New metrics admitting the principal Killing–Yano tensor

Valeri P. Frolov,[1,*] Pavel Krtouš,[2,†] and David Kubizňák[3,‡]
[1]Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2G7, Canada
[2]Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, Prague 18000, Czech Republic
[3]Perimeter Institute, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada

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It is believed that in any number of dimensions the off-shell Kerr–NUT–(A)dS metric represents a unique geometry admitting the principal (rank 2, non-degenerate, closed conformal Killing–Yano) tensor. The original proof relied on the Euclidean signature and therein natural assumption that the eigenvalues of the principal tensor have gradients of spacelike character. In this paper we evade this common wisdom and construct new classes of Lorentzian (and other signature) off-shell metrics admitting the principal tensor with "null eigenvalues," uncovering so a much richer structure of spacetimes with principal tensor in four and higher dimensions. A few observations regarding the Kerr–Schild ansatz are also made.

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I. INTRODUCTION

It is a remarkable property of the *Kerr–NUT–(A)dS family* of spacetimes [1,2] that they admit a hidden symmetry encoded in the principal Killing–Yano tensor. This is true in four [3,4] as well as in higher dimensions [5]. Many of the basic characteristics of this family can be directly linked to this tensor and derived from its very existence. For example, the principal tensor generates a "Killing tower" of symmetries that guarantee complete integrability of geodesic motion and separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations. The type D property of these spacetimes can easily be derived from the integrability condition for the principal tensor, and its eigenvectors are intrinsically related to the principal null directions and a possibility to cast the metric in the Kerr–Schild form, we refer the interested reader to a recent review [6].

The *principal tensor* is a non-degenerate, closed conformal Killing–Yano 2-form. In D number of spacetime dimensions (in what follows we assume $D \geq 4$), it obeys an equation

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \qquad \xi_a = \frac{1}{D-1}\nabla^b h_{ba}, \tag{1}$$

or in the language of differential forms

$$\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{h} = \boldsymbol{X}\wedge\boldsymbol{\xi}, \qquad \boldsymbol{\xi} = \frac{1}{D-1}\boldsymbol{\nabla}\cdot\boldsymbol{h}, \tag{2}$$

where $\boldsymbol{X}$ is an arbitrary vector field. Since $\boldsymbol{h}$ is closed, there exists at least locally a potential 1-form $\boldsymbol{b}$ such that

$$\boldsymbol{h} = \boldsymbol{d}\boldsymbol{b}. \tag{3}$$

The condition of nondegeneracy means that the principal tensor has the maximal possible (matrix) rank and possesses the maximal number of functionally independent eigenvalues.

The *classification* of higher-dimensional metrics admitting the principal tensor has been attempted in [7–10]—the off-shell Kerr–NUT–(A)dS metric was found to be a unique solution admitting the principal tensor. As a starting point of the derivation, the authors considered a Riemannian metric $\boldsymbol{g}$ in

$$D = 2n + \epsilon, \tag{4}$$

dimensions, with $\epsilon = 0, 1$ in even, odd dimensions, and the principal tensor was written in the *Darboux frame*,

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu,$$
$$\boldsymbol{g} = \sum_{\mu=1}^{n}(\boldsymbol{e}^\mu\boldsymbol{e}^\mu\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu) + \varepsilon\hat{\boldsymbol{e}}^0\hat{\boldsymbol{e}}^0. \tag{5}$$

Here, $(\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu, \hat{\boldsymbol{e}}^0)$ is an orthonormal frame and the quantities x_μ are related to the "eigenvalues" of the 2-form $\boldsymbol{h}$. The condition of nondegeneracy of the principal tensor requires that there are exactly n nonvanishing and functionally independent eigenvalues x_μ whose gradients are linearly independent. Obviously, the Euclidean signature is at least formally assumed and the spatial character of the gradients,

*frolov@ualberta.ca
†Pavel.Krtous@utf.mff.cuni.cz
‡dkubiznak@perimeterinstitute.ca

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$$(dx_\mu)^2 > 0, \tag{6}$$

naturally follows.

Provided these assumptions, it was shown in [7–10] that the most general metric admitting the principal tensor takes the *canonical off-shell Kerr–NUT–(A)dS form*. In even dimensions it reads

$$g_{2n} = \sum_{\mu=1}^{n}\left[\frac{U_\mu}{X_\mu}dx_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{j=0}^{n-1}A_\mu^{(j)}d\psi_j\right)^2\right]. \tag{7}$$

In odd dimensions we have two possibilities

$$g_{2n+1}^{(1)} = g_{2n} + \frac{c}{A^{(n)}}\left(\sum_{k=0}^{n}A^{(k)}d\psi_k\right)^2, \tag{8}$$

$$g_{2n+1}^{(2)} = g_{2n} + cA^{(n)}d\psi_n^2. \tag{9}$$

Here, c is an arbitrary parameter, the functions $A^{(k)}$, $A_\mu^{(j)}$, and U_μ are "symmetric polynomials" of coordinates x_μ:

$$A^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k}}^{n} x_{\nu_1}^2\ldots x_{\nu_k}^2, \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\ldots,\nu_j=1\\ \nu_1<\ldots<\nu_j\\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2\ldots x_{\nu_j}^2,$$

$$U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2), \tag{10}$$

and each metric function X_μ is a function of a single coordinate x_μ:

$$X_\mu = X_\mu(x_\mu). \tag{11}$$

To indicate that these functions are not specified, we call such a metric *off-shell*. The Darboux frame (5) in these coordinate reads

$$e^\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}}dx_\mu, \quad \hat{e}^\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}}\sum_{j=0}^{n-1}A_\mu^{(j)}d\psi_j,$$

$$\hat{e}^0_{(1)} = \left(\frac{c}{A^{(n)}}\right)^{\frac{1}{2}}\sum_{k=0}^{n}A^{(k)}d\psi_k, \quad \hat{e}^0_{(2)} = (cA^{(n)})^{\frac{1}{2}}d\psi_n. \tag{12}$$

