Modular symmetry and non-Abelian discrete flavor symmetries in string compactification

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We study the modular symmetry in magnetized D-brane models on T^2. Non-Abelian flavor symmetry D_4 in the model with magnetic flux $M = 2$ (in a certain unit) is a subgroup of the modular symmetry. We also study the modular symmetry in heterotic orbifold models. The T^2/Z_4 orbifold model has the same modular symmetry as the magnetized brane model with $M = 2$, and its flavor symmetry D_4 is a subgroup of the modular symmetry.

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I. INTRODUCTION

Non-Abelian discrete flavor symmetries play an important role in particle physics. In particular, many models with various finite groups have been studied in order to explain quark and lepton masses and their mixing angles. (See for review [1–3].) Those symmetries may be useful for dark matter physics and multi-Higgs models.

Superstring theory is a promising candidate for unified theory including gravity. It has been shown that some non-Abelian discrete flavor symmetries appear in superstring theory with certain compactifications. Heterotic string theory on toroidal Z_N orbifolds can realize non-Abelian flavor symmetries, e.g., D_4, and $\Delta(54)$ [4]. (See also [5,6].)[1] Furthermore, magnetized D-brane models within the framework of type II superstring theory can lead to similar flavor symmetries [8–12]. Intersecting D-brane models are T-dual to magnetized D-brane models. Then, one can realize the same aspects in intersecting D-brane models as in the magnetized ones.[2]

On the other hand, superstring theories on tori and orbifolds have modular symmetry. Recently, the behavior of zero modes under modular transformation was studied in magnetized D-brane models in Ref. [14]. (See also [15].) Also, the behavior of twisted sectors under modular transformation was already studied in Refs. [16–18]. These modular transformations also act nontrivially on flavors and transform mutually flavors from each other. The remarkable difference is that modular transformation also acts on Yukawa couplings as well as higher order couplings, while those couplings are trivial singlets under the usual non-Abelian symmetries.

The purpose of this paper is to study more how modular transformation is represented by zero modes in magnetized D-brane models, and to discuss relations between modular transformation and non-Abelian flavor symmetries in magnetized D-brane models. Intersecting D-brane models have the same aspects as magnetized D-brane models, because they are T-dual to each other. Furthermore, intersecting D-brane models in type II superstring theory and heterotic string theory have similarities, e.g., in two-dimensional conformal field theory. Thus, here we study modular symmetry and non-Abelian discrete flavor symmetries in heterotic orbifold models, too.

This paper is organized as follows. In Sec. II, we study the modular symmetries in magnetized D-brane models and the relation to the D_4 flavor symmetry. In Sec. III, we study the modular symmetries in heterotic orbifold models. Section IV is the conclusion and discussions. We give brief reviews on non-Abelian discrete flavor symmetries in magnetized D-brane models and heterotic orbifold models in Appendixes A and B, respectively.

II. MODULAR TRANSFORMATION IN MAGNETIZED D-BRANE MODELS

In this section, we study the modular transformation of zero-mode wave functions in magnetized D-brane models.

A. Zero-mode wave function

Here, we give a brief review on zero-mode wave functions on torus with magnetic flux [15]. For simplicity, we concentrate on T^2 with $U(1)$ magnetic flux. The complex coordinate on T^2 is denoted by $z = x^1 + \tau x^2$, where τ is

[1] In Ref. [7], a relation between gauge symmetries and non-Abelian flavor symmetries is discussed at the enhancement point.
[2] See also [13].

the complex modular parameter, and x^1 and x^2 are real coordinates. The metric on T^2 is given by

$$g_{\alpha\beta} = \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = (2\pi R)^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{1}$$

We identify $z \sim z+1$ and $z \sim z+\tau$ on T^2.

On T^2, we introduce the $U(1)$ magnetic flux F,

$$F = i\frac{\pi M}{\mathrm{Im}\tau}(dz \wedge d\bar{z}), \tag{2}$$

which corresponds to the vector potential,

$$A(z) = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}(\bar{z}dz). \tag{3}$$

Here we concentrate on vanishing Wilson lines.

In the above background, we consider the zero-mode equation for the spinor field with the $U(1)$ charge $q=1$,

$$i\not{D}\Psi = 0. \tag{4}$$

The spinor field on T^2 has two components,

$$\Psi(z,\bar{z}) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{5}$$

The magnetic flux should be quantized such that M is an integer. Either ψ_+ or ψ_- has zero modes exclusively for $M \neq 0$. For example, we set M to be positive. Then, ψ_+ has M zero modes, while ψ_- has no zero mode. Hence, we can realize a chiral theory. Their zero-mode profiles are given by

$$\psi^{j,M}(z) = \mathcal{N} e^{i\pi M z\frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \cdot \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(Mz, M\tau), \tag{6}$$

with $j = 0, 1, \ldots, (M-1)$, where ϑ denotes the Jacobi theta function,

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{l\in\mathbf{Z}} e^{\pi i(a+l)^2\tau} e^{2\pi i(a+l)(\nu+b)}. \tag{7}$$

Here, $\mathcal{N}$ denotes the normalization factor given by

$$\mathcal{N} = \left(\frac{2\mathrm{Im}\tau M}{\mathcal{A}^2}\right)^{1/4}, \tag{8}$$

with $\mathcal{A} = 4\pi^2 R^2 \mathrm{Im}\tau$.

The ground states of scalar fields also have the same profiles as $\psi^{j,M}$. Thus, the Yukawa coupling including one scalar and two spinor fields can be computed by using these zero-mode wave functions. Zero mode wave functions satisfy the following relation:

$$\begin{aligned} \psi^{i,M}\psi^{j,M} = {} & \mathcal{A}^{-1/2}(2\mathrm{Im}\tau)^{1/4}\left(\frac{MN}{M+N}\right)^{1/4} \\ & \times \sum_m \psi^{i+j+Mm,M+N} \\ & \cdot \vartheta \begin{bmatrix} \frac{Ni-Mj+MNm}{MN(M+N)} \\ 0 \end{bmatrix}(0, MN(M+N)\tau). \end{aligned} \tag{9}$$

By use of this relation, their Yukawa couplings are given by the wave function overlap integral,

$$\begin{aligned} Y_{ijk} &= y\int d^2z \psi^{i,M}\psi^{j,N}(\psi^{k,M'})^* \\ &= y\left(\frac{2\mathrm{Im}\tau}{\mathcal{A}^2}\right)^{1/4} \sum_{m\in Z_{M'}} \delta_{k,i+j+Mm} \\ &\quad \cdot \vartheta \begin{bmatrix} \frac{Ni-Mj+MNm}{MNM'} \\ 0 \end{bmatrix}(0, MNM'\tau), \end{aligned} \tag{10}$$

where y is constant. This Yukawa coupling vanishes for $M' \neq M+N$. Similarly, we can compute higher order couplings using the relation (9) [19]. In the above equation, the Kronecker delta $\delta_{k,i+j+Mm}$ implies the coupling selection rule. For $g = \gcd(M,N,M')$, nonvanishing Yukawa couplings appear only if

$$i + j = k \pmod{g}. \tag{11}$$

Hence, we can define Z_g charges in these couplings [8].

B. Modular transformation

Here, we study modular transformation. First, we give a brief review on results of modular transformation [14]. (See also [15].) Then, we will study in more detail.

The T^2 is constructed by $\mathbb{R}^2/\Lambda$, and the lattice Λ is spanned by the vectors (α_1, α_2), where $\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$. However, the same lattice can be described by another basis,

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}, \tag{12}$$

where a, b, c, d are integers satisfying $ad - bc = 1$. That is the $SL(2,Z)$ transformation.

