Conformal window 2.0: The large N_f safe story

Oleg Antipin[1] and Francesco Sannino[2,3]

[1]*Rudjer Boskovic Institute, Division of Theoretical Physics, Bijenička 54, HR-10000 Zagreb, Croatia*
[2]*CP[3]-Origins and Danish IAS, University of Southern Denmark, Campusvej 55, Odense M - DK-5230, Denmark*
[3]*CERN, Theoretical Physics Department, 1211 Geneva 23, Switzerland*

(Received 18 September 2017; published 8 June 2018)

We extend the phase diagram of SU(N) gauge-fermion theories as a function of the number of flavors and colors to the region in which asymptotic freedom is lost. We argue, using large N_f results, for the existence of an ultraviolet interacting fixed point at a sufficiently large number of flavors opening up to a second ultraviolet conformal window in the number of flavors vs colors phase diagram. We first review the state-of-the-art for the large N_f beta function and then estimate the lower boundary of the ultraviolet window. The theories belonging to this new region are examples of safe non-Abelian quantum electrodynamics, termed here *safe QCD*. Therefore, according to Wilson, they are fundamental. An important critical quantity is the fermion mass anomalous dimension at the ultraviolet fixed point that we determine at leading order in $1/N_f$. We discover that its value is comfortably below the bootstrap bound. We also investigate the Abelian case and find that at the potential ultraviolet fixed point the related fermion mass anomalous dimension has a singular behavior suggesting that a more careful investigation of its ultimate fate is needed.

DOI: 10.1103/PhysRevD.97.116007

The discovery of asymptotic freedom [1,2] has been a landmark in our understanding of fundamental interactions. By fundamental we mean that, following Wilson [3,4], these theories are valid at arbitrary short and long distance scales. Asymptotic freedom has therefore guided a great deal of Standard Model (SM) extensions. Likewise the discovery of four-dimensional asymptotically safe field theories [5] constitutes an important alternative to asymptotic freedom. It has opened the door to new ways to generalize the Standard Model [6–11] with impact in dark matter physics and cosmology [10]. The essential feature of an asymptotically safe completion of the SM is that it tames its high energy behavior dynamically uplifting to the status of a truly fundamental field theory. In practice this means that theory does not have a physical cutoff and that the UV theory is mapped into an interacting conformal field theory. What kind of theories can be asymptotically safe and what are their fundamental features? We know already that scalar field theories (Higgs-like) are unsafe, and that quantum electrodynamics is unsafe as well, at least in perturbation theory.

In the original construction [5] elementary scalars and their induced Yukawa interactions played a crucial role in helping make the overall gauge-Yukawa theory safe. Here we will investigate, instead, the ultraviolet fate of gauge-fermion theories at a finite number of colors but a very large number of flavors of both Abelian and non-Abelian nature.

We start by considering an $SU(N_c)$ gauge theory with N_f fermions transforming according to a given representation of the gauge group. We will assume that asymptotic freedom is lost, meaning that the number of flavors is larger than $N_f^{\rm AF} > 11C_G/(4T_R)$, where the first coefficient of the beta function changes sign. We do not need to specify the fermion representation, but will give explicit examples later. In any case, for normalization purposes, we recall that in the fundamental representation the relevant group theory coefficients are $C_G = N_c$, $C_R = (N_c^2 - 1)/2N_c$, and $T_R = 1/2$. At the one-loop order the theory is simultaneously free in the infrared (non-Abelian QED) and trivial, meaning that the only sensible way to take the continuum limit (i.e., sending the Landau pole induced cutoff to infinity) is for the theory to become noninteracting. At two loops, in a pioneering work, Caswell [12] demonstrated that the sign of the second coefficient of the gauge beta function is such that an UV interacting fixed point (asymptotic safety) cannot arise when the number of flavors is just above the value for which asymptotic freedom is lost. This observation immediately implies that for gauge-fermion theories triviality can be replaced by safety only above a new critical number of flavors. To investigate this possibility a logical limit to consider is the large N_f one at a fixed number of colors [13–16]. This will be the focus of our work.

