Broken phase solitons in the baby Skyrme model

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The baby Skyrme model is studied in the presence of an isospin chemical potential. The inclusion of the chemical potential leads to an asymmetric vacuum. As a result, the residual $SO(2)$ symmetry of the model is broken into the discrete $\mathbb{Z}_2$ group. Nontrivial topological field configurations are found to be allowed in the model. We obtained numerical solutions for the solitons on top of the new vacuum by minimizing the energy functional using a simulated annealing algorithm. All the calculations were performed by relaxing the rotational symmetry inherent to the hedgehog approximation. The corresponding soliton solutions exhibit discrete symmetries imposed by the new vacuum.

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I. INTRODUCTION

The Skyrme model is a nonlinear theory for $SU(2)$ valued scalar fields in $3+1$ dimensions which admits topologically stable soliton solutions [1]. There is a good number of related models sharing analogous topological properties which are directly derived from a generalization of the mathematical structure of the Skyrme model. Published in the literature are realizations of the general picture where the static fields are considered maps between two Riemannian manifolds. Such is the case of the so-called baby Skyrme model, which can be formally obtained from a consistent truncation of the physical and the target space of its parent $SU(2)$ Skyrme model [2]. In that specific model the physical space is two-dimensional euclidean space, and the target space is the two-sphere S^2.

Topological nontrivial soliton solutions in the baby Skyrme model have proven to be relevant in the description of many condensed-matter systems, such as ferromagnetic quantum Hall systems, liquid crystals and helical magnets [3,4].

The existence of topological excitations in the Skyrme-type models is quite often analyzed in a physical phase where the vacuum field configuration exhibits a global continuous symmetry from a subgroup of the full symmetry group of the energy functional. Guided by physical motivations one can however envisage different ways to destabilize the symmetric vacuum, thus triggering a spontaneous breaking of the symmetry in the theory. One can for instance endow the underlying scalar field theory with chemical potentials associated with the conserved charges corresponding to the global symmetries. Roughly speaking, the mechanism provides the essential ingredients for Bose-Einstein condensation and thus sets the proper conditions to produce a broken physical phase. This idea was explained in detail in Ref. [5]; though in the framework of electroweak theory the guiding lines developed there are entirely general and can be perfectly imported to Skyrme-type models [6,7].

The phenomenon of symmetry breaking in the baby Skyrme model has been studied from different perspectives, namely that of spinning and isospinning Skyrmions [8–11]. For instance, authors in [9,10] go beyond the axially symmetric solution ansatz for the baby Skyrme model by allowing the isospining solitons to deform, thus breaking the symmetries of the static configurations. On these grounds, previous works highlighted the fact that in both $3+1$ and $2+1$ dimensions, a spinning Skyrmion becomes unstable above a certain value of the angular velocity where it starts radiating the excess energy and angular momentum [11,12]. As a result, the spinning Skyrmion slows down until its stability is effectively restored. The authors in [8] present a situation where the radiation is inhibited by a given mechanism revealing numerical solutions with spontaneously broken symmetry, although the physical arguments behind this numerical result remain unclear. Nonetheless, the result is suggestive, and it is presumably related to a change in the vacuum structure of the theory.

In this work we explore the possibility of having stable topological solitons in the baby Skyrme model where an isospin chemical potential destabilizes the symmetric vacuum. The isospin chemical potential is coupled to the Skyrme field in such a way as to reproduce the coupling of a $U(1)$ isovector gauge field. Gauging the Skyrme field has been proposed previously [13], and the idea has been used in different physical contexts. Here the motivation for gauging is to destabilize the symmetric vacuum, resulting in possible new genuine solutions with discrete symmetries describing solitons in a condensation phase.

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In the next section we recall the main features of the original baby Skyrme model, and then we develop the description of the model in the presence of an isospin chemical potential. We then construct the topological solitons on top of an asymmetric field configuration which minimizes the energy functional in the broken phase. The numerical calculations in different topological sectors have been performed by relaxing the axial symmetry on Skyrme configurations and varying the isospin chemical potential parameter μ_I.