The modular parameter $\tau = \alpha_2/\alpha_1$ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{13}$$

under (12). This transformation includes two important generators, S and T,

$$S\colon\ \tau \to -\frac{1}{\tau}, \tag{14}$$

$$T\colon\ \tau \to \tau + 1. \tag{15}$$

They satisfy

$$S^2 = \mathbb{I}, \qquad (ST)^3 = \mathbb{I}. \tag{16}$$

On top of that, if we impose the algebraic relation,

$$T^N = \mathbb{I}, \tag{17}$$

that corresponds to the congruence subgroup of modular group, $\Gamma(N)$. For example, it is found that $\Gamma(2) \simeq S_3$, $\Gamma(3) \simeq A_4$, $\Gamma(4) \simeq S_4$, and $\Gamma(5) \simeq A_5$. Since group A_4 has the symmetry of the tetrahedron, it is often called the tetrahedral group $T = A_4$. Also, it may be useful to mention about $\Delta(3N^2) \simeq (Z_N \times Z_N) \rtimes Z_3$ and $\Delta(6N^2) \simeq (Z_N \times Z_N) \rtimes S_3$. We find that $S_3 \simeq \Delta(6)$, $A_4 \simeq \Delta(12)$, and $S_4 \simeq \Delta(24)$.

Following [14], we restrict ourselves to even magnetic fluxes M ($M > 0$). Under S, the zero-mode wave functions transform as [14,15]

$$\psi^{j,M} \to \frac{1}{\sqrt{M}} \sum_k e^{2\pi i jk/M} \psi^{k,M}. \tag{18}$$

On the other hand, the zero-mode wave functions transform as [14]

$$\psi^{j,M} \to e^{\pi i j^2/M} \psi^{j,M}, \tag{19}$$

under T. Generically, the T transformation satisfies

$$T^{2M} = \mathbb{I}, \tag{20}$$

on the zero modes, $\psi^{j,M}$. Furthermore, in Ref. [14] it is shown that

$$(ST)^3 = e^{\pi i/4}, \tag{21}$$

on the zero modes, $\psi^{j,M}$.

In what follows, we study more concretely.

1. Magnetic flux $M = 2$

Let us study the case with the magnetic flux $M = 2$ concretely. There are two zero modes, $\psi^{0,2}$ and $\psi^{1,2}$. The S transformation acts on these zero modes as

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix} \to S_{(2)} \begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \qquad S_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{22}$$

The T transformation acts as

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix} \to T_{(2)} \begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \qquad T_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{23}$$

They satisfy the following algebraic relations:

$$S_{(2)}^2 = \mathbb{I}, \qquad T_{(2)}^4 = \mathbb{I}, \qquad (S_{(2)}T_{(2)})^3 = e^{\pi i/4}\mathbb{I}. \tag{24}$$

They construct a closed algebra with the order 192, which we denote here by $G_{(2)}$. By such an algebra, the modular transformation is represented by two zero modes, $\psi^{0,2}$ and $\psi^{1,2}$. We find that $(S_{(2)}T_{(2)})^3$ is a center. Indeed, there are eight center elements and their group is Z_8. Other diagonal elements correspond to Z_4, which is generated by $T_{(2)}$. Here, we denote

$$a = (S_{(2)}T_{(2)})^3, \qquad a' = T_{(2)}. \tag{25}$$

The diagonal elements are represented by $a^m a'^n$, i.e., $Z_8 \times Z_4$.

Here, we examine the right coset Hg for $g \in G_{(2)}$, where H is the above $Z_8 \times Z_4$, i.e., $H = \{a^m a'^n\}$. There would be $6 (= 192/(8 \times 4))$ cosets. Indeed, we obtain the following six cosets:

$$H, \qquad HS_{(2)}, \qquad HS_{(2)}T_{(2)}^k, \qquad HS_{(2)}T_{(2)}^2 S_{(2)}, \tag{26}$$

with $k = 1, 2, 3$. By simple computations, we find $HS_{(2)}T_{(2)}^k S_{(2)} \sim HS_{(2)}T_{(2)}^{4-k}S_{(2)}$ and $HS_{(2)}T_{(2)}^2 S_{(2)}T \sim HS_{(2)}T_{(2)}^2 S_{(2)}$.

Furthermore, we would make a (non-Abelian) subgroup with the order 6 by choosing properly six elements such that we pick one element up from each coset and their algebra is closed. The non-Abelian group with the order 6 is unique, i.e., S_3. For example, we may be able to obtain the Z_3 generator from $HS_{(2)}T_{(2)}$ because $(S_{(2)}T_{(2)})^3 \in H$. That is, we define

$$b = a^m a'^n S_{(2)}T_{(2)}. \tag{27}$$

Then, we require $b^3 = \mathbb{I}$. There are three solutions, $(m, n) = (3, 2), (5, 0)$ mod (8,4). Similarly, we can obtain the Z_2 generator, e.g., from $HS_{(2)}T_{(2)}^2 S_{(2)}$ because $(S_{(2)}T_{(2)}^2 S_{(2)})^2 \in H$. Then, we define

$$c = a^{m'} a'^{n'} S_{(2)}T_{(2)}^2 S_{(2)}. \tag{28}$$

We find $c^2 = \mathbb{I}$ when $n' = -m'$ mod 4. On top of that, we require $(bc)^2 = \mathbb{I}$, and that leads to the conditions $n = -m' - 1$ mod 4 and $m = m' + 2$ mod 8. As a result, there are six solutions, $(m, n, m') = (3, 2, 1), (3, 2, 5), (5, 0, 3), (5, 0, 7)$ with $n' = -m'$ mod 4.

For example, for $(m,n,m')=(3,2,5)$ we write

$$b=\frac{1}{\sqrt{2}}\begin{pmatrix}\rho^3 & \rho^{-3}\\ \rho^{-1} & \rho^{-3}\end{pmatrix},\qquad c=\begin{pmatrix}0 & \rho^{-3}\\ \rho^3 & 0\end{pmatrix}. \tag{29}$$

The six elements of the subgroup are written explicitly,

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\quad \begin{pmatrix}0 & \rho^{-3}\\ \rho^3 & 0\end{pmatrix},\quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & i\\ -i & 1\end{pmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix}\rho^3 & \rho^{-3}\\ \rho^{-1} & \rho^{-3}\end{pmatrix},\quad \frac{1}{\sqrt{2}}\begin{pmatrix}\rho^{-3} & \rho\\ \rho^3 & \rho^3\end{pmatrix}, \tag{30}$$

where $\rho=e^{2\pi i/8}$. They correspond to $S_3\simeq\Gamma(2)\simeq\Delta(6)$ because they satisfy the following algebraic relations:

$$c^2=b^3=(bc)^2=\mathbb{I}. \tag{31}$$

Moreover, they satisfy the following algebraic relation with $Z_8\times Z_4$:

$$b^{-1}ab^1=a,\qquad cac=a,\qquad b^{-1}a'b=a,$$
$$ca'c^{-1}=a^2a'^3. \tag{32}$$

Thus, the algebra of $G_{(2)}$ is isomorphic to $(Z_8\times Z_4)\rtimes S_3$.

We have started by choosing $HS_{(2)}T^2_{(2)}S_{(2)}$ for a candidate of the Z_2 generator. We can obtain the same results by starting with $HS_{(2)}$ for a candidate of the Z_2 generator.

