Spin-dependent distribution functions for relativistic hydrodynamics of spin-$\frac{1}{2}$ particles

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Recently advocated expressions for the phase-space dependent spin-$\frac{1}{2}$ density matrices of particles and antiparticles are analyzed in detail and reduced to forms linear in the Dirac spin operator. This allows for a natural determination of the spin-polarization vectors of particles and antiparticles by evaluating the trace of products of the spin density matrices and the Pauli matrices. We demonstrate that the total spin-polarization vector obtained in this way agrees with the Pauli-Lubański four-vector, constructed from an appropriately chosen spin tensor and boosted to the particle rest frame. We further show that several forms of the spin tensor used in the literature yield the same Pauli-Lubański four-vector.

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I. INTRODUCTION

The idea that the global angular momentum of the hot and dense matter created in heavy-ion collisions may be reflected in the polarization of Λ hyperons and vector mesons has triggered broad interest in studies of the possible relation between vorticity and polarization [1–9] (see Ref. [10] for a recent review). The study of vorticity has gained widespread interest also because it is an important ingredient in studies of theories that deal with the production of false QCD vacuum states and chiral symmetry restoration [11]. In studies of vorticity, polarization and related topics, the role of the spin-orbit coupling [1,3,12,13], the polarization of rigidly rotating fluids in global equilibrium [4,14,15], the kinetics of spin [16–18], and anomalous hydrodynamics [19,20] have been explored. We also note the recent work based on the Lagrangian formulation of hydrodynamics [21,22].

Indeed, in noncentral heavy-ion collisions, a fireball is created with large global angular momentum, which may generate spin polarization in a way that resembles the Einstein-de Haas [23] and Barnett [24] effects. Since such collisions are well described by relativistic hydrodynamic models [25–27], it is of interest to include polarization explicitly in a hydrodynamic framework. So far, polarization effects have been taken into account only at the end of the hydrodynamic expansion, i.e., on the freeze-out hypersurface where a connection between vorticity and polarization was assumed [14,15]. In such approaches, the preceding dynamics of the polarization, from the initial stages of the collision until the freeze-out, is not accounted for.

Recently, a new hydrodynamic framework was constructed [28], which fully incorporates spin degrees of freedom in a perfect-fluid approach. This approach is based on the local-equilibrium, spin-dependent phase-space distribution functions $f^{\pm}(x,p)$, put forward in Ref. [15]. In this work, we study formal aspects connected with the calculation of thermodynamic and hydrodynamic quantities

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using the functions $f^{\pm}(x,p)$. We reduce the original exponential form to an expression linear in the Dirac spin operator $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. This allows for a straightforward determination of the spin-polarization vectors of particles and antiparticles by evaluating the trace of the product of the phase-space densities with Pauli matrices. We show that the total spin-polarization vector obtained in this way agrees with the Pauli-Lubański (PL) four-vector[1] [29], constructed from the spin tensor used in [28] and boosted to the particle rest frame. Interestingly, other forms of the spin tensors used in the literature yield the same PL four-vector (with the exception of the Belinfante construction, which leads to a vanishing spin tensor). This indicates that the framework put forward in [28] provides a consistent extension of ideal hydrodynamics for the treatment of spin-polarized fluids. The starting point of our approach is a phase-space distribution function, where positions and momenta are treated classically, whereas the spins are treated quantum mechanically. This is similar in spirit to a semiclassical transport theory for particles with spin [30,31].

Conventions and notation: We use the following conventions and notation for the metric tensor, the four-dimensional Levi-Civita's tensor, and the scalar product in Minkowski space: $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, $\epsilon^{0123} = -\epsilon_{0123} = 1$, $a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu$. Three-vectors are shown in bold font and a dot is used to denote the scalar product of both four- and three-vectors, e.g., $a^\mu b_\mu = a \cdot b = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$. For the three-dimensional Levi-Civita tensor ϵ_{ijk}, with $\epsilon_{123} = +1$, we do not distinguish between lower and upper components, note that $\epsilon_{0123} = -\epsilon_{123} = -1$. The symbol $\mathbb{1}$ is used for a two-by-two or four-by-four unit matrix. On the other hand, we distinguish the traces in spin and spinor spaces by using the symbols tr_2 and tr_4, respectively.

The components of the four-momentum of a particle with mass m are $p^\mu = (E_p, \boldsymbol{p})$, with E_p being the on-mass-shell energy, $E_p = \sqrt{m^2 + \boldsymbol{p}^2}$, and the components of the four-velocity of the fluid element are $u^\mu = (u^0, \boldsymbol{u})$. The quantities defined in the particle rest frame are marked by an asterisk, those defined in the local fluid rest frame (LFRF) are labeled with a circle, while unlabeled quantities refer to the laboratory frame (LAB). Using this convention, the symbol $\boldsymbol{u}_*$ denotes the components of the fluid three-velocity seen in the particle rest frame, whereas $\boldsymbol{p}_\circ$ denotes the components of a particle three-momentum in the local fluid rest frame.[2] The sign and labeling conventions for the Dirac bispinors are given in Appendix A. Except for Appendix B, where we temporarily switch to the chiral representation, all calculations are done using the Dirac representation for the gamma matrices. Throughout the text we use natural units with $c = \hbar = k_B = 1$.