II. BABY SKYRMION IN THE PRESENCE OF AN ISOSPIN CHEMICAL POTENTIAL

We start by considering the Skyrme-type Lagrangian density in two spatial dimensions,

$$\mathcal{L} = -\frac{F^2}{4}\mathrm{Tr}\{L_\alpha L^\alpha\} + \frac{K^2}{16}\mathrm{Tr}\{[L_\alpha, L_\beta]^2\} - \frac{m^2}{2}\mathrm{Tr}\{\mathbf{1} + i\tau_3 U\}, \tag{1}$$

where $U \in SU(2)$, and the Maurier-Cartan operator L_α is defined by $L_\alpha = U^\dagger \partial_\alpha U$.

From the Lagrangian density (1) one can derive the baby Skyrme model by restricting the Skyrme field to an equatorial 2-sphere of the $SU(2)$ target space. In the baby Skyrme formulation the basic field is a real three-component vector $\vec{\Phi} \equiv (\Phi_1, \Phi_2, \Phi_3)$ which satisfies the constraint $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \mathbf{1}$. The associated restricted Skyrme field is $U = i\vec{\Phi}\cdot\vec{\tau}$.

The field $\vec{\Phi}$ is defined in the three-dimensional Minkowski space $\mathbb{M}^3$. Together with the constraint $\vec{\Phi}\cdot\vec{\Phi} = \mathbf{1}$ it defines a map of the physical space into the field space,

$$\vec{\Phi}(t, x_1, x_2) : \mathbb{M}^3 \mapsto S^2. \tag{2}$$

The target manifold S^2 is the 2-sphere of unit radius.

Substituting the restricted Skyrme field into the Lagrangian (1) results in the baby Skyrme Lagrangian density,

$$\mathcal{L} = \frac{F^2}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} - \frac{K^2}{4}\{(\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi})^2 - (\partial_\mu\vec{\Phi}\cdot\partial_\nu\vec{\Phi})(\partial^\mu\vec{\Phi}\cdot\partial^\nu\vec{\Phi})\} - m^2(1-\Phi_3), \tag{3}$$

or, equivalently,

$$\mathcal{L} = \frac{F^2}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} - \frac{K^2}{4}|\partial_\mu\vec{\Phi}\times\partial_\nu\vec{\Phi}|^2 - m^2(1 - \vec{n}\cdot\vec{\Phi}), \tag{4}$$

where $\vec{n} = (0, 0, 1)$.

The classical solutions in this model were first studied in detail in Refs. [2,11,14]. The finite energy configurations require that the field $\vec{\Phi}$ tends to the constant field $\vec{n}$ at spatial infinity,

$$\lim_{|\vec{x}|\to\infty}\vec{\Phi}(\vec{x}) = \vec{n}. \tag{5}$$

As a result the fields $\vec{\Phi}$ at a fixed time can be regarded as maps from the compacted physical space $\mathbb{R}^2 \cup \{\infty\}$ to S^2. Thus topologically $\vec{\Phi}(\vec{x})$ defines a map,

$$\vec{\Phi}(x_1, x_2) : S^2 \longmapsto S^2. \tag{6}$$

A given configuration can thus be considered as a representative of a homotopy class in $\pi_2(S^2) \simeq \mathbb{Z}$, labeled by an integer topological degree,

$$N = \frac{1}{4\pi}\int d^2x \vec{\Phi}\cdot\partial_1\vec{\Phi}\times\partial_2\vec{\Phi}. \tag{7}$$

The size stability of the field configurations in each topological sector is ensured by the interplay among the different terms in the Lagrangian, which scale either as negative or as positive powers of a spatial dilation factor.

The fourth order term in the field derivatives, or Skyrme term, breaks the scale invariance of the underlying σ model. According to the Derrick's theorem [15], the contribution with no derivatives is further needed to prevent the soliton from collapsing under the rescaling $\vec{\Phi}(\vec{x}) \to \vec{\Phi}(\vec{x}/\lambda)$, where $\lambda < 1$. This contribution becomes a mass term for the field $\vec{\Phi}$ when it is regarded as a small fluctuation around the vacuum given by Eq. (13). In the $(3+1)$ dimensional model, which has been widely used as an effective description for strong interactions, this term gives the pions a tree level mass.

The baby Skyrme model (4) is invariant under the symmetry group,

$$G = E_2 \times SO(2) \times P. \tag{8}$$

Here, P is a combined reflection in both space and the target space S^2, namely,

$$P : (x_1, x_2) \mapsto (x_1, -x_2) \quad \text{and} \quad (\Phi_1, \Phi_2, \Phi_3) \mapsto (\Phi_1, -\Phi_2, \Phi_3). \tag{9}$$

The model inherits the $O(3)$ symmetry group from the linear σ model which is applicable to some condensed-matter systems. A particular selection in (4) of the constant field $\vec{n}$ at spatial infinity breaks $O(3)$ into $H \equiv SO(2)$.