2. Magnetic flux $M=4$

Similarly, we study the case with the magnetic flux $M=4$. There are four zero modes, $\psi^{i,M}$ with $i=0,1,2,3$. The S and T transformations are represented by $\psi^{i,M}$,

$$S_{(4)}=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{pmatrix},$$
$$T_{(4)}=\begin{pmatrix}1 & & & \\ & e^{\pi i/4} & & \\ & & -1 & \\ & & & e^{\pi i/4}\end{pmatrix}. \tag{33}$$

This is a reducible representation. In order to obtain irreducible representations, we use the flowing basis,

$$\begin{pmatrix}\psi^{0.4}\\ \psi^{1,4}_{+}\\ \psi^{2,4}\end{pmatrix}=\begin{pmatrix}\psi^{0.4}\\ \frac{1}{\sqrt{2}}(\psi^{1,4}+\psi^{3,4})\\ \psi^{2,4}\end{pmatrix},$$
$$\psi^{1,4}_{-}=\frac{1}{\sqrt{2}}(\psi^{1,4}-\psi^{3,4}). \tag{34}$$

This is nothing but zero modes on the T^2/Z_2 orbifold [20]. The former corresponds to Z_2 even states, while the latter corresponds to the Z_2 odd state. Note that $(ST)^3$ transforms the lattice basis $(\alpha_1,\alpha_2)\to(-\alpha_1,-\alpha_2)$. Thus, it is reasonable that the zero modes on the T^2/Z_2 orbifold correspond to the irreducible representations.

The S and T representations by the Z_2 odd zero mode are quite simple, and these are represented by

$$S_{(4-)}=i,\qquad T_{(4)-}=e^{\pi i/4}. \tag{35}$$

Their closed algebra is Z_8.

On the other hand, the S and T transformations are represented by the Z_2 even zero- modes,

$$S_{(4)+}=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix},$$
$$T_{(4)+}=\begin{pmatrix}1 & & \\ & e^{\pi i/4} & \\ & & -1\end{pmatrix}. \tag{36}$$

They satisfy the following algebraic relation:

$$(S_{(4)+})^2=\mathbb{I},\qquad (T_{(4)+})^8=\mathbb{I},\qquad (S_{(4)+}T_{(4)+})^3=e^{\pi i/4}\mathbb{I}. \tag{37}$$

We denote the closed algebra of $S_{(4)+}$ and $T_{(4)+}$ by $G_{(4)+}$. Its order is equal to 768, and it includes the center element $(S_{(4)+}T_{(4)+})^3$, i.e., Z_8. Other diagonal elements correspond to Z_8, which is generated by $T_{(4)+}$. Again, we denote $a=(S_{(4)+}T_{(4)+})^3$ and $a'=T_{(4)+}$, and the diagonal elements are written by $a^ma'^n$, i.e., $Z_8\times Z_8$.

Similar to the case with $M=2$, we examine the coset structure, Hg. Indeed, there are the following 12 cosets:

$$H,\quad HS_{(4)+},\quad HS^k_{(4)+},\quad HS_{(4)+}T^{\ell}_{(4)+}S_{(4)+}, \tag{38}$$

where $k=1,\ldots,7$ and $\ell=2,4,6$. By simple computation, we find that

$$HS_{(4)+}T^k_{(4)+}S_{(4)+}\sim HS_{(4)+}T^{8-k}_{(4)+},$$
$$HS_{(4)+}T^{\ell}_{(4)+}S_{(4)+}T\sim HS_{(4)+}T^{8-\ell}_{(4)+}S_{(4)+} \tag{39}$$

for $k=$ odd and $\ell=$ even.

We make a subgroup with the order 12 by choosing properly 12 elements such that we pick one element up from each coset and their algebra is closed. The non-Abelian group with the order 12 are D_6, Q_6, and A_4. Among them, A_4 would be a good candidate. Indeed, we can obtain the Z_3 generator from $HS_{(4)+}T_{(4)+}$, gain. That is, we define

$$t = a^m a'^n S_{(4)+} T_{(4)+}. \tag{40}$$

The solutions for $t^3 = \mathbb{I}$ are obtained by $(m, n) = (1, 4)$, (3, 6), (5, 0), and (7, 2). We also define

$$s = a^{m'} a'^{n'} S_{(4)+} T^4_{(4)+} S_{(4)+}. \tag{41}$$

The solutions for $s^2 = \mathbb{I}$ are obtained by $(m', n') = (0, 0)$, (0, 4), (4, 0), and (4, 4). These two generators satisfy $(st)^3 = \mathbb{I}$ if $(m', n') = (0, 4)$, and (4, 0), i.e.,

$$s = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & -1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}. \tag{42}$$

As a result, they satisfy

$$s^2 = t^3 = (st)^3 = \mathbb{I}. \tag{43}$$

That is the A_4 algebra.

3. Large magnetic flux M

For larger magnetic fluxes, S and T transformations are represented by zero modes $\psi^{j,M}$, but those are reducible representations. The irreducible representations are obtained in the T^2/Z_2 orbifold basis,

$$\psi^{j,M}_{\pm} = \frac{1}{\sqrt{2}}(\psi^{j,M} \pm \psi^{M-j,j}). \tag{44}$$

The representations of $T_{(M)}$ are simply obtained by

$$T_{(M)+}\begin{pmatrix} \psi^{0,M} \\ \psi^{1,M}_+ \\ \vdots \\ \psi^{j,M}_+ \\ \vdots \\ \psi^{M/2,M} \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & e^{\pi i/M} & & & & \\ & & \ddots & & & \\ & & & e^{\pi i j^2/M} & & \\ & & & & \ddots & \\ & & & & & e^{\pi i M/4} \end{pmatrix} \times \begin{pmatrix} \psi^{0,M} \\ \psi^{1,M}_+ \\ \vdots \\ \psi^{j,M}_+ \\ \vdots \\ \psi^{M/2,M} \end{pmatrix} \tag{45}$$

and

$$T_{(M)-}\begin{pmatrix} \psi^{1,M}_- \\ \vdots \\ \psi^{j,M}_- \\ \vdots \\ \psi^{M/2-1,M}_- \end{pmatrix} = \begin{pmatrix} e^{\pi i/M} & & & & \\ & \ddots & & & \\ & & e^{\pi i j^2/M} & & \\ & & & \ddots & \\ & & & & e^{\pi (M/2-1)^2/M} \end{pmatrix} \begin{pmatrix} \psi^{1,M}_- \\ \vdots \\ \psi^{j,M}_- \\ \vdots \\ \psi^{M/2-1,M}_- \end{pmatrix}. \tag{46}$$

Both correspond to Z_{2M}.