The principal tensor is given by

$$h = \sum_{\mu=1}^{n}x_\mu dx_\mu \wedge \left(\sum_{k=0}^{n-1}A_\mu^{(k)}d\psi_k\right) = \sum_\mu x_\mu e^\mu \wedge \hat{e}^\mu, \tag{13}$$

and can be derived from the potential

$$b = \frac{1}{2}\sum_{k=0}^{n-1}A^{(k+1)}d\psi_k. \tag{14}$$

This single object generates a whole Killing tower of explicit and hidden symmetries, including Killing vectors, rank-2 Killing tensors, and increasing rank Killing–Yano tensors, see [6].

When the vacuum Einstein equations are imposed,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0, \tag{15}$$

the metric functions take the following on-shell form [2,9–11]:

$$\begin{aligned} \text{even}\, D:\ X_\mu &= \sum_{k=0}^{n}c_k x_\mu^{2k} - 2b_\mu x_\mu, \\ \text{odd}\, D:\ X_\mu^{(1)} &= \sum_{k=1}^{n}c_k x_\mu^{2k} - 2b_\mu - \frac{c}{x_\mu^2}, \\ X_\mu^{(2)} &= \sum_{k=1}^{n}c_k x_\mu^{2k} - 2b_\mu, \end{aligned} \tag{16}$$

where the parameter c_n is related to the cosmological constant,

$$\Lambda = \frac{1}{2}(-1)^n(D-1)(D-2)c_n, \tag{17}$$

while other parameters c_k, b_μ, and (in odd dimensions) c are related to rotations, mass, and NUT parameters, see [6] for discussion.

Although this result has been derived assuming the formal Euclidean signature, it has been used also for the Lorentzian case. Indeed, using a suitable Wick rotation (see following sections) and choosing carefully coordinate ranges, the metric (7)–(9) with metric functions (16) can be a Lorentzian solution of Einstein's equations [6]. For this reason the uniqueness of the metric admitting the principal tensor has been usually formulated regardless of the signature.

However satisfactory this uniqueness result is in the Euclidean signature, in what follows we will show that it is no longer true in the spacetimes of Lorentzian and other signatures. In particular, considering such signatures, the principal tensor may possess null eigenvalues, that is eigenvalues characterized by a null gradient. Such a possibility has not been considered in the original construction presented in [7–10]. This allows one to construct new canonical metric elements with the principal tensor. In this paper we present several such new metrics, uncovering the fact that the classification of corresponding metrics is far from complete and a much richer structure of spacetimes with principal tensor may exist in four and higher dimensions.

Our paper is organized as follows. We start in four dimensions and show that a complete set of Lorentzian metrics admitting the principal tensor, classified long time ago by Dietz and Rudiger [12] and Taxiarchis [13], can be obtained by a certain limiting procedure starting from the Euclidean off-shell canonical element (7). Still in four dimensions, Sec. III is devoted to a few observations about the Kerr–Schild ansatz in various signatures and its connections to the construction of new canonical metrics. Both considerations are then extended to higher dimensions in Sec. IV. Section V is devoted to conclusions.

II. CANONICAL METRICS IN 4D

A. Euclidean signature

In four dimensions, and denoting by $(x, y, \tau, \psi) = (x_1, x_2, \psi_0, \psi_1)$ and $X = X_1$, $Y = X_2$, the unique Euclidean metric (7) reads

$$\boldsymbol{g}_E = \frac{X}{y^2 - x^2}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \frac{Y}{x^2 - y^2}(\boldsymbol{d}\tau + x^2\boldsymbol{d}\psi)^2 + \frac{y^2 - x^2}{X}\boldsymbol{d}x^2 + \frac{x^2 - y^2}{Y}\boldsymbol{d}y^2, \tag{18}$$

where ranges of coordinates x, y must be chosen such that $x^2 < y^2$, $X > 0$, and $Y < 0$. The principal tensor is determined from

$$\boldsymbol{b} = \frac{1}{2}[(x^2 + y^2)\boldsymbol{d}\tau + x^2y^2\boldsymbol{d}\psi]. \tag{19}$$

Of course, both the eigenvalues $\{x, y\}$ are spacelike.

The metric is a solution of vacuum Einstein equations, provided we set

$$X = c_0 + c_1x^2 + c_2x^4 - 2b_xx, \quad Y = c_0 + c_1y^2 + c_2y^4 - 2b_yy, \qquad c_2 = \Lambda/3. \tag{20}$$

One can check that a suitable choice of coordinate ranges can be achieved by restricting x and y to lie in between the roots of polynomials X and Y, respectively.

B. Lorentzian signature

To obtain the canonical metric (18) in the Lorentzian signature, additionally to a suitable choice of coordinate ranges, we must perform the following Wick rotation:

$$x = ir, \qquad \Delta_r = -X, \qquad \Delta_y = -Y, \tag{21}$$

to get

$$\boldsymbol{g}_L = -\frac{\Delta_r}{\Sigma}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2, \qquad \Sigma = r^2 + y^2, \tag{22}$$

while the principal tensor reads

$$\boldsymbol{b} = \frac{1}{2}[(y^2 - r^2)\boldsymbol{d}\tau - r^2y^2\boldsymbol{d}\psi]. \tag{23}$$

Both its eigenvalues $\{r, y\}$ are non-null, that is have non-null gradients. For a proper choice of coordinate ranges, the angular coordinate y is spacelike, and the causal character of r depends on a sign of the metric function Δ_r, reflecting whether we are below or above the horizon. $\{\tau, \psi\}$ are Killing time and angular coordinates, respectively, with their causal character depending on signs of the metric functions Δ_r and Δ_y.