We shall analyze the baby Skyrme model in the presence of an isospin chemical potential μ_I, following a completely general prescription to include it into the model. The coupling of μ_I modifies the vacuum structure giving rise to a new phase where the global $SO(2)$ symmetry of the original model is spontaneously broken. This corresponds to the regime where it becomes energetically favorable to condense field excitations of mass m out of the vacuum, that is to say, when $\mu_I > m$. We are interested in studying the existence of topological solitons on top the new vacuum.

The coupling of the isospin chemical potential μ_I to the Skyrme field U mimics the coupling of a $U(1)$ isovector gauge field [16,17]. It is introduced in the Lagrangian density (1) by performing the replacement,

$$\partial_\alpha U \to \partial_\alpha U - i\frac{\mu_I}{2}[\tau_3, U]g_{\alpha 0}, \tag{10}$$

where τ_3 is the third Pauli matrix and $g_{\alpha\beta}$ is the metric tensor in Minkowski space $\mathbb{M}^3$. The coupling of the Skyrme field to a gauge field is based on the idea first introduced in [13] and underlying a version of the Skyrme model where the dynamics is governed by a Maxwell term.

Upon the substitution (10), the Lagrangian density in the presence of isospin chemical potential results in

$$\begin{aligned}\mathcal{L} = &-\frac{F^2}{4}\mathrm{Tr}\{L_\alpha L^\alpha\} + \frac{K^2}{16}\mathrm{Tr}\{[L_\alpha, L_\beta]^2\} - i\frac{m^2}{2}\mathrm{Tr}\{\tau_3 U\}\\ &+\frac{i\mu_I F^2}{4}\mathrm{Tr}\{wL_0\} - \frac{i\mu_I K^2}{4}\mathrm{Tr}\{wL_\alpha[L_0, L_\alpha]\}\\ &+\frac{\mu_I^2 F^2}{16}\mathrm{Tr}\{w^2\} - \frac{\mu_I^2 K^2}{32}\mathrm{Tr}\{[w, L_i]^2\},\end{aligned} \tag{11}$$

with $w = U^\dagger \tau_3 U - \tau_3$.

The field configuration which minimizes the energy is modified by the inclusion of the isospin chemical potential. In order to find the vacuum energy of the static Skyrme field we need to consider the following potential terms of the full Lagrangian presented in (11),

$$V = -\frac{F^2}{8}\mu_I^2\mathrm{Tr}\{1 - \tau_3 U \tau_3 U^\dagger\} + i\frac{m^2}{2}\mathrm{Tr}\{\tau_3 U\}. \tag{12}$$

Notice that the first term in (12) lowers the energy if U aligns along the τ_1 or τ_2 directions, while the second term favors the τ_3 direction.

The configurations that minimize the energy are found by using an ansatz which allows for a general rotation between the τ_3 and τ_1 or τ_2 directions. A convenient way for writing the vacuum is

$$U = i\{\tau_3 \cos\alpha + \sin\alpha(\tau_1 \cos\Phi_0 + \tau_2 \sin\Phi_0)\}, \tag{13}$$

where the angle α provides the rotation between the τ_3 and τ_1 or τ_2 isospin directions.

The vacuum is not unique because there is a $U(1)$ degeneracy corresponding to rotations generated by $g_{\alpha 0}\tau_3$. The arbitrary phase Φ_0, which appears in Eq. (13), reflects the infinite degeneracy of the vacuum.

Substituting the ansatz (13) in (12), it is found that the minimum occurs at $\alpha = 0$ in the phase where $|\mu_I| < m$ and $\cos\alpha = \frac{m^2}{F^2\mu_I^2}$ in the phase where $|\mu_I| > m$.

