c\mathtt{E}_d, \qquad (82)$$

with the Levi-Cività tensor,

$$\epsilon^{abcd} = -\frac{1}{\sqrt{-g}}[abcd], \qquad (83)$$

where $g$ is the determinant of the spacetime metric $g_{ab}$, $[abcd]$ is the completely antisymmetric Levi-Cività symbol, and $2^*F^{ab} = -\epsilon^{abcd}F_{cd}$ holds. We use here the sign convention of Ref. [37]. Both the electric and magnetic fields satisfy the orthogonality relations $\mathtt{E}^a\mathtt{N}_a = \mathtt{B}^a\mathtt{N}_a = 0$.

Using the field strength tensor and its dual, Maxwell's equations are written as

$$\nabla^*_b F^{ab} = 0, \qquad \nabla_b F^{ab} = \mathcal{J}^a. \qquad (84)$$

According to Ohm's law (see Sec. V), the electric four-current $\mathcal{J}^a$ can be expressed as

$$\mathcal{J}^a = \rho_{\mathrm{el}}u^a + \sigma F^{ab}u_b, \qquad (85)$$

with the proper charge density $\rho_{\mathrm{el}}$ measured by the comoving observer with $u^a$ and $\sigma$ the electric conductivity.

### 2. Ideal MHD condition

In the limit of infinite conductivity $\sigma$ but finite current, the electric field $e^a$ measured by the comoving observer $u^a$, has to vanish,

$$e^a = F^{ab}u_b \equiv 0. \qquad (86)$$

This equality holds by use of expression (82) taking $\mathtt{N}^a = u^a$, $\mathtt{B}^a = b^a$ and $\mathtt{E}^a = e^a$.

### 3. Energy-momentum tensor

The total energy-momentum tensor of magnetohydrodynamics (MHD) is expressed as the sum of the ideal fluid part,

$$T^{ab}_{\mathrm{fluid}} = \rho_0 h u^a u^b + g^{ab}p, \qquad (87)$$

plus the standard electromagnetic energy-momentum tensor,

$$T^{ab}_{\mathrm{em}} = F^{ac}F^b{}_c - \frac{1}{4}g^{ab}F_{cd}F^{cd}. \qquad (88)$$

Using the ideal MHD condition (86) and expressing the field strength tensor via (82), the electromagnetic energy-momentum tensor in terms of the magnetic field is

$$T^{ab}_{\mathrm{em}} = \left(u^a u^b + \frac{1}{2}g^{ab}\right)b^2 - b^a b^b, \qquad (89)$$

and the total energy-momentum tensor is given by

$$T^{ab} = \rho_0 h^* u^a u^b + p^* g^{ab} - b^a b^b, \qquad (90)$$

with $h^* = h + b^2/\rho_0$ and $p^* = p + b^2/2$. In Eq. (89), we used as a shorthand $b^2 = b^a b_a$.

### 4. Covariant PDE system of GRMHD

The equations of GRMHD are the conservation of the number of particles

$$\nabla_a(\rho_0 u^a) = 0, \qquad (91)$$

the conservation of energy-momentum

$$\nabla_b T^{ab} = 0, \qquad (92)$$

and the Maxwell equations

$$\nabla_b{}^*F^{ab} = 0. \qquad (93)$$

## B. 3 + 1 decomposition of the PDE system

The $3+1$ decomposition needs a bit more care since we have a constrained system. For convenience, we will use ${}^{(u)}\gamma^b{}_a$, $u^a$ to decompose the equations given in Sec. IV A 4. Afterward, we will add to each equation some parametrized combination of the Gauss constraint. A concrete choice of the constraint addition parameters results in a set of evolution equations which we call a *formulation of GRMHD*. We will focus here on two specific formulations. The first of these is essentially that of Ref. [26], but without the artificial expansion of variables through the definition of the algebraic constraints $u^a u_a = -1$ and $u^a b_a = 0$, which are satisfied *a priori* in our approach. The second formulation corresponds to the flux-balance law system used in numerics by Refs. [30,36]. We arrive at the second by matching the values of the formulation parameters with the literature to obtain the desired form of the field equations. We also want to stress that we neither consider in this work formulations using the magnetic four-potential instead of the magnetic field as in Refs. [38,39] nor systems with divergence cleaning as in Ref. [35].

The eight equations determining the time evolution of the GRMHD system are

$$\nabla_a(\rho_0 u^a) = 0, \qquad {}^{(\mathrm{u})}\gamma_{ab}\nabla_c T^{bc} = 0,$$
$$u_b \nabla_c T^{bc} = 0, \qquad {}^{(\mathrm{u})}\gamma_{ab}\nabla_c {}^*F^{bc} = 0, \tag{94}$$

together with an equation of state $p = p(\rho_0, \varepsilon)$ and the Gauss constraint

$$0 = u_c \nabla_b {}^*F^{bc} = {}^{(\mathrm{u})}\gamma^{bc}\nabla_b b_c. \tag{95}$$

The magnetic four-vector $b^a$ can be split in the lowercase as

$$n_a b^a = -(v_a \hat{b}^a), \qquad \gamma^a{}_b b^b = \hat{b}^a, \tag{96}$$

and we have $b^a = (\hat{b}^c v_c) n^a + \hat{b}^a$ with $n_a \hat{b}^a = 0$. Furthermore, we introduce the Eulerian magnetic field vector $B^a$ as

$$\hat{b}_a = \frac{1}{W}\mathbb{g}_{ab}B^b = \frac{1}{W}B_a + (B^b \hat{v}_b) v_a,$$
$$B^a = W(\mathbb{g}^{-1})^{ab}\hat{b}_b = W\hat{b}^a - (\hat{b}^c \hat{v}_c) v^a, \tag{97}$$

where the lowercase Gauss constraint reads

$$\gamma^{ab}\nabla_a B_b = 0. \tag{98}$$

Taking Eq. (94), a straightforward calculation similar to that for GRHD in Sec. III C provides evolution equations for the pressure,

$$\nabla_u p = -c_s^2 \rho_0 h {}^{(\mathrm{u})}\gamma^d{}_c (\mathbb{g}^{-1})^{ce}\nabla_d \hat{v}_e + S^{(p)}$$
$$+ \omega^{(p)}({}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e + S^{(\mathrm{c})}); \tag{99}$$

the boost vector,

$${}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u \hat{v}_c = -\left(\frac{b^d b_a}{\rho_0^2 h h^*} + \frac{{}^{(\mathrm{u})}\gamma^d{}_a}{\rho_0 h^*}\right)\nabla_d p$$
$$+ \frac{2}{\rho_0 h^*} {}^{(\mathrm{u})}\gamma^{[b}{}_a b^{d]} {}^{(\mathrm{u})}\gamma_{bc}(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e$$
$$+ S_a^{(\hat{\mathbf{v}})} + \omega_a^{(\hat{v})}({}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e + S^{(\mathrm{c})}); \tag{100}$$

the magnetic field,

$${}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u \perp b_c = 2{}^{(\mathrm{u})}\gamma_{ab}{}^{(\mathrm{u})}\gamma^{[b}{}_c b^{d]}(\mathbb{g}^{-1})^{ce}\nabla_d \hat{v}_e + S_a^{(\perp \mathbf{b})}$$
$$+ \omega_a^{(\perp b)}({}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e + S^{(\mathrm{c})}); \tag{101}$$

and finally the specific internal energy,

$$\nabla_u \varepsilon = -\frac{p}{\rho_0} {}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \hat{v}_e + S^{(\varepsilon)}$$
$$+ \omega^{(\varepsilon)}({}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e + S^{(\mathrm{c})}). \tag{102}$$

By Eq. (95), we also obtain the Gauss constraint

$${}^{(\mathrm{u})}\gamma^{ac}\nabla_a b_c = {}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d \perp b_e + S^{(\mathrm{c})}. \tag{103}$$

The sources are given by

$$S^{(p)} = -c_s^2 W \rho_0 h {}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d n_e,$$
$$S_a^{(\hat{\mathbf{v}})} = -W {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{be}\nabla_u n_e + \frac{2W}{\rho_0 h^*} {}^{(\mathrm{u})}\gamma^{[b}{}_a b^{e]} V_b b^d \nabla_d n_e,$$
$$S_a^{(\perp \mathbf{b})} = 2W {}^{(\mathrm{u})}\gamma_{ab}{}^{(\mathrm{u})}\gamma^{[b}{}_c b^{d]}(\mathbb{g}^{-1})^{ce}\nabla_d n_e$$
$$+ 2W {}^{(\mathrm{u})}\gamma^e{}_{[a}V_{b]} b^b \nabla_u n_e,$$
$$S^{(\varepsilon)} = -\frac{Wp}{\rho_0} {}^{(\mathrm{u})}\gamma^d{}_c(\mathbb{g}^{-1})^{ce}\nabla_d n_e,$$
$$S^{(\mathrm{c})} = (W V^d b^e - W(b^c V_c){}^{(\mathrm{u})}\gamma^{de})\nabla_d n_e.$$

The auxiliary magnetic vector $\perp b_c$ is defined by the relation

$${}^{(\mathrm{u})}\gamma_{ac}(\mathbb{g}^{-1})^{cd}\nabla_b \perp b_d \coloneqq {}^{(\mathrm{u})}\gamma_{ac}(\mathbb{g}^{-1})^{cd}\nabla_b \hat{b}_d$$
$$+ V_a b_d (\mathbb{g}^{-1})^{de}\nabla_b \hat{v}_e. \tag{104}$$

As usual, square brackets around indices denote antisymmetrization, so that $2\hat{v}^{[a}b^{b]} = \hat{v}^a b^b - \hat{v}^b b^a$. In the system (99)–(102), we already added multiples of the Gauss constraint (103) connected to coefficients $\omega^{(p)}, \omega_a^{(\hat{v})}, \omega_a^{(\perp b)}$, and $\omega^{(\varepsilon)}$.

### C. Prototype algebraic constraint free formulation

In the following subsections, we proceed with the characteristic analysis for the prototype algebraic constraint free formulation of Eqs. (99)–(102) by setting $\omega^{(p)} = 0$, $\omega_a^{(\hat{v})} = 0$, $\omega_a^{(\perp b)} = 0$, and $\omega^{(\varepsilon)} = 0$. The resulting system is connected to the augmented system of equations in Ref. [26] as follows: take the equations of Ref. [26], project the momentum equation and the evolution equation for the magnetic field with ${}^{(\mathrm{u})}\gamma^a{}_b$ orthogonal to the four-velocity of the fluid, change the evolved variables to $(p, \hat{v}_a, \perp b_a, \varepsilon)$, and replace the derivative of $p$ in the evolution equation for the magnetic field using the evolution equation for $p$. After this, one obtains our principal symbol. The fact that Anile and Pennisi [26] work exclusively in RMHD is of no consequence, since in our notation the principal symbol in GRMHD is fundamentally the same as that of RMHD.

As previously mentioned, the equations become very lengthy in the lowercase frame. As such, we were not able to find a choice of variables in which the principal symbol takes a nice and easy form. Nevertheless, by applying the strategy of Sec. II F, we were able to derive for the prototype system all lowercase characteristic quantities, such as eigenvalues, eigenvectors, and characteristic variables, which are displayed in Sec. IV E, including a discussion of degeneracies that may occur. Our analysis of the flux-balance law formulation of GRMHD is given afterward in Sec. IV F.