The Kerr–NUT–(A)dS solution of vacuum Einstein equations is recovered upon introducing the mass $M = -ib_x$ and the NUT parameter $N = b_y$, yielding

$$\Delta_r = -c_0 + c_1r^2 - \frac{1}{3}\Lambda r^4 - 2Mr, \quad \Delta_y = -c_0 - c_1y^2 - \frac{1}{3}\Lambda y^4 + 2Ny. \tag{24}$$

In order to write this solution in the standard form, e.g., [14], one has to further introduce the rotation parameter a, $c_0 = -a^2$, set $c_1 = 1 - a^2\Lambda/3$, and perform the following coordinate transformation to the Boyer–Lindquist coordinates (t, ϕ, r, θ):

$$y = a\cos\theta, \qquad \psi = \phi/a, \qquad \tau = t - a\phi. \tag{25}$$

However, as discovered by Dietz and Rudiger [12] and Taxiarchis [13], there is yet another off-shell canonical spacetime besides (22), that admits the principal tensor given by (23). It reads[1]

$$\boldsymbol{g}_{L'} = \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 + 2\boldsymbol{d}r(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi). \tag{26}$$

The eigenvalue r of the principal tensor is now null everywhere, that is

$$r\colon\ (\boldsymbol{d}r)^2 = 0. \tag{27}$$

Despite this fact, the principal tensor is still nondegenerate in the sense of definition in [7–10] and generates (in a standard way) both isometries $\boldsymbol{\partial}_\tau$ and $\boldsymbol{\partial}_\phi$ and a nontrivial Killing tensor in this spacetime. The metric (26) becomes a

[1] As per usual, in all expressions for the metric we assume a symmetric tensor product in off-diagonal terms.

solution of vacuum (with $\Lambda = 0$) Einstein equations provided we set

$$\Delta_y = 2Ny. \tag{28}$$

We expect it to describe some kind of the NUT charged Aichelberg–Sexl solution.

Although originally derived in a different way [12,13], let us now demonstrate that the new canonical metric (26) can in fact be obtained by a certain limit starting from the metric (22). To this purpose, we rewrite (22) as

$$\begin{aligned} \boldsymbol{g}_L &= -\frac{\Delta_r}{\Sigma}\left(\left(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi\right)^2 - \frac{\Sigma^2}{\Delta_r^2}\boldsymbol{d}r^2\right) \\ &\quad + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 \\ &= -\frac{\Delta_r}{\Sigma}\boldsymbol{l}\boldsymbol{l} + 2\boldsymbol{d}r\boldsymbol{l} + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2, \end{aligned} \tag{29}$$

where we introduced a null vector

$$\boldsymbol{l} \equiv \boldsymbol{d}\tau + y^2\boldsymbol{d}\psi + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r. \tag{30}$$

Defining new coordinates

$$\boldsymbol{d}\hat{\tau} = \boldsymbol{d}\tau + \frac{r^2}{\Delta_r}\boldsymbol{d}r, \qquad \boldsymbol{d}\hat{\psi} = \boldsymbol{d}\psi + \frac{\boldsymbol{d}r}{\Delta_r}, \tag{31}$$

we find that

$$\boldsymbol{l} = \boldsymbol{d}\hat{\tau} + y^2\boldsymbol{d}\hat{\psi}, \tag{32}$$

while $(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi) = (\boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi})$. That is, we obtained

$$\boldsymbol{g}_L = -\frac{\Delta_r}{\Sigma}\boldsymbol{l}\boldsymbol{l} + 2\boldsymbol{d}r\boldsymbol{l} + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi})^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2, \tag{33}$$

with $\boldsymbol{l}$ given by (32). It is now obvious that one can take the limit

$$\Delta_r \to 0, \tag{34}$$

the metric remains nondegenerate, while r becomes a null coordinate and we recover the canonical metric (26). One can easily check that neither the coordinate transformation (31) nor the limit (34) affect (apart from a gauge term) the potential $\boldsymbol{b}$, (23).

Let us note that if one were to perform the limit $\Delta_r \to 0$ for the class of on-shell vacuum solutions of Einstein's equations, more care would have to be taken. Namely, the limit $\Delta_r \to 0$ effectively sets all the constants, apart from N, equal to zero, and influences thus also other metric functions, effectively recovering (28). However, since the metric functions also determine coordinate ranges, these ranges can degenerate in the limit and a suitable rescaling of coordinates has to be performed to resolve such a degeneracy. The limits of this type and the corresponding scaling of coordinates have been recently studied in [15].

C. Other signature

Let us now perform an additional Wick rotation

$$y = iz, \qquad \Delta_z = -\Delta_y, \qquad \Sigma = r^2 - z^2. \tag{35}$$

The metric (22) then gives

$$\begin{aligned} \boldsymbol{g}_{LL} &= -\frac{\Delta_r}{\Sigma}(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi)^2 - \frac{\Delta_z}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 \\ &\quad + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\Delta_z}\boldsymbol{d}z^2, \end{aligned} \tag{36}$$

while (26) yields

$$\boldsymbol{g}_{LL'} = \frac{\Sigma}{\Delta_z}\boldsymbol{d}z^2 - \frac{\Delta_z}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 + 2\boldsymbol{d}r(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi). \tag{37}$$

Both these metrics admit the principal tensor given by

$$\boldsymbol{b} = \frac{1}{2}[-(z^2 + r^2)\boldsymbol{d}\tau + r^2z^2\boldsymbol{d}\psi]. \tag{38}$$

The only difference is that in the latter case the eigenvalue r is null.