On the other hand, the $S_{(M)\pm}$ transforms

$$S_{(M)\pm}\psi^{j,M}_{\pm} = \frac{1}{\sqrt{2M}} \sum_k (e^{2\pi jk/M} \pm e^{2\pi i(M-j)k/M,M})\psi^{k,M}. \tag{47}$$

This representation is also written by

$$S_{(M)\pm}\psi^{j,M}_{\pm} = \frac{1}{\sqrt{2M}} \sum_k (e^{2\pi (M-j)(M-k)/M} \pm e^{2\pi ij(M-k)/M,M})\psi^{M-k,M}. \tag{48}$$

Thus, the S transformation is represented on the T^2/Z_2 orbifold basis by

$$S_{(M)\pm}\psi^{j,M}_{\pm} = \frac{1}{\sqrt{M}} \sum_{k \le M/2} (e^{2\pi jk/M} \pm e^{2\pi i(M-j)k/M})\psi^{k,M}_{\pm}. \tag{49}$$

These are written by

$$S_{(M)+}\psi^{j,M}_{+} = \frac{2}{\sqrt{M}} \sum_{k \le M/2} \cos(2\pi jk/M)\psi^{j,M}_{+},$$
$$S_{(M)-}\psi^{j,M}_{-} = \frac{2i}{\sqrt{M}} \sum_{k \le M/2} \sin(2\pi jk/M)\psi^{j,M}_{-}. \tag{50}$$

For example, for $M = 6$, S and T are represented by Z_2 even zero modes,

$$S_{(6)+}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}1 & \sqrt{2} & \sqrt{2} & 1\\ \sqrt{2} & 1 & -1 & -\sqrt{2}\\ \sqrt{2} & -1 & -1 & \sqrt{2}\\ 1 & -\sqrt{2} & \sqrt{2} & -1\end{pmatrix}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}, \tag{51}$$

$$T_{(6)+}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}=\begin{pmatrix}1 & & & \\ & e^{\pi i/6} & & \\ & & e^{2\pi i/3} & \\ & & & e^{3\pi i/2}\end{pmatrix}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}, \tag{52}$$

while S and T are represented by Z_2 odd zero mode,

$$S_{(6)-}\begin{pmatrix}\psi_-^{1,6}\\ \psi_-^{2,6}\end{pmatrix}=\frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\psi_-^{1,6}\\ \psi_-^{2,6}\end{pmatrix}, \tag{53}$$

$$T_{(6)-}\begin{pmatrix}\psi_-^{1,6}\\ \psi_-^{2,6}\end{pmatrix}=\begin{pmatrix}e^{\pi i/6} & 0\\ 0 & e^{2\pi i/3}\end{pmatrix}\begin{pmatrix}\psi_-^{1,6}\\ \psi_-^{2,6}\end{pmatrix}. \tag{54}$$

C. Non-Abelian discrete flavor symmetries

In Ref. [8], it is shown that the models with $M=2$ as well as even magnetic fluxes have the D_4 flavor symmetry. See Appendix A. One of the Z_2 elements in D_4 corresponds to $(T_{(2)})^2$ on the zero modes, $\psi^{0,2}$ and $\psi^{1,2}$, i.e.,

$$Z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=(T_{(2)})^2. \tag{55}$$

In addition, the permutation Z_2^C element in D_4 corresponds to $S_{(2)}T_{(2)}T_{(2)}S_{(2)}$, i.e.,

$$C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=S_{(2)}T_{(2)}T_{(2)}S_{(2)}. \tag{56}$$

Thus, the D_4 group, which includes the eight elements (A8), is a subgroup of $G_{(2)}\simeq(Z_8\times Z_4)\rtimes S_3$.

However, there is the difference between the modular symmetry and the D_4 flavor symmetry, which is studied in Ref. [8]. The modular symmetry transforms the Yukawa couplings, while the Yukawa couplings are invariant under the D_4 flavor symmetry. In order to study this point, here we examine the Yukawa couplings among $\psi^{i,2}$, $\psi'^{j,2}$, and $\psi^{k,4}$. Both $\psi^{i,2}$ and $\psi'^{j,2}$ are D_4 doublets, and their tensor product $\mathbf{2}\times\mathbf{2}$ is expanded by

$$\mathbf{2}\times\mathbf{2}=\mathbf{1}_{++}+\mathbf{1}_{+-}+\mathbf{1}_{-+}+\mathbf{1}_{--}. \tag{57}$$

Thus, the products $\psi^{i,2}\psi'^{j,2}$ correspond to four singlets,

$$\mathbf{1}_{++}:\ \psi^{0,2}\psi'^{0,2}\pm\psi^{1,2}\psi'^{1,2},\qquad \mathbf{1}_{-\pm}:\ \psi^{0,2}\psi'^{1,2}\pm\psi^{1,2}\psi'^{0,2}. \tag{58}$$

On the other hand, by use of Eq. (9), the products $\psi^{i,2}\psi'^{j,2}$ are expanded by $\psi^{k,4}$. For example, we can expand as

$$\begin{aligned}\psi^{0,2}\psi'^{0,2}\pm\psi^{1,2}\psi'^{1,2}\sim&\,(Y^{(0)}(16\tau)+Y^{(8/16)}(16\tau)\\&\pm(Y^{(4/16)}(16\tau)+Y^{(12/16)}(16\tau)))\\&\times(\psi^{0,4}\pm\psi^{2,4})\end{aligned} \tag{59}$$

up to constant factors, where

$$Y^{(j/M)}(M\tau)=\mathcal{N}\cdot\vartheta\begin{bmatrix}\frac{j}{M}\\ 0\end{bmatrix}(0,M\tau). \tag{60}$$

Note that $Y^{(j/M)}(M\tau)=Y^{(1-j/M)}(M\tau)$. It is found that

$$\begin{aligned}(T_{(4)})^2(\psi^{0,4}\pm\psi^{2,4})&=(\psi^{0,4}\pm\psi^{2,4}),\\(S_{(4)}T_{(4)}T_{(4)}S_{(4)})(\psi^{0,4}\pm\psi^{2,4})&=\pm(\psi^{0,4}\pm\psi^{2,4}).\end{aligned} \tag{61}$$

Thus, the zero modes $\psi^{0,4}\pm\psi^{2,4}$ are indeed D_4 singlets, $\mathbf{1}_{+\pm}$ when we identify $(T_{(4)})^2$ and $(S_{(4)}T_{(4)}T_{(4)}S_{(4)})$ as Z_2 and Z_2^C of D_4. In this sense, the D_4 flavor symmetry is a subgroup of the modular symmetry. Also, it is found that the above Yukawa couplings, $Y^{(m/4)}(16\tau)$, with $m=0$, 1, 2, 3 are invariant under T^2 and $STTS$ transformation.

Similarly, we can expand

$$\begin{aligned}\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2}\sim&\,(Y^{(2/16)}(16\tau)+Y^{(6/16)}(16\tau))\\&\times(\psi^{1,4}+\psi^{3,4}),\end{aligned} \tag{62}$$

up to constant factors. It is found that

$$(T_{(2)})^2(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2})=-(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2}). \tag{63}$$

On the other hand, we obtain

$$(T_{(4)})^2(\psi^{1,4}+\psi^{3,4})=i(\psi^{1,4}+\psi^{3,4}). \tag{64}$$

In addition, we find

$$\begin{aligned}T^2:\ (Y^{(2/16)}(16\tau)+Y^{(6/16)}(16\tau))&\to i(Y^{(2/16)}(16\tau)\\&+Y^{(6/16)}(16\tau)).\end{aligned} \tag{65}$$

Thus, the T^2 transformation is consistent between left- and right-hand sides in (62). However, when we interpret T^2 as Z_2 of the D_4 flavor symmetry, we face inconsistency, because Yukawa couplings are not invariant and $(\psi^{1,4}+\psi^{3,4})$ has transformation behavior different from $(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2})$. We can make this consistent by

defining Z_2 of the D_4 on $(\psi^{1,4}+\psi^{3,4})$ such that its transformation absorbs the phase of Yukawa couplings under T^2 transformation. Then, the mode $(\psi^{1,4}+\psi^{3,4})$ exactly corresponds to the D_4 singlet, $\mathbf{1}_{-+}$. We find that $(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2})$ is invariant under $S_{(2)}T_{(2)}T_{(2)}S_{(2)}$, and $(\psi^{1,4}+\psi^{3,4})$ is also invariant under $S_{(4)}T_{(4)}T_{(4)}S_{(4)}$. That is consistent. Therefore, the D_4 flavor symmetry is a subgroup of the modular symmetry on $\psi^{j,2}$ $(j=0,\ 1)$. However, when the model includes couplings to zero modes with larger M, we have to modify their modular symmetries such that coupling constants are invariant under the flavor symmetry. Then, we can define the D_4 flavor symmetry.

