Analysis of the Yukawa gravitational potential in $f(R)$ gravity. II. Relativistic periastron advance

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Alternative theories of gravity may serve to overcome several shortcomings of the standard cosmological model but, in their weak field limit, general relativity must be recovered so as to match the tight constraints at the Solar System scale. Therefore, testing such alternative models at scales of stellar systems could give a unique opportunity to confirm or rule them out. One of the most straightforward modifications is represented by analytical $f(R)$-gravity models that introduce a Yukawa-like modification to the Newtonian potential thus modifying the dynamics of particles. Using the geodesics equations, we have illustrated the amplitude of these modifications. First, we have integrated numerically the equations of motion showing the orbital precession of a particle around a massive object. Second, we have computed an analytic expression for the periastron advance of systems having their semimajor axis much shorter than the Yukawa-scale length. Finally, we have extended our results to the case of a binary system composed of two massive objects. Our analysis provides a powerful tool to obtain constraints on the underlying theory of gravity using current and forthcoming data sets.

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I. INTRODUCTION

Does general relativity (GR) need to be modified to overcome the shortcomings at ultraviolet and infrared scales? This is one of the fundamental questions that still needs to be answered. As it is well known, GR is very well established on the Solar System scale [1–3], and forms the basis of the *concordance* cosmological model. Although, in the last decades many observational data sets have emerged confirming the model further [4–15], some shortcomings have brought questions about whether GR is the true effective theory of gravity. First, GR is not a quantum theory and it cannot provide a description of the Universe at quantum scales [16,17]. Second, GR cannot explain the emergence of the large scale structure and the accelerated expansion of the Universe without adding two extra components to the total energy density budget, namely dark matter (DM) and dark energy (DE). The dynamical effects of these two components are evident at both galactic/extragalactic and cosmological scales, but their fundamental nature, whether particles or scalar fields, is completely unknown [18–26]. These problems have been interpreted as a breakdown of GR, and many alternative theories of gravity have been proposed [27–32]. In brief, there are two possible approaches to describe all observational data sets from planetary to cosmological scales: the first is to preserve GR by adding extra particles and/or scalar fields; the second is to modify the geometrical description of the space-time. Both must be tested in all possible astronomical scenarios in order to understand at which scales their contributions become significant. Let us note that some of these modified theories have been ruled out using the recent discovery of the electromagnetic counterpart associated to the emission of the gravitational waves [33–40]. Such a discovery opens new avenues to test modified theories of gravity further, and those tracks must be explored.

The simplest prescription to modify GR is to generalize the Einstein-Hilbert Lagrangian to an arbitrary function of the Ricci scalar, $f(R)$. Then, one should take care of the fact that, in the weak field limit, any alternative relativistic theory of gravity must reproduce GR in order to recover the tight constraints at the Solar System scale [1–3]. Here, we are interested in the post-Newtonian limit to describe the motion of test-particles (and more in general, of a system).

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In models where unknown particles/scalar fields are added to GR, in order to recover the Solar System bounds, one has to require that such scalar fields are screened in a high density environment. However, these mechanisms are imposed *ad-hoc* to avoid that scalar fields dominate the dynamics of small scale systems. In the case of $f(R)$-gravity the gravitational potential is modified by a Yukawa-like term related to a new characteristic scale length of the system that appears because one has to solve forth (instead of second), order field equations, and this new scale length can act automatically as a screening mechanism [41].

Some of the most promising objects to test the underlying theory of gravity are pulsars. These objects are very dense and rapidly rotating (up to hundred times per second) neutron stars emitting gamma radiation beams or X-rays. They act as a very precise clock and any deviation in their pulse from the one predicted by GR can be detected. These deviations can be related to the violation of the strong equivalence principle and the variation of the gravitational constant. Both circumstances have been investigated using binary systems composed by a pulsar and another massive object (such as a neutron star or a white dwarf) that produces these anomalies in the pulse [42]. Anyway, these deviations can also be interpreted as a signature of an alternative theory of gravity [43–48]. Forthcoming observations will increase the current point source sensitivity and resolution by combining different facilities such as large telescopes apertures, adaptive optics, and near infrared (NIR) interferometry, and they will allow to detect pulsars with orbital period in scales as low as one year. Therefore, the measure of the periastron shift will became one of the most promising tools to test GR and alternative theories [49–52]. The most rigorous test of alternative theories would be provided by a pulsar orbiting near a supermassive black hole (SMBH) [53]. In such a case, we would not only expect the largest deviations from GR, but we could also measure the properties of the black hole (BH). A pulsar-BH system has not been found yet, but the prospects of finding one such can increase enormously within the curved space-time around Sagittarius A* (Sgr A*), the SMBH at the center of the Milky Way [54,55].