The existence of topological solitons corresponding to $\alpha = 0$ has been analyzed for the $(3+1)$D Skyrme model in Ref. [16,17]. In order to describe the soliton in the pion condensation phase, a suitable ansatz for a finite energy configuration is

$$\tilde{U} = U_0^{1/2} U(t) U_0^{1/2}, \tag{14}$$

where $U_0^{1/2}$ is given by

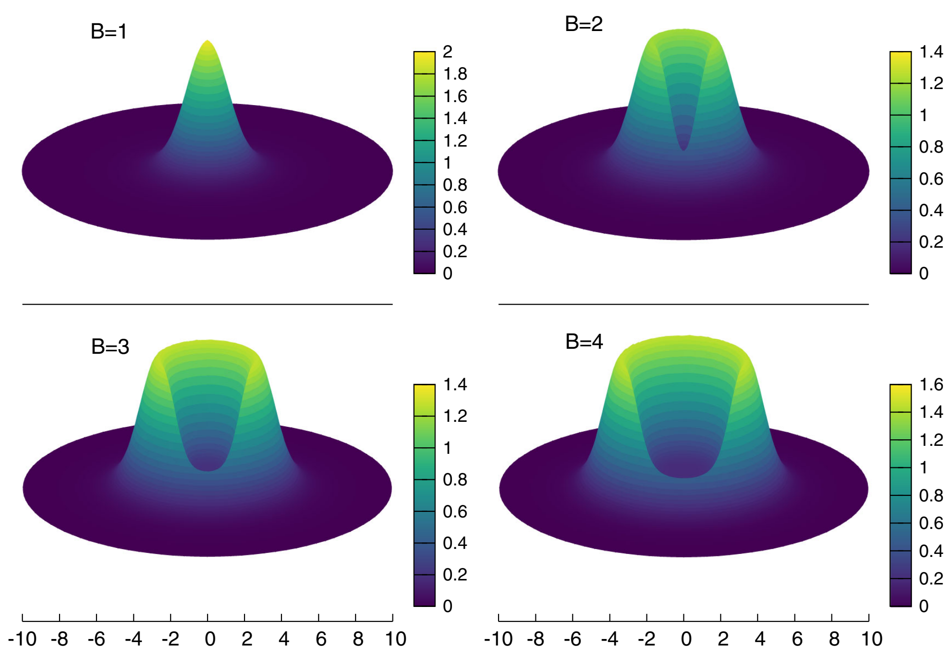

FIG. 1. Energy density contour plots for the baby skyrmion B = 1, B = 2, B = 3 and B = 4 corresponding to $\mu_I = 0$.

$$U_0^{1/2} = i\left\{\tau_3 \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}(\tau_1 \cos\Phi_0 + \tau_2 \sin\Phi_0)\right\}. \quad (15)$$

The static energy solution is found by minimizing

$$M(\mu_I) = M_0 - \frac{1}{2}\Lambda\mu_I^2, \quad (16)$$

where

$$M_0 = -\int d^2x\left[\frac{F^2}{4}\mathrm{Tr}\{L_iL_i\} + \frac{K^2}{16}\mathrm{Tr}\{[L_i, L_j]^2\} - i\frac{m^2}{2}\mathrm{Tr}\{\tau_3\cdot(\tilde{U} - \tilde{U}|_{\mathrm{vac}})\}\right], \quad (17)$$

and

$$\Lambda = \frac{1}{8}\int d^2x\left[\left(F^2\mathrm{Tr}\{\tilde{\omega}^2 - \tilde{\omega}^2|_{\mathrm{vac}}\} + \frac{K^2}{4}\mathrm{Tr}\{[\tilde{\omega}, \tilde{L}_i]^2\}\right)\right], \quad (18)$$