We note that $\boldsymbol{g}_{LL}$ is a vacuum solution of the Einstein equations provided

$$\begin{aligned} \Delta_r &= -c_0 + c_1r^2 - \frac{1}{3}\Lambda r^4 - 2Mr, \\ \Delta_z &= c_0 - c_1z^2 + \frac{1}{3}\Lambda z^4 - 2Nz. \end{aligned} \tag{39}$$

while $\boldsymbol{g}_{LL'}$ requires

$$\Delta_z = -2Nz. \tag{40}$$

Let us now apply the procedure above, with the aim to make also the eigenvalue y null. Starting from (37), we can write

$$\begin{aligned} \boldsymbol{g}_{LL'} &= -\frac{\Delta_z}{\Sigma}\left(\left(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi\right)^2 - \frac{\Sigma^2}{\Delta_z^2}\boldsymbol{d}z^2\right) + 2\boldsymbol{d}r(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi) \\ &= -\frac{\Delta_z}{\Sigma}\boldsymbol{m}\boldsymbol{m} + 2\boldsymbol{d}z\,\boldsymbol{m} + 2\boldsymbol{d}r(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi), \end{aligned} \tag{41}$$

where we introduced a null vector

$$\boldsymbol{m} \equiv \boldsymbol{d}\tau - r^2\boldsymbol{d}\psi + \frac{\Sigma}{\Delta_z}\boldsymbol{d}z. \tag{42}$$

Defining new coordinates

$$\boldsymbol{d}\hat{\tau} = \boldsymbol{d}\tau - \frac{z^2}{\Delta_z}\boldsymbol{d}z, \qquad \boldsymbol{d}\hat{\psi} = \boldsymbol{d}\psi - \frac{\boldsymbol{d}z}{\Delta_z}, \tag{43}$$

we find that

$$\boldsymbol{m} = \boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi}, \tag{44}$$

while $(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi) = (\boldsymbol{d}\hat{\tau} - z^2\boldsymbol{d}\hat{\psi})$. This allows one to take the limit

$$\Delta_z \to 0, \tag{45}$$

recovering the new metric

$$\boldsymbol{g}_{L'L'} = 2\boldsymbol{d}z(\boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi}) + 2\boldsymbol{d}r(\boldsymbol{d}\hat{\tau} - z^2\boldsymbol{d}\hat{\psi}), \tag{46}$$

Of course, such a metric is just a flat metric. Nevertheless, it possesses the same nondegenerate principal tensor, given by (38), with now two null eigenvalues:

$$(\boldsymbol{d}r)^2 = 0, \qquad (\boldsymbol{d}z)^2 = 0. \tag{47}$$

Although the corresponding $\boldsymbol{h} = \boldsymbol{d}\boldsymbol{b}$ is reducible (can be written as a skew symmetric product of Killing vectors), it is nondegenerate in the sense of [7–10] and gives rise to two Killing vectors and a (reducible) Killing tensor by a standard construction.

To conclude, in this (double minus) signature, there are three possible canonical metrics, (36), (37), and (46) that all admit the principal tensor (38).

III. VARIATIONS ON THE KERR–SCHILD FORM

In this section we obtain the Kerr–Schild form [16,17] of the canonical metric elements. As we shall see, the above described method for generating the new off-shell canonical metrics shares many features with the procedure for obtaining such a form.

A. Lorentzian signature

We start in the Lorentzian signature, from the Lorentzian metric $\boldsymbol{g}_L$ written in the form (33). Instead of setting $\Delta_r \to 0$ as we did in the previous section to obtain the metric (26), we now split

$$\Delta_r = \tilde{\Delta}_r - f, \tag{48}$$

where both $\tilde{\Delta}_r$ and f are *arbitrary* functions of coordinate r. By applying the "inverse transformation" to (31) on the tilde part,

$$\boldsymbol{d}\hat{\tau} = \boldsymbol{d}\tilde{\tau} + \frac{r^2}{\tilde{\Delta}_r}\boldsymbol{d}r, \qquad \boldsymbol{d}\hat{\psi} = \boldsymbol{d}\tilde{\psi} + \frac{\boldsymbol{d}r}{\tilde{\Delta}_r}, \tag{49}$$

we hence recover the following *off-shell Kerr–Schild form* of the canonical metric:

$$\boldsymbol{g}_L = \tilde{\boldsymbol{g}}_L + \frac{f}{\Sigma}\boldsymbol{l}\boldsymbol{l}, \tag{50}$$

where $\tilde{\boldsymbol{g}}_L$ is the canonical metric (22) with $\Delta_r \to \tilde{\Delta}_r$ and $\boldsymbol{l}$ is a null vector with respect to both $\boldsymbol{g}_L$ and $\tilde{\boldsymbol{g}}_L$ metrics:

$$\boldsymbol{l} = \boldsymbol{d}\tilde{\tau} + y^2\boldsymbol{d}\tilde{\psi} + \frac{\Sigma}{\tilde{\Delta}_r}\boldsymbol{d}r, \qquad \Sigma = r^2 + y^2,$$

$$\tilde{\boldsymbol{g}}_L = -\frac{\tilde{\Delta}_r}{\Sigma}(\boldsymbol{d}\tilde{\tau} + y^2\boldsymbol{d}\tilde{\psi})^2 + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tilde{\tau} - r^2\boldsymbol{d}\tilde{\psi})^2 + \frac{\Sigma}{\tilde{\Delta}_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2. \tag{51}$$

Moreover, the potential $\boldsymbol{b}$ reads

$$\boldsymbol{b} = \frac{1}{2}[(y^2 - r^2)\boldsymbol{d}\tilde{\tau} - r^2y^2\boldsymbol{d}\tilde{\psi}], \tag{52}$$

and generates the principal tensor for both the metric $\boldsymbol{g}_L$ and the metric $\tilde{\boldsymbol{g}}_L$. In other words, one can add to the canonical metric $\tilde{\boldsymbol{g}}_L$ a term $\frac{f}{\Sigma}\boldsymbol{l}\boldsymbol{l}$, with arbitrary $f(r)$, and the metric still admits the same principal tensor. We believe that this observation is new and quite interesting.