In order to be measurable with current instruments, pulsars with short orbital periods would need to be discovered, such pulsars would orbit at distances inside a 10 AU radius circle centered at Sgr A*. In particular, an ideal pulsar would be one spinning a few hundred times per second. Searches are currently undergoing with the BlackHoleCam[1] and Event Horizon Telescope (EHT) Collaboration[2] [54,56–66]. EHT is a project to create a large telescope array consisting of a global network of radio telescopes and combining data from several Very-Long-Baseline Interferometry (VLBI) stations around the Earth. The aim is to observe the immediate environment of the Galactic Center, as well as the even larger BH in Messier 87 (M87), with angular resolution comparable to the BH's event horizon [60]. These facilities, together with current and forthcoming Pulsar Timing Array (PTA) observatories [67], will give us a unique opportunity to test alternative theories of gravity using the orbital motion of a test particle around a massive object as well as the motion of a binary system. Hereby, we are indeed currently building the theoretical facilities needed to test $f(R)$-gravity.

The aim is to demonstrate the capability of the Yukawa-like gravitational potential of explaining the dynamics of the particles at the Galactic center. The study of the periastron shift is complementary to other studies on the time variation of the orbital period in $f(R)$ gravity that have been used to constrain the graviton mass [44,45]. Although the periastron shift has been studied in a sort of semiclassical approach where the Yukawa-potential has been considered to describe the gravitational force in the Newtonian classical dynamics [68–72], the full relativistic approach is needed to take into account the geodesic structure of the space-time, and to investigate how particles dynamics is affected. The systems that we will examine are somewhat idealized, compared to real astrophysical sources. For example, we neglect tidal effects that become important only when the mean separation of the two objects is of the order of their radius. This allow us to understand the essence of the physical mechanism with minimal complications, and to form the basis for a more detailed study of realistic sources in alternative theories of gravity. The paper is divided as follows: in Sec. II we briefly review the post-Newtonian limit of an analytic $f(R)$ model showing how the Yukawa-like gravitational potential arises; in Sec. III, we introduce the geodesic motion in $f(R)$ gravity computing the geodesic equation and the canonical momenta; in Sec. IV, we solve numerically the geodesic equation illustrating the effect of the Yukawa-potential on the orbital precession; in Sec. V, we compute an analytic formula for the periastron advance and apply it to toy models; finally in Sec. VI, we give our conclusion and remarks.

II. POST-NEWTONIAN LIMIT AND YUKAWA-LIKE GRAVITATIONAL POTENTIALS

Here we summarize the main steps that lead to the modification of the gravitational potential in the post-Newtonian limit of the $f(R)$-gravity. The natural starting point is to consider a general fourth order gravity action:

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m], \tag{1}$$

where $f(R)$ is an analytic function of Ricci scalar, g is the determinant of the metric $g_{\mu\nu}$, $\mathcal{X} = 16\pi G/c^4$ is the coupling constant and $\mathcal{L}_m$ describes the standard fluid-matter

[1] https://blackholecam.org.
[2] http://www.eventhorizontelescope.org.

Lagrangian. For $f(R) = R$, the Hilbert-Einstein action of GR is restored.

Varying the action in Eq. (1) with respect to the metric tensor we obtain the following field equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) = \frac{\mathcal{X}}{2}T_{\mu\nu}, \quad (2)$$

and their trace

$$3\Box f'(R) + f'(R)R - 2f(R) = \frac{\mathcal{X}}{2}T. \quad (3)$$

Here primes indicate derivatives with respect to the Ricci curvature, and $\Box$ is the usual d'Alembert operator. The next step is the fairly common practice to make a conformal transformation to pass from the Jordan frame to the Einstein frame, in which the field equations are reduced from fourth order partial differential equations to second order ones, and a scalar field arises from the extra degrees of freedom. On the one hand, this operation simplifies the calculations and requires to introduce a mechanism to screen the scalar field in high density environments (short distances) [73–75]. On the other hand, the two frame are mathematically equivalent but their physical equivalence is, nowadays, under debate [28,76,77]. To be sure of the physical equivalence one should reproduce the results in both frames and compare them. The alternative is to stay in Jordan frame accepting the idea of having to handle with the fourth order field equations in Eq. (2), and regarding to the extra degrees of freedom of the theory as free parameters to be constrained with the data. This approach avoids the need of introducing a screening mechanism because of the scale dependence of the theory. Thus, hereafter, all calculations will be performed in the Jordan frame.