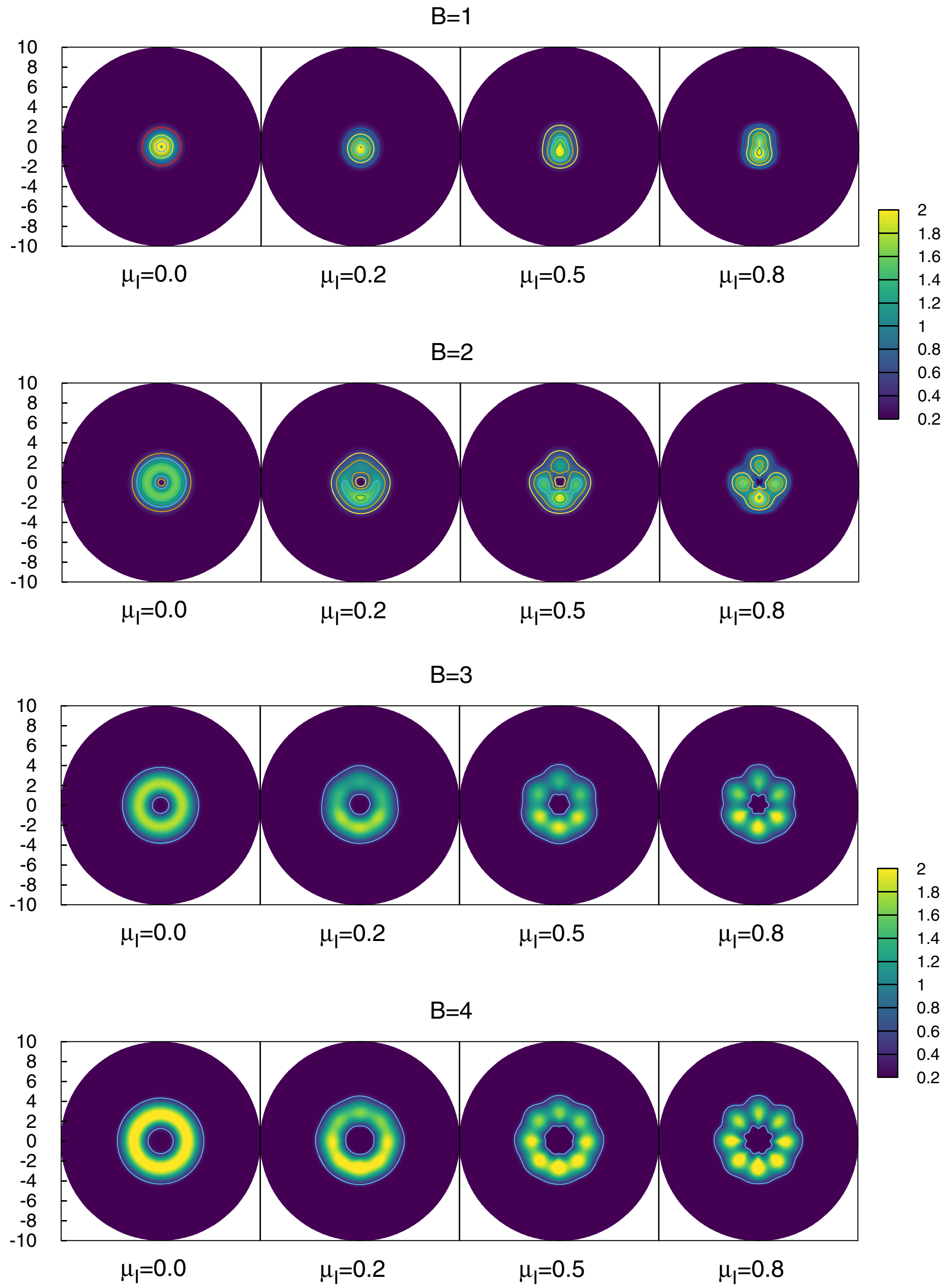

FIG. 2. Energy density contour plots for the $B = 1$, 2, 3 and $B = 4$ soliton solutions in the baby Skyrme model, including the isospin chemical potential. The results correspond to different values of μ_I.

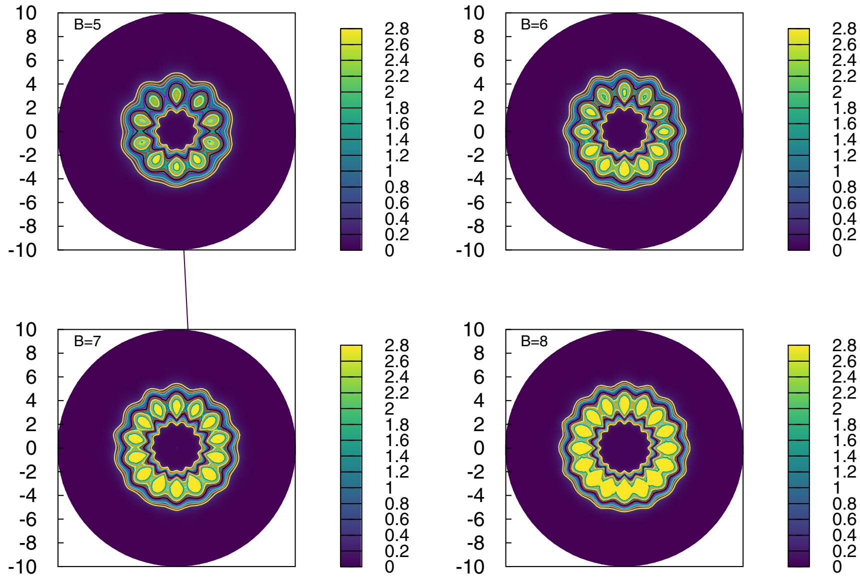

FIG. 3. Energy density contour plots for the baby skyrmion in the presence of isospin chemical potential corresponding to B = 5, B = 6, B = 7 and B = 8 and $\mu_I = 0.8$.