Note, however, that this is no longer true at the level of Killing tensors. Namely, $\tilde{\boldsymbol{g}}_L$ admits the following Killing tensor:

$$\tilde{\boldsymbol{k}}_L = \frac{1}{\Sigma}[y^2\tilde{\Delta}_r(\boldsymbol{d}\tilde{\tau} + y^2\boldsymbol{d}\tilde{\psi})^2 + r^2\Delta_y(\boldsymbol{d}\tilde{\tau} - r^2\boldsymbol{d}\tilde{\psi})^2] + \Sigma\left[\frac{r^2\boldsymbol{d}y^2}{\Delta_y} - \frac{y^2\boldsymbol{d}r^2}{\tilde{\Delta}_r}\right]. \tag{53}$$

Whereas, the metric $\boldsymbol{g}_L$ has

$$\boldsymbol{k}_L = \tilde{\boldsymbol{k}}_L - \frac{fy^2}{\Sigma}\boldsymbol{l}\boldsymbol{l}. \tag{54}$$

Note that such a Killing tensor again takes the "Kerr–Schild form."

In particular, we may choose

$$f = 2Mr, \qquad \tilde{\Delta}_r = -c_0 + c_1r^2 - \frac{1}{3}\Lambda r^4,$$
$$\Delta_y = -c_0 - c_1y^2 - \frac{1}{3}\Lambda y^4, \tag{55}$$

in which case the metric $\tilde{\boldsymbol{g}}_L$ describes a pure (A)dS space, while the whole metric $\boldsymbol{g}_L$ is the Kerr–(A)dS spacetime, written as a linear in mass deformation of the (A)dS space [16,17]. In this case, $\tilde{\boldsymbol{k}}_L$ is a reducible Killing tensor of the (A)dS space, and becomes a nonreducible Killing tensor of the Kerr–(A)dS spacetime, $\boldsymbol{k}_L$, upon adding the $-\frac{fy^2}{\Sigma}\boldsymbol{ll}$ term.

B. Other signature

Similarly, starting from the Lorentzian canonical element (50), while performing the Wick rotation (35), repeating the steps (29)–(44), splitting

$$\Delta_z = \tilde{\Delta}_z - g, \tag{56}$$

and finally applying the inverse transformation to (43), we obtain the *double off-shell Kerr–Schild form*

$$\boldsymbol{g}_{LL} = \tilde{\boldsymbol{g}}_{LL} + \frac{f(r)}{\Sigma}\boldsymbol{ll} + \frac{g(z)}{\Sigma}\boldsymbol{mm}, \tag{57}$$

where $\Sigma = r^2 - z^2$, and

$$\begin{aligned}
\boldsymbol{l} &= \boldsymbol{d}\tau - z^2\boldsymbol{d}\psi + \frac{\Sigma}{\tilde{\Delta}_r}\boldsymbol{d}r,\\
\boldsymbol{m} &= \boldsymbol{d}\tau - r^2\boldsymbol{d}\psi + \frac{\Sigma}{\tilde{\Delta}_z}\boldsymbol{d}z,\\
\tilde{\boldsymbol{g}}_{LL} &= -\frac{\tilde{\Delta}_r}{\Sigma}(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi)^2 - \frac{\tilde{\Delta}_z}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2\\
&\quad + \frac{\Sigma}{\tilde{\Delta}_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\tilde{\Delta}_z}\boldsymbol{d}z^2.
\end{aligned} \tag{58}$$

We also have

$$\boldsymbol{b} = \frac{1}{2}[-(z^2 + r^2)\boldsymbol{d}\tau + r^2z^2\boldsymbol{d}\psi] \tag{59}$$

for the potential of the principal tensor of both $\boldsymbol{g}_{LL}$ and $\tilde{\boldsymbol{g}}_{LL}$.

In particular, choosing

$$\begin{aligned}
f &= 2Mr, &\quad \tilde{\Delta}_r &= -c_0 + c_1r^2 - \frac{1}{3}\Lambda r^4,\\
g &= 2\tilde{N}z, &\quad \tilde{\Delta}_z &= c_0 - c_1z^2 + \frac{1}{3}\Lambda z^4,
\end{aligned} \tag{60}$$

we recover the special case of the on-shell multi-Kerr–Schild form discussed in all dimensions in [18]. Note that the Kerr–NUT–(a)dS metric is written here as a linear in "mass" and linear in "NUT charge" deformation of the (A) dS space.

Another interesting choice is given by

$$\begin{aligned}
f &= \frac{1}{3}\Lambda r^4, &\quad \tilde{\Delta}_r &= -c_0 + c_1r^2 - 2Mr,\\
g &= -\frac{1}{3}\Lambda z^4, &\quad \tilde{\Delta}_z &= c_0 - c_1z^2 - 2\tilde{N}z.
\end{aligned} \tag{61}$$

This means that the Kerr–NUT–(A)dS metric can also be understood as a linear in Λ deformation of the Kerr–NUT metric, perhaps an interesting observation unnoticed in the literature. Of course, this result may be combined with the above special case, to get Kerr–NUT–(A)dS as a linear deformation of flat space in all: mass, NUT, and Λ.