II. SPIN-DEPENDENT DISTRIBUTION FUNCTIONS

A. Basic definitions

In this work we analyze the phase-space distribution functions for spin-$\frac{1}{2}$ particles and antiparticles in local equilibrium, introduced in Ref. [15]. To include spin degrees of freedom, the standard scalar functions are generalized to two-by-two matrices in spin space for each value of the space-time position x and four-momentum p,

$$[f^+(x,p)]_{rs} \equiv f^+_{rs}(x,p) = \bar{u}_r(p) X^+ u_s(p), \qquad (1)$$

$$[f^-(x,p)]_{rs} \equiv f^-_{rs}(x,p) = -\bar{v}_s(p) X^- v_r(p). \qquad (2)$$

Here $u_r(p)$ and $v_r(p)$ are Dirac bispinors (with the spin indices r and s running from 1 to 2), and the normalization[3] $\bar{u}_r(p) u_s(p) = \delta_{rs}$ and $\bar{v}_r(p) v_s(p) = -\delta_{rs}$. Note the minus sign and different ordering of spin indices in Eq. (2) compared to Eq. (1). The objects $f^{\pm}(x,p)$ are two-by-two Hermitian matrices with the matrix elements defined by Eqs. (1) and (2).

Following Refs. [15,28], we introduce the four-by-four matrices

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x) p^\mu\right] M^{\pm}, \qquad (3)$$

where

$$M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]. \qquad (4)$$

In Eqs. (3) and (4), $\beta^\mu = u^\mu/T$ and $\xi = \mu/T$, with the temperature T, chemical potential μ and the fluid four velocity u^μ (normalized to unity). The quantity $\omega_{\mu\nu}$ is the spin-polarization tensor. For the sake of simplicity, we restrict ourselves to classical Boltzmann statistics in this work.[4]

B. Polarization tensor

The antisymmetric polarization tensor $\omega_{\mu\nu}$ is defined by the tensor decomposition

[1] In the particle rest frame, the PL four-vector does not change sign under reflections and is, therefore, often referred to as a pseudo-four-vector.

[2] For a particle with four-momentum p in the laboratory frame, the particle rest frame is obtained by boosting from LAB by the three-velocity $\mathbf{v}_p = \boldsymbol{p}/E_p$, while the local fluid rest frame is reached by a boost from LAB by $\mathbf{v} = \boldsymbol{u}/u^0$. The boosts considered in this work are all canonical or pure boosts [32]. Their explicit form is given in Sec. IV B.

[3] To simplify the notation, the factor $2m$ appearing explicitly in the normalization conditions used in Refs. [15,28] (with m being the particle mass) is here included in the definition of the bispinors.

[4] We note that by performing an analytic continuation of the polarization tensor, $\omega_{\mu\nu} \to -i\omega_{\mu\nu}$, the matrix M^+ becomes a representation of the Lorentz transformation $S(\Lambda)$ with $\Lambda^\mu_{\ \nu} = g^\mu_\nu + \omega^\mu_{\ \nu}$.

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \omega^\beta, \tag{5}$$

where $k \cdot u = \omega \cdot u = 0$ and

$$k_\mu = \omega_{\mu\nu} u^\nu, \qquad \omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\nu\alpha}u^\beta. \tag{6}$$

We note that k_μ and ω_μ are spacelike four-vectors with only three independent components. In early works on fluids with spin [33], the so-called Frenkel condition, $k_\mu = 0$, was introduced. We shall return to this condition below.

The dual polarization tensor is defined by the expression

$$\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta} = \omega_\mu u_\nu - \omega_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta}k^\alpha u^\beta. \tag{7}$$

Using Eqs. (5) and (7) one easily finds

$$\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = k \cdot k - \omega \cdot \omega, \qquad \frac{1}{2}\tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = 2k \cdot \omega,$$
$$\frac{1}{2}\tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu} = \omega \cdot \omega - k \cdot k. \tag{8}$$

It is instructive to introduce another parametrization of the polarization tensor, which uses electric- and magnetic-like three-vectors[5] in LAB, $\boldsymbol{e} = (e^1, e^2, e^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$. In this case we write (following the sign conventions of Ref. [34])

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \tag{9}$$

Using Eq. (9) in Eq. (6) one finds

$$k^\mu = (k^0, \boldsymbol{k}) = (\boldsymbol{e} \cdot \boldsymbol{u}, u^0 \boldsymbol{e} + \boldsymbol{u} \times \boldsymbol{b}),$$
$$\omega^\mu = (\omega^0, \boldsymbol{\omega}) = (\boldsymbol{b} \cdot \boldsymbol{u}, u^0 \boldsymbol{b} - \boldsymbol{u} \times \boldsymbol{e}). \tag{10}$$

In the LFRF, where $u^0 = 1$ and $\boldsymbol{u} = 0$, we have $\boldsymbol{k} = \boldsymbol{e}$ and $\boldsymbol{\omega} = \boldsymbol{b}$ (i.e., $\boldsymbol{k}_\circ = \boldsymbol{e}_\circ$ and $\boldsymbol{\omega}_\circ = \boldsymbol{b}_\circ$). In order to switch from $\omega_{\mu\nu}$ to the dual tensor $\tilde{\omega}_{\mu\nu}$, one replaces $\boldsymbol{e}$ by $\boldsymbol{b}$ and $\boldsymbol{b}$ by $-\boldsymbol{e}$. Using Eq. (10), one finds

$$\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = \boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{e} \cdot \boldsymbol{e}, \qquad \frac{1}{2}\tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = -2\boldsymbol{e} \cdot \boldsymbol{b},$$
$$\frac{1}{2}\tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu} = \boldsymbol{e} \cdot \boldsymbol{e} - \boldsymbol{b} \cdot \boldsymbol{b}. \tag{11}$$

[5]Unlike real magnetic fields, which act on the magnetic moments, $\boldsymbol{b}$ act on the spins and consequently induces the same spin polarization for particles and antiparticles [see Eq. (28) in Sec. III].