where we have introduced $\tilde{w} = \tilde{U}^\dagger \tau_3 \tilde{U} - \tau_3$ and $\tilde{L}_i = U_0^{1/2\dagger} L_i U_0^{1/2}$.

The presence of the isospin chemical potential breaks the $SO(2)$ symmetry to the remanent discrete group $\mathbb{Z}_2$.

The full breaking pattern is thus

$$O(3) \to SO(2) \to \mathbb{Z}_2. \quad (19)$$

III. STATIC SOLITON SOLUTIONS

In order to find static solutions in various homotopy classes, numerical methods are needed to solve the classical equations derived from the functional energy of the static configurations. Alternatively, we can resort to numerical optimization methods to find the field configurations that minimize the energy functional (16) in every topological sector. For the numerical analysis we fix the parameters of the baby Skyrme model to those used in the first works [2,11], that is to say, $m = 1/\sqrt{10}$ and $F = K = 1$.

The Skyrme field U can be cast in spherical coordinates by proposing for the Skyrme field the following form:

$$U = (\sin[F(r,\phi)]\cos[B\Phi(r,\phi)], \sin[F(r,\phi)]\sin[B\Phi(r,\phi)], \cos[F(r,\phi)]), \quad (20)$$

where B is the topological charge and $F(r,\phi)$ and $\Phi(r,\phi)$ satisfy the following boundary conditions,

$$\left.\frac{\partial F}{\partial \phi}\right|_{\phi=0} = 0, \qquad \left.\frac{\partial F}{\partial \phi}\right|_{\phi=2\pi} = 0,$$
$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=0} = 0, \qquad \left.\frac{\partial \Phi}{\partial r}\right|_{r=\infty} = 0,$$
$$\Phi(r,0) = 0, \qquad \Phi(r,2\pi) = 2\pi B,$$
$$F(0,\phi) = \pi, \qquad F(\infty,\phi) = 0. \quad (21)$$

In Fig. 1 we present the energy density for soliton solutions in some topological sectors for model (4), which is equivalent to the case where $\mu_I = 0$ in (11). These results were obtained by implementing a simulated annealing method to find the minimum of the static energy functional (16) on a grid containing 100^2 lattice points. This method has proven to be robust in determining the minimal energy configurations of the Skyrme model in 2D and 3D (see for example [18,19]). Results in Fig. 1 for the topological sectors $B = 1, 2, 3$ and 4 clearly reflect the symmetric phase in the model. The soliton masses in 4π units are found to be $M_1 = 1.567$, $M_2 = 2.942$, $M_3 = 4.487$, and $M_4 = 6.163$, which are in good agreement with those elsewhere [2,9,20].

For nonzero values of the chemical potential μ_I, the axial symmetry of the energy density is broken and reduced to discrete symmetries. The corresponding results for different topological sectors are displayed in Figs. 2 and 3. In Fig. 2 we present the energy density contour plots for a sequence of values of μ_I.

IV. CONCLUDING REMARKS

We have examined a gauged version of the baby Skyrme model motivated by the inclusion of an isospin chemical potential μ_I. A new vacuum structure arises by virtue of the coupling of the isospin chemical potential to the Skyrme field. As a consequence, the rotational symmetry of the original baby Skyrme model is broken.

Regarding the physical content of the model, the variation of μ_I sets the possibility to condense field excitations out of the vacuum. We have applied numerical techniques to look for the minimal energy soliton solutions of the modified baby Skyrme model on top of the new vacuum. The corresponding solutions in different topological sectors exhibit discrete symmetries and remain stable by their topological nature.

The results open the possibility for a good number of analyses about the physics of solitons in the baby Skyrme model on top of the condensate vacuum. The features discussed here for the model in $2+1$ dimensions might also serve as guidelines to investigate the Skyrme model in $3+1$ dimensions, where the presence of an isospin chemical potential could provide a description of a pion condensate, in agreement with the findings of chiral perturbation theory [6].

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