### D. Uppercase formulation

Writing Eqs. (99)–(102) with $\omega^{(p)} = 0$, $\omega_a^{(\hat{v})} = 0$, $\omega_a^{(\perp b)} = 0$, and $\omega^{(\varepsilon)} = 0$ in a vectorial form with state vector $\mathbf{U} = (p, \hat{v}_a, \perp b_a, \varepsilon)^T$,

$$\mathbf{B}^{\mathrm{u}} \nabla_u \mathbf{U} = \mathbf{B}^p \nabla_p \mathbf{U} + \boldsymbol{\mathcal{S}}, \tag{105}$$

we identify

$$\mathbf{B}^{\mathrm{u}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc} & 0 & 0 \\ 0 & 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{106}$$

and the uppercase spatial part

$$\mathbf{B}^p = \begin{pmatrix} 0 & -c_s^2\rho_0 h^{(\mathrm{u})}\gamma^p{}_c(\mathbb{g}^{-1})^{ce} & 0 & 0 \\ f^p{}_a & 0 & l^{pe}{}_a & 0 \\ 0 & 2^{(\mathrm{u})}\gamma_{ab}{}^{(\mathrm{u})}\gamma^{[b}{}_c b^{p]}(\mathbb{g}^{-1})^{ce} & 0 & 0 \\ 0 & -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^p{}_c(\mathbb{g}^{-1})^{ce} & 0 & 0 \end{pmatrix}, \tag{107}$$

with shorthands

$$l^{pe}{}_a = \frac{2}{\rho_0 h^*}{}^{(\mathrm{u})}\gamma^{[b}{}_a b^{p]}{}^{(\mathrm{u})}\gamma_{bc}(\mathbb{g}^{-1})^{ce},$$
$$f^p{}_a = -\left(\frac{b^p b_a}{\rho_0^2 h h^*} + \frac{{}^{(\mathrm{u})}\gamma^p{}_a}{\rho_0 h^*}\right) \tag{108}$$

and source vector $\boldsymbol{\mathcal{S}} = (S^{(p)}, S_a^{(\hat{\mathbf{v}})}, S_a^{(\perp\mathbf{b})}, S^{(\varepsilon)})^T$. A straightforward calculation shows that $\mathbb{1} + \mathbf{B}^V$ is invertible for all $V^a V_a < 1$.

#### 1. 2+1 decomposition

Let $S_a$ be an arbitrary unit spatial 1-form and ${}^{\mathrm{Q}}\!\perp^b{}_a$ be the associated orthogonal projector. Let $\mathbb{s}_a$ and ${}^{\mathrm{q}}\!\perp^b{}_a$ be their lowercase projected versions (see Tables II and III). Decomposing ${}^{(\mathrm{u})}\gamma^b{}_a$ and $\gamma^b{}_a$ against $S_a$ and $\mathbb{s}_a$, respectively, Eq. (105) can be written as

$$(\nabla_u \mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} \simeq \mathbf{P}^S (\nabla_S \mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{B}}}, \tag{109}$$

with the principal symbol $\mathbf{P}^S = \mathbf{B}^S =$

$$\begin{pmatrix} 0 & -c_s^2\rho_0 h & 0^B & 0 & 0^B & 0 \\ -\frac{(b^S)^2+\rho_0 h}{\rho_0^2 h h^*} & 0 & 0^B & 0 & -\frac{b^B}{\rho_0 h^*} & 0 \\ -\frac{b^S b_A}{\rho_0^2 h h^*} & 0_A & 0^B{}_A & 0_A & \frac{b^S}{\rho_0 h^*}{}^{\mathrm{Q}}\!\perp^B{}_A & 0_A \\ 0 & 0 & 0^B & 0 & 0^B & 0 \\ 0_A & -b_A & b^{S\mathrm{Q}}\!\perp^B{}_A & 0_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & 0 & 0^B & 0 \end{pmatrix}. \tag{110}$$

The characteristic polynomial $P_\lambda$ for the principal symbol (110) can be written as

$$P_\lambda = \frac{\lambda^2}{(\rho_0 h^*)^2} P_{\text{Alfvén}} P_{\text{mgs}}, \tag{111}$$

with the quadratic polynomial for Alfvén waves

$$P_{\text{Alfvén}} = -(b^S)^2 + \lambda^2 \rho_0 h^* \tag{112}$$

and the quartic polynomial for magnetosonic waves

$$P_{\text{mgs}} = (\lambda^2 - 1)(\lambda^2 b^2 - (b^S)^2 c_s^2) + \lambda^2(\lambda^2 - c_s^2)\rho_0 h. \tag{113}$$

Solving (111) provides us with different kinds of speeds of waves propagating in the $S^a$-direction. All speeds are real, and the system is strongly hyperbolic, as will be seen later. The entropic waves have speed

$$\lambda_{(\mathrm{e})} = 0. \tag{114}$$

The constraint waves have the same speed, given by

$$\lambda_{(\mathrm{c})} = 0. \tag{115}$$

The Alfvén waves are given by solving $P_{\text{Alfvén}} = 0$, which results in the two different speeds

$$\lambda_{(\mathrm{a}\pm)} = \pm \frac{b^S}{\sqrt{\rho_0 h^*}}. \tag{116}$$

Solving the quartic equation $P_{\text{mgs}} = 0$, we obtain four different speeds of the magnetosonic waves, two slow magnetosonic waves,

$$\lambda_{(\mathrm{s}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}} - \sqrt{\zeta_{\mathrm{S}}^2 - \xi_{\mathrm{S}}}}, \qquad (117)$$

and two fast magnetosonic waves,

$$\lambda_{(\mathrm{f}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}} + \sqrt{\zeta_{\mathrm{S}}^2 - \xi_{\mathrm{S}}}}, \qquad (118)$$

where we employ the shorthands

$$\zeta_{\mathrm{S}} = \frac{(b^2 + c_s^2[(b^S)^2 + \rho_0 h])}{2\rho_0 h^*}, \qquad \xi_{\mathrm{S}} = \frac{(b^S)^2 c_s^2}{\rho_0 h^*}. \qquad (119)$$

Please note that the index "S" in $\zeta_{\mathrm{S}}$ and $\xi_{\mathrm{S}}$ is not a contraction with a vector but rather a reminder that we used for $2+1$ decomposition the vector $S^a$. Since $(b^S)^2 \le b^2$ and $c_s^2 \le 1$, all eigenvalues have absolute value smaller than or equal to one, and relation $|\lambda_{\mathrm{u}}||V| < 1$, required for application of the formalism of Sec. II F, is satisfied for all boost velocities. Thus, we are allowed to use the recovering procedure for arbitrary boost velocities.

The left eigenvectors corresponding to $\lambda_{(\mathrm{e})}, \lambda_{(\mathrm{c})}, \lambda_{(\mathrm{a}\pm)}$, and $\lambda_{(\mathrm{m}\pm)}$ with $\mathrm{m} = \mathrm{s}, \mathrm{f}$ being

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^A & 0 & 0^A & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0^A & 1 & 0^A & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & \mp {}^{(\mathrm{S})}\epsilon^{AC} b_C \sqrt{\rho_0 h^*} & 0 & -{}^{(\mathrm{S})}\epsilon^{AC} b_C & 0 \end{pmatrix},$$
$$\begin{pmatrix} \frac{\rho_0 h^*(\lambda_{(\mathrm{m}\pm)})^2 - b^2}{c_s^2 \rho_0 h} & \frac{(b^S)^2 - \rho_0 h^* (\lambda_{(\mathrm{m}\pm)})^2}{\lambda_{(\mathrm{m}\pm)}} & \frac{b^S b^A}{\lambda_{(\mathrm{m}\pm)}} & 0 & b^A & 0 \end{pmatrix}, \qquad (120)$$

respectively. We defined the antisymmetric uppercase two and three Levi-Cività tensors as ${}^{(\mathrm{S})}\epsilon^{AB} = S_d {}^{(\mathrm{u})}\epsilon^{dAB} = u_c S_d {}^{\mathrm{Q}}\!\perp^A{}_a {}^{\mathrm{Q}}\!\perp^B{}_b \epsilon^{cdab}$. The right eigenvectors can be obtained by inverting of the matrix of left eigenvectors or by solving the eigenvalue problem and can be expressed as

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \\ 0_B \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ \mp \frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}} b^C \\ 0 \\ -{}^{(\mathrm{S})}\epsilon_{BC} b^C \\ 0 \end{pmatrix}, \qquad (121)$$

for entropy, constraint, and Alfvén waves, and

$$\begin{pmatrix} \frac{c_s^2 \rho_0^2 h}{p} \\ -\frac{\rho_0 \lambda_{(\mathrm{m}\pm)}}{p} \\ \frac{\rho_0 \lambda_{(\mathrm{m}\pm)}}{p b^S b_\perp^2} [(b^S)^2 + \rho_0 h^* ((\lambda_{(\mathrm{m}\pm)})^2 - 2\zeta_{\mathrm{S}})] b_B \\ 0 \\ \frac{\rho_0}{b_\perp^2 p} [b^2 + \rho_0 h^* ((\lambda_{(\mathrm{m}\pm)})^2 - 2\zeta_{\mathrm{S}})] b_B \\ 1 \end{pmatrix} \qquad (122)$$

for the four magnetosonic waves with $\mathrm{m} = \mathrm{s}, \mathrm{f}$. We introduced in the magnetosonic eigenvectors the orthogonal magnetic field vector $b_\perp^a = {}^{\mathrm{Q}}\!\perp^a{}_b b^b$ with $b_\perp^2 = b_\perp^a b_a^\perp = b^A b_A$. For the moment, we have a complete set of eigenvectors for real eigenvalues. Nevertheless, we have to check if any of the eigenvalues may change their multiplicity and, if so, whether or not a complete set of eigenvectors is still available. The situation where *a priori* distinct eigenvalues coincide and their muliplicities change is called a degenerate state or for short a degeneracy. To show strong hyperbolicity of the system, we have to show that for each possible degenerate state a complete set of eigenvectors still exists. For the augmented system of RMHD, this was already described in Refs. [26,27,29,31]. A full account was furthermore given by Ref. [30]. We also want to mention that in the Appendix of Ref. [29] the eigenvalues and right eigenvectors in the fluid rest frame are given for seven variables in a one-dimensional analysis of RMHD. They are obtained by explicitly setting (locally) the spatial entries of the four-velocity to zero, which is ultimately quite similar to our approach.

### 2. *Degeneracy analysis of the uppercase*

For the prototype algebraic constraint free formulation of GRMHD, just as in the augmented system of Ref. [26], two different types of degeneracies can occur. For degeneracy type I, $b^S$ is equal to zero, whereas for degeneracy type II, the magnetic field four-vector is parallel to $S^a$, so that $b_\perp^a = {}^{\mathrm{Q}}\!\perp^a{}_b b^b = 0$ holds. To describe the different situations properly, we write the magnetic field four-vector as

$$b^a = b^S S^a + b_\perp^a, \qquad b^2 = (b^S)^2 + b_\perp^2. \qquad (123)$$

First, we note that the polynomials (112) and (113) have solutions

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{a}\pm)} = \pm\sqrt{\rho_0 h^*}, \qquad (124)$$

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm)} = \pm\sqrt{\left(\rho_0 h + \frac{b^2}{c_s^2}\right) + \rho_0 h \left(1 - \frac{1}{c_s^2}\right)\frac{\lambda_{(\mathrm{m}\pm)}^2}{1 - \lambda_{(\mathrm{m}\pm)}^2}}$$
$$= \pm\sqrt{(b^S)^2 + \left(\rho_0 h + \frac{b^2}{c_s^2}\right) - \rho_0 h^* \frac{\lambda_{(\mathrm{m}\pm)}^2}{c_s^2}}, \qquad (125)$$

which are well defined even for degeneracies.