C. Spin matrices $M^\pm$

In Appendix B, we show that the exponential dependence of the distribution function on $\Sigma^{\mu\nu}$ given in Eq. (4), which is defined in terms of a power series, can be resummed. This results in an expression for $M^\pm$, linear in $\Sigma^{\mu\nu}$,

$$M^\pm = \mathbb{1}\left[\Re(\cosh z) \pm \Re\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right]$$
$$+ i\gamma_5\left[\Im(\cosh z) \pm \Im\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right], \tag{12}$$

where $\mathbb{1}$ is a unit matrix and

$$z = \frac{1}{2\sqrt{2}}\sqrt{\omega_{\mu\nu}\omega^{\mu\nu} + i\omega_{\mu\nu}\tilde{\omega}^{\mu\nu}}$$
$$= \frac{1}{2}\sqrt{k \cdot k - \omega \cdot \omega + 2ik \cdot \omega}. \tag{13}$$

It was demonstrated in Ref. [28] that a consistent thermodynamic description of particles with spin is obtained for real z. In this case, z can be interpreted as the spin chemical potential Ω divided by T. Here we follow this approach and restrict our considerations to the case where

$$k \cdot \omega = \boldsymbol{e} \cdot \boldsymbol{b} = 0,$$
$$k \cdot k - \omega \cdot \omega = \boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{e} \cdot \boldsymbol{e} \geq 0. \tag{14}$$

Consequently, in what follows, we replace z by a real number ζ in Eq. (12) and use

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\omega_{\mu\nu}\Sigma^{\mu\nu}, \tag{15}$$

where[6]

$$\zeta = \frac{\Omega}{T} = \frac{1}{2}\sqrt{k \cdot k - \omega \cdot \omega} = \frac{1}{2}\sqrt{\boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{e} \cdot \boldsymbol{e}}. \tag{16}$$

At this point it is convenient to introduce the rescaled quantities:

$$\bar{\omega}_{\mu\nu} = \frac{\omega_{\mu\nu}}{2\zeta}, \qquad \bar{\tilde{\omega}}_{\mu\nu} = \frac{\tilde{\omega}_{\mu\nu}}{2\zeta}, \qquad \bar{k}_\mu = \frac{k_\mu}{2\zeta}, \qquad \bar{\omega}_\mu = \frac{\omega_\mu}{2\zeta}, \tag{17}$$

[6]We note that since $M^\pm$ is an even function of ζ, we can, without loss of generality, choose the positive root in Eq. (16). The direction of the polarization is determined by the elements of the polarization tensor $\omega^{\mu\nu}$, in particular by $\boldsymbol{b}$ [cf. Eq. (31)].

which satisfy the following normalization conditions:

$$\frac{1}{2}\bar{\omega}_{\mu\nu}\bar{\omega}^{\mu\nu} = 1, \quad \frac{1}{2}\bar{\tilde{\omega}}_{\mu\nu}\bar{\tilde{\omega}}^{\mu\nu} = -1, \quad \bar{k}\cdot\bar{k} - \bar{\omega}\cdot\bar{\omega} = 1. \tag{18}$$

D. Observables

The matrix distribution functions, given in Eqs. (1) and (2), can be used to obtain the energy-momentum tensor [35]

$$T^{\mu\nu} = \kappa \int \frac{d^3p}{2E_p} p^\mu p^\nu \mathrm{tr}_4(X^+ + X^-) \tag{19}$$

and the spin tensor [14]

$$S^{\lambda,\mu\nu} = \kappa \int \frac{d^3p}{2E_p} p^\lambda \,\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\mu\nu}]. \tag{20}$$

Here $\kappa = g/(2\pi)^3$ with g accounting for internal degrees of freedom different from spin (for example, color or isospin).

For the discussion of the PL four-vector in Sec. IV it is convenient to introduce the total particle current

$$\mathcal{N}^\mu = \kappa \int \frac{d^3p}{2E_p} p^\mu [\mathrm{tr}_4(X^+) + \mathrm{tr}_4(X^-)], \tag{21}$$

which sums the contributions from particles and antiparticles, and the net conserved charge current

$$N^\mu = \kappa \int \frac{d^3p}{2E_p} p^\mu [\mathrm{tr}_4(X^+) - \mathrm{tr}_4(X^-)], \tag{22}$$

which is the difference between the particle and antiparticle currents.