Here, we give a comment on the T^2/Z_2 orbifold. The T^2/Z_2 orbifold basis gives the irreducible representations of the modular symmetry. The D_4 flavor symmetry is defined through the modular symmetry, as above. That is the reason why the D_4 flavor symmetry remains on the T^2/Z_2 orbifold [11,12].

III. HETEROTIC ORBIFOLD MODELS

Intersecting D-brane models in type II superstring theory is T-dual to magnetized D-brane models. Thus, intersecting D-brane models also have the same behavior under modular transformation as magnetized D-brane models. Furthermore, intersecting D-brane models in type II superstring theory and heterotic string theory on orbifolds have similarities, e.g., in two-dimensional conformal field theory. For example, computations of three-point couplings as well as n-point couplings are similar to each other. Here, we study modular symmetry in heterotic orbifold models. Using results in Refs. [16–18], we compare the modular symmetries in heterotic orbifold models with non-Abelian flavor symmetries and also the modular symmetries in the magnetized D-brane models, which have been derived in the previous section.

A. Twisted sector

Here, we give a brief review on heterotic string theory on orbifolds. The orbifold is the division of the torus T^n by the Z_N twist θ, i.e., T^n/Z_N. Since the T^n is constructed by $\mathbb{R}^n/\Lambda$, the Z_N twist θ should be an automorphism of the lattice Λ. Here, we focus on two-dimensional orbifolds, T^2/Z_N. The six-dimensional orbifolds can be constructed by products of two-dimensional ones. All of the possible orbifolds are classified as T^2/Z_N with $N=2,\ 3,\ 4,\ 6$.

On orbifolds, there are fixed points, which satisfy the following condition:

$$x^i=(\theta^n x)^i+\sum_k m_k\alpha_k^i, \tag{66}$$

where x^i are real coordinates, α_k^i are two lattice vectors, and m_k are integers for $i,\ k=1,\ 2$. Thus, the fixed points can be represented by corresponding space group elements $(\theta^n,\sum_k m_k\alpha_k^i)$, or in short $(\theta^n,(m_1,m_2))$.

The heterotic string theory on orbifolds has localized modes at fixed points, and these are the so-called twisted strings. These twisted states can be labeled by use of fixed points, $\sigma_{\theta,(m_1,m_2)}$. All of the twisted states $\sigma_{\theta,(m_1,m_2)}$ have the same spectrum, if discrete Wilson lines vanish. Thus, the massless modes are degenerate by the number of fixed points.

On the T^2/Z_2 orbifold, there are four fixed points, which are denoted by $(\theta,(0,0))$, $(\theta,(1,0))$, $(\theta,(0,1))$, $(\theta,(1,1))$. The corresponding twisted states are denoted by $\sigma_{\theta,(m,n)}$ for $m,\ n=0,\ 1$.

On the T^2/Z_3 orbifold, α_1 and α_2 correspond to the $SU(3)$ simple roots, and they are identified with each other by the Z_3 twist. Thus, three fixed points on the T^2/Z_3 orbifold are represented by the space group elements, $(\theta,m\alpha_1)$ for $m=0,\ 1,\ 2$, or in short (θ,m). The corresponding twisted states are denoted by $\sigma_{\theta,m}$ for $m=0,1,2$.

Similarly, we can obtain the fixed points and twisted states on the T^2/Z_4, where α_1 and α_2 correspond to the $SO(4)$ simple roots and they are identified with each other by the Z_4 twist. For the Z_4 twist θ, two fixed points satisfy Eq. (66), and these can be represented by $(\theta,m\alpha_1)$ for $m=0,\ 1$, or in short (θ,m). Then, the first twisted states are denoted by $\sigma_{\theta,m}$ for $m=0,\ 1$. In addition, for θ^2, there are four points, which satisfy Eq. (66), and these can be denoted by $(\theta^2,(m,n))$ for $m,\ n=0,\ 1$. Indeed, these correspond to the four fixed points on the T^2/Z_2 orbifold. Then, the second twisted states are denoted by $\sigma_{\theta^2,(m,n)}$ for $m,\ n=0,\ 1$. However, the fixed points $(\theta^2,(1,0))$ and $(\theta^2,(0,1))$ transform each other under the Z_4 twist θ. Thus, the Z_4 invariant states are written by [21]

$$\sigma_{\theta^2,(0,0)},\qquad \sigma_{\theta^2,+},\qquad \sigma_{\theta^2,(1,1)}, \tag{67}$$

while $\sigma_{\theta^2,-}$ transforms to $-\sigma_{\theta^2,-}$ under the Z_4 twist, where

$$\sigma_{\theta^2,\pm}=\frac{1}{\sqrt{2}}(\sigma_{\theta^2,(1,0)}\pm\sigma_{\theta^2,(0,1)}). \tag{68}$$

Similarly, we can obtain the fixed points on T^2/Z_6. There is a fixed point $(\theta,0)$ for the Z_6 twist θ, and a single twisted state $\sigma_{\theta,0}$. The second twisted sector has three fixed points (θ^2,m) $(m=0,\ 1,\ 2)$, which correspond to the three fixed points on the T^2/Z_3 orbifold. The two fixed points $(\theta^2,1)$ and $(\theta^2,2)$ transform each other by the Z_6 twist, while $(\theta^2,0)$ is invariant. Thus, we can write the Z_6-invariant θ^2-twisted states by

$$\sigma_{\theta^2,0},\qquad \sigma_{\theta^2,+}, \tag{69}$$

while $\sigma_{\theta^2,-}$ transforms to $-\sigma_{\theta^2,-}$ under the Z_6 twist, where

$$\sigma_{\theta^2,\pm}=\frac{1}{\sqrt{2}}(\sigma_{\theta^2,1}\pm\sigma_{\theta^2,2}). \tag{70}$$

The third twisted sector has four fixed points, which correspond to the fixed points on T^2/Z_2, and the corresponding θ^3 twisted states. Their linear combinations are Z_6 eigenstates similar to the second twisted states. Since the first twisted sector has the single fixed point and twisted state, the modular symmetry as well as non-Abelian discrete flavor symmetry is rather trivial. We do not discuss the T^2/Z_6 orbifold itself.

B. Modular symmetry

In Ref. [16], modular symmetry in heterotic string theory on orbifolds was studied in detail. Here we use those results.