Non-Abelian large-N_f beta function review:— Using the conventions of [13,15], the standard beta function reads

$$\beta(\alpha) \equiv \frac{\partial \ln \alpha}{\partial \ln \mu} = -b_1 \frac{\alpha}{\pi} + \cdots, \qquad \alpha = \frac{g^2}{4\pi}, \tag{1}$$

with g the gauge coupling. At large N_f it is conveniently expressed in terms of the normalized coupling $A \equiv N_f T_R \alpha/\pi$, and expanding in $1/N_f$ we can write

$$\frac{3}{2}\frac{\beta(A)}{A} = 1 + \sum_{i=1}^{\infty} \frac{H_i(A)}{N_f^i}, \tag{2}$$

where the leading identity term corresponds to the one-loop result and constitutes the zeroth order term in the $1/N_f$ expansion. If the functions $|H_i(A)|$ were finite, then in the large N_f limit the zeroth order term would prevail and the Landau pole would be inevitable. This, however, is not the case. The occurrence of a divergent structure in the $H_i(A)$ functions renders the situation worth investigating.

According to the large N_f limit each function $H_i(A)$ resums an infinite set of Feynman diagrams at the same order in N_f with A kept fixed. Let us make this point explicit for the leading $H_1(A)$ term. The nth-loop beta function coefficients b_n for $n \geq 2$ are polynomials in $T_R N_f$ of lowest degree 0 and highest degree $n-1$:

$$b_n = \sum_{k=0}^{n-1} b_{n,k} (T_R N_f)^k. \tag{3}$$

The coefficient with the highest power of $T_R N_f$ will be $b_{n,n-1}$, and this is the coefficient contributing to $H_1(A)$ at the nth-loop order. Moreover, it was shown in [17] that the $b_{n,n-1}$ terms are invariant within the scheme transformations that are independent of N_f (as appropriate for the large-N_f limit).

Now, the nth-loop beta function will have an interacting UV fixed point (UVFP) when the following equation has a physical zero [16]:

$$b_1 + \sum_{k=2}^{n} b_k \alpha^{k-1} = 0 \quad \text{where } b_1 = \frac{\beta_0}{2} = \frac{11 C_G}{6} - \frac{2 T_R N_f}{3}. \tag{4}$$

This expression simplifies at large N_f. In fact when truncated at a given perturbative order $n\,\mathrm{max}$ one finds that the highest loop beta function coefficient $b_{n\,\mathrm{max}}$ contains just the highest power of $(T_R N_f)^{n\,\mathrm{max}-1}$ multiplied by the coefficient $b_{n\,\mathrm{max},n\,\mathrm{max}-1}$, as it can be seen from Eq. (3). Since this highest power of $(T_R N_f)^{n\,\mathrm{max}-1}$ dominates in the $N_f \to \infty$ limit and since in this limit $b_1 < 0$, the criterion for the existence of a UV zero in the n max-loop beta function becomes [16]

$$\text{for } N_f \to \infty, \quad \beta(\alpha) \text{ has an UVFP only if } b_{n\,\mathrm{max},n\,\mathrm{max}-1} > 0.$$

In perturbation theory, only the first few coefficients $b_{n,n-1}$ are known but, remarkably, it is possible to resum the perturbative infinite sum to obtain $H_1(A)$. From the results in [13,14]

$$\begin{aligned} H_1(A) &= -\frac{11}{4}\frac{C_G}{T_R} + \int_0^{A/3} I_1(x) I_2(x) dx, \\ I_1(x) &= \frac{(1+x)(2x-1)^2(2x-3)^2 \sin(\pi x)^3 \Gamma(x-1)^2 \Gamma(-2x)}{(x-2)\pi^3}, \\ I_2(x) &= \frac{C_R}{T_R} + \frac{(20-43x+32x^2-14x^3+4x^4)}{4(2x-1)(2x-3)(1-x^2)} \frac{C_G}{T_R}. \end{aligned} \tag{5}$$

By inspecting $I_1(x)$ and $I_2(x)$ one notices that the C_G term in I_2 has a pole in the integrand at $x = 1$ ($A = 3$). This corresponds to a logarithmic singularity in $H_1(A)$ that will cause the beta function to have a UV zero already to this order in the $1/N_f$ expansion and, by solving the $1 + H_1(A)/N_f = 0$ condition, this nontrivial UV fixed point occurs at [5]:

$$A^* = 3 - \exp\left[-k \frac{N_f}{N_c} + l\right], \tag{6}$$

where $k = 16 T_R$ and $l = 18.49 - 5.26\, C_R/C_G$.