E. Stationary vortex

In Fig. 1 we show the vectors $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{k}$, and $\boldsymbol{\omega}$ for the stationary vortex studied in Ref. [4]. In this case $\boldsymbol{e} = (0,0,0)$ and $\boldsymbol{b} = -\tilde{\Omega}/T_0(0,0,1)$, while $\boldsymbol{k} = -\tilde{\Omega}^2(\gamma/T_0)(x,y,0)$ and $\boldsymbol{\omega} = \gamma\boldsymbol{b} = -\tilde{\Omega}(\gamma/T_0)(0,0,1)$. Here $\tilde{\Omega}$ and T_0 are constant parameters corresponding to the angular momentum and central temperature of the vortex. The hydrodynamic flow is given by the four-vector u^μ [28],

$$u^0 = \gamma, \quad u^1 = -\gamma\tilde{\Omega}y, \quad u^2 = \gamma\tilde{\Omega}x, \quad u^3 = 0, \tag{23}$$

where $\gamma = 1/\sqrt{1-\tilde{\Omega}^2 r^2}$, and $r = \sqrt{x^2+y^2}$ is the distance from the center of the vortex in the transverse plane. Since the flow velocity cannot exceed the speed of light, the flow profile Eq. (23) may be realized only within a cylinder of radius $R < 1/\tilde{\Omega}$ (illustrated by the green circle in Fig. 1).

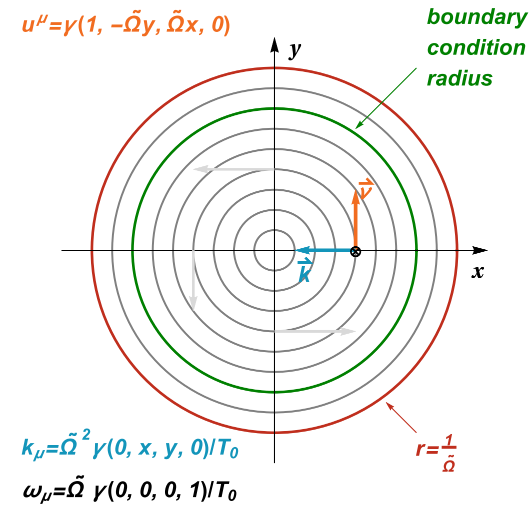

FIG. 1. Hydrodynamic flow and polarization variables for the global thermodynamic equilibrium state studied in [4].

III. SPIN POLARIZATION THREE-VECTOR

We expect that the spin observables are represented by Pauli matrices $\boldsymbol{\sigma}$ and that the expectation values of $\boldsymbol{\sigma}$ provide information on the polarization of spin-$\frac{1}{2}$ particles in their rest frame. Since we consider Dirac bispinors obtained by the so-called *canonical Lorentz boosts* applied to states with zero momentum, we refer to the resulting spin distributions and particle rest frames as the canonical ones (differing from other definitions by a rotation).

In the following, we start with Eqs. (1) and (2), and derive a decomposition of the distribution functions $f^\pm(x,p)$ in terms of Pauli matrices. The decomposition introduces a polarization vector $\boldsymbol{P}(x,p)$, which can be interpreted as a spatial part of the polarization four-vector $P^\mu(x,p) = (0,\boldsymbol{P})$, with a vanishing zeroth component. The average polarization vector $\boldsymbol{\mathcal{P}}(x,p)$ is normalized by the trace of the distribution functions. In Sec. IV we demonstrate that $\boldsymbol{\mathcal{P}}(x,p)$ agrees with the spatial part of the PL four-vector $(\pi^0(x,p), \boldsymbol{\pi}(x,p))$, obtained from the spin tensor employed in [28], boosted to the canonical rest frame of particles with the LAB four-momentum p^μ.

Using Eqs. (3) and (15), the spin-dependent distribution functions given in Eqs. (1) and (2), can be rewritten in a form linear in the Dirac spin tensor,

$$f^+_{rs}(x,p) = e^{\xi - p\cdot\beta}\left[\cosh(\zeta)\delta_{rs} + \frac{\sinh(\zeta)}{2\zeta}\bar{u}_r(p)\omega_{\alpha\beta}\Sigma^{\alpha\beta}u_s(p)\right], \tag{24}$$

$$f^-_{rs}(x,p) = e^{-\xi - p\cdot\beta}\left[\cosh(\zeta)\delta_{rs} + \frac{\sinh(\zeta)}{2\zeta}\bar{v}_s(p)\omega_{\alpha\beta}\Sigma^{\alpha\beta}v_r(p)\right]. \tag{25}$$

To proceed further we use the two identities:

$$\bar{u}_r(p)\Sigma^{0i}u_s(p) = \bar{v}_s(p)\Sigma^{0i}v_r(p) = -\frac{1}{2m}\epsilon_{ijk}p^j\sigma^k_{rs}, \tag{26}$$

$$\bar{u}_r(p)\Sigma^{ij}u_s(p) = \bar{v}_s(p)\Sigma^{ij}v_r(p) = \frac{E_p}{2m}\epsilon_{ijk}\left(\sigma^k_{rs} - p^k\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}_{rs}}{E_p(E_p+m)}\right). \tag{27}$$