Following [78,79], the post-Newtonian (PN) limit of $f(R)$ gravity can be computed assuming a general spherically symmetric metric:

$$ds^2 = g_{tt}(x^0, r)dx^{02} - g_{rr}(x^0, r)dr^2 - r^2 d\Omega^2, \quad (4)$$

where $x^0 = ct$ and $d\Omega^2$ is the solid angle. For the sake of simplicity, following [41] we set $c = 1$ (it will be restored in the next sections). Then, let us add perturbations of the metric tensor with respect to a Minkowskian background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and assume an $f(R)$ Lagrangian expandable in Taylor series:

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n$$
$$\simeq f_0 + f'_0 R + f''_0 R^2 + f'''_0 R^3 + \cdots. \quad (5)$$

Inserting the Eq. (5) into field equations (2)–(3) and expanding them up to orders $\mathcal{O}(0)$, $\mathcal{O}(2)$ and $\mathcal{O}(4)$, one obtains

$$\begin{aligned}
&f'_0 rR^{(2)} - 2f'_0 g^{(2)}_{tt,r} + 8f''_0 R^{(2)}_{,r} - f'_0 r g^{(2)}_{tt,rr} + 4f''_0 rR^{(2)} = 0,\\
&f'_0 rR^{(2)} - 2f'_0 g^{(2)}_{rr,r} + 8f''_0 R^{(2)}_{,r} - f'_0 r g^{(2)}_{tt,rr} = 0,\\
&2f'_0 g^{(2)}_{rr} - r[f'_0 rR^{(2)} - f'_0 g^{(2)}_{tt,r} - f'_0 g^{(2)}_{rr,r}\\
&\quad + 4f''_0 R^{(2)}_{,r} + 4f''_0 rR^{(2)}_{,rr}] = 0,\\
&f'_0 rR^{(2)} + 6f''_0[2R^{(2)}_{,r} + rR^{(2)}_{,rr}] = 0,\\
&2g^{(2)}_{rr} + r[2g^{(2)}_{tt,r} - rR^{(2)} + 2g^{(2)}_{rr,r} + rg^{(2)}_{tt,rr}] = 0. \qquad (6)
\end{aligned}$$

Using the trace equation [the fourth in system (6)], one gets the following general solution:

$$g^{(2)}_{tt} = \delta_0 - \frac{\delta_1}{f'_0 r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{3} + \frac{\delta_3(t)\lambda^3 e^{r/\lambda}}{6r}, \quad (7)$$

$$g^{(2)}_{rr} = -\frac{\delta_1}{f'_0 r} - \frac{\delta_2(t)\lambda^2(1 + r/\lambda)e^{-r/\lambda}}{3r} + \frac{\delta_3(t)\lambda^3(1 - r/\lambda)e^{r/\lambda}}{6r}, \quad (8)$$

$$R^{(2)} = \delta_2(t)\frac{e^{-r/\lambda}}{r} + \frac{\delta_3(t)\lambda e^{r/\lambda}}{2r}, \quad (9)$$

where $\lambda \doteq \sqrt{-6f''_0/f'_0}$, the constant δ_0 can be neglected, the δ_1 is an arbitrary constant, and $\delta_2(t)$ and $\delta_3(t)$ are completely arbitrary functions of time which, since the differential equations in the system (6) contain only spatial derivatives, can be fixed to constant values. Let us note that on the limit $f(R) \to R$, for a pointlike mass M, we recover the standard weak field limit when $\delta_1 = GM$. Finally, requiring that the metric must be asymptotically flat [Yukawa growing mode in the system of Eqs. (7)–(9) are discarded] one obtains

$$g_{tt}(x^0, r) = 1 - \frac{GM}{f'_0 r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{3}, \quad (10)$$

$$g_{rr}(x^0, r) = 1 + \frac{GM}{f'_0 r} + \frac{\delta_2(t)\lambda^2(1 + r/\lambda)e^{-r/\lambda}}{3r}, \quad (11)$$

$$R = \frac{\delta_2(t)e^{-r/\lambda}}{r}. \quad (12)$$

The metric in Eqs. (10) and (11) also contains the solution of the modified gravitational potential. Specifically, remembering that $g_{00} = 1 + 2\Phi_{\mathrm{grav}} = 1 + g^{(2)}_{tt}$ [80], one can extract the expression for the gravitational potential in $f(R)$-gravity:

$$\Phi = -\frac{GM}{f'_0 r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{6r}. \quad (13)$$