For a type I degeneracy in the uppercase where $b^S = 0$ and $b^2 = b_\perp^2$, the eigenvalues become

$$\lambda_{(\mathrm{e})} = \lambda_{(\mathrm{c})} = \lambda_{(\mathrm{a}\pm)} = \lambda_{(\mathrm{s}\pm)} = 0, \quad \lambda_{(\mathrm{f}\pm)} = \pm\frac{\sqrt{b^2 + c_s^2\rho_0 h}}{\sqrt{\rho_0 h^*}}, \tag{126}$$

and

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{s}\pm)} = \pm\sqrt{\rho_0 h + \frac{b^2}{c_s^2}}, \qquad \left.\frac{b^S}{\lambda}\right|_{(\mathrm{f}\pm)} = 0 \tag{127}$$

hold.

For type II degeneracy, namely when $b_\perp^a = 0$ and $b^2 = (b^S)^2$, we have

$$\lambda_{(\mathrm{s}\pm)} = \lambda_{(\mathrm{a})}^{\pm} = \pm\frac{|b^S|}{\sqrt{\rho_0 h^*}}, \qquad \lambda_{(\mathrm{f}\pm)} = \pm c_s, \quad \text{if } c_s^2 > \frac{(b^S)^2}{\rho_0 h^*},$$
$$\lambda_{(\mathrm{f}\pm)} = \lambda_{(\mathrm{a})}^{\pm} = \pm\frac{|b^S|}{\sqrt{\rho_0 h^*}}, \qquad \lambda_{(\mathrm{s}\pm)} = \pm c_s, \quad \text{if } c_s^2 < \frac{(b^S)^2}{\rho_0 h^*} \tag{128}$$

and get

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm\neq\mathrm{a}\pm)} = \pm\frac{b^S}{c_s}, \qquad \left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm=\mathrm{a}\pm)} = \pm\sqrt{\rho_0 h^*}. \tag{129}$$

To classify the corresponding waves with equal speed properly (see Refs. [29,30]), we defined $\lambda_{(\mathrm{a})}^{\pm}$ with $\lambda_{(\mathrm{a})}^{+} \geq \lambda_{(\mathrm{a})}^{-}$ such that $\lambda_{(\mathrm{a})}^{\pm} = \lambda_{(\mathrm{a}\pm)}$ for $b^S \geq 0$ or $\lambda_{(\mathrm{a})}^{\pm} = \lambda_{(\mathrm{a}\mp)}$ for $b^S < 0$ holds. The special case $(b^S)^2 = c_s^2\rho_0 h^*$ is called a type II′ degeneracy where $\lambda_{(\mathrm{s}\pm)} = \lambda_{(\mathrm{a})}^{\pm} = \lambda_{(\mathrm{f}\pm)} = \pm c_s$. Note that type I and type II degeneracies may occur simultaneously, in which case we recover the pure GRHD decoupled from the magnetic field evolution as a limiting case. On the other hand, since we insist that $c_s > 0$, it is not possible for type I and type II′ degeneracies to occur simultaneously.

### 3. Renormalized uppercase left eigenvectors

We rescale the Alfvén and magnetosonic eigenvectors in a way analogous to Ref. [30]. The procedure can also be found in the provided notebook [13]. The rescaled eigenvectors are

$$\text{entropy:} \left(-\tfrac{p}{c_s^2\rho_0^2 h} \quad 0 \quad 0^A \quad 0 \quad 0^A \quad 1\right),$$
$$\text{constraint:} \left(0 \quad 0 \quad 0^A \quad 1 \quad 0^A \quad 0\right),$$
$$\text{Alfvén:} \left(0 \quad 0 \quad \pm{}^{(S)}\epsilon^{AC}\sqrt{\rho_0 h^*}\tfrac{b_C^\perp}{|b_\perp|} \quad 0 \quad {}^{(S)}\epsilon^{AC}\tfrac{b_C^\perp}{|b_\perp|} \quad 0\right); \tag{130}$$

the magnetosonic left eigenvectors which have eigenvalues closer to the Alfvén eigenvalues,

$$\left(\tfrac{\mathcal{H}(\lambda^2-1)}{\rho_0 h} \quad (1-c_s^2)\mathcal{H}\lambda \quad \left(\tfrac{b^S}{\lambda}\right)\tfrac{b_\perp^A}{|b_\perp|} \quad 0 \quad \tfrac{b_\perp^A}{|b_\perp|} \quad 0\right)_{(\mathrm{m}\pm)}; \tag{131}$$

and the other two magnetosonic left eigenvectors,

$$\left(\tfrac{1}{c_s^2\rho_0 h} \quad \tfrac{(1-c_s^2)\lambda}{c_s^2(\lambda^2-1)} \quad \left(\tfrac{b^S}{\lambda}\right)\mathcal{F}^A \quad 0 \quad \mathcal{F}^A \quad 0\right)_{(\mathrm{m}\pm)}, \tag{132}$$

with abbreviations

$$\mathcal{H} = \frac{|b_\perp|}{c_s^2 - \lambda_{(\mathrm{m}\pm)}^2}, \tag{133}$$

$$\mathcal{F}^A = \frac{b_\perp^A}{(\rho_0 h^* \lambda_{(\mathrm{m}\pm)}^2 - b^2)}, \tag{134}$$

where for type II and even for type II′ degeneracy we take ${Q_1}^a$ and ${Q_2}^a$ such that in the degenerate limit we have

$$\frac{b_C^\perp}{|b_\perp|} = \frac{1}{\sqrt{2}}(Q_{1C} + Q_{2C}), \tag{135}$$

$$\mathcal{H} = 0, \tag{136}$$

$$\mathcal{F}^A = 0^A. \tag{137}$$

Here, some comments are in order. In Eqs. (135)–(137), we are just making a canonical choice for how to represent the complete set of eigenvectors under a type II or type II′ degenerate limit. Note that for type II degeneracies $\mathcal{H}$ and $\mathcal{F}^A$ vanish automatically. For type II′ degeneracies, depending on how the limit is taken, their values may not vanish, but the form (131) and (132) with $\mathcal{H} = \mathcal{F}^A = 0$ can nevertheless be obtained by taking appropriate linear combinations of the resulting eigenvectors.

### 4. Renormalized uppercase right eigenvectors

The right eigenvectors are obtained in the same way and with the same abbreviations. The entropy, constraint, and Alfvén eigenvectors are given by

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \\ 0_B \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \pm{}^{(S)}\epsilon_{BC}\frac{b^C}{|b_\perp|} \\ 0 \\ {}^{(S)}\epsilon_{BC}\sqrt{\rho_0 h^*}\frac{b^C}{|b_\perp|} \\ 0 \end{pmatrix}; \tag{138}$$

the magnetosonic eigenvectors corresponding to the eigenvalues closer to the Alfvén eigenvalues are

$$\begin{pmatrix} c_s^2\rho_0 h\mathcal{H} \\ -\mathcal{H}\lambda \\ -\left(\frac{b^S}{\lambda}\right)\frac{b_B^\perp}{|b_\perp|} \\ 0 \\ \frac{\rho_0 h}{(\lambda^2-1)}\frac{b_B^\perp}{|b_\perp|} \\ \frac{p}{\rho_0}\mathcal{H} \end{pmatrix}_{(\mathrm{m}\pm)}; \tag{139}$$

and the other two magnetosonic eigenvectors are

$$\begin{pmatrix} c_s^2\rho_0 h \\ -\lambda \\ c_s^2(1-\lambda^2)\left(\frac{b^S}{\lambda}\right)\mathcal{F}_B \\ 0 \\ c_s^2\rho_0 h\mathcal{F}_B \\ \frac{p}{\rho_0} \end{pmatrix}_{(\mathrm{m}\pm)}; \tag{140}$$

respectively.

#### 5. Characteristic variables

The characteristic variables valid for all degeneracies are

$$\begin{aligned} \hat{\mathrm{U}}_{\mathrm{e}} &= \delta\epsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p, \qquad \hat{\mathrm{U}}_{\mathrm{c}} = (\delta\perp b)_{\hat{\mathbb{S}}}, \\ \hat{\mathrm{U}}_{\mathrm{a}\pm} &= \pm{}^{(\mathrm{S})}\epsilon^{AC}\sqrt{\rho_0 h^*}\frac{b_C^\perp}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{A}}} + {}^{(\mathrm{S})}\epsilon^{AC}\frac{b_C^\perp}{|b_\perp|}(\delta\perp b)_{\hat{\mathbb{A}}}, \\ \hat{\mathrm{U}}_{\mathrm{m}_1\pm} &= \frac{\mathcal{H}(\lambda^2_{(\mathrm{m}_1\pm)}-1)}{\rho_0 h}\delta p + (1-c_s^2)\mathcal{H}\lambda_{(\mathrm{m}_1\pm)}(\delta\hat{v})_{\hat{\mathbb{S}}} \\ &\quad + \left(\frac{b^S}{\lambda}\right)_{(\mathrm{m}_1\pm)}\frac{b_\perp^A}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{A}}} + \frac{b_\perp^A}{|b_\perp|}(\delta\perp b)_{\hat{\mathbb{A}}}, \\ \hat{\mathrm{U}}_{\mathrm{m}_2\pm} &= \frac{1}{c_s^2\rho_0 h}\delta p + \frac{(1-c_s^2)\lambda_{(\mathrm{m}_2\pm)}}{c_s^2(\lambda^2_{(\mathrm{m}_2\pm)}-1)}(\delta\hat{v})_{\hat{\mathbb{S}}} \\ &\quad + \left(\frac{b^S}{\lambda}\right)_{(\mathrm{m}_2\pm)}\mathcal{F}^A(\delta\hat{v})_{\hat{\mathbb{A}}} + \mathcal{F}^A(\delta\perp b)_{\hat{\mathbb{A}}}, \end{aligned} \tag{141}$$

with $\{\mathrm{m}_1, \mathrm{m}_2\}$ equal to $\{\mathrm{s}, \mathrm{f}\}$ or $\{\mathrm{f}, \mathrm{s}\}$. Note that, since the resulting similarity transform matrix $\mathbf{T}_S$ and inverse $\mathbf{T}_S^{-1}$ always exist and have bounded components, the regularity condition (3) is fulfilled. This shows that the prototype algebraic constraint free system is in the uppercase strongly hyperbolic. Since all the eigenvalues have absolute values smaller than or equal to 1, the system must also be strongly hyperbolic in the lowercase frame.

### E. Lowercase formulation

We know already that the prototype algebraic constraint free formulation is strongly hyperbolic. Nevertheless, the lowercase eigenvalues and eigenvectors would be important if we were to employ the system numerically, and therefore we derive them in this subsection.

#### 1. Recovering the lowercase quantities

To obtain the lowercase eigenvalues and eigenvectors as well as the characteristic variables, we use the procedure described in Sec. II F. The recovery will be done in several steps.