Using Eqs. (26) and (27), a straightforward calculation yields

$$\bar{u}_r(p)\omega_{\alpha\beta}\Sigma^{\alpha\beta}u_s(p) = \bar{v}_s(p)\omega_{\alpha\beta}\Sigma^{\alpha\beta}v_r(p) = P^0\delta_{rs} - \boldsymbol{P}\cdot\boldsymbol{\sigma}_{rs}, \tag{28}$$

where $P^0 = 0$ and the three-vector $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \frac{1}{m}\left[u^0\left(E_p\boldsymbol{\omega} - \boldsymbol{p}\times\boldsymbol{k} - \frac{\boldsymbol{p}\cdot\boldsymbol{\omega}}{E_p+m}\boldsymbol{p}\right) - \omega^0\left(E_p\boldsymbol{u} - \frac{\boldsymbol{p}\cdot\boldsymbol{u}}{E_p+m}\boldsymbol{p}\right) + k^0(\boldsymbol{p}\times\boldsymbol{u}) + (\boldsymbol{p}\cdot\boldsymbol{\omega})\boldsymbol{u} - (\boldsymbol{p}\cdot\boldsymbol{u})\boldsymbol{\omega} - \left(E_p(\boldsymbol{k}\times\boldsymbol{u}) - \frac{\boldsymbol{p}\cdot(\boldsymbol{k}\times\boldsymbol{u})}{E_p+m}\boldsymbol{p}\right)\right] \tag{29}$$

or

$$\boldsymbol{P} = \frac{1}{m}\left[E_p\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right], \tag{30}$$

depending whether we use the parametrization given in Eq. (5) or (9), respectively. We note that the expression on the right-hand side of Eq. (30) is just the field $\boldsymbol{b}$ in the particle rest frame [34]. We summarize this finding by writing

$$\boldsymbol{P} = \boldsymbol{b}_*. \tag{31}$$

Thus, the polarization is determined by the field $\boldsymbol{b}$ in the canonical rest frame of the particle.

Using Eq. (28) in Eqs. (24) and (25) we then find

$$f^\pm(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]. \tag{32}$$

In the next step, we define the average polarization vector $\boldsymbol{\mathcal{P}}$ by the formula

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\frac{\mathrm{tr}_2[(f^+ + f^-)\boldsymbol{\sigma}]}{\mathrm{tr}_2[f^+ + f^-]} = -\frac{1}{4\zeta}\tanh(\zeta)\boldsymbol{P}. \tag{33}$$

Using Eq. (16), we obtain an alternative expressions

$$\boldsymbol{\mathcal{P}} = -\frac{1}{2}\tanh\left[\frac{1}{2}\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}\right]\frac{\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}, \tag{34}$$

where we have used the property that the quantity $\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{e}\cdot\boldsymbol{e}$ is independent of the choice of the Lorentz frame.

IV. PAULI-LUBAŃSKI FOUR-VECTOR

A. Phase-space density

Starting from the definition of the Pauli-Lubański four-vector $\Pi_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}J^{\nu\alpha}p^\beta$ (where $J^{\nu\alpha}$ is angular momentum), and following Ref. [15], we introduce the phase-space density of Π_μ defined by the following expression:

$$E_p\frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Delta\Sigma_\lambda(x)E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}. \tag{35}$$

Here $\Delta\Sigma_\lambda$ denotes a space-time element of the fluid and $E_p dJ^{\lambda,\nu\alpha}/d^3p$ denotes the invariant angular momentum phase-space density of particles with four-momentum p. Using definitions introduced in Ref. [28], analogous to Eqs. (19) and (20), we find

$$E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}p^\lambda(x^\nu p^\alpha - x^\alpha p^\nu)\mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2}p^\lambda\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\nu\alpha}]. \tag{36}$$

Clearly, the orbital part in Eq. (36) does not contribute to the density of Π_μ. Hence we find

$$E_p\frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Delta\Sigma_\lambda(x)E_p\frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}, \tag{37}$$

where

$$E_p\frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}p^\lambda\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\nu\alpha}]. \tag{38}$$

Performing the traces in Eq. (38), one obtains

$$E_p\frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p} = \kappa e^{-p\cdot\beta}\cosh(\xi)\frac{\sinh(\zeta)}{\zeta}p^\lambda\omega^{\nu\alpha}. \tag{39}$$

Now, substituting Eq. (39) into Eq. (37) we find

$$E_p\frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Delta\Sigma_\lambda(x)\frac{w(x,p)}{4\zeta}p^\lambda\omega^{\nu\alpha}\frac{p^\beta}{m} = -\Delta\Sigma\cdot p\frac{w(x,p)}{4m\zeta}\tilde{\omega}_{\mu\beta}p^\beta, \tag{40}$$

where $w(x,p) = 4\kappa e^{-p\cdot\beta}\cosh(\xi)\sinh(\zeta)$. Since we are interested in the polarization effect per particle, it is convenient to introduce the particle density in the volume $\Delta\Sigma$ defined with the help of Eq. (21). This leads to the expression

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2}\Delta\Sigma\cdot p\,\mathrm{tr}_4(X^+ + X^-) = 4\kappa\Delta\Sigma\cdot p e^{-p\cdot\beta}\cosh(\xi)\cosh(\zeta). \tag{41}$$