Let notice that the standard Newtonian potential is recovered only in the particular case $f(R) = R$ while it is not so for generic analytic $f(R)$ models. Equation (13) can be straightforwardly recast as (for more details see [32,41])

$$\Phi(r) = -\frac{GM}{(1+\delta)r}(1+\delta e^{-\frac{r}{\lambda}}), \tag{14}$$

by defining $1+\delta = f_0'$, and assuming that δ_1 is quasiconstant, and it is related to δ as follows through

$$\delta_2 = -\frac{6GM}{\lambda^2}\frac{\delta}{1+\delta}. \tag{15}$$

Equation (14) deserves some comments. If $\delta = 0$ then the Newtonian potential is recovered. Next, the first term is the Newtonian potential generated by a pointlike mass $\frac{M}{1+\delta}$. And, the second term is the Yukawa-like modification of the gravitational potential with a scale length, λ, related to the above coefficient of the Taylor expansion of the gravitational Lagrangian. The parameter λ naturally arises from the theory, and acts as a screening mechanism. It makes the Yukawa correction be negligible at small scales while relevant at galactic, extragalactic and cosmological scales providing a possible way to explain galaxy rotation curves, cluster of galaxies and the accelerated expansion of the Universe without requiring dark matter and/or dark energy [27,28,81–85].

Understanding the amplitude of these corrections to the gravitational potential at the scale of the stellar systems is one of the most important tools that could be used to observationally confirm or rule out these alternative approaches to GR.

III. GEODESIC MOTION IN F(R)-GRAVITY

Let us apply the Euler-Lagrange equations to find the geodesics equations of motion associated to the line element given in Eqs. (10) and (11). After some manipulations, they can be recast into the following form

$$ds^2 = [1+\Phi(r)]dt^2 - [1-\Psi(r)]dr^2 - r^2 d\Omega, \tag{16}$$

where the two potentials $\Phi(r)$ and $\Psi(r)$ are given by

$$\Phi(r) = -\frac{2GM(\delta e^{-\frac{r}{\lambda}}+1)}{rc^2(\delta+1)}, \tag{17}$$

$$\Psi(r) = \frac{2GM}{rc^2}\left[\frac{(\delta e^{-\frac{r}{\lambda}}+1)}{(\delta+1)} + \frac{(\frac{\delta r e^{-\frac{r}{\lambda}}}{\lambda}-2)}{(\delta+1)}\right], \tag{18}$$

with the speed of light having been reinstated. Note that the potential $\Psi(r)$ can be rewritten as

$$\Psi(r) = \Phi(r) + \delta\Phi(r), \tag{19}$$

where the term $\delta\Phi(r)$ representing an extra contribution to the total gravitational potential. Since we are interested in

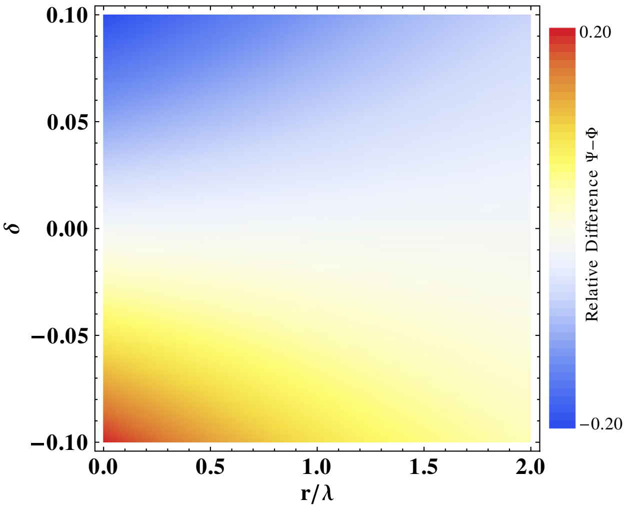

FIG. 1. Relative difference of the two gravitational fields $\Phi(r)$ and $\Psi(r)$ as a function of the parameters of the strength and the scale length of the Yukawa term in Eq. (14). Here, we have used $G = M = c = 1$.

small scale systems,[3] we have verified whether such contribution is negligible or not. In Fig. 1, we show the region plot of the ratio $(\Psi(r) - \Phi(r))/\Phi(r)$. Since such ratio is almost insensitive to the scale length λ, the latter has been kept fixed to the confidence value of 5000 AU [68–70,86]. The color bar on the figure indicates the relative change of the two potentials. We have varied δ from −0.1 to 0.1 showing that the departure of $\Psi(r)$ from $\Phi(r)$ is ~20% for $\delta = \pm 0.1$, while it decreases to ~2% for $\delta \sim \pm 0.01$. To explain binary systems in the framework of $f(R)$ gravity, we need very small departure from GR, which means $|\delta| \ll 0.1$ [44,45]. Thus, hereafter, we will assume $\Psi(r) \sim \Phi(r)$.