1. T^2/Z_4 orbifold

The S and T transformations are represented by the first twisted sectors of T^2/Z_4 orbifold as [16]

$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\end{pmatrix} \to S_{Z_4}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\end{pmatrix}, \qquad S_{Z_4}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$
$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\end{pmatrix} \to T_{Z_4}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\end{pmatrix}, \qquad T_{Z_4}=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}. \tag{71}$$

These are exactly the same as representations of $S_{(2)}$ and $T_{(2)}$ on two-zero modes, $\psi^{0,2}$ and $\psi^{1,2}$ in the magnetized model with magnetic flux $M=2$. Hence, the twisted sectors on the T^2/Z_4 orbifold has the same behavior of modular symmetry as the magnetized model with magnetic flux $M=2$. Indeed, the twisted sectors have the D_4 flavor symmetry and two twisted states, $\sigma_{\theta,0}$ and $\sigma_{\theta,1}$, correspond to the D_4 doublet [4]. The whole flavor symmetry of the T^2/Z_4 orbifold model is slightly larger than D_4. (See Appendix B.) The T^2/Z_4 orbifold model has the Z_4 symmetry, which transforms the first twisted sector,

$$\sigma_{\theta,m} \to e^{\pi i/2}\sigma_{\theta,m}, \tag{72}$$

for $m=0$, 1 and the second twisted sector,

$$\sigma_{\theta^2,(m,n)} \to e^{\pi i}\sigma_{\theta^2,(m,n)}, \tag{73}$$

for m, $n=0$, 1. The above Z_4 transformation (72) is nothing but $(S_{Z_4}T_{Z_4})^6$ as clearly seen from Eq. (24). Thus, the whole flavor symmetry originates from the modular symmetry.

The second twisted sectors correspond to D_4 singlets, $\mathbf{1}_{\pm 1,\pm}$ [4], as

$$\mathbf{1}_{++}\colon\ \sigma_{\theta^2,(0,0)} \pm \sigma_{\theta^2,(1,1)}, \qquad \mathbf{1}_{-\pm}\colon\ \sigma_{\theta^2,\pm}, \tag{74}$$

up to coefficients. Compared with the results in Sec. II C, the D_4 behavior of the second twisted states corresponds to one of the zero modes $\psi^{m,4}$ with magnetic flux $M=4$. Their correspondence can be written as

$$\sigma_{\theta^2,(0,0)} \sim \psi^{0,4}, \qquad \sigma_{\theta^2,(1,1)} \sim \psi^{2,4},$$
$$\sigma_{\theta^2,(1,0)} \sim \psi^{1,4}, \qquad \sigma_{\theta^2,(1,0)} \sim \psi^{3,4}. \tag{75}$$

The above correspondence can also been seen from the Yukawa couplings. By use of operator product expansion, we obtain the following relations [16]:

$$\sigma_{\theta,0}\sigma_{\theta,0} \sim Y_{0,0}(\sigma_{\theta^2,(0,0)}+\sigma_{\theta^2,(1,1)}),$$
$$\sigma_{\theta,1}\sigma_{\theta,1} \sim Y_{1,1}(\sigma_{\theta^2,(0,0)}+\sigma_{\theta^2,(1,1)}),$$
$$\sigma_{\theta,0}\sigma_{\theta,1}+\sigma_{\theta,1}\sigma_{\theta,0} \sim Y_{0,1}\sigma_{\theta^2,+} \tag{76}$$

up to constants. The second twisted state $\sigma_{\theta^2,-}$ cannot couple with the first twisted sectors. Using results in Ref. [16], it is found that

$$(T_{Z_4})^2\begin{pmatrix}Y_{0,0}\\ Y_{1,1}\\ Y_{0,1}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}Y_{0,0}\\ Y_{1,1}\\ Y_{0,1}\end{pmatrix}. \tag{77}$$

This is the same as behavior of the Yukawa couplings under T^2 studied in Sec. II C.

2. T^2/Z_2 orbifold

Here, let us study the T^2/Z_2 orbifold in a way similar to the previous section on the T^2/Z_4. The S transformation is represented by the four twisted states on the T^2/Z_2 orbifold [16],

$$\begin{pmatrix}\sigma_{\theta,(0,0)}\\ \sigma_{\theta,(0,1)}\\ \sigma_{\theta,(1,0)}\\ \sigma_{\theta,(1,1)}\end{pmatrix} \to S_{Z_2}\begin{pmatrix}\sigma_{\theta,(0,0)}\\ \sigma_{\theta,(0,1)}\\ \sigma_{\theta,(1,0)}\\ \sigma_{\theta,(1,1)}\end{pmatrix},$$
$$S_{Z_2}=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}. \tag{78}$$

Also the T transformation is represented as

$$\begin{pmatrix}\sigma_{\theta,(0,0)}\\ \sigma_{\theta,(0,1)}\\ \sigma_{\theta,(1,0)}\\ \sigma_{\theta,(1,1)}\end{pmatrix} \to T_{Z_2}\begin{pmatrix}\sigma_{\theta,(0,0)}\\ \sigma_{\theta,(0,1)}\\ \sigma_{\theta,(1,0)}\\ \sigma_{\theta,(1,1)}\end{pmatrix},$$
$$T_{Z_2}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}. \tag{79}$$

The representation S_{Z_2} is similar to S_{Z_4} and $S_{(2)}$. Indeed, we find that $S_{Z_2}=S_{(2)}\otimes S_{(2)}$. However, the representation T_{Z_2} is different from T_{Z_4} and $T_{(2)}$.

The matrices S_{Z_2} and T_{Z_2} satisfy the following relations:

$$(S_{Z_2})^2 = (T_{Z_2})^2 = (S_{Z_2}T_{Z_2})^6 = \mathbb{I}. \tag{80}$$

These correspond to the D_6. Indeed, the order of closed algebra including S_{Z_2} and T_{Z_2} is equal to 12. At any rate, these matrices are reducible. We change the basis in order to obtain irreducible representations,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\sigma_{\theta,(0,0)}\\ \sigma_{\theta,(1,0)}\\ \sigma_{\theta,(0,1)}\\ \sigma_{\theta,(1,1)}\end{pmatrix}. \tag{81}$$

Then, σ_1 and σ_2 correspond to the D_6 doublet, while σ_3 and σ_4 correspond to the D_6 singlets. For example, $S_{Z_2}T_{Z_2}$ and T_{Z_2} are represented by

$$S_{Z_2}T_{Z_2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix}\cos(2\pi/6) & -\sin(2\pi/6) & 0 & 0\\ \sin(2\pi/6) & \cos(2\pi/6) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\times\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix},$$

$$T_{Z_2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}. \tag{82}$$

It is found that σ_3 and σ_4 correspond to $\mathbf{1}_{--}$ and $\mathbf{1}_{-+}$.

The twisted sector on the T^2/Z_2 orbifold has the flavor symmetry $(D_4 \times D_4)/Z_2$. However, this flavor symmetry seems independent of the above D_6, because they do not include any common elements. The twisted sector on the S^1/Z_2 orbifold has the flavor symmetry D_4. The flavor symmetry of the T^2/Z_2 orbifold is obtained as a kind of product, $D_4 \times D_4$, although two D_4 groups have a common Z_2 element. Thus, the flavor symmetry of T^2/Z_2 originates from the product of symmetries of the one-dimensional orbifold. On the other hand, the modular symmetry appears in two or more dimensions, but not in one dimension. Hence, these symmetries would be independent. When we include the above D_6 as low-energy effective field theory in addition to the flavor symmetry $(D_4 \times D_4)/Z_2$, low-energy effective field theory would have larger symmetry including D_6 and $(D_4 \times D_4)/Z_2$, although Yukawa couplings as well as higher order couplings transform nontrivially under D_6.