*Step one.*—First of all, we take the calculated uppercase eigenvalues (114)–(118) and replace the vector $S^a$ by $S^a_\lambda = (S^a - W(\lambda - WV^S))/N$, whereby we obtain the new uppercase eigenvalues

$$\lambda^{\mathrm{u}}_{(\mathrm{e})} = 0, \tag{142}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{c})} = 0, \tag{143}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{a}\pm)} = \pm\frac{b^{S_\lambda}}{\sqrt{\rho_0 h^*}}, \tag{144}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{s}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}_\lambda} - \sqrt{\zeta^2_{\mathrm{S}_\lambda} - \xi_{\mathrm{S}_\lambda}}}, \tag{145}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{f}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}_\lambda} + \sqrt{\zeta^2_{\mathrm{S}_\lambda} - \xi_{\mathrm{S}_\lambda}}}, \tag{146}$$

where we used the shorthands

$$\zeta_{\mathrm{S}_\lambda} = \frac{(b^2 + c_s^2[(b^{S_\lambda})^2 + \rho_0 h])}{2\rho_0 h^*}, \qquad \xi_{\mathrm{S}_\lambda} = \frac{(b^{S_\lambda})^2 c_s^2}{\rho_0 h^*}, \tag{147}$$

and the magnetic field vector in the new direction $S^a_\lambda$ becomes

$$b^{S_\lambda} = b^a S_a^\lambda = \frac{1}{N}(b^S - W(b^a V_a)(\lambda - WV^S)), \tag{148}$$

$$N = \sqrt{(W\lambda - W^2V^S)^2 + 1 + (V^S)^2W^2 - \lambda^2}. \tag{149}$$

We want to reiterate that the relation $WV^S = -v^{\mathbb{s}}$ holds and is used at several points in this paper.

*Step two.*—We calculate now the lowercase eigenvalues by use of Eq. (33), that is,

$$\frac{1}{N}W(\lambda - WV^S) = \lambda_{\mathrm{u}}[S^a_\lambda]. \tag{150}$$

For example, taking $\lambda_{\mathrm{u}}[S^a_\lambda] = \lambda^{\mathrm{u}}_{(\mathrm{e})} = 0$, we arrive with the lowercase entropy wave speed $\lambda_{(\mathrm{e})} = WV^S$, the normalization factor $N$ becomes unity, and $S^a$ and $S^a_\lambda$ are identical.

*Step three.*—We now transform the uppercase left eigenvectors for $S^a_\lambda$ in the lowercase left eigenvectors for the state vector,

$$(\delta\mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} = (\delta p, (\delta\hat{v})_{\hat{\mathbb{s}}}, (\delta\hat{v})_{\hat{\mathbb{A}}}, (\delta\bot b)_{\hat{\mathbb{s}}}, (\delta\bot b)_{\hat{\mathbb{A}}}, \delta\varepsilon)^T.$$

The transformation is $\lambda$ dependent and therefore has to be done in each eigenspace independently. We take Eq. (44), that is,

$$\mathbf{l}^{\mathrm{n}}_{\lambda}|_{\mathbb{s}} = \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}_\lambda}(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}_\lambda})\mathbf{T}_\lambda, \tag{151}$$

where we use the eigenvectors $\mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]|_{\mathbf{S}}$ explicitly written in (120) and replace all basis vectors with the ones associated with $S^a_\lambda$. The matrices $(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}})$ and $(\mathbb{1} + \mathbf{B}^V|_{\mathbf{S}_\lambda})$ for bases $\mathbf{S} = (S^a, {Q_1}^a, {Q_2}^a)$ and $\mathbf{S}_\lambda = (S^a_\lambda, {Q_1}^a_\lambda, {Q_2}^a_\lambda)$ can be found in the notebook [13].

To obtain the basis transformation $\mathbf{T}_\lambda$, we need to give a little more details: writing $S^a_\lambda$ in the basis $\mathbf{S}$, we get

$$\begin{aligned}
{S_\lambda}^a &= c_S S^a + c_1 {Q_1}^a + c_2 {Q_2}^a,\\
c_S &= \frac{1 + (W^2V^S - W\lambda)V^S}{N},\\
c_1 &= \frac{(W^2V^S - W\lambda)V^{Q_1}}{N},\\
c_2 &= \frac{(W^2V^S - W\lambda)V^{Q_2}}{N}.
\end{aligned} \tag{152}$$

This relation defines a rotation of the basis, so we are able to build a transformation matrix which is an element of SO(3). By denoting ${Q_1}^a_\lambda$ and ${Q_2}^a_\lambda$ as rotated basis vectors $Q^a_1$ and $Q^a_2$, respectively, the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} c_S & c_1 & c_2 \\ -c_1 & \frac{c_S c_1^2 + c_2^2}{c_1^2 + c_2^2} & \frac{(c_S - 1)c_1 c_2}{c_1^2 + c_2^2} \\ -c_2 & \frac{(c_S - 1)c_1 c_2}{c_1^2 + c_2^2} & \frac{c_1^2 + c_S c_2^2}{c_1^2 + c_2^2} \end{pmatrix}, \tag{153}$$

such that

$$\begin{pmatrix} S^a_\lambda \\ {Q_1}^a_\lambda \\ {Q_2}^a_\lambda \end{pmatrix} = \mathbf{R} \begin{pmatrix} S^a \\ {Q_1}^a \\ {Q_2}^a \end{pmatrix}. \tag{154}$$

Since $\mathbf{R} \in \mathrm{SO}(3)$, we may transpose to invert $\mathbf{R}^T = \mathbf{R}^{-1}$. The associated lowercase bases obey the same transformation, since we just have to multiply Eq. (154) with ${\gamma^b}_a$. The transformation matrix is taken to be $\mathbf{T}_\lambda = \mathrm{diag}(1, \mathbf{R}, \mathbf{R}, 1)$. The derivative of the state vector transforms like

$$\mathbb{1}(\nabla_z\mathbf{U})_{\mathbf{S}} = \mathbf{T}^T_\lambda(\nabla_z\mathbf{U})_{\mathbf{S}_\lambda} \tag{155}$$

for any vector $z^a$.

*Step four.*—For a last step, we have to calculate the right eigenvectors by Eq. (45), so we arrive with

$$\mathbf{r}^{\mathrm{n}}_{\lambda}|_{\mathbb{s}} = \mathbf{T}^T_\lambda \mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]|_{\mathbf{S}_\lambda}. \tag{156}$$

For this, we will take the right eigenvectors $\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]|_{\mathbf{S}}$ given in (122) and replace the basis vectors.

### 2. Definitions and formulas

Let us first define some new relations and quantities:

$$\begin{aligned}
a &:= N\lambda_{\mathrm{u}} = W\lambda - W^2V^S = W\lambda + \hat{v}^{\mathbb{s}},\\
\mathcal{B} &:= Nb^a S^\lambda_a = b^S - (b^a V_a)a\\
&= b^S + (b^a V_a)W(V^S W - \lambda),\\
\mathcal{G} &:= 1 + (V^S)^2 W^2 - \lambda^2,\\
N^2 &= a^2 + \mathcal{G}.
\end{aligned} \tag{157}$$

These definitions are motivated by those in Refs. [26,30] in regard to the covariant approach of characteristic analysis shown by Eq. (46).

In analogy to the uppercase, we write the magnetic field four-vector as

$$b^a = b^{S_\lambda} S^a_\lambda + b^a_\bot, \qquad b^2 = (b^{S_\lambda})^2 + b^2_\bot, \tag{158}$$

with

$$|b_\bot|^2 = b^2 - (b^{S_\lambda})^2 = b^a_\bot b^\bot_a. \tag{159}$$

Please note that we nevertheless still use capital letters for contraction with ${}^{\mathrm{Q}}\bot$, e.g., $b^A_\bot = {}^{\mathrm{Q}}{\bot^A}_a b^a_\bot$. In general, $b^S_\bot \neq 0$ is not vanishing. These definitions are taken for all lowercase characteristic quantities. Since $b^a_\bot$ is orthogonal to $S^a_\lambda$, we use the relation $b^S_\bot = a(b^a_\bot V_a)$ several times.

### 3. Entropy wave

Taking $\lambda_{\mathrm{u}} = 0$ as in (142), we arrive at the lowercase eigenvalue

$$\lambda_{(\mathrm{e})} = WV^S. \tag{160}$$

In this case, we have $N = 1$ and $S^a_\lambda = S^a$, and the left and right eigenvectors for entropy waves remain the same,

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^A & 0 & 0^A & 1 \end{pmatrix}, \tag{161}$$

$$\begin{pmatrix} 0 & 0 & 0_B & 0 & 0_B & 1 \end{pmatrix}^T. \tag{162}$$

### 4. Constraint wave

Taking $\lambda_{\mathrm{u}} = 0$ as in (143), we arrive at the lowercase eigenvalue

$$\lambda_{(\mathrm{c})} = WV^S. \tag{163}$$

In this case, we have $N = 1$ and $S^a_\lambda = S^a$, and the left and right eigenvectors for constraint waves become

$$\begin{pmatrix} 0 & (b^C V_C) & -b^S V^A & 1 & 0^A & 0 \end{pmatrix}, \qquad (164)$$

$$\begin{pmatrix} 0 & 0 & 0_B & 1 & 0_B & 0 \end{pmatrix}^T, \qquad (165)$$

respectively.

### 5. Alfvén waves

For Alfvén waves, we obtain by taking (144) the lowercase eigenvalues

$$\lambda_{(\mathrm{a}\pm)} = \frac{b^S + V^S W^2[(b^a V_a) \pm \sqrt{\rho_0 h^*}]}{W[(b^a V_a) \pm \sqrt{\rho_0 h^*}]}. \qquad (166)$$

They coincide up to a minus sign and factor $W$ (due to our choice of the spatial vector) with the literature [30]. The already rescaled left and right eigenvectors to $\lambda_{(\mathrm{a}\pm)}$ are

$$\begin{pmatrix} \pm \frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}} \frac{V^B b_\perp^C}{|b_\perp|} \\ b^{S\,(\mathrm{S})}\epsilon_{BC} \frac{b_\perp^B V^C}{|b_\perp|} \\ ((b^a V_a) \pm \sqrt{\rho_0 h^*})^{(\mathrm{u})}\epsilon^A{}_{bc} \frac{N S^b_{\lambda(\mathrm{a}\pm)} b_\perp^c}{|b_\perp|} \\ 0 \\ -{}^{(\mathrm{S})}\epsilon^A{}_B \left( \frac{b_\perp^B}{|b_\perp|} \pm \frac{|b_\perp| V^B}{\sqrt{\rho_0 h^*}} \right) \\ 0 \end{pmatrix}^T \qquad (167)$$

and

$$\begin{pmatrix} 0 \\ \frac{b^S}{\sqrt{\rho_0 h^*}} {}^{(\mathrm{S})}\epsilon_{AC} \frac{b_\perp^A V^C}{|b_\perp|} \\ \frac{(b^b V_b) \pm \sqrt{\rho_0 h^*}}{\sqrt{\rho_0 h^*}} {}^{(\mathrm{u})}\epsilon_{Bac} \frac{N S^a_{\lambda(\mathrm{a}\pm)} b_\perp^c}{|b_\perp|} \\ \pm b^{S\,(\mathrm{S})}\epsilon_{AC} \frac{b_\perp^A V^C}{|b_\perp|} \\ (\sqrt{\rho_0 h^*} \pm (b^b V_b))^{(\mathrm{u})}\epsilon_{Bac} \frac{N S^a_{\lambda(\mathrm{a}\pm)} b_\perp^c}{|b_\perp|} \\ 0 \end{pmatrix}, \qquad (168)$$

respectively.