Performing a Taylor expansion of the integrand in Eq. (5) and integrating term-by-term we can obtain the nth-loop coefficients $b_{n,n-1}$ and check our criteria above for the existence of the safe fixed point. This procedure was performed in [17] up to 18th-loop order where it was also checked that the first four loops agree with the known perturbative results. It was found that, even though up to the 12th-loop order the resulting coefficients are scattered between the positive and negative values, starting from the 13th-loop order all $b_{n,n-1}$ are positive for the fundamental, two-index representations and symmetric/antisymmetric rank-3 tensors. This supports the possible existence of the UV fixed point. These results have been confirmed, extended, and employed to build the first realistic asymptotically safe extensions of the SM [7,9,10].

This concludes our review of the large N_f beta function and its use to investigate the UV fate of non-Abelian QED theories. If these theories are safe, we will call them *Safe QCD* [18]. We move now to provide a careful investigation and prediction of the safe large N_f quark mass anomalous dimension.

Safe large N_f mass anomalous dimension and bootstrap:— We start by summarizing the general expression for the large N_f mass anomalous dimension [13]

$$\gamma_m(A) \equiv -\frac{\partial \ln m}{\partial \ln \mu} = \sum_{i=1}^{\infty} \frac{G_i(A)}{N_f^i}, \tag{7}$$

$$G_1(A) = \frac{C_R}{2T_R} \frac{A(1-2A/9)\Gamma(4-2A/3)}{\Gamma(1+A/3)[\Gamma(2-A/3)]^2\Gamma(3-A/3)}. \tag{8}$$

We immediately note that the first singularity in the expression for $\gamma_m(A)$ appears at $A = 15/2$ while the first singularity of the beta function occurs at the smaller value of $A = 3$.

Inserting the UVFP value from Eq. (6) into Eq. (8) and taking the limit of $N_f \to \infty$ with N_c fixed, we achieve the UV fixed point for $A^* \to 3$ up to exponentially small corrections yielding

$$\gamma_m^*(A^*) \overset{N_f\to\infty}{\longrightarrow} \frac{C_R}{2T_R N_f} = \frac{(N_c^2-1)}{2N_c N_f}, \tag{9}$$

where in the last equation we specialized to the case of the fundamental representation. At relatively large N_c, Eq. (9) simplifies to $\gamma_m^*(A) \to N_c/2N_f$. In Fig. 1 we plot the full $\gamma_m^*(A^*)$ as a function of N_f for distinct values of N_c and nicely reproduce Eq. (9) at the large N_f right-handed side of the plot.

Now, using the complete four-loop beta function, a few perturbative terms of the higher $1/N_f$ order expansion functions $H_{2,3,4}(A)$ can be extracted. Requiring these functions to be sufficiently small for $0 < A \leq 3$ it was argued in [15] that values of $N_f \gtrsim 10\ N_c$ or more are needed for the $1/N_f$ expansion to be applicable. The reason is that starting with $N_f \gtrsim N_f^{\mathrm{AF}} = 5.5\ N_c$, there is always an interval ΔA where $r(A) \equiv (H_2/N_f + H_3/N_f^2 + H_4/N_f^3)/H_1 > 1$ so that subleading $1/N_f$ terms are large. As N_f increases, this interval increases until it reaches its maximum at $N_f \approx 10\ N_c$. After this value, the interval starts to decrease and eventually, after $N_f \approx 50\ N_c$, $r(A) < 1$ for all $0 < A \leq 3$. Taking $N_f \gtrsim 10\ N_c$ as a rough estimate for the $1/N_f$ validity region, this implies that only values of $\gamma_m^*(A) \lesssim 1/20$ are acceptable. The regions where the $1/N_f$ expansion holds, for a fixed N_c, are shaded in Fig. 1.