Thus, the line element in Eq. (16) becomes

$$ds^2 = [1+\Phi(r)]dt^2 - [1-\Phi(r)]dr^2 - r^2 d\Omega. \tag{20}$$

To compute the geodesic equations, we use the Euler-Lagrange equations:

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \tag{21}$$

that are equivalent to the geodesic equations [80]

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0. \tag{22}$$

For the line element in Eq. (20), the nonzero Levi-Civita connections are

$$\Gamma^1_{11} = -\frac{R_S[(e^{\frac{r}{\lambda}}+\delta)\lambda + \delta r]}{\lambda r[2R_S(e^{\frac{r}{\lambda}}+\delta) + e^{\frac{r}{\lambda}}(1+\delta)r]}, \tag{23}$$

[3] Here "small scales" means stellar system scales.

$$\Gamma^1_{22} = -\frac{e^{\frac{r}{\lambda}}(1+\delta)r^2}{2R_S(e^{\frac{r}{\lambda}}+\delta)+e^{\frac{r}{\lambda}}(1+\delta)r}, \tag{24}$$

$$\Gamma^1_{33} = -\frac{e^{\frac{r}{\lambda}}(1+\delta)r^2\sin^2\theta}{2R_S(e^{\frac{r}{\lambda}}+\delta)+e^{\frac{r}{\lambda}}(1+\delta)r}, \tag{25}$$

$$\Gamma^1_{00} = \frac{R_S[(e^{\frac{r}{\lambda}}+\delta)\lambda+\delta r]}{\lambda r[2R_S(e^{\frac{r}{\lambda}}+\delta)+e^{\frac{r}{\lambda}}(1+\delta)r]}, \tag{26}$$

$$\Gamma^2_{21} = \frac{1}{r}, \tag{27}$$

$$\Gamma^2_{33} = -\cos\theta\sin\theta, \tag{28}$$

$$\Gamma^3_{31} = \frac{1}{r}, \tag{29}$$

$$\Gamma^3_{32} = \cot\theta, \tag{30}$$

$$\Gamma^0_{01} = \frac{R_S[(e^{\frac{r}{\lambda}}+\delta)\lambda+\delta r]}{\lambda r[e^{\frac{r}{\lambda}}(1+\delta)r-2R_S(e^{\frac{r}{\lambda}}+\delta)]}. \tag{31}$$

Here, we have introduced the definition of the general relativistic Schwarzschild radius: $R_S = GM/c^2$ and we have eliminated the proper time. Finally, the geodesics equations are

$$\ddot{r} = \Delta^{-1}[R_S(\dot{r}^2-\dot{t}^2)(\delta(\lambda+r)+e^{\frac{r}{\lambda}}\lambda) + e^{\frac{r}{\lambda}}\lambda(1+\delta)r^3(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)], \tag{32}$$

$$\ddot{\theta} = \cos\theta\sin\theta\dot{\phi}^2 - \frac{2\dot{r}\,\dot{\theta}}{r}, \tag{33}$$

$$\ddot{\phi} = -\frac{2\dot{\phi}}{r}[\dot{r}+\cot\theta r\dot{\theta}], \tag{34}$$

$$\ddot{t} = \Delta^{-1}[2R_S[(e^{\frac{r}{\lambda}}+\delta)\lambda+\delta r]\dot{r}\,\dot{t}], \tag{35}$$

where, for the sake of convenience, we have defined

$$\Delta \equiv \lambda r[2R_S\delta + e^{\frac{r}{\lambda}}(2R_S-(1+\delta)r)]. \tag{36}$$

The above equations can be integrated numerically to obtain the orbital motion and precession of a two-body system. Although this represents a powerful tool to study the orbital motion of the stars around a massive object, such as the S-stars around the SMBH at the center of the Milky way galaxy, an analytical solution to predict the periastron advance would be more convenient for studies of binary systems of neutron stars and/or white dwarfs. To this aim, we must define the Lagrangian associated to the metric elements of Eq. (20)

$$2\mathcal{L} = [1+\Phi(r)]\dot{t}^2 - [1-\Phi(r)]\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2. \tag{37}$$

Then, the canonical momenta are

$$p_t \equiv \frac{\partial\mathcal{L}}{\partial\dot{t}} = [1+\Phi(r)]\dot{t}, \tag{38}$$

$$p_r \equiv \frac{\partial\mathcal{L}}{\partial\dot{r}} = -[1-\Phi(r)]\dot{r}, \tag{39}$$

$$p_\theta \equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -r^2\dot{\theta}, \tag{40}$$

$$p_\phi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -r^2\sin^2\theta\dot{\phi}. \tag{41}$$