### 6. Magnetosonic waves

The uppercase slow and fast magnetosonic eigenvalues are defined in (145) and (146). Inserting one of these eigenvalues into Eq. (150), one can show after some manipulations (given in the notebook [13]) that the lowercase magnetosonic eigenvalues are solutions of the quartic equation

$$\mathcal{N}_4 = \rho_0 h \left( \frac{1}{c_s^2} - 1 \right) a^4 - \left( \rho_0 h + \frac{b^2}{c_s^2} \right) a^2 \mathcal{G} + \mathcal{B}^2 \mathcal{G} = 0, \qquad (169)$$

where $\mathcal{N}_4$ is the same polynomial as obtained by Ref. [26]. We have computed analytic expressions for the magnetosonic eigenvalues. Explicitly written out, however, they are rather long, and hence a numerical computation relying on the characteristic information may be better served by using some root finder.

The rescaled left and right magnetosonic eigenvectors with eigenvalues closer to the Alfvén speeds can be expressed as

$$\begin{pmatrix} -\frac{\mathcal{G}}{\rho_0 h} \frac{(b_\perp^a V_a)}{|b_\perp|} \left( \frac{\mathcal{B}}{a} \right) - \frac{(1 - a V^S)\mathcal{G}}{\rho_0 h} \mathcal{F} \\ a(a^2 + \mathcal{G}) \left( (1 - c_s^2)\mathcal{F} + \left[ \left( \frac{\mathcal{B}}{a} \right) + (b^a V_a) \right] \frac{(b_\perp^a V_a)}{|b_\perp|} \right) \\ (a^2 + \mathcal{G}) \left[ \left( \frac{\mathcal{B}}{a} \right) + (b^a V_a) \right] \frac{b_\perp^A}{|b_\perp|} \\ 0 \\ a \frac{b_\perp^S V^A}{|b_\perp|} + (1 - a V^S) \frac{b_\perp^A}{|b_\perp|} \\ 0 \end{pmatrix}^T_{(\mathrm{m}\pm)} \qquad (170)$$

and

$$\begin{pmatrix} -c_s^2 \rho_0 h \mathcal{G}(a^2 + \mathcal{G})\mathcal{F} \\ \mathcal{G}\left( \frac{\mathcal{B}}{a} \right) \frac{b_\perp^S}{|b_\perp|} + a(1 - a V^S)\mathcal{G}\mathcal{F} \\ \mathcal{G}\left( \frac{\mathcal{B}}{a} \right) \frac{b_B^\perp}{|b_\perp|} - a^2 \mathcal{G}\mathcal{F} V_B \\ (a^2 + \mathcal{G}) \frac{\rho_0 h}{|b_\perp|} b_\perp^S \\ (a^2 + \mathcal{G}) \frac{\rho_0 h}{|b_\perp|} b_B^\perp \\ -\frac{p}{\rho_0} (a^2 + \mathcal{G})\mathcal{G}\mathcal{F} \end{pmatrix}_{(\mathrm{m}\pm)}. \qquad (171)$$

The remaining two left and right magnetosonic lowercase eigenvectors are given by

$$\begin{pmatrix} -\frac{\mathcal{G}}{\rho_0 h} \left( \frac{\mathcal{B}}{a} \right) (\mathcal{C}^a V_a) + \frac{(1 - a V^S)}{c_s^2 \rho_0 h (a^2 + \mathcal{G})} \\ \left( 1 - \frac{1}{c_s^2} \right) \frac{a}{\mathcal{G}} + a(a^2 + \mathcal{G}) \left[ \left( \frac{\mathcal{B}}{a} \right) + (b^a V_a) \right] (\mathcal{C}^b V_b) \\ (a^2 + \mathcal{G}) \left[ \left( \frac{\mathcal{B}}{a} \right) + (b^a V_a) \right] \mathcal{C}^A \\ 0 \\ a \mathcal{C}^S V^A + (1 - a V^S) \mathcal{C}^A \\ 0 \end{pmatrix}^T_{(\mathrm{m}\pm)} \qquad (172)$$

and

$$\begin{pmatrix} \rho_0 h \\ \left(\frac{\mathcal{B}}{a}\right)\mathcal{G}\mathcal{C}^S - \frac{a(1-aV^S)}{c_s^2(a^2+\mathcal{G})} \\ \left(\frac{\mathcal{B}}{a}\right)\mathcal{G}\mathcal{C}_B + \frac{a^2}{c_s^2(a^2+\mathcal{G})}V_B \\ (a^2+\mathcal{G})\rho_0 h\mathcal{C}^S \\ (a^2+\mathcal{G})\rho_0 h\mathcal{C}_B \\ \frac{p}{c_s^2\rho_0} \end{pmatrix}_{(\mathrm{m}\pm)}. \tag{173}$$

Here, we took the definitions

$$\mathcal{C}^a = \frac{b_\perp^a}{a^2\rho_0 h - \mathcal{G}b^2}, \qquad \mathcal{F} = \frac{|b_\perp|}{c_s^2(a^2+\mathcal{G}) - a^2}, \tag{174}$$

where for type II and type II′ degeneracies we take

$$\mathcal{C}^a = 0, \qquad \mathcal{F} = 0, \tag{175}$$

and

$$\frac{b_C^\perp}{|b_\perp|} = \frac{1}{\sqrt{2}}(Q_{1C}^{\lambda} + Q_{2C}^{\lambda}). \tag{176}$$

### 7. Degeneracies in lowercase GRMHD

In the lowercase frame, the degeneracy analysis is performed just as in the uppercase setting. One has only to replace the vector $S^a$ by $S^a_\lambda$ and the corresponding orthogonal basis vectors as well. We then have for type I degeneracy that $b^a S^\lambda_a$ is equal to zero. In this case, the entropic wave, the constraint wave, the two Alfvén waves, and the two slow magnetosonic waves propagate at the same speed ($\lambda_{(\mathrm{e})} = \lambda_{(\mathrm{c})} = \lambda_{(\mathrm{a}\pm)} = \lambda_{(\mathrm{s}\pm)} = -v^{\mathbb{s}}$). For type II degeneracy, the tangential magnetic field vector, $b^a_\perp = {}^{Q_\lambda}\perp^a{}_b b^b$, ${}^{Q_\lambda}\perp^a{}_b = {}^{(\mathrm{u})}\gamma^a{}_b - S^a_\lambda S^\lambda_b$, vanishes. In this case, one of the Alfvén waves and one of the magnetosonic waves of the appropriate class (here denoted by a superscript as in Ref. [30]) have the same speed ($\lambda^+_{(\mathrm{a})} = \lambda^+_{(\mathrm{s})}$ or $\lambda^-_{(\mathrm{a})} = \lambda^-_{(\mathrm{s})}$ or $\lambda^+_{(\mathrm{a})} = \lambda^+_{(\mathrm{f})}$ or $\lambda^-_{(\mathrm{a})} = \lambda^-_{(\mathrm{f})}$). In the type II′ degeneracy, one Alfvén wave and the slow and fast magnetosonic waves of the appropriate class travel at the same speed ($\lambda^+_{(\mathrm{a})} = \lambda^+_{(\mathrm{s})} = \lambda^+_{(\mathrm{f})}$ or $\lambda^-_{(\mathrm{a})} = \lambda^-_{(\mathrm{s})} = \lambda^-_{(\mathrm{f})}$). In the uppercase, we have for type II and type II′ degeneracies that both Alfvén speeds are degenerate at the same time. Replacing $S^a$ by $S^a_\lambda$ leads to different SO(3)-transformations for different values of $\lambda$. Therefore, in the lowercase, this cannot be fulfilled in general. A more detailed description and derivation can be found in Ref. [40].

### 8. Characteristic variables

The characteristic variables valid for all degeneracies are

$$\begin{aligned} \hat{\mathrm{U}}_0 &= \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p, \\ \hat{\mathrm{U}}_{\mathrm{c}} &= (\delta\perp b)_{\hat{\mathbb{s}}} + (b^A V_A)(\delta\hat{v})_{\hat{\mathbb{s}}} - b^S V^A(\delta\hat{v})_{\hat{\mathbb{A}}}, \\ \hat{\mathrm{U}}_{\mathrm{a}\pm} &= \pm\frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}}\frac{V^B b_\perp^C}{|b_\perp|}\delta p + b^{S(\mathrm{S})}\epsilon_{BC}\frac{b_\perp^B V^C}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{s}}} \\ &\quad + ((b^a V_a) \pm \sqrt{\rho_0 h^*})\frac{N S^b_{\lambda_{(\mathrm{a}\pm)}} b^c_\perp}{|b_\perp|}{}^{(\mathrm{u})}\epsilon^A{}_{bc}(\delta\hat{v})_{\hat{\mathbb{A}}} \\ &\quad - \left(\frac{b_\perp^B}{|b_\perp|} \pm \frac{|b_\perp|V^B}{\sqrt{\rho_0 h^*}}\right){}^{(\mathrm{S})}\epsilon^A{}_B(\delta\perp b)_{\hat{\mathbb{A}}}, \end{aligned} \tag{177}$$

for entropy, constraint, and Alfvén waves, and

$$\begin{aligned} \hat{\mathrm{U}}_{\mathrm{m}_1\pm} &= -\left(\frac{\mathcal{G}}{\rho_0 h}\frac{(b_\perp^a V_a)}{|b_\perp|}\left(\frac{\mathcal{B}}{a}\right) + \frac{(1-aV^S)\mathcal{G}}{\rho_0 h}\mathcal{F}\right)\delta p \\ &\quad + a(a^2+\mathcal{G})(1-c_s^2)\mathcal{F}(\delta\hat{v})_{\hat{\mathbb{s}}} \\ &\quad + N^2\left[\left(\frac{\mathcal{B}}{a}\right) + (b^a V_a)\right]\left(\frac{b_\perp^S}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{s}}} + \frac{b_\perp^A}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{A}}}\right) \\ &\quad + \left(a\frac{b_\perp^S V^A}{|b_\perp|} + (1-aV^S)\frac{b_\perp^A}{|b_\perp|}\right)(\delta\perp b)_{\hat{\mathbb{A}}}, \\ \hat{\mathrm{U}}_{\mathrm{m}_2\pm} &= \left(\frac{(1-aV^S)}{c_s^2\rho_0 h(a^2+\mathcal{G})} - \frac{\mathcal{G}}{\rho_0 h}\left(\frac{\mathcal{B}}{a}\right)(\mathcal{C}^a V_a)\right)\delta p \\ &\quad + \left(1-\frac{1}{c_s^2}\right)\frac{a}{\mathcal{G}}(\delta\hat{v})_{\hat{\mathbb{s}}} \\ &\quad + N^2\left[\left(\frac{\mathcal{B}}{a}\right) + (b^a V_a)\right](C^S(\delta\hat{v})_{\hat{\mathbb{s}}} + C^A(\delta\hat{v})_{\hat{\mathbb{A}}}) \\ &\quad + (aC^S V^A + (1-aV^S)\mathcal{C}^A)(\delta\perp b)_{\hat{\mathbb{A}}}, \end{aligned} \tag{178}$$

for magnetosonic waves, with $\{\mathrm{m}_1, \mathrm{m}_2\}$ equal to $\{\mathrm{s},\mathrm{f}\}$ or $\{\mathrm{f},\mathrm{s}\}$. The functions on the right-hand side of $\hat{\mathrm{U}}_{\mathrm{m}\pm}$ are evaluated with the corresponding eigenvalue.