3. T^2/Z_3 orbifold

The S and T transformations are represented by the first twisted sectors of the T^2/Z_3 orbifold as [16]

$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix} \to S_{Z_3}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix},$$

$$S_{Z_3} = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{2\pi i/3} & e^{-2\pi i/3}\\ 1 & e^{-2\pi i/3} & e^{2\pi i/3}\end{pmatrix},$$

$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix} \to T_{Z_3}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix},$$

$$T_{Z_3} = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{2\pi i/3} & 0\\ 0 & 0 & e^{2\pi i/3}\end{pmatrix}. \tag{83}$$

These forms look similar to S and T transformations in magnetized models (18) and (19). Indeed, they correspond to submatrices of $S_{(6)}$ and $T_{(6)}$ in the magnetized models with the magnetic flux $M = 6$. Alternatively, in Ref. [17] the following S and T representations were studied[3]

$$S'_{Z_3} = -\frac{i}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{2\pi i/3} & e^{-2\pi i/3}\\ 1 & e^{-2\pi i/3} & e^{2\pi i/3}\end{pmatrix},$$

$$T'_{Z_3} = \begin{pmatrix}e^{2\pi i/3} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{84}$$

At any rate, the above representations are reducible representations. Thus, we use the flowing basis,

$$\begin{pmatrix}\sigma_+\\ \sigma_0\\ \sigma_-\end{pmatrix}, \tag{85}$$

where $\sigma_\pm = (\sigma_1 \pm \sigma_-)/\sqrt{2}$. The (σ_+, σ_0) is a doublet, while σ_- is a singlet. The former corresponds to the Z_6 invariant states among the θ^2 twisted sector on the T^2/Z_6 orbifold. Similarly, σ_- is the θ^2 twisted state, which transforms $\sigma_- \to -\sigma_-$ under the Z_6 twist. Alternatively, we can say that the doublet (σ_+, σ_0) corresponds to Z_2 even states and the singlet σ_- is the Z_2 odd states, where the Z_2 means the π rotation of the lattice vectors, $(\alpha_1, \alpha_2) \to (-\alpha_1, -\alpha_2)$. This point is similar to the aspect in magnetized D-brane models, where irreducible representations

[3]See also Ref. [18].

correspond to the T^2/Z_2 orbifold basis. Also, note that the first twisted states of the T^2/Z_4 orbifold correspond already to the Z_2-invariant basis.

For example, we represent S'_{Z_3} and T'_{Z_3} on the above basis [17],

$$S'_{Z_3} = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \qquad T'_{Z_3} = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & 1 \end{pmatrix}, \quad (86)$$

on the doublet $(\sigma_+, \sigma_0)^T$, while σ_- is the trivial singlet. Here, we define

$$Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{T}_{Z_3} = ZT'_{Z_3}. \quad (87)$$

Then, they satisfy the following algebraic relations [17,18]:

$$(S'_{Z_3})^2 = (\tilde{T}_{Z_3})^3 = (S'_{Z_3}\tilde{T}_{Z_3})^3 = Z, \qquad Z^2 = \mathbb{I}. \quad (88)$$

This group is the so-called T', which is the binary extension of $A_4 = T$.

The non-Abelian discrete flavor symmetry on the T^2/Z_3 orbifold is $\Delta(54)$, and the three twisted states correspond to the triplet of $\Delta(54)$. Thus, this modular symmetry seems independent of the $\Delta(54)$ flavor symmetry.

Two representations are related as

$$S'_{Z_3} = -iS_{Z_3}, \qquad T'_{Z_3} = e^{2\pi i/3}(T_{Z_3})^{-1}. \quad (89)$$

When we change phases of S, T, and ST, the group such as $(Z_N \times Z_M)\rtimes H$ in Secs. II and III would change to $(Z_{N'} \times Z_{M'})\rtimes H$.

IV. CONCLUSION

We have studied the modular symmetries in magnetized D-brane models. Representations due to zero modes on T^2 are reducible except the models with the magnetic flux $M = 2$. Irreducible representations are provided by zero modes on the T^2/Z_2, i.e., Z_2 even states and odd states. It is reasonable because $(ST)^3$ transforms the lattice vectors (α_1, α_2) to $(-\alpha_1, -\alpha_2)$. The orders of modular groups are large, and in general, they include the Z_8 symmetry as the center. The D_4 flavor symmetry is a subgroup of the modular group, which is represented in the models with the magnetic flux $M = 2$. The system including zero modes with $M = 2$, $M = 4$, and larger even M, also includes the D_4 flavor symmetry, when we define transformations of couplings in a proper way.

We have also studied the modular symmetries in heterotic orbifold models. The heterotic model on the T^2/Z_4 has exactly the same representation as the magnetized model with $M = 2$, and the modular symmetry includes the D_4 flavor symmetry. The representation due to the twisted states on the T^2/Z_3 orbifold is reducible, similar to representations due to zero modes in magnetized D-brane models on T^2. Their irreducible representations correspond to Z_2 even and odd states, similar to those in magnetized D-brane models. Thus, the $\Delta(54)$ flavor symmetry seems independent of the modular symmetry in the T^2/Z_3 orbifold models. Note that the first twisted states on the T^2/Z_4 are Z_2-invariant states. In this sense, we find that the modular symmetry is the symmetry on the Z_2 orbifold in both heterotic orbifold models and magnetized D-brane models. The symmetries, which remain under the Z_2 twist, can be realized as the modular symmetry.

We have set vanishing Wilson lines. It would be interesting to extend our analysis to magnetized D-brane models with discrete Wilson lines on orbifolds [22]. It would also be interesting to extend our analysis on zero modes to higher Kaluza-Klein modes [23].

Four-dimensional low energy-effective field theory is modular invariant [24–26]. Anomalies of the modular symmetry were studied [27,28], and they have important aspects [29–31]. The non-Abelian flavor symmetries such as D_4 can be anomalous. (See for anomalies of non-Abelian discrete symmetries, e.g., [2,32,33].) In certain models, the modular symmetries are related with the non-Abelian flavor symmetry D_4. It would be interesting to study their anomaly relations.

We also give a comment on phenomenological application. Recently, the mixing angles in the lepton sector were studied in the models, whose flavor symmetries are congruence subgroups, $\Gamma(N)$ [34,35]. In those models, the couplings are nontrivial representations of $\Gamma(N)$ and modular functions. Our models show massless modes represent larger finite groups. It would be interesting to apply our results to derive realistic lepton mass matrices as well as quark mass matrices.

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APPENDIX A: NON-ABELIAN DISCRETE FLAVOR SYMMETRY IN MAGNETIZED D-BRANE MODELS

In this appendix, we give a brief review on non-Abelian discrete flavor symmetries in magnetized D-brane models [8].

As mentioned in Sec. II A, the Yukawa couplings as well as higher order couplings have the coupling selection rule (11). That is, we can define Z_g charges for zero modes. Such Z_g transformation is represented on $\psi^{i,M=g}$ by

$$Z = \begin{pmatrix} 1 & & & & \\ & \rho & & & \\ & & \rho^2 & & \\ & & & \ddots & \\ & & & & \rho^{g-1} \end{pmatrix}, \tag{A1}$$

where $\rho = e^{2\pi i/g}$. Furthermore, their effective field theory has the following permutation symmetry:

$$\psi^{i,g} \to \psi^{i+1,g}, \tag{A2}$$

and such permutation can be represented by

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{A3}$$

This is another Z_g^C symmetry. However, these two generators do not commute with each other,

$$CZ = \rho ZC. \tag{A4}$$

Thus, the flavor symmetry corresponds to the closed algebra including Z and C. Its diagonal elements are given by $Z^m Z'^n$, i.e., $Z_g \times Z'_g$ where

$$Z' = \begin{pmatrix} \rho & & \\ & \ddots & \\ & & \rho \end{pmatrix}, \tag{A5}$$

and the full group corresponds to $(Z_g \times Z'_g) \rtimes Z_g^C$.