Similarly, starting from (37), we arrive at the following off-shell Kerr–Schild metric:

$$\boldsymbol{g}_{LL'} = \tilde{\boldsymbol{g}}_{LL'} + \frac{g(z)}{\Sigma}\boldsymbol{mm}, \tag{62}$$

where $\Sigma = r^2 - z^2$ and

$$\begin{aligned}
\boldsymbol{m} &= \boldsymbol{d}\tau - r^2\boldsymbol{d}\psi + \frac{\Sigma}{\tilde{\Delta}_z}dz,\\
\tilde{\boldsymbol{g}}_{LL'} &= \frac{\Sigma}{\tilde{\Delta}_z}\boldsymbol{d}z^2 - \frac{\tilde{\Delta}_z}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2\\
&\quad + 2\boldsymbol{d}r(\boldsymbol{d}\tau - z^2\boldsymbol{d}\psi).
\end{aligned} \tag{63}$$

This metric also admits the principal tensor, given by the potential (59).

To summarize, we have seen that the procedures for obtaining the new canonical metric element and that for obtaining the Kerr–Schild form are quite similar. Namely, the key ingredient is to write the original metric in terms of a null vector, and perform a coordinate transformation so that the metric is linear in the associated metric function Δ. The Kerr–Schild form then corresponds to the appropriate split $\Delta = \tilde{\Delta} - f$, whereas the null limit amounts to setting $\Delta \to 0$.

IV. HIGHER-DIMENSIONAL GENERALIZATIONS

It is obvious that the above presented four-dimensional results can be straightforwardly generalized to higher dimensions. Namely, starting from the Euclidean metric elements (7)–(9), one can perform up to n Wick rotations of the eigenvalues x_μ and take their "null limit," to produce a family of new canonical metrics of various signatures. Also the trick for obtaining the off-shell multi-Kerr–Schild form works as expected.

In what follows, let us limit ourselves to the case of Lorentzian signature and write the two canonical elements and the corresponding off-shell Kerr–Schild form. We first illustrate on a five-dimensional example and then proceed to a general dimension.

A. 5d case

Let us start from the 5-dimensional Euclidean metrics (8), (9), denoting by $(x, y, \tau, \psi, \phi) = (x_1, x_2, \psi_0, \psi_1, \psi_2)$ and $X = X_1, Y = X_2$. To obtain the Lorentzian version we perform the Wick rotation (21). This yields

$$\begin{aligned} \boldsymbol{g}_L^{(1)} = & -\frac{\Delta_r}{\Sigma}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 \\ & + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 \\ & - \frac{c}{r^2y^2}(\boldsymbol{d}\tau + (y^2 - r^2)\boldsymbol{d}\psi - r^2y^2\boldsymbol{d}\phi)^2, \end{aligned} \tag{64}$$

respectively,

$$\begin{aligned} \boldsymbol{g}_L^{(2)} = & -\frac{\Delta_r}{\Sigma}(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2 \\ & + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 - cr^2y^2\boldsymbol{d}\phi^2, \end{aligned} \tag{65}$$

where $\Sigma = r^2 + y^2$, and assuming $c < 0$, $\Delta_y > 0$. The principal tensor is given by the potential

$$\boldsymbol{b} = \frac{1}{2}[(y^2 - r^2)\boldsymbol{d}\tau - r^2y^2\boldsymbol{d}\psi], \tag{66}$$

both its eigenvalues $\{r, y\}$ are spacelike.

The metric becomes a vacuum solution provided we set

$$\begin{aligned} \Delta_r^{(1)} &= -\frac{c}{r^2} + c_1r^2 - \frac{1}{6}\Lambda r^4 + 2M, \\ \Delta_y^{(1)} &= \frac{c}{y^2} - c_1y^2 - \frac{1}{6}\Lambda y^4 + 2N, \end{aligned} \tag{67}$$

respectively

$$\begin{aligned} \Delta_r^{(2)} &= c_1r^2 - \frac{1}{6}\Lambda r^4 + 2M, \\ \Delta_y^{(2)} &= c_1y^2 - \frac{1}{6}\Lambda y^4 + 2N, \end{aligned} \tag{68}$$

where we have denoted $M = b_x$ and $N = b_y$.

Note that

$$\boldsymbol{g}_L^{(1,2)} + \frac{f(r)}{\Sigma}\boldsymbol{l}\boldsymbol{l}, \qquad \boldsymbol{l} = \boldsymbol{d}\tau + y^2\boldsymbol{d}\psi + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r, \tag{69}$$

where $f(r)$ is an arbitrary function, are again the off-shell Kerr–Schild metrics with the same principal tensor (66).

By performing the transformation

$$\boldsymbol{d}\tau = \boldsymbol{d}\hat{\tau} - \frac{r^2}{\Delta_r}\boldsymbol{d}r, \qquad \boldsymbol{d}\psi = \boldsymbol{d}\hat{\psi} - \frac{\boldsymbol{d}r}{\Delta_r}, \tag{70}$$

complemented with

$$\boldsymbol{d}\phi = \boldsymbol{d}\hat{\phi} - \frac{1}{r^2\Delta_r}\boldsymbol{d}r, \tag{71}$$

for the metric (64) and

$$\boldsymbol{d}\phi = \boldsymbol{d}\hat{\phi}, \tag{72}$$

for the metric (65), we can take the limit $\Delta_r \to 0$ to obtain

$$\begin{aligned} \boldsymbol{g}_{L'}^{(1)} = & \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi})^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 + 2\boldsymbol{d}r(\boldsymbol{d}\hat{\tau} + y^2\boldsymbol{d}\hat{\psi}) \\ & - \frac{c}{r^2y^2}(\boldsymbol{d}\hat{\tau} + (y^2 - r^2)\boldsymbol{d}\hat{\psi} - r^2y^2\boldsymbol{d}\hat{\phi})^2, \end{aligned} \tag{73}$$