Next, if we write the Euler-Lagrange equations for the time component we obtain

$$\frac{d}{ds}[(1+\Phi(r))\dot{t}] = 0. \tag{42}$$

The latter implies there is a conserved quantity we will call energy:

$$p_t \equiv [1+\Phi(r)]\dot{t} \equiv E. \tag{43}$$

Then, we find the ϕ component of the Euler-Lagrange equation

$$\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0, \tag{44}$$

which also leads us to define a conserved quantity:

$$p_\phi \equiv r^2\sin^2\theta\dot{\phi} \equiv L, \tag{45}$$

where L is the angular momentum per unit mass of the two bodies. From the equation for the θ component we find

$$\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \neq 0, \tag{46}$$

which is not a conserved quantity. Thus, the θ equation reads:

$$\frac{d}{ds}(r^2\dot{\theta}) = r^2\dot{\phi}^2\sin\theta\cos\theta. \tag{47}$$

Finally we need to compute the r equation, which is quite involved because of the heavy explicit dependence on r in the metric.

Since we want to study the orbits, as a first step we may simplify the problem by using its symmetries. Therefore, we fix the coordinate system so that the orbit of the particle lies on the plane $(r-\phi)$, and fix the θ coordinate to be $\pi/2$ so that $\dot{\theta} = 0$. Since we are interested on studying only timelike geodesics [87], we use the constants of motion defined in the above equations to obtain the following identity:

$$E^2[1+\Phi(r)]^{-1} - \frac{L^2}{r^2} - \frac{[\Phi(r)-1]^2}{1-\Phi(r)}\dot{r}^2 = 1. \tag{48}$$

Finally, by solving Eq. (48), we get an explicit equation for $\dot{r}^2$:

$$\dot{r}^2 = \frac{L^2[\Phi(r)+1] - E^2 r^2}{r^2[\Phi(r)-1][\Phi(r)+1]}. \tag{49}$$

The equations we have built are needed to compute the periastron shift discussed/calculated in Sec. V.

IV. NUMERICAL SOLUTIONS OF THE GEODESIC EQUATIONS OF MOTION

In order to show how the Yukawa correction to the Newtonian potential affects the orbital motion, we solve numerically the geodesic equations (32)–(35). Those parametric differential equations are nonlinear, thus, in order to have a well-posed Cauchy problem, we have to define the initial and boundary conditions. We solve the Cauchy problem, as in the classical case, with the initial conditions $\dot{r}(\tau=0)=0$, $\dot{\phi}(\tau=0)=0$, $\dot{\theta}(\tau=0)=0$, and $\theta(\tau=0)=\pi/2$, obtaining a not planar solution ($\ddot{\theta}\neq 0$). We perform a set of simulations varying the parameters $\{M, E, \epsilon, \dot{r}(\tau=0), \dot{\phi}(\tau=0), \dot{\theta}(\tau=0)\}$ to account for the high nonlinearity of the geodesic equations, and to obtain a set of parameters that guarantee the stability of the solution. Once the numerical integration of the geodesic equation has been optimized, we are able to highlight the specific contributions of Yukawa-correction to orbital motion. Usually, one uses the orbital motion and the pulsar timing to study the properties of the SMBH at the center of the Milky Way (for detailed explanations on pulsar timing and other pulsar observing techniques see [42]). Here, we are going to use an inverse approach. We fix *a priori* the parameters of the SMBH to study the orbital motion of a pulsar-like object. Specifically, we consider the SMBH at the center of the Milky Way galaxy, SgrA*, having a mass $M=(4.5\pm 0.6)\times 10^6\ M_\odot$ [88] and located at a distance of $R_0\sim 8$ kpc from the Sun [89]. For convenience, we have fixed the scale length $\lambda = 5000AU$ [68], and set $G=c=1$. Thus, all results in the figures are given in physical units.

In Fig. 2, we illustrate the phase portrait of $\dot{r}(\tau)$ versus $r(\tau)$ for the GR solution ($\delta=0$, black line), and for $\delta=-0.1$ and $\delta=0.1$ shown in red and blue lines, respectively. For both values of the δ the orbit assumes a stable configuration and, the Yukawa correction term induces departures form the configuration of the orbits obtained in GR. Specifically, for $\delta=-0.1$, the semi major axis is shorter, while for $\delta=0.1$ is longer, than the GR one ($\delta=0.1$).