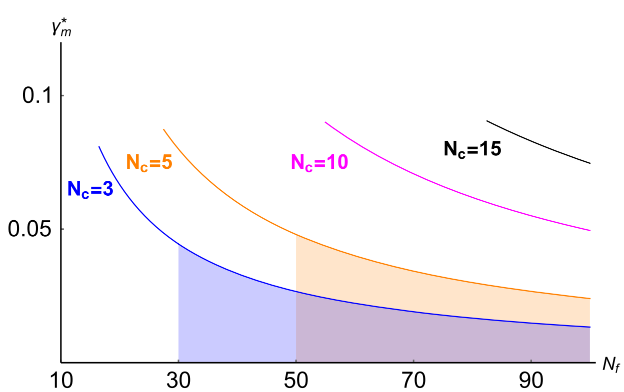

FIG. 1. The anomalous dimension at the UVFP for $N_c = 3, 5, 10, 15$ (from left to right) for the fermions in the fundamental representation. The shaded regions start from $N_f = 30(50)$ for $N_c = 3(5)$ and correspond to the $1/N_f$ validity regions that for $N_c = 10, 15$ start for $N_f = 100, 150$, respectively.

This prediction can be confronted with the prediction from the powerful a-theorem named after the real coefficient a entering the vacuum expectation value of the trace of the energy-momentum tensor for a locally flat metric $g_{\mu\nu}$. Recall that in four dimensions and for a general quantum field theory the vacuum expectation value of the trace of the energy-momentum tensor Θ suffers from the Weyl anomaly which, in full generality, can be written as

$$\langle\Theta\rangle = \langle\Theta_\mu^\mu\rangle = cW^2(g_{\mu\nu}) - aE_4(g_{\mu\nu}) + \frac{1}{9}bR^2 + \tilde{b}\Box R + \cdots, \tag{10}$$

where c, a, b, and $\tilde{b}$ are functions of the couplings of the theory, R is the Ricci scalar, $E_4(g_{\mu\nu})$ is the Euler density, and $W(g_{\mu\nu})$ is the Weyl tensor. The dots represent contributions coming from operators that can be constructed out of the fields defining the theory. The change of a along the renormalization group (RG) flow is directly related to the underlying dynamics of the theory via the beta functions, and the a-theorem states that the quantity $\Delta a \equiv (a^{\mathrm{UV}} - a^{\mathrm{IR}}) > 0$ is for any RG flow between physical fixed points. To four-loop order, needed for our comparison [with $\alpha_g^* = \alpha^*/(4\pi)$ at the UV zero of the four-loop beta function] is

$$\Delta a = -2\chi_{gg}\left[\beta_0\alpha_g^* + \frac{(\alpha_g^*)^2}{2}(\beta_1 + B\beta_0) + \frac{(\alpha_g^*)^3}{3}(\beta_2 + B\beta_1 + C\beta_0) + \frac{(\alpha_g^*)^4}{4}(\beta_3 + B\beta_2 + C\beta_1 + D\beta_0)\right], \tag{11}$$

where the beta function coefficients β_{0-3} can be found in [16]. The coefficients B, C, D enter the quantity χ which plays the role of the gauge coupling metric [19]

$$\chi = \frac{\chi_{gg}}{\alpha_g^2}(1 + B\alpha_g + C\alpha_g^2 + D\alpha_g^3), \quad \chi_{gg} = \frac{N_c^2 - 1}{128\pi^2} \tag{12}$$

with $B = 17C_G - \frac{20}{3}N_f T_R$ and $C = C_G^2(\frac{19447}{108} + 22\zeta(3)) - \frac{4}{27}C_G N_f T_R(667 + 378\zeta(3)) + \frac{2}{27}N_f T_R[50N_f T_R + 9C_R \times (-107 + 72\zeta(3))]$. The four-loop coefficient D is not known; however, for a rough estimate in the large N_f limit, we use the leading N_f^3 term for the closely related metric χ_a [19], $D = \frac{32}{729}N_f^3 T_R^3(-109 + 432\zeta(3))$. The two metrics differ from the three-loop order, and to this order the numerical difference is negligible. From the $\Delta a > 0$ condition we obtain $N_f \gtrsim 30\ N_c$ which is roughly near the four-loop beta function estimate made above [20].