$$\begin{aligned} \boldsymbol{g}_{L'}^{(2)} = & \frac{\Delta_y}{\Sigma}(\boldsymbol{d}\hat{\tau} - r^2\boldsymbol{d}\hat{\psi})^2 + \frac{\Sigma}{\Delta_y}\boldsymbol{d}y^2 \\ & + 2\boldsymbol{d}r(\boldsymbol{d}\hat{\tau} + y^2\boldsymbol{d}\hat{\psi}) - cr^2y^2\boldsymbol{d}\hat{\phi}^2, \end{aligned} \tag{74}$$

respectively, both admitting the same principal tensor (66). The first metric, $\boldsymbol{g}_L^{(1)}$ does not solve the vacuum equations for any choice of $\Delta_y^{(1)}$, whereas the second one is a solution provided we set

$$\Delta_y^{(2)} = 2N. \tag{75}$$

B. General dimension

To write the Lorentzian canonical metrics in a general dimension, we perform the following Wick rotation:

$$x_n = ir, \qquad X_n = -\Delta, \qquad U_n = \Sigma, \tag{76}$$

leaving all other X_μ's (contrary to the previous sections) unchanged. We also understand all functions having x_n-dependence replaced by ir, for example,

$$\Sigma = \prod_{\nu=1}^{n-1}(x_\nu^2 + r^2). \tag{77}$$

The Lorentzian canonical element in even dimensions, (7), then reads

$$\begin{aligned} \boldsymbol{g}_{L2n} = & -\frac{\Delta}{\Sigma}\left(\sum_{j=0}^{n-1}A_n^{(j)}\boldsymbol{d}\psi_j\right)^2 + \frac{\Sigma}{\Delta}\boldsymbol{d}r^2 \\ & + \sum_{\mu=1}^{n-1}\left[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{j=0}^{n-1}A_\mu^{(j)}\boldsymbol{d}\psi_j\right)^2\right]. \end{aligned} \tag{78}$$

In odd dimensions, corresponding to (8) and (9), we have

$$\boldsymbol{g}_{L2n+1}^{(1)} = \boldsymbol{g}_{L2n} + \frac{c}{A^{(n)}}\left(\sum_{k=0}^{n}A^{(k)}\boldsymbol{d}\psi_k\right)^2, \tag{79}$$

$$\boldsymbol{g}_{L2n+1}^{(2)} = \boldsymbol{g}_{L2n} + cA^{(n)}\boldsymbol{d}\psi_n^2. \tag{80}$$

All of them admit the principal tensor given by (14),

$$\boldsymbol{b} = \frac{1}{2}\sum_{k=0}^{n-1} A^{(k+1)} \boldsymbol{d}\psi_k. \tag{81}$$

The metrics become vacuum solutions provided we set

$$\begin{aligned}
\text{even}\, D\colon\ \Delta &= -\sum_{k=0}^{n} c_k(-r^2)^k - 2Mr,\\
\text{odd}\, D\colon\ \Delta^{(1)} &= -\sum_{k=1}^{n} c_k(-r^2)^k + 2M - \frac{c}{r^2},\\
\Delta^{(2)} &= -\sum_{k=1}^{n} c_k(-r^2)^k + 2M,
\end{aligned} \tag{82}$$

while other X_μ's ($\mu = 1, \ldots, n-1$) are given in (16).

As in the lower-dimensional cases, we introduce a null vector

$$\boldsymbol{l} = \sum_{j=0}^{n-1} A_n^{(j)} \boldsymbol{d}\psi_j + \frac{\Sigma}{\Delta}\boldsymbol{d}r, \tag{83}$$

and change coordinates as

$$\boldsymbol{d}\hat{\psi}_j = \boldsymbol{d}\psi_j + \frac{r^{2(n-1-j)}}{\Delta}\boldsymbol{d}r, \tag{84}$$

with the expression for $\hat{\psi}_n$ modified to just $\hat{\psi}_n = \psi_n$ in the second odd-dimensional case (80). Here, $\hat{\tau} \equiv \hat{\psi}_0$ plays a role of time coordinate. Such a change satisfies $\sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\psi_j = \sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\hat{\psi}_j$ for $\mu = 1, \ldots, n-1$, $\sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\psi_k = \sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\hat{\psi}_k$, and $\boldsymbol{l}$ simplifies to

$$\boldsymbol{l} = \sum_{j=0}^{n-1} A_n^{(j)} \boldsymbol{d}\hat{\psi}_j. \tag{85}$$

Here we have extensively used the identity

$$\sum_{j=0}^{n-1} A_\mu^{(j)} \frac{(-x_\nu^2)^{n-1-j}}{U_\nu} = \delta_\mu^\nu. \tag{86}$$

The metric takes the form

$$\begin{aligned}
\boldsymbol{g}_{L2n} &= -\frac{\Delta}{\Sigma}\boldsymbol{l}\boldsymbol{l} + 2\boldsymbol{l}\boldsymbol{d}r\\
&\quad + \sum_{\mu=1}^{n-1}\left[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\hat{\psi}_j\right)^2\right],
\end{aligned} \tag{87}$$

supplemented by (79) or (80) with $\hat{\psi}_j$ instead of ψ_j in the odd-dimensional case.