The PL vector per particle is then obtained by dividing Eq. (40) by Eq. (41),

$$\pi_\mu(x,p) \equiv \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)} = -\frac{\tanh(\zeta)}{4m\zeta}\tilde{\omega}_{\mu\beta}p^\beta = -\frac{\tanh(\zeta)}{4m\zeta}(\omega_\mu p\cdot u - u_\mu p\cdot\omega + \epsilon_{\mu\rho\sigma\beta}k^\rho u^\sigma p^\beta), \tag{42}$$

where in the second line we have used the definition of the dual polarization tensor given in Eq. (7). Using the rescaled quantities, defined in Eq. (17), we finally arrive at

$$\pi_\mu(x,p) = -\frac{\tanh(\zeta)}{2m}\tilde{\bar{\omega}}_{\mu\beta}p^\beta = -\frac{\tanh(\zeta)}{2m}(\bar{\omega}_\mu p\cdot u - u_\mu p\cdot\bar{\omega} + \epsilon_{\mu\rho\sigma\beta}\bar{k}^\rho u^\sigma p^\beta). \tag{43}$$

B. Boost to particle rest frame

In order to transform the four-vector π^μ to the local rest frame of a particle with momentum p, we use the canonical boost [32]

$$\Lambda^\mu_\nu(-\mathbf{v}_p) = \begin{bmatrix} \frac{E_p}{m} & -\frac{p^1}{m} & -\frac{p^2}{m} & -\frac{p^3}{m} \\ -\frac{p^1}{m} & 1+\alpha_p p^1p^1 & \alpha_p p^1 p^2 & \alpha_p p^1 p^3 \\ -\frac{p^2}{m} & \alpha_p p^2 p^1 & 1+\alpha_p p^2p^2 & \alpha_p p^2 p^3 \\ -\frac{p^3}{m} & \alpha_p p^3 p^1 & \alpha_p p^3 p^2 & 1+\alpha_p p^3 p^3 \end{bmatrix}, \tag{44}$$

where $\mathbf{v}_p = \boldsymbol{p}/E_p$ and $\alpha_p = 1/(m(E_p+m))$. Using Eq. (43), we can express the time and space components of $\pi^\mu = (\pi^0, \boldsymbol{\pi})$ in the LAB frame in the three-vector notation

$$\pi^0 = -\frac{\tanh\zeta}{4\zeta m}(u^0\boldsymbol{p}\cdot\boldsymbol{\omega} - \omega^0\boldsymbol{p}\cdot\boldsymbol{u} + \boldsymbol{k}\cdot(\boldsymbol{p}\times\boldsymbol{u})), \tag{45}$$

$$\boldsymbol{\pi} = -\frac{\tanh\zeta}{4\zeta m}(\boldsymbol{\omega} p\cdot u - \boldsymbol{u} p\cdot\omega + k^0\boldsymbol{p}\times\boldsymbol{u} - u^0\boldsymbol{p}\times\boldsymbol{k} - E_p\boldsymbol{k}\times\boldsymbol{u}). \tag{46}$$

By applying the Lorentz transformation Eqs. (44)–(46) we finally arrive at

$$\pi_*^0 = 0 \tag{47}$$

and

$$\boldsymbol{\pi}_* = \boldsymbol{\mathcal{P}} = -\frac{1}{4\zeta}\tanh(\zeta)\boldsymbol{P}. \tag{48}$$

Due to the Lorentz four-vector character of π_μ, we have $\pi_\mu\pi^\mu = \pi^*_\mu\pi^\mu_* = -\boldsymbol{\mathcal{P}}^2$.

V. OTHER SPIN-TENSOR FORMS

A. Independence of the PL four-vector

Another form for the spin tensor, which can be used to construct the PL four-vector, is given by

$$S^{\lambda,\mu\nu}_{\rm can} = \kappa\int\frac{d^3p}{2E_p}(p^\lambda\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\mu\nu}] - p^\mu\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\lambda\nu}] + p^\nu\mathrm{tr}_4[(X^+ - X^-)\Sigma^{\lambda\mu}]). \tag{49}$$

Equation (49) was derived in Ref. [15] and corresponds to the canonical spin tensor, obtained directly by applying Noether's theorem to the Dirac Lagrangian. This form of the spin tensor differs from Eq. (20) by two additional terms containing p^μ and p^ν in the integrand. When inserted in Eq. (37), such terms yield bilinears in the momentum, symmetric in two of the indices that are contracted with the Levi-Civita tensor. Consequently, their contribution to the PL four-vector vanishes.

Yet another version of the spin tensor, introduced in the textbook by de Groot, van Leeuven, and van Weert [35], reads

$$S^{\lambda,\mu\nu}_{\rm GLW} = \kappa\int\frac{d^3p}{E_p}p^\lambda(\mathrm{tr}_2[f^+(x,p)\Sigma^{\mu\nu}_+] + \mathrm{tr}_2[f^-(x,p)\Sigma^{\mu\nu}_-]), \tag{50}$$

where

$$[\Sigma^{\mu\nu}_+]_{rs} = \bar{u}_r(p)\Sigma^{\mu\nu}u_s(p), \quad [\Sigma^{\mu\nu}_-]_{rs} = \bar{v}_s(p)\Sigma^{\mu\nu}v_r(p). \tag{51}$$

By changing the trace over spin indices to a trace in spinor space and using the commutation relation

$$[\Sigma^{\mu\nu}, p_\alpha\gamma^\alpha] = ip^\nu\gamma^\mu - ip^\mu\gamma^\nu \tag{52}$$

we find

$$S_{\rm GLW}^{\lambda,\mu\nu} = \kappa \int \frac{d^3p}{2E_p} p^\lambda \bigg({\rm tr}_4[(X^+ - X^-)\Sigma^{\mu\nu}] + \frac{i}{2m^2} {\rm tr}_4[(X^+ - X^-) p_\alpha \gamma^\alpha (\gamma^\mu p^\nu - \gamma^\nu p^\mu)] \bigg). \tag{53}$$

We again notice that only the first term of the integrand in Eq. (53) contributes to the PL four-vector. Thus, the resulting PL four-vector is identical for all three forms of the spin tensor.