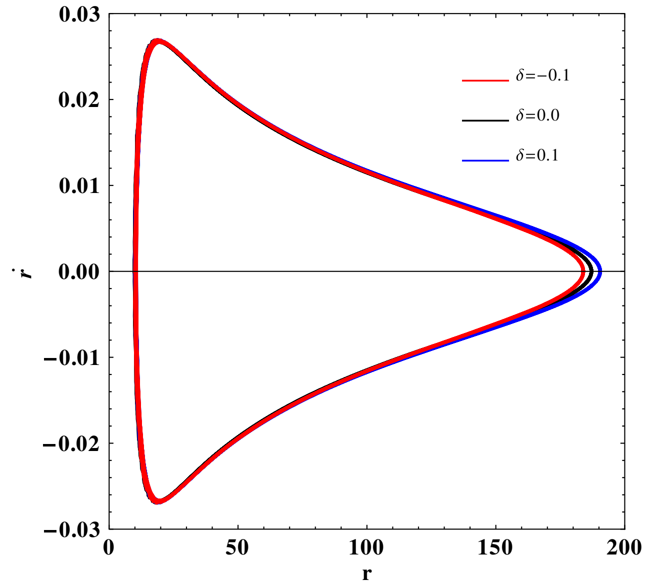

FIG. 2. Phase space diagram of a closed orbit in the Yukawa potential.

The orbital precession is easily discernible drawing orbits. Thus, in Figs. 3 and 4, we illustrate the periastron advance for both $\delta=-0.1$ and $\delta=0.1$ with a comparison with the general relativistic one. Let us note that the effect of the Yukawa-term is always to enhance the orbital precession while its sign can change from being positive ($\delta>0$) to being negative ($\delta<0$). The numerical integration of geodesic equations qualitatively confirms previous results found in semi-classical approaches [68–71]. This

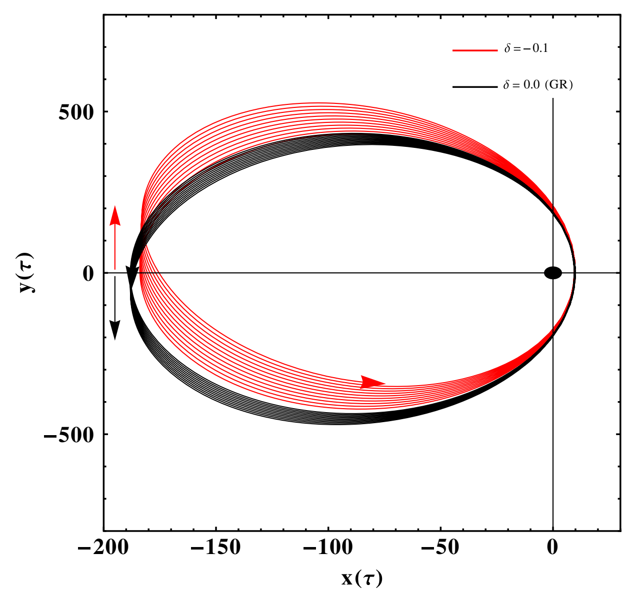

FIG. 3. Numerical solution of the geodesic equation illustrating the periastron advance in the Yukawa-potential. Here, we compare the GR solution ($\delta=0$) and the one for $\delta=-0.1$. The black dot point indicates the central object.

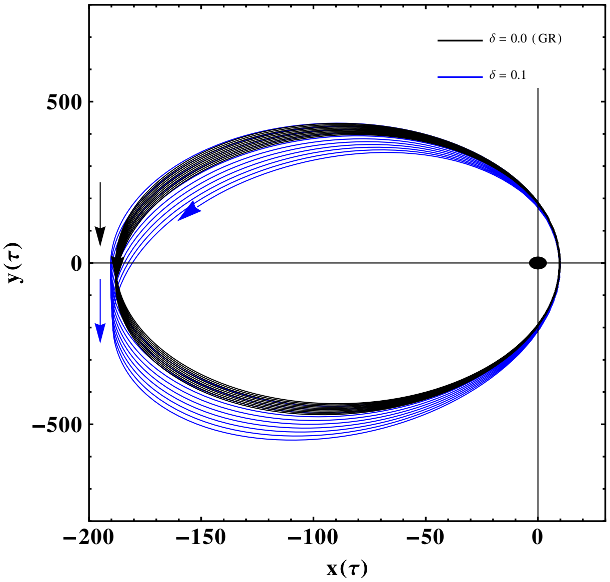

FIG. 4. The plot follows the conventions adopted for Fig. 3, while comparing the GR with the Yukawa solution for $\delta = 0.1$.

effect is due to the exponential term in the gravitational potential and it is negligible in binary systems. Nevertheless, it becomes viewable when simulating an object orbiting around a SMBH on scales comparable with λ, and it can be used to reduce further the parameter space of $f(R)$ gravity as previously suggested by [68–71].

V. PERIASTRON SHIFT IN YUKAWA-LIKE POTENTIAL

The most suitable candidates to test theories of gravity are binary systems constituted by a SMBH and an orbiting star [90]. Even just finding one normal pulsar around the BH will be phenomenally interesting to test alternative theories of gravity. Generally speaking, an orbit closes if the angle ϕ sweeps out exactly 2π in the passage between two successive inner or two successive outer radial turning points. If the orbits precess, ϕ changes by more than 2π between successive radial turning points.