## F. Weak hyperbolicity of the flux-balance law formulation of GRMHD

We want now to analyze whether or not the flux-balance law formulation of GRMHD as in Ref. [33] is strongly hyperbolic. To do so, we need to find the values for the formulation parameters such that a linear combination of Eqs. (99)–(102) is equal to the system in the form of Refs. [30,41], up to the use of the same evolved variables. In fact, several flux-balance law formulations exist, but remarkably, in our variables, they differ only by a linear combination of the conservation of particle number equation.

To reproduce the flux-balance law formulation given in Ref. [36], we worked in computer algebra and found the linear combination of our equations that reproduced the flux-balance ones. This was done ignoring all derivatives of the normal vector $n^a$. In our analysis, we may ignore all derivatives of the normal vector anyway since they only contribute to the source vector and do not affect the principal part. The coefficients are then

$$\omega^{(p)} = \frac{\kappa}{\rho_0}(b^c V_c), \qquad \omega_a^{(\hat{v})} = \frac{1}{\rho_0 h} b_a,$$
$$\omega_a^{(\perp b)} = -V_a, \qquad \omega^{(\epsilon)} = \frac{1}{\rho_0}(b^c V_c). \tag{179}$$

Proceeding in the same way as for the previous formulation, the principal symbol $\mathbf{P}^S$ becomes

$$\begin{pmatrix} 0 & -c_s^2\rho_0 h & 0^B & \frac{\kappa}{\rho_0}(b^c V_c) & 0^B & 0 \\ -\frac{(b^S)^2+\rho_0 h}{\rho_0^2 h h^*} & 0 & 0^B & \frac{b^S}{\rho_0 h} & -\frac{b^B}{\rho_0 h^*} & 0 \\ -\frac{b^S b_A}{\rho_0^2 h h^*} & 0_A & 0^B{}_A & \frac{b_A}{\rho_0 h} & \frac{b^S}{\rho_0 h^*}{}^{\mathrm{Q}}\!\perp^B{}_A & 0_A \\ 0 & 0 & 0^B & -V^S & 0^B & 0 \\ 0_A & -b_A & b^{S\mathrm{Q}}\!\perp^B{}_A & -V_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & \frac{1}{\rho_0}(b^c V_c) & 0^B & 0 \end{pmatrix}; \tag{180}$$

the characteristic polynomial is then of the form

$$P_\lambda = \frac{1}{(\rho_0 h^*)^2}\lambda(\lambda + V^S)P_{\text{Alfvén}}P_{\text{mgs}}, \tag{181}$$

where $P_{\text{Alfvén}}$ and $P_{\text{mgs}}$ coincide with the polynomials given earlier in Eqs. (112) and (113). As expected, the eigenvalue associated with the constraint has changed from zero, in the previous formulation, to $-V^S$. Therefore, new degeneracies have to be considered, for example, when the constraint and entropic speeds collide. This occurs when $V^S = 0$, in which case we find that the principal symbol is not diagonalizable. Hence, the system is only weakly hyperbolic and has an ill-posed IVP. To get an intuitive idea of what precisely goes wrong, we may consider the left eigenvectors associated with the entropy and constraint waves in generic directions and then consider a limiting direction with $V^S \to 0$. These are

$$\begin{pmatrix} -\frac{p\rho_0}{c_s^2\rho_0^2 h - \kappa p}\frac{V^S}{(b^c V_c)} & 0 & 0^A & 1 & 0^A & \frac{c_s^2\rho_0^3 h}{c_s^2\rho_0^2 h - \kappa p}\frac{V^S}{(b^c V_c)} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0^A & 1 & 0^A & 0 \end{pmatrix},$$

respectively, with eigenvalues $\lambda_{(\mathrm{e})} = 0$ and $\lambda_{(\mathrm{c})} = -V^S$. Both right eigenvectors can be found in our scripts but are suppressed here because the constraint eigenvector is quite lengthy. Taking the limit $V^S \to 0$, we immediately arrive at the conclusion that the geometric multiplicity is only 1 as the two vectors become coincident. The eigenvector can not be rescaled as for the earlier degeneracies since only some entries in the left entropy eigenvector become zero; the limit of the principal symbol is truly problematic. This degeneracy was unfortunately overlooked in Ref. [36], although there the focus was rather on the convexity of the system as opposed to hyperbolicity. Nevertheless, we have explicitly checked in our notebooks [13] that, taking the lowercase matrices from Ref. [36] and deriving the left eigenvectors of the entropy and constraint waves, the exact same problem is present. Deriving the right constraint eigenvector in the lowercase frame is much worse than in the uppercase, however, so we only evaluated the left ones. We want to stress that using the matrices of Ref. [36] is a completely independent calculation and underlines the weak hyperbolicity of the system. Somewhat interestingly, in the Newtonian limit, the flux-balance formulation, see, for example, Refs. [42,43], suffers from the same degeneracy and is also only weakly hyperbolic.

It should be explicitly noted that in more than one spatial dimension the condition $V^S = 0$ will certainly be satisfied everywhere in space for some $S_a$. One should therefore avoid thinking that the breakdown of hyperbolicity happens only on a set of measure zero in spacetime. Rather the generic situation is that when the flow is nontrivial there are specific bad directions everywhere in spacetime which obstruct the well-posedness of the initial value problem. The fact that only specific directions are problematic may make the effect in numerical work hard to identify. In particular, many tests of GRMHD are focused on one-dimensional (nonsmooth) solutions, and by construction, such experiments are insensitive to the breakdown identified here. This will be studied in greater detail in future work.

We stress again that the result does not automatically apply to formulations evolving the magnetic four-potential [38,39] nor systems with divergence cleaning [35]. It would naturally be desirable to perform a similar analysis for those systems also.

## V. HYPERBOLICITY OF RGRMHD

In this section, we want to investigate the evolution equations used in the literature for RRMHD [44–48] and [49–53] describing RGRMHD and show that these two systems are weakly hyperbolic and therefore have ill-posed IVPs. In this section, we will use the lowercase frame exclusively. As in Sec. IV, we use Lorentz-Heaviside units where vacuum permittivity and vacuum permeability are equal to 1. We start by deriving the equations of motion for the state vector $\mathbf{U}$.

### A. Equations of RGRMHD

As with earlier, we want to derive the evolution equations and are primarily concerned with their mathematical structure. Interesting physical facts, particularly those related to Ohm's law, will be sidelined in our discussion.

#### 1. Augmented Maxwell equations

As in the beginning of the last section about GRMHD, we take the following definition of the field strength tensor for a generic Eulerian observer with four-velocity $n^a$,

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abcd} n_c B_d, \tag{182}$$

$$^*F^{ab} = n^a B^b - n^b B^a - \epsilon^{abcd} n_c E_d, \tag{183}$$

with the Levi-Cività tensor,

$$\epsilon^{abcd} = -\frac{1}{\sqrt{-g}}[abcd]; \tag{184}$$

the Levi-Cività symbol $[abcd]$; $[0123] = 1$ and

$$\epsilon^{abcd} n_a = \epsilon^{bcd} = \frac{1}{\sqrt{\gamma}}[bcd], \tag{185}$$

where we follow the definition and convention by Ref. [37]. Please note that in this convention $2^*F^{ab} = -\epsilon^{abcd} F_{cd}$ holds.

To control the constraints during the evolution, the augmented scalar fields $\psi$ and $\phi$ are introduced, see for example [45,47,53], and hence the Maxwell equations become

$$\nabla_b (F^{ab} - g^{ab}\psi) = \mathcal{J}^a - \frac{1}{\tau} n^a \psi, \tag{186}$$

$$\nabla_b (^*F^{ab} - g^{ab}\phi) = -\frac{1}{\tau} n^a \phi. \tag{187}$$

Note that in the literature the notation $\kappa = \tau^{-1}$ is normally employed. The electric four-current is split against $n^a$ and $\gamma^b{}_a$ defined by

$$\mathcal{J}^a \coloneqq q n^a + J^a, \qquad n_a J^a = 0. \tag{188}$$

Proceeding with a $3+1$ decomposition of (186) and (187) using (188), we arrive at the equations

$$\gamma^a{}_b \mathcal{L}_n E^b = \epsilon^{abc} D_b B_c - \gamma^{ab} D_b \psi + S^a_{(\mathbf{E})}, \tag{189}$$

$$\gamma^a{}_b \mathcal{L}_n B^b = -\epsilon^{abc} D_b E_c - \gamma^{ab} D_b \phi + S^a_{(\mathbf{B})}, \tag{190}$$

$$\mathcal{L}_n \psi = -D_a E^a - \frac{1}{\tau}\psi + q, \tag{191}$$

$$\mathcal{L}_n \phi = -D_a B^a - \frac{1}{\tau}\phi, \tag{192}$$

with sources

$$S^a_{(\mathbf{E})} = \frac{1}{\alpha} B_c \epsilon^{abc} D_b \alpha + K E^a - J^a,$$
$$S^a_{(\mathbf{B})} = -\frac{1}{\alpha} E_c \epsilon^{abc} D_b \alpha + K B^a.$$

The constant $\tau$ is the timescale for the exponential driving of Eqs. (191) and (192) toward the constraints

$$D_a E^a = q, \tag{193}$$

$$D_a B^a = 0, \tag{194}$$

respectively. The three-current $J^a$ is given by generalized Ohm's law; see below. We must to stress that, although $J^a$ is inside the 'source' term, it *could* contain derivatives of the evolved variables. Such terms would then of course contribute to the principal part.

As a consequence of the antisymmetry of the field strength tensor, we have additionally a conservation law for the electric charge, $\nabla_a \mathcal{J}^a = 0$, that is in the $3+1$ language

$$\mathcal{L}_n q = -\gamma^{ab} D_a J_b - \frac{1}{\alpha} J^b D_b \alpha + K q. \tag{195}$$

#### 2. Energy-momentum tensor

The energy-momentum tensor $T^{ab}$ of RGRMHD contains an ideal fluid component,

$$T^{ab}_{\mathrm{mat}} = \rho_0 h u^a u^b + p g^{ab}, \tag{196}$$

plus the standard electromagnetic energy-momentum tensor,

$$T^{ab}_{\mathrm{em}} = F^{ac} F^b{}_c - \frac{1}{4} g^{ab} F_{cd} F^{cd}, \tag{197}$$

with a field strength tensor defined in (182). Writing $F^{ab}$ in terms of $E^a$ and $B^a$, we obtain

$$T^{ab}_{\mathrm{em}} = \frac{1}{2}(B_c B^c + E_c E^c)(\gamma^{ab} + n^a n^b) - B^a B^b - E^a E^b + (n^a \epsilon^{bcd} + n^b \epsilon^{acd}) E_c B_d. \tag{198}$$

#### 3. Generalized Ohm's law

The generalized Ohm's law provides us with an expression for the spatial current $J^a$. Explanations about the physical validity and form of $J^a$ can be found in the literature [53,54]. We consider here an equation for $J^a$ which is of the form

$$J^a = q v^a + \tilde{J}^a, \qquad \tilde{J}^a = \tilde{J}^a(p, v_b, \varepsilon, E_c, B_d), \tag{199}$$

where $\tilde{J}^a$ contains no derivatives of the matter and electromagnetic variables nor second-order or higher derivatives of the metric tensor. This fairly general choice of $J^a$ includes the particular form used in the literature mentioned above, that is,

$$J^a = qv^a + W\sigma(E^a + \epsilon^{abc} v_b B_c - (v_b E^b) v^a), \tag{200}$$

where $\sigma$ is the conductivity of the fluid and is permitted to be an arbitrary function of the evolved variables besides the charge density $q$.