The asymptotic behavior in Eq. (9) holds also for the other matter representations. For example, for the adjoint representation we have $\exp[-k\frac{N_f}{N_c}+l] \ll 1$ for any $N_f \geq 1$ and thus

$$\gamma_m^*(A) \approx \frac{C_R}{2T_R N_f} = \frac{1}{2N_f}. \qquad (13)$$

Also, in contrast with the fundamental representation case, we find that for $N_f \gtrsim 7$ the $1/N_f$ expansion is trustable *independently* of the value of N_c. The reason for this is that for the adjoint representation $C_G = C_R = T_R = N_c$, and therefore, up to the negligible N_c dependence in the fourth-order group invariants appearing at the fourth-loop order in the beta function, the N_c dependence in $H_{1,2,3,4}(A)$ cancels completely. For the N_c dependence of the traditional conformal window we refer to [21]. This means that the large N_f UVFPs will have $\gamma_m^*(A) \lesssim 1/14$, a result close to $\gamma_m^*(A) \lesssim 1/20$ for the fundamental representation.

We now confront our predictions for the safe anomalous mass dimensions with the bound coming from the conformal bootstrap. These derive from imposing crossing symmetry constraints on the four-point function of a scalar (meson) operator Φ_{ij} transforming according to the bifundamental representation of the $SU(N_f) \times SU(N_f)$ global symmetry group. From the work of Nakayama [22] the bounds are $\gamma_m^* < 1.79$ for $N_f = 8$ and $\gamma_m^* < 1.88$ for $N_f = 100$. Clearly the values of the safe anomalous dimensions lie comfortably below this bound.[1]

Conformal window 2.0: We now use the information acquired above to delineate the complete, in N_c and N_f, phase diagram for an $SU(N_c)$ gauge theory with fermionic matter in a given representation. We use as reference the line where asymptotic freedom is lost, i.e., $N_f^{\mathrm{AF}} = 11C_G/(4T_R)$. As it is well known, decreasing N_f slightly below this value one achieves the perturbative Banks-Zaks infrared fixed point (IRFP) that at two loops yields $\alpha^* = -b_1/b_2$. This analysis has been extended to the maximum known order in perturbation theory [16,23,24] and constitutes the state-of-the-art in this field. As we continue to lower the number of flavors, the IRFP becomes strongly coupled and at some critical N_f^{IRFP}, is lost. The lower boundary of the conformal window has been estimated analytically in different ways [25] and tested via lattice simulations [26]. Just above the loss of asymptotic freedom, as mentioned in the introduction, Caswell [12] demonstrated that no perturbative UVFP can emerge. By continuity there should be a region in color-flavor space where the resulting theory is non-Abelian QED with an unavoidable Landau pole. We will refer collectively to this region as *Unsafe QCD*. Here the theories can be viewed as low energy effective field theories with a trivial IRFP. We then expect a critical value of the number of flavors N_f^{Safe} above which we achieve safety. This region extends to infinite values of N_f, i.e., the *Safe QCD* region. Given that for the fundamental representation, the leading $1/N_f$ expansion is applicable only for $N_c \lesssim N_f/10$ while for the adjoint representation we find $N_f \gtrsim 7$ for any N_c it is sensible to use these as the first estimate of the lower boundary of the *Safe QCD* region. Altogether, these constraints define the corresponding phase diagrams depicted in Fig. 2. We conclude this discussion by commenting on the status of large N_f super QCD. Using exact nonperturbative results it has been demonstrated that super QCD cannot be safe for any N_f [27].

On Abelian safety:— Singularly interesting is the ultimate UV fate of Abelian gauge theories. We investigate this by first rescaling the gauge coupling $A \equiv N_f\alpha/\pi$ in Eq. (1). This results in the $U(1)$ large N_f β-function

$$\frac{3}{2}\frac{\beta(A)}{A} = 1 + \sum_{i=1}^{\infty}\frac{F_i(A)}{N_f^i}, \qquad F_1(A) = \int_0^{A/3} I_1(x)dx, \qquad (14)$$

with $I_1(x)$ the same as in the non-Abelian case. Performing a Taylor expansion of the integrand in Eq. (14) and integrating term-by-term as for the non-Abelian case Shrock [17] obtained the nth loop coefficients $b_{n,n-1}$ with explicit results up to the 24th-loop order. Different from the non-Abelian case one finds, till the 24th order, alternating signs for $b_{n,n-1}$ indicating a worse convergence for the Abelian with respect to the non-Abelian case. Nevertheless with this information we cannot yet exclude the possible existence of a stable UVFP. What we can, however, still determine at the would be fixed point is the correspondent fermion anomalous dimension. The latter is related to the function $F_1(A)$ [13,15] as follows:

$$\gamma_m(A) = \frac{2A}{N_f}\frac{9}{(3-2A)(3+A)}\frac{dF_1(A)}{dA} + \mathcal{O}(1/N_f^2). \qquad (15)$$