B. Large m/T limit of the de Groot-van Leeuwen-van Weert spin tensor

In this section we consider the large m/T limit of the spin tensor[7] introduced in Ref. [35]. This exercise is instructive, since the result is simple and sheds some light on the relevance of the Frenkel condition. We introduce the symbol $S_\Delta^{\lambda,\mu\nu}$ for the second term in Eq. (53):

$$S_{\rm GLW}^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} + S_\Delta^{\lambda,\mu\nu}. \tag{54}$$

Then, using the identity

$$\begin{aligned} &{\rm tr}_4[(a + b\omega_{\rho\sigma}\Sigma^{\rho\sigma}) p_\alpha \gamma^\alpha (\gamma^\mu p^\nu - \gamma^\nu p^\mu)] \\ &= 4ibp_\alpha (p^\nu \omega^{\mu\alpha} - p^\mu \omega^{\nu\alpha}), \end{aligned} \tag{55}$$

where a and b are arbitrary scalars, we find

$$S_\Delta^{\lambda,\mu\nu} = \frac{\kappa \sinh(\zeta)\cosh(\xi)}{m^2\zeta} \int \frac{d^3p}{E_p} e^{-\beta\cdot p} p^\lambda p_\alpha (p^\mu \omega^{\nu\alpha} - p^\nu \omega^{\mu\alpha}). \tag{56}$$

The integral over momentum in Eq. (56) can be decomposed into tensors constructed from the four-velocity u^μ and the metric tensor $g^{\mu\nu}$. After contraction with the polarization tensor $\omega^{\mu\nu}$, we find

$$\begin{aligned} S_\Delta^{\lambda,\mu\nu} = &\frac{g \sinh(\zeta)\cosh(\xi)}{m^2\zeta} T[\varepsilon_{(0)}(T) + P_{(0)}(T)] s_\Delta^{\lambda,\mu\nu} \\ &+ \frac{g \sinh(\zeta)\cosh(\xi)}{T\zeta} P_{(0)}(T)(k^\nu u^\lambda u^\mu - k^\mu u^\lambda u^\nu). \end{aligned} \tag{57}$$

Here $\varepsilon_{(0)}(T)$ and $P_{(0)}(T)$ are the energy density and pressure[8] of classical spinless particles with mass m computed at the temperature T [28], while the tensor $s_\Delta^{\lambda,\mu\nu}$ is defined by

$$\begin{aligned} s_\Delta^{\lambda,\mu\nu} = &2u^\lambda(\omega^{\mu\nu} + 3(k^\nu u^\mu - k^\mu u^\nu)) + \omega^{\mu\lambda}u^\nu - \omega^{\nu\lambda}u^\mu \\ &+ k^\mu g^{\lambda\nu} - k^\nu g^{\lambda\mu}. \end{aligned} \tag{58}$$

In the limit $m \ll T$, $\varepsilon_{(0)}(T) \approx mn_{(0)}(T)$, while $P_{(0)}(T) = Tn_{(0)}(T)$ for Boltzmann statistics. Thus, the first term on the right-hand side of Eq. (57) is of order T/m compared to the second one, and thus negligible in the large m/T limit. Moreover, the term in the second line of Eq. (57) exactly cancels the part of $S^{\lambda,\mu\nu}$ depending on the four-vector k [see Eqs. (5) and (27) in [28]]. Consequently, the large m/T limit of Eq. (50) yields

$$S_{\rm GLW}^{\lambda,\mu\nu} = g\frac{\sinh(\zeta)}{\zeta}\cosh(\xi) n_{(0)}(T) u^\lambda \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta \tag{59}$$

which is independent of k. Interestingly, this result is identical to that obtained by imposing the Frenkel condition $k^\mu = 0$ on Eq. (27) in [28].

VI. SUMMARY AND CONCLUSIONS

In this work we have studied the properties of the spin density matrices used in recent formulations of relativistic hydrodynamics of particles with spin $\frac{1}{2}$. We showed that the total polarization vector, obtained by calculating the trace of the product of spin density matrices and the Pauli matrices, agrees with the Pauli Lubański four-vector obtained from the spin tensor used in Ref. [28]. Consequently, this scheme results in a natural determination of the spin-polarization vectors of both particles and antiparticles. We also found that two other common forms of the spin tensor yield the same polarization vector, thus demonstrating that the form used in [28] represents a consistent approximation for the spin tensor in the case where positions and momenta are treated classically, while the spin is described in terms of a spin density matrix.

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[7]In the canonical spin tensor, the three terms appearing in Eq. (49) have the same dependence on m/T. Hence, in that case the large m/T limit yields no new insight. This is different for the de Groot-van Leeuwen-van Weert spin tensor.

[8]In the definition of thermodynamic functions, such as $\varepsilon_{(0)}(T)$ or $P_{(0)}(T)$, the factor $(2\pi)^3$ in the denominator of the momentum integration measure are included. Hence, the factor κ in Eq. (57) has been replaced by g.

APPENDIX A: DIRAC SPINORS

The conventions for labels and signs of bispinors used in this work are as follows:

$$u_s(p) = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix} 1 & \varphi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m} & \varphi_s \end{pmatrix},$$
$$v_s(p) = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m} & \chi_s \\ 1 & \chi_s \end{pmatrix}, \qquad \text{(A1)}$$

with

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_2 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad \text{(A2)}$$

The spin operator $\Sigma^{\mu\nu}$ is defined by the expression

$$\Sigma^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \qquad \text{(A3)}$$

which in the Dirac representation gives

$$\Sigma^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \Sigma^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \text{(A4)}$$

with σ^i being the ith Pauli matrix.

APPENDIX B: SPIN MATRICES $M^\pm$

In this section we present details of the calculation of the matrix $M^\pm$, which is defined by the exponential function of the Dirac spin operator, see Eq. (4). To do this calculation most easily we first switch to the chiral representation of the Dirac matrices, where $\Sigma^{\mu\nu}$ is block diagonal, and then move to the local rest frame of the fluid element, where $u^\mu_\circ = \Lambda^\mu{}_\nu u^\nu = (1,0,0,0)$. Calculation of the exponential function in Eq. (4) in the chiral representation with $u^\mu_\circ = (1,0,0,0)$ is reduced to the well-known calculation of the exponential function of a linear combination of the Pauli matrices. Once it is done, we come back to the LAB frame (from the local rest frame of the fluid element) and perform a unitary transformation back to the Dirac representation.

With $\mathcal{S}$ denoting the transformation matrix that corresponds to the Lorentz transformation Λ, we have

$$\mathcal{S}\gamma^\mu\mathcal{S}^{-1} = \Lambda^\mu{}_\nu\gamma^\nu, \qquad \text{(B1)}$$

$$\mathcal{S}\Sigma^{\mu\nu}\mathcal{S}^{-1} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Sigma^{\alpha\beta}, \qquad \text{(B2)}$$

and

$$M^\pm_\circ = \mathcal{S}M^\pm\mathcal{S}^{-1} = \mathcal{S}\exp\left(\pm\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\mathcal{S}^{-1}$$
$$= \exp\left(\pm\frac{1}{2}\omega_{\mu\nu}\mathcal{S}\Sigma^{\mu\nu}\mathcal{S}^{-1}\right) = \exp\left(\pm\frac{1}{2}\omega^\circ_{\mu\nu}\Sigma^{\mu\nu}\right). \qquad \text{(B3)}$$

Working in the chiral representation, we use

$$\Sigma^{0i} = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad \Sigma^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \qquad \text{(B4)}$$

In the fluid rest frame $\omega^\circ_{0i} = k^i_\circ$ and $\omega^\circ_{ij} = -\epsilon_{ijk}\omega^k_\circ$, thus we have

$$\pm\frac{1}{2}\omega^\circ_{\mu\nu}\Sigma^{\mu\nu} = \begin{pmatrix} \pm\boldsymbol{z}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \pm(\boldsymbol{z}\cdot\boldsymbol{\sigma})^\dagger \end{pmatrix}, \qquad \text{(B5)}$$

where $\boldsymbol{z} = (-\boldsymbol{\omega}_\circ + i\boldsymbol{k}_\circ)/2$. Consequently, using the method for exponentiating the Pauli matrices we obtain

$$M^\pm_\circ = \exp\left[\begin{pmatrix} \pm\boldsymbol{z}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \pm(\boldsymbol{z}\cdot\boldsymbol{\sigma})^\dagger \end{pmatrix}\right]$$
$$= \begin{pmatrix} \cosh z \pm \frac{\sinh z}{z}\boldsymbol{z}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \cosh z^\star \pm \frac{\sinh z^\star}{z^\star}(\boldsymbol{z}\cdot\boldsymbol{\sigma})^\dagger \end{pmatrix}, \qquad \text{(B6)}$$

with $z^2 = \boldsymbol{z}\cdot\boldsymbol{z}$. Introducing the γ_5 matrix in the chiral representation and using Eq. (B5) one can further simplify Eq. (B6) to

$$M^\pm_\circ = \mathbb{1}\left(\Re(\cosh z) \pm \Re\left(\frac{\sinh z}{z}\right)\frac{1}{2}\omega^\circ_{\mu\nu}\Sigma^{\mu\nu}\right)$$
$$+ i\gamma_5\left(\Im(\cosh z) \pm \Im\left(\frac{\sinh z}{z}\right)\frac{1}{2}\omega^\circ_{\mu\nu}\Sigma^{\mu\nu}\right). \qquad \text{(B7)}$$

As this equation is manifestly Lorentz covariant, we may drop the symbol $\circ$ denoting that it has been derived in the local fluid rest frame. Moreover, as it has a form expressed in terms of the Dirac matrices, it is valid in any representation, including the Dirac one.

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