#### 4. Hydrodynamical equations

To obtain the evolution equations for $p$, $v_a$, and $\varepsilon$, we take the conservation of the number of particles and the conservation of energy momentum,

$$\nabla_a(\rho_0 u^a) = 0, \tag{201}$$

$$\nabla_a(T^{ab}) = 0, \tag{202}$$

and proceed with the $3+1$ split. After combining the equations, using the Maxwell equations and introducing the speed of sound, we arrive at the evolution equation for the pressure,

$$\begin{aligned}\mathcal{L}_n p = {}& (c_s^2 - 1) v^p W_{c_s}^2 D_p p - c_s^2 \rho_0 h W_{c_s}^2 \gamma^{pc} D_p v_c \\ & - c_2 (E^b v_b) \gamma^{pc} D_p E_c - c_2 (B^b v_b) \gamma^{pc} D_p B_c \\ & + (c_1 E^p - c_2 \epsilon^{bdp} B_b v_d) D_p \psi \\ & + (c_1 B^p + c_2 \epsilon^{bdp} E_b v_d) D_p \phi + S^{(p)};\end{aligned} \tag{203}$$

for the fluid velocity,

$$\begin{aligned}\gamma^b{}_a \mathcal{L}_n v_b = {}& -\frac{1}{W^2\rho_0 h}(\gamma^p{}_a + (c_s^2 - 1) W_{c_s}^2 v^p v_a) D_p p + \left(\frac{c_s^2 W_{c_s}^2}{W^2} v_a \gamma^{pc} - v^p \gamma^c{}_a\right) D_p v_c \\ & + \frac{1}{W^2\rho_0 h}(E_a + c_2 (E^b v_b) v_a) \gamma^{pc} D_p E_c + \frac{1}{W^2\rho_0 h}(B_a + c_2 (B^b v_b) v_a) \gamma^{pc} D_p B_c \\ & + \frac{1}{W^2\rho_0 h}(\gamma_{ad} + c_2 v_a v_d) \epsilon^{bdp} B_b D_p \psi - \frac{1}{W^2\rho_0 h}(\gamma_{ad} + c_2 v_a v_d) \epsilon^{bdp} E_b D_p \phi \\ & - c_5 v_a E^p D_p \psi - c_5 v_a B^p D_p \phi + S^{(\mathrm{v})};\end{aligned} \tag{204}$$

and for the internal specific energy,

$$\begin{aligned}\mathcal{L}_n \varepsilon = {}& \frac{p W_{c_s}^2}{W^2 \rho_0^2 h} v^p D_p p - \frac{p W_{c_s}^2}{\rho_0} \gamma^{pc} D_p v_c - v^p D_p \varepsilon \\ & - c_4 (E^b v_b) \gamma^{pc} D_p E_c - c_4 (B^b v_b) \gamma^{pc} D_p B_c \\ & + (c_3 E^p - c_4 \epsilon^{bdp} B_b v_d) D_p \psi \\ & + (c_3 B^p + c_4 \epsilon^{bdp} E_b v_d) D_p \phi + S^{(\varepsilon)};\end{aligned} \tag{205}$$

with sources

$$\begin{aligned} S^{(p)} = {}& c_1 (E^b J_b) + c_2 \epsilon^{bcd} B_b J_c v_d \\ & + W_{c_s}^2 c_s^2 \rho_0 h (\mathbb{g}^{-1})^{bc} K_{bc}, \\ S^{(\mathrm{v})} = {}& \frac{1}{W^2 \rho_0 h} (\gamma_{ad} + c_2 v_a v_d) \epsilon^{bde} B_b J_e \\ & - c_5 (E^d J_d) v_a - \frac{1}{\alpha} (\mathbb{g}^{-1})^c{}_a D_c \alpha \\ & - c_s^2 \frac{W_{c_s}^2}{W^2} (\mathbb{g}^{-1})^{bc} K_{bc} v_a - K_{bc} v^b v^c v_a, \\ S^{(\varepsilon)} = {}& c_3 (E^b J_b) + c_4 \epsilon^{bcd} B_b J_c v_d \\ & + \frac{W_{c_s}^2 p}{\rho_0} (\mathbb{g}^{-1})^{bc} K_{bc}, \end{aligned} \tag{206}$$

where we have employed the shorthands,

$$\begin{aligned} c_1 &= \frac{W_{c_s}^2}{W^2 \rho_0} (\kappa W^2 + c_s^2 (W^2 - 1) \rho_0), \\ c_2 &= W_{c_s}^2 \left(\frac{\kappa}{\rho_0} + c_s^2\right), \\ c_3 &= \frac{W_{c_s}^2}{W^2 \rho_0^2 h} (p (W^2 - 1) + (\chi - \chi W^2 + h W^2) \rho_0), \\ c_4 &= \frac{W_{c_s}^2}{W^2 \rho_0^2 h} (p W^2 + (\chi - \chi W^2 + h W^2) \rho_0), \\ c_5 &= \frac{W_{c_s}^2}{W^2 \rho_0^2 h} (\kappa + \rho_0). \end{aligned} \tag{207}$$

The system of Eqs. (203), (204), (205), (189), (190), (191), (192), and (195) is identical to the system of evolution equations in Ref. [53], as was explicitly checked up to source terms.

### B. Analysis with evolution of $q$

In this subsection, we want to analyze the characteristic structure of equations used in Refs. [45–49]. As always, we $2+1$ decompose the equations, this time using an arbitrary unit spatial 1-form $s_a$, $s_a s^a = 1$, $s_a n^a = 0$ and denote the orthogonal projector by ${}^{q}\perp^{b}{}_{a} \coloneqq \gamma^{b}{}_{a} - s^{b}s_{a}$. Taking the state vector to be $\mathbf{U} = (p, v_a, \varepsilon, q, E_a, B_a, \psi, \phi)^T$, we write Eqs. (203), (204), (205), (195), (189), (190), (191), and (192) for the 14 components of $\mathbf{U}$ in matrix form:

$$\mathbf{A}^n \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}}. \tag{208}$$

The form of the matrices is easily obtained from the system of equations and so is not explicitly given. A simple $2+1$ decomposition of this equation yields the principal symbol in the form

$$\mathbf{P}^s = \mathbf{A}^s = \begin{pmatrix} \mathbf{A}_{6\times 6} & \mathbf{B}_{6\times 8} \\ \mathbf{0}_{8\times 6} & \mathbf{C}_{8\times 8} \end{pmatrix}, \tag{209}$$

where $\mathbf{B}_{6\times 8}$ contains the coefficients of spatial derivatives with respect to the variables $(E_a, B_a, \psi, \phi)$ in the time evolution of $(p, v_a, \varepsilon, q)$. The matrix $\mathbf{C}_{8\times 8}$ is the submatrix of the electromagnetic variables $(E_a, B_a, \psi, \phi)$. The matrix $\mathbf{A}_{6\times 6}$ can be written as

$$\mathbf{A}_{6\times 6} = \begin{pmatrix} \mathbf{A}_{5\times 5} & \mathbf{0}_{5\times 1} \\ \mathbf{A}_{1\times 5} & -v^s \end{pmatrix}, \tag{210}$$

with $\mathbf{A}_{5\times 5} = \mathbf{P}^s_{\mathrm{HD}}$ the principal symbol of the pure hydrodynamical sector, explicitly given by (A3) and

$$\mathbf{A}_{1\times 5} = \begin{pmatrix} -\frac{\partial J^s}{\partial p} & -s_c \frac{\partial J^s}{\partial v_c} & -{}^{q}\perp^{B}{}_{A} \frac{\partial J^s}{\partial v_A} & -\frac{\partial J^s}{\partial \varepsilon} \end{pmatrix}. \tag{211}$$

Since the principal symbol (209) is block triangular, the eigenvalues are given by those of $\mathbf{A}_{6\times 6}$ and $\mathbf{C}_{8\times 8}$, these are

$$\begin{aligned} \mathbf{A}_{6\times 6}\colon\ \lambda &= -v^s, \quad (\text{multiplicity } 4), \\ \lambda &= \lambda_{(\pm)}, \quad [\text{see (A4)}], \end{aligned} \tag{212}$$

$$\mathbf{C}_{8\times 8}\colon\ \lambda = \pm 1, \quad (\text{multiplicity } 4). \tag{213}$$

Continuing the characteristic analysis, it can be shown that only 13 eigenvectors exist. The eigenspace of the eigenvalue $\lambda = -v^s$, with algebraic multiplicity 4, has only geometric multiplicity 3. For example, the linearly independent right eigenvectors can be chosen as

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \\ 0 \\ \mathbf{0}_{8\times 1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 0 \\ 1 \\ \mathbf{0}_{8\times 1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ {}^{(s)}\epsilon_{BC}{}^{q}\perp^{C}{}_{A} \frac{\partial J^s}{\partial v_A} \\ 0 \\ 0 \\ \mathbf{0}_{8\times 1} \end{pmatrix}, \tag{214}$$

where we defined the antisymmetric lowercase two-Levi-Cività tensor for $s_a$ as ${}^{(s)}\epsilon^{AB} = n_c s_d {}^{q}\perp^{A}{}_{a} {}^{q}\perp^{B}{}_{b} \epsilon^{cdab}$. This result is contrary to an earlier analysis presented in Ref. [55]. The earlier analysis is erroneous since the three vectors called $r_{\lambda_{H0}}$ corresponding to $\lambda = -v^s$ are not eigenvectors. The explicit error is that the ninth component of these vectors may not be zero, since they produce cross-terms with the $A_{qH}$ (corresponding to our $\mathbf{A}_{1\times 5}$ part of the principal symbol). To substantiate our result, we performed a Jordan decomposition of the principal symbol (209). The Jordan normal form $\mathbf{J}[\mathbf{P}^s]$ of (209) can be written as

$$\mathbf{J}[\mathbf{P}^s] = \mathrm{diag}(\lambda_{(+)}, \lambda_{(-)}, \mathbf{J}_{v^s}, -\mathbb{1}_2 v^s, -\mathbb{1}_4, \mathbb{1}_4), \tag{215}$$

with

$$\mathbf{J}_{v^s} = \begin{pmatrix} -v^s & 1 \\ 0 & -v^s \end{pmatrix}, \tag{216}$$

and confirms that $\mathbf{P}^s$ is not diagonalizable. Therefore, the system of equations is weakly hyperbolic and has an ill-posed IVP.

It should be mentioned that for the special subcase $\tilde{J}^a \equiv 0$ the system is strongly hyperbolic. More generally, if $\tilde{J}^a$ does not depend on $v^b$ (more precisely, if $\frac{\partial J^s}{\partial v_A}$ vanishes identically), then the system is strongly hyperbolic. For the current in Eq. (200), these two cases coincide.

### C. Analysis without evolution of $q$

Next, we consider the system but suppress the $q$ variable. This analysis is for the system of equations used in Refs. [50–53]. We set $\psi$ to zero, the set of equations reduces to 12 evolution equations (203), (204), (205), (189), (190), and (192) for the components of the state vector $\mathbf{U} = (p, v_a, \varepsilon, E_a, B_a, \phi)^T$, and Eq. (191) becomes the standard Gauss constraint $D_a E^a = q$. This equation is not a constraint in the PDE sense; it is now rather the definition used to obtain $q$.

Since now we do not evolve $q$ by the conservation of charge equation (195), we have to replace all $q$'s by $D_a E^a$. Therefore, in Eqs. (203), (204), (205), and (189), we replace $J^a$ by use of Eq. (199) with

$$J^a = v^a \gamma^{pc} D_p E_c + \tilde{J}^a, \tag{217}$$

where the first term will contribute to the principal symbol.