Clearly, splitting the metric function $\Delta = \tilde{\Delta} - f(r)$ yields the off-shell Kerr–Schild form

$$\boldsymbol{g}_L = \tilde{\boldsymbol{g}}_L + \frac{f(r)}{\Sigma}\boldsymbol{l}\boldsymbol{l}, \tag{88}$$

where $\tilde{\boldsymbol{g}}_L$ is given by (78)–(80) with Δ replaced by $\tilde{\Delta}$. Both $\boldsymbol{g}_L$ and $\tilde{\boldsymbol{g}}_L$ admit the same principal tensor. (If other signatures were considered, we would recover the off-shell multi-Kerr–Schild form, generalizing the results in [18].)

In the metric (87) we can easily perform the limit $\Delta \to 0$, making the eigenvalue r null, and obtaining the new canonical elements:

$$\begin{aligned}
\boldsymbol{g}_{L'2n} &= 2\boldsymbol{d}r\left(\sum_{j=0}^{n-1} A_n^{(j)}\boldsymbol{d}\hat{\psi}_j\right)\\
&\quad + \sum_{\mu=1}^{n-1}\left[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\hat{\psi}_j\right)^2\right],
\end{aligned} \tag{89}$$

and

$$\boldsymbol{g}^{(1)}_{L'2n+1} = \tilde{\boldsymbol{g}}_{L2n} + \frac{c}{A^{(n)}}\left(\sum_{k=0}^{n} A^{(k)}\boldsymbol{d}\hat{\psi}_k\right)^2, \tag{90}$$

$$\boldsymbol{g}^{(2)}_{L'2n+1} = \tilde{\boldsymbol{g}}_{L2n} + cA^{(n)}\boldsymbol{d}\hat{\psi}_n^2. \tag{91}$$

These metrics become the vacuum solutions of Einstein equations (with $\Lambda = 0$) provided we set

$$\begin{aligned}
\text{even}\, D\colon\ X_\mu &= -2b_\mu x_\mu,\\
\text{odd}\, D\colon\ X_\mu^{(2)} &= -2b_\mu x_\mu,
\end{aligned} \tag{92}$$

whereas there is no solution for $X_\mu^{(1)}$.

Let us finally note that by introducing the following veilbein ($\mu = 1, \ldots, n-1$):

$$\begin{aligned}
&\boldsymbol{k} = \boldsymbol{d}r, \qquad \boldsymbol{l} = \sum_{j=0}^{n-1} A_1^{(j)}\boldsymbol{d}\hat{\psi}_j,\\
&\boldsymbol{e}^\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}}\boldsymbol{d}x_\mu, \qquad \hat{\boldsymbol{e}}^\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}}\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\hat{\psi}_j,\\
&\hat{\boldsymbol{e}}^0_{(1)} = \left(\frac{c}{A^{(n)}}\right)^{\frac{1}{2}}\sum_{k=0}^{n} A^{(k)}\boldsymbol{d}\hat{\psi}_k, \qquad \hat{\boldsymbol{e}}^0_{(2)} = \sqrt{cA^{(n)}}\boldsymbol{d}\hat{\psi}_n,
\end{aligned} \tag{93}$$

where $\boldsymbol{k}$ and $\boldsymbol{l}$ are null, with $\boldsymbol{k}\cdot\boldsymbol{l} = 1$, the new metrics (89)–(91), together with their principal tensor can be written as

$$\boldsymbol{h} = -r\,\boldsymbol{k}\wedge\boldsymbol{l} + \sum_{\mu=1}^{n-1} x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu, \tag{94}$$

$$\boldsymbol{g}_{L'} = 2\,\boldsymbol{k}\boldsymbol{l} + \sum_{\mu=2}^{n}(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu) + \varepsilon\,\hat{\boldsymbol{e}}^0\hat{\boldsymbol{e}}^0, \tag{95}$$

which is a "null Lorentzian version" of the Darboux frame (5).

V. CONCLUSIONS

It has been believed that the classification of metrics admitting the principal tensor, that is a *nondegenerate* closed conformal Killing–Yano 2-form, is completed and uniquely leads to the off-shell Kerr–NUT–(A)dS metric (reviewed in the Introduction). However, as shown in this paper, this is not true unless the signature of the metric is Euclidean. The original classification assumed the Euclidean signature for the metric and the canonical Darboux form for the principal tensor, see Eq. (5). Consequently, also the eigenvalues of the principal tensor were assumed to have spacelike character.

In this paper we have shown how to construct new canonical elements whose characteristic feature is that one or more of the eigenvalues of the principal tensor are null. Among these, the new Lorentzian canonical elements (89)–(91) are perhaps of the biggest interest, generalizing the 4-dimensional metric element constructed in [12,13]. These metrics can formally be constructed by a procedure similar to obtaining the Kerr–Schild form. Namely, one introduces a null vector and performs a coordinate transformation such that the metric becomes linear in the corresponding metric function. Such a metric function can then be sent to zero, making the associated eigenvalue null, while other functions remain unspecified. Going back to the canonical coordinates recovers the new metric elements (89)–(91). This is a clever trick for how to switch off one of the metric functions without making the metric singular. In this sense, the new metrics can be considered as a "special case" of the off-shell Kerr–NUT–(A)dS metric.

The presented results reopen the problem of classifying the metrics admitting the principal tensor. Of special importance to physics are of course the metrics with Lorentzian signature. While in this paper we uncovered some new such metrics, the classification is far for complete. In this paper we simply concentrated on the metrics for which the principal tensor takes the "null Lorentzian Darboux form" (95), with r a null eigenvalue. However, in the Lorentzian signature there is many more possibilities for the canonic form of a non-degenerate 2-form, see e.g., [19]. For this reason the problem of classifying all Lorentzian metrics admitting the principal tensor still remains open and will be discussed elsewhere [20].

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