Different from the non-Abelian case, the singularities in the $\gamma_m(A)$ and $\beta(A)$ happen at the same value of A since the resummation of the fermion bubbles is shared by both functions. Also, Eq. (15) relates the strength of the singularities with the logarithmic singularities in $F_1(A)$ manifested as simple poles in $G_1(A)$. The resulting UVFP to leading order in $1/N_f$ occurs at [15]: $A^* = \frac{15}{2} - 0.0117e^{-15\pi^2 N_f/7}$. Inserting this value into Eq. (15) we obtain

[1] We thank Nakayama for providing the $N_f = 100$ bootstrap value.

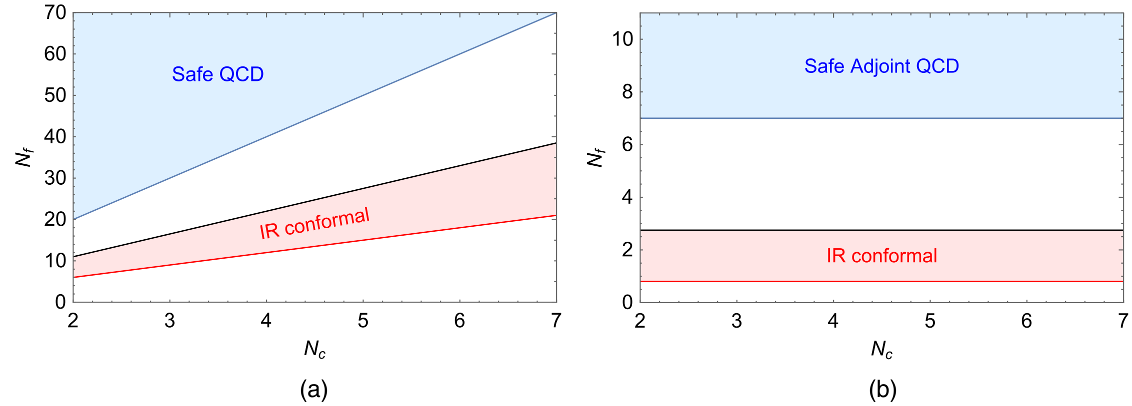

FIG. 2. Phase diagram of $SU(N_c)$ gauge theories with fermionic matter in the fundamental (left panel) and adjoint (right panel) representations. The shaded areas depict the corresponding conformal windows where the theories develop an IRFP (light red region) or an UVFP (light blue region). The estimate of the lower boundary of the IRFP conformal window is taken from [16].

$$\gamma_m^*(A^*) \approx \frac{e^{15\pi^2 N_f/7}}{2\pi^2 N_f \times 0.0117}. \tag{16}$$

The exponential proximity of the fixed point to the pole generates the exponential growth in the number of flavors of the mass anomalous dimension. For the physical case of $N_f \geq 1$, the corresponding $\gamma_m^*(A^*)$ exceeds the unitary bound that requires $\gamma_m(A^*) \leq 2$. This result suggests that the existence of an UVFP stemming from the resummation procedure for the Abelian case must be taken with a grain of salt, and more work is needed to disentangle the ultimate ultraviolet fate of Abelian gauge theories.