Writing the system of equations in matrix form and decomposing against $s_a$, $s_a s^a = 1$, and ${}^{q}\perp^{b}{}_{a}$, we obtain

$$\mathbf{A}^n \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}}, \tag{218}$$

with the principal symbol

$$\mathbf{P}^s = \mathbf{A}^s = \begin{pmatrix} \mathbf{A}_{5\times 5} & \mathbf{B}_{5\times 7} \\ \mathbf{0}_{7\times 5} & \mathbf{C}_{7\times 7} \end{pmatrix}. \tag{219}$$

Again, $\mathbf{B}_{5\times7}$ contains the coefficients of spatial derivatives with respect to the variables $(E_a, B_a, \phi)$ in the time evolution of $(p, v_a, \varepsilon)$, and $\mathbf{A}_{5\times5} = \mathbf{P}^s_{\mathrm{HD}}$ is the principal symbol of the pure hydrodynamical sector, explicitly given in (A3). The matrix $\mathbf{C}_{7\times7}$ is the submatrix of the electromagnetic variables $(E_a, B_a, \phi)$, explicitly given by

$$\mathbf{C}_{7\times7} = \begin{pmatrix} -v^s & 0 & 0 & 0 & 0 & 0 & 0 \\ -v^{q_1} & 0 & 0 & 0 & 0 & -1 & 0 \\ -v^{q_2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{220}$$

The 12 eigenvalues of (219) are given by the ones of $\mathbf{A}_{5\times5}$ and $\mathbf{C}_{7\times7}$; these are

$$\mathbf{A}_{5\times5}\colon \lambda = -v^s, \qquad \text{(multiplicity 3)},$$
$$\lambda = \lambda_{(\pm)}, \qquad \text{[see (A4)]}, \tag{221}$$

$$\mathbf{C}_{7\times7}\colon \lambda = \pm 1, \qquad \text{(multiplicity 3)},$$
$$\lambda = -v^s, \qquad \text{(multiplicity 1)}. \tag{222}$$

As in the previous case, the eigenspace of the eigenvalue $\lambda = -v^s$ with algebraic multiplicity 4 has only geometric multiplicity 3. A set of right eigenvectors is

$$\begin{pmatrix} \mathbf{0}_{2\times1} \\ 1 \\ \mathbf{0}_{9\times1} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{0}_{3\times1} \\ 1 \\ \mathbf{0}_{8\times1} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{0}_{4\times1} \\ 1 \\ \mathbf{0}_{7\times1} \end{pmatrix}. \tag{223}$$

The Jordan normal form $\mathbf{J}[\mathbf{P}^s]$ of (219) is given by

$$\mathbf{J}[\mathbf{P}^s] = \mathrm{diag}(\lambda_{(+)}, \lambda_{(-)}, -\mathbb{1}_2 v^s, \mathbf{J}_{v^s}, -\mathbb{1}_3, \mathbb{1}_3), \tag{224}$$

with

$$\mathbf{J}_{v^s} = \begin{pmatrix} -v^s & 1 \\ 0 & -v^s \end{pmatrix}. \tag{225}$$

Therefore, the system of equations is also only weakly hyperbolic when the charge density variable $q$ is not evolved. The result also holds for $\phi = 0$, so that equation (192) reduces to the usual constraint $D_a B^a = 0$, and we evolve the 11 variables $(p, v_a, \varepsilon, E_a, B_a)$. In this case, a pair of eigenvalues $\lambda = \pm 1$ changes to the single eigenvalue $\lambda = 0$.

For the special subcase $\tilde{J}^a \equiv 0$, the system is strongly hyperbolic. This happens because in that case $q$ is algebraically related to the rest mass density $\rho_0$ and may thus be seen as a source term. Then, the algebraic multiplicity of $\lambda = -v^s$ changes to 3, and a complete set of eigenvectors can be found. From the physical point of view, the relevance of this model to compact binaries is, however, unclear to us. Note that we have not considered in this section general formulations of RGRMHD and that our calculations apply only to those formulations implemented. It is possible that these systems can be cured by a carefully chosen constraint addition.

A final comment is reserved for the special case of charged dust. In this model, $p = \varepsilon = 0$, and the charge density is proportional to the mass density with constant of proportionality equal to the specific charge. The system of equations for variables $(\rho_0, v_i, E_i, B_i)$ decouples into two parts: first, the evolution equations for $(\rho_0, v_i)$, which were already found to be weakly hyperbolic in Sec. III B, and second the electromagnetic equations, which can be given in a symmetric hyperbolic form; see Ref. [37]. The whole system is thus only weakly hyperbolic. In Ref. [56], it is shown that a different formulation of charged dust using $(v_i, E_i, B_i)$ as variables is strongly hyperbolic. In the authors' system, $\rho_0$ is obtained by the Gauss constraint equation relating the divergence of the electric field with the charge density. Under this treatment, however, the minimal coupling condition with the gravitational field equations, see Eq. (7), breaks. Therefore, away from the Cowling approximation, the full coupled system must be considered fresh.

## VI. CONCLUSION

Motivated by applications in numerical relativity, and in particular by the wish for the computation of accurate gravitational waveforms in compact binary spacetimes, we have revisited hyperbolicity of several popular relativistic fluid models. Our main technical achievement has been to bring about the DF formalism [9,10] to these matter models in a systematic way. This allowed us to arrive at a tractable form of even GRMHD, which is notorious for its complicated characteristic structure. The key idea was to use a Lagrangian frame in the analysis. In this frame, the principal symbol takes the simplest possible form and can be easily analyzed. Afterward, we could translate the results into the desired frame using the developed formalism.

Along the way, we arrived at several disconcerting results. That a commonly used formulation of GRMHD, plus those of RGRMHD, is only weakly hyperbolic is clearly a huge shortcoming that must be overcome if we are ever to obtain numerical results with meaningful error estimates for binary systems involving magnetic fields. One might wonder why the problem has not been discovered earlier on the basis of numerical work. The effect of ill-posedness on the errors in approximation is a subtle issue, however, and without very careful convergence testing can be easily overlooked, particularly when considering very complicated data. One aspect of this is that canonical test beds often focus on one-dimensional tests, which would not be suitable for identifying the issue identified for GRMHD. That said, it is important to realize that, although we are motivated by numerical applications, the analysis presented here is for the continuum PDE system. Thus, no numerical method, no matter how sophisticated, can circumvent our results, and therefore the equations *do* have to be altered. An obvious step in this direction would be to use our prototype algebraic constraint free formulation of GRMHD, which is at least strongly hyperbolic. This formulation cannot be written in flux-balance law form, but it fails only by the addition of constraint terms, so there is reason to be optimistic that existing codes can be easily modified to overcome this worst possible problem of ill-posedness of the IVP. There is hope that formulations using the four-potential, or those with divergence cleaning, are strongly hyperbolic. Thus, another possibility would be to affirm this and, if so, move wholesale to such systems. For RGRMHD, more work is needed.

We started the paper by stressing the well-known fact that the stellar surface is *also* a terrible problem in numerical relativity. Even in the case of GRHD, which does not have the same problems as flux-balance law GRMHD, the formally singular nature of the surface prevents clean convergence in simulations of even the most simple spacetimes. We expect that before this problem can be solved a much deeper understanding of the underlying initial free boundary value problem will be needed. So far, nothing in our treatment does anything whatsoever to alleviate this. We do think, however, that by carefully choosing the complete uppercase frame it may be possible to make progress by building on the present work. Sadly, this remains a distant goal.

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## APPENDIX: GRHD USING THE BOOST VECTOR

It holds that

$$\gamma^b{}_a \mathcal{L}_n \hat{v}_b = W \mathbb{g}^b{}_a \mathcal{L}_n v_b + \hat{v}_a K_{cd} \hat{v}^c \hat{v}^d. \tag{A1}$$

Using the state vector $\mathbf{U} = (p, v_a, \varepsilon)$ and an arbitrary unit spatial 1-form $s_a$ with $s_a s^a = 1$, $s_a n^a = 0$, and denoting the orthogonal projector by ${}^{\mathrm{q}}\!\perp^b{}_a := \gamma^b{}_a - s^b s_a$, the system of equations reads

$$(\mathcal{L}_n \mathbf{U})_{s,A} \simeq \mathbf{P}^s (D_s \mathbf{U})_{s,B}, \tag{A2}$$

and the principal symbol $\mathbf{P}^s$ is given by

$$\begin{pmatrix} W_{c_s}^2 (c_s^2 - 1) v^s & -W_{c_s}^2 c_s^2 \rho_0 h & 0^B & 0 \\ -\frac{1 + (c_s^2 - 1)(v^s)^2 W_{c_s}^2}{W^2 \rho_0 h} & W_{c_s}^2 (c_s^2 - 1) v^s & 0^B & 0 \\ \frac{(1 - c_s^2) W_{c_s}^2}{W^2 \rho_0 h} v^s v_A & \frac{c_s^2 W_{c_s}^2}{W^2} v_A & -v^s {}^{\mathrm{q}}\!\perp^B{}_A & 0_A \\ \frac{p W_{c_s}^2}{W^2 \rho_0^2 h} v^s & -\frac{p W_{c_s}^2}{\rho_0} & 0^B & -v^s \end{pmatrix}, \tag{A3}$$

with eigenvalues for material and acoustic waves

$$\lambda_{(0,1,2)} = -v^s,$$
$$\lambda_{(\pm)} = -\frac{1}{1 - c_s^2 v^2} \left( (1 - c_s^2) v^s \pm \frac{c_s}{W} \sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2} \right), \tag{A4}$$

respectively. They coincide with the literature [7]. The corresponding left eigenvectors are given by

$$\left( -\frac{p}{c_s^2 \rho_0^2 h} \quad 0 \quad 0^A \quad 1 \right),$$
$$\left( \frac{1}{W^2 \rho_0 h} v_C \quad v^s v_C \quad (1 - (v^s)^2) {}^{\mathrm{q}}\!\perp^A{}_C \quad 0 \right),$$
$$\left( \pm \frac{\sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2}}{c_s \rho_0 h W} \quad 1 \quad 0^A \quad 0 \right). \tag{A5}$$

For the same variables and order, the right eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ {}^{\mathrm{q}}\!\perp^C{}_B \\ 0 \end{pmatrix}, \tag{A6}$$

and

$$\begin{pmatrix} \frac{c_s^2 \rho_0^2 h}{p} (1 - (v^s)^2) \\ \pm \frac{c_s \rho_0}{p W} \sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2} (1 - (v^s)^2) \\ -\frac{c_s \rho_0}{p W} \left( \frac{c_s}{W} \pm v^s \sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2} \right) v_B \\ 1 - (v^s)^2 \end{pmatrix}, \tag{A7}$$

in agreement with the ones given in Ref. [7] up to the chosen set of variables and the spatial vector $s^a$. The characteristic variables corresponding to the speeds $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are given by

$$\hat{\mathrm{U}}_0 = \delta\varepsilon - \frac{p}{c_s^2 \rho_0^2 h} \delta p,$$
$$\hat{\mathrm{U}}_A = (\delta v)_A + v^s (v_A (\delta v)_s - v_s (\delta v)_A) + \frac{1}{\rho_0 h W^2} \hat{v}_A \delta p,$$
$$\hat{\mathrm{U}}_\pm = (\delta v)_s \pm \frac{\sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2}}{c_s \rho_0 h W} \delta p. \tag{A8}$$

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