To obtain an analytic formula for the periastron advance we need to obtain the orbits $r = r(\phi)$. Thus, we replace the variable τ by ϕ with the aid of the angular momentum law Eq. (45) and of Eq. (43), and we obtain

$$\left(\frac{dr}{d\phi}\right)^2 = -\frac{r^2[r^2(\Phi(r) - E^2 + 1) + L^2(\Phi(r) + 1)]}{L^2[\Phi(r)^2 - 1]},$$

that explicitly assumes the following form

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{2\delta GMe^{-\frac{r}{\lambda}}}{c^2(\delta+1)L^2 r} + \frac{2GM}{c^2(\delta+1)L^2 r} + \frac{2\delta GMe^{-\frac{r}{\lambda}}}{c^2(\delta+1)r^3} + \frac{2GM}{c^2(\delta+1)r^3} + \frac{E^2}{L^2} - \frac{1}{L^2} - \frac{1}{r^2}. \tag{50}$$

Let us perform the change of variable $u = 1/r$, so that the previous equation reads

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - [\Phi(u) + 1][L^2u^2 + 1]}{L^2u^4[\Phi(u)^2 - 1]}. \tag{51}$$

After some simplifications and imposing $(du/d\phi)^2 = 0$ we obtain

$$\frac{2\delta GMue^{-\frac{1}{\lambda u}}}{c^2(\delta+1)L^2} + \frac{2GMu}{c^2(\delta+1)L^2} + \frac{2\delta GMu^3e^{-\frac{1}{\lambda u}}}{c^2(\delta+1)} + \frac{2GMu^3}{c^2(\delta+1)} + \frac{E^2}{L^2} - \frac{1}{L^2} - u^2 = 0. \tag{52}$$

The most fruitful way to proceed is to rewrite the previous equation in terms of orbital parameters. We introduce the eccentricity e and the *latus rectum* l of the orbit, and we define the parameter $\mu \equiv M/l$. By definition, we use the ansatz that

$$u = \frac{1 + e\cos\chi}{l}, \tag{53}$$

where χ is the so called *relativistic anomaly*. Thus, $\chi = 0$ and 2π correspond to successive periastron passages, and $\chi = \pi$ at intermediate apoastron. Then, inserting Eq. (53) in Eq. (52), we obtain

$$\left(\frac{d\chi}{d\phi}\right)^2 = [1 - (e^2+3)\mu + 2\mu(e\cos\chi + 1)^2]\Upsilon + (e^2 - 1)(1 - 4\mu)\mu^2 - \mu^2(e\cos\chi + 1)^2, \tag{54}$$

where we have defined the auxiliary variable

$$\Upsilon = \frac{2\mu^2(e\cos\chi + 1)}{\delta + 1}(\Upsilon_1 + 1), \tag{55}$$

$$\Upsilon_1 = \delta\left(\frac{1}{2\lambda^2\mu^2(e\cos\chi+1)^2} - \frac{1}{\lambda\mu(e\cos\chi+1)} + 1\right). \tag{56}$$

Note that as we want to get an analytical solution and to study very close orbiting binary objects, we have expanded in Taylor series the exponential $e^{-\frac{1}{\lambda u}}$ up to the second order. Therefore, the use of the previous formula is restricted to the cases in which the semimajor axis of the orbit is much lower than the Yukawa scale length. It is also important to note that when $\delta = 0$ one recovers the well-known results of GR [87]

$$\left(\frac{d\chi}{d\phi}\right)^2 = 1 - 2\mu(3 + e\cos\chi), \tag{57}$$

that leads to the likewise well-known result

$$\Delta\phi_{GR} = \frac{6\pi GM}{ac^2(1 - e^2)}. \tag{58}$$

TABLE I. Values of periastron advance for different objects. In the table are reported the measured values of the eccentricity e, semi-major axis a in meters, the general relativistic periastron advance, and the predicted values of $\Delta\phi$ for $\delta = \pm 0.01$ from Eq. (59).

Toy model	e	a (10^{11} m)	$\Delta\phi_{GR}$ (°/orbit)	$\Delta\phi_{\delta=-0.01}$ (°/orbit)	$\Delta\phi_{\delta=0.01}$ (°/orbit)
A	0.678	14.96	8.880 59	8.970 53	8.792 42
B	0.786	7.48	25.1087	25.3642	24.8583
C	0.888	1.496	226.918	229.303	224.580

The integration of Eq. (54) can be performed trivially, and finally it