Furthermore, the zero modes $\psi^{i,M=gn}$ with the magnetic flux $M = gn$ also represent $(Z_g \times Z'_g) \rtimes Z_g^C$. The zero modes, $\psi^{i,M=gn}$, have Z_g charges (mod g). Under C, they transform as

$$\psi^{i,M=gn} \to \psi^{i+n,M=gn}. \tag{A6}$$

For example, the model with $g = 2$ has the D_4 flavor symmetry. The zero modes,

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \tag{A7}$$

correspond to the D_4 doublet **2**, where eight D_4 elements are represented by

TABLE I. D_4 representation.

Magnetic flux M	D_4 representations
2	**2**
4	$\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$, $\mathbf{1}_{--}$
6	$3 \times \mathbf{2}$

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A8}$$

In addition, when the model has the zero modes $\psi^{i,4}$ ($i = 0$, 1, 2, 3), the zero modes $\psi^{0,4}$ and $\psi^{2,4}$ ($\psi^{1,4}$ and $\psi^{3,4}$) transform each other under C, and they have Z_2 charge even (odd). Thus, $\psi^{0,4} \pm \psi^{2,4}$ correspond to $\mathbf{1}_{+\pm}$ of D_4 representations, while $\psi^{1,4} \pm \psi^{3,4}$ correspond to $\mathbf{1}_{-\pm}$. Furthermore, among the zero modes $\psi^{i,6}$ ($i = 0$, 1, 2, 3, 4, 5), the zero modes $\psi^{i,6}$ and $\psi^{i+3,6}$ transform each other under C. Hence, three pairs of zero modes,

$$\begin{pmatrix} \psi^{0,6} \\ \psi^{3,6} \end{pmatrix}, \quad \begin{pmatrix} \psi^{1,6} \\ \psi^{4,6} \end{pmatrix}, \quad \begin{pmatrix} \psi^{2,6} \\ \psi^{5,6} \end{pmatrix}, \tag{A9}$$

correspond to three D_4 doublets. These results are shown in Table I.

APPENDIX B: NON-ABELIAN DISCRETE FLAVOR SYMMETRY IN HETEROTIC ORBIFOLD MODELS

Here, we give a brief review on non-Abelian discrete flavor symmetries in heterotic orbifold models [4].

The twisted string x^i on the orbifold satisfy the following boundary condition:

$$x^i(\sigma = 2\pi) = (\theta^n x(\sigma = 0))^i + \sum_k m_k \alpha_k^i, \tag{B1}$$

similar to Eq. (66). Thus, the twisted string can be characterized by the space group element $g = (\theta^n, \sum_k m_k \alpha_k^i)$. The product of the two space group elements (θ^{n_1}, v_1) and (θ^{n_2}, v_2) is computed as

$$(\theta^{n_1}, v_1)(\theta^{n_2}, v_2) = (\theta^{n_1}\theta^{n_2}, v_1 + \theta^{n_1} v_2). \tag{B2}$$

The space group element g belongs to the same conjugacy class as hgh^{-1}, where h is any space group element on the same orbifold.

Now, let us consider the couplings among twisted strings corresponding to space group elements (θ^{n_k}, v_k). Their couplings are allowed by the space group invariance if the following condition,

$$\prod_k (\theta^{n_k}, v_k) = (1, 0), \tag{B3}$$

is satisfied up to the conjugacy class. That includes the point group selection rule, $\prod_k \theta^{n_k} = 1$, which is the Z_N invariance on the Z_N orbifold. We can define discrete Abelian symmetries from the space group invariance as well as the point group invariance. These symmetries together with geometrical symmetries of orbifolds become non-Abelian discrete flavor symmetries in heterotic orbifold models. We show them explicitly on concrete orbifolds.

1. S^1/Z_2 orbifold

The S^1/Z_2 orbifold has two fixed points, which are denoted by the space group elements, $(\theta, m\alpha)$ with $m=0, 1$, where α is the lattice vector. In short, we denote them by (θ, m) and the corresponding twisted states are denoted by $\sigma_{(\theta,m)}$. These states transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}, \tag{B4}$$

under the Z_2 twist. In addition, the space group invariance requires $\sum_k m_k = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k (\theta, m_k)$ with $m_k = 0, 1$. Hence, we can define another Z_2 symmetry, under which $\sigma_{(\theta,0)}$ is even, while $\sigma_{(\theta,1)}$ is odd. That is, another Z_2 transformation can be written by

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B5}$$

Furthermore, there is the geometrical permutation symmetry, which exchanges two fixed points with each other. Such a permutation is represented by

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B6}$$

The closed algebra including Eqs. (B4), (B5), and (B6) is $D_4 \simeq (Z_2 \times Z_2) \rtimes Z_2$.

2. T^2/Z_3 orbifold

As shown in Sec. III, the T^2/Z_3 orbifold has three fixed points denoted by (θ, m) with $m = 0, 1, 2$, and the corresponding twisted states are denoted by $\sigma_{(\theta,m)}$. The Z_3 twist transforms

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \to \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix}. \tag{B7}$$

The space group invariance requires $\sum_k m_k = 0 \pmod 3$ for the couplings corresponding to the product of the space group elements $\prod_k (\theta, m_k)$ with $m_k = 0, 1, 2$. Then, we can define another Z_3 symmetry, under which $\sigma_{(\theta,m)}$ transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix}. \tag{B8}$$

There is also the permutation symmetry of the three fixed points, that is, S_3. Thus, the flavor symmetry is $\Delta(54) \simeq (Z_3 \times Z_3) \rtimes S_3$.

3. T^2/Z_4 orbifold

As shown in Sec. III, the T^2/Z_4 orbifold has two θ fixed points denoted by (θ, m) with $m = 0, 1$, and the corresponding twisted states are denoted by $\sigma_{(\theta,m)}$. The Z_4 twist transforms

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \to \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B9}$$

The space group invariance requires $\sum_k m_k = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k (\theta, m_k)$ with $m_k = 0, 1$. Then, we can define another Z_2 symmetry, under which $\sigma_{(\theta,m)}$ transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B10}$$

There is also the permutation symmetry of the two fixed points. Thus, the flavor symmetry is almost the same as the one on the S^1/Z_2 orbifold. The difference is the Z_4 twist, although its squire is nothing but the Z_2 twist. Hence, the flavor symmetry can be written as $(D_4 \times Z_4)/Z_2$.

4. T^2/Z_2 orbifold

As shown in Sec. III, the T^2/Z_4 orbifold has two θ fixed points denoted by $(\theta, (m, n))$ with $m, n = 0, 1$, and the corresponding twisted states are denoted by $\sigma_{\theta,(m,n)}$. The space group invariance requires $\sum_k m_k = \sum_j n_j = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k (\theta, (m_k, n_j))$ with $m_k, n_j = 0, 1$. There are two independent permutation symmetries between $(\theta, (0, n))$ and $(\theta, (1, n))$, and $(\theta, (m, 0))$ and $(\theta, (m, 1))$. Thus, this structure seems be a product of two one-dimensional orbifolds, S^1/Z_2. However, the Z_2 twist is commented such as $\sigma_{\theta,(m,n)} \to -\sigma_{\theta,(m,n)}$. Thus, the flavor symmetry can be written by $(D_4 \times D_4)/Z_2$.

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