Concluding, we briefly reviewed the salient large N_f results for non-Abelian gauge-fermion theories. These lead to the possible existence of an UVFP when asymptotic freedom is lost. To further test the emergence of *Safe QCD*–like theories we determined the related safe mass anomalous dimension. We discovered that this important quantity is controllably small. In particular for the fundamental representation we find that $\gamma_m^*(A) \lesssim 1/20$ and for the adjoint case $\gamma_m^*(A) \lesssim 1/14$. In fact the safe anomalous dimension decreases with N_f at finite N_c for the fundamental representation and independently of N_c for the adjoint representation. The so determined anomalous dimensions are comfortably within the current bootstrap bounds. Our results lend support to the existence of two distinct regions in the color-flavor plane when asymptotic freedom is lost. The region contiguous to the loss of asymptotic freedom is *unsafe* with the theory being non-Abelian QED in the IR and featuring an incurable Landau pole in the UV; and a second region starts above a new critical number of flavor lines where safety is reached. The overall picture is summarized by the 2.0 upgraded version of the conformal window [28,29] of Fig. 2. For the $U(1)$ gauge theory the discovered exponential growth in the number of flavors of the safe mass anomalous dimension leaves unanswered the question of whether these theories can be safe at a large number of flavors.

Our results constitute a step forward in delineating and understanding theories of fundamental interactions *à la* Wilson. The conformal window 2.0 completes and dramatically changes the landscape of important quantum field theories such as QCD as a function of the number of colors and flavors [28,29]. The results presented in this work constitute an essential stepping stone for phenomenological extensions of the SM and offer important guidance when searching and testing asymptotic safety via first principle lattice simulation.

The work of O. A. and F. S. are partially supported, respectively, by the Croatian Science Foundation (HRZZ) project "Terascale Physics for the LHC and Cosmos" as well as the H2020 CSA Twinning Project No. 692194, RBI-T-WINNING, and the Danish National Research Foundation Grant No. DNRF:90.

[1] D. J. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973).
[2] H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).
[3] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).
[4] K. G. Wilson, Phys. Rev. B **4**, 3184 (1971).
[5] D. F. Litim and F. Sannino, J. High Energy Phys. 12 (2014) 178.
[6] S. Abel and F. Sannino, Phys. Rev. D **96**, 056028 (2017).
[7] S. Abel and F. Sannino, Phys. Rev. D **96**, 055021 (2017).

[8] G. M. Pelaggi, F. Sannino, A. Strumia, and E. Vigiani, Front. Phys. **5**, 49 (2017).
[9] R. Mann, J. Meffe, F. Sannino, T. Steele, Z. W. Wang, and C. Zhang, Phys. Rev. Lett. **119**, 261802 (2017).
[10] G. M. Pelaggi, A. D. Plascencia, A. Salvio, F. Sannino, J. Smirnov, and A. Strumia, arXiv:1708.00437.
[11] A. D. Bond, G. Hiller, K. Kowalska, and D. F. Litim, J. High Energy Phys. 08 (2017) 004.
[12] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).
[13] A. Palanques-Mestre and P. Pascual, Commun. Math. Phys. **95**, 277 (1984).
[14] J. A. Gracey, Phys. Lett. B **373**, 178 (1996).
[15] B. Holdom, Phys. Lett. B **694**, 74 (2010).
[16] C. Pica and F. Sannino, Phys. Rev. D **83**, 035013 (2011).
[17] R. Shrock, Phys. Rev. D **89**, 045019 (2014).
[18] F. Sannino, arXiv:1511.09022.
[19] I. Jack and H. Osborn, Nucl. Phys. **B343**, 647 (1990).
[20] O. Antipin and F. Sannino (to be published).
[21] G. Bergner, T. A. Ryttov, and F. Sannino, J. High Energy Phys. 12 (2015) 054.
[22] Y. Nakayama, J. High Energy Phys. 07 (2016) 038.
[23] T. A. Ryttov and R. Shrock, Phys. Rev. D **83**, 056011 (2011).
[24] T. A. Ryttov and R. Shrock, Phys. Rev. D **94**, 105015 (2016).
[25] F. Sannino, Acta Phys. Pol. B **40**, 3533 (2009).
[26] C. Pica, Proc. Sci. LATTICE2016 (**2016**) 015 [arXiv:1701.07782].
[27] K. Intriligator and F. Sannino, J. High Energy Phys. 11 (2015) 023.
[28] F. Sannino and K. Tuominen, Phys. Rev. D **71**, 051901 (2005).
[29] D. D. Dietrich and F. Sannino, Phys. Rev. D **75**, 085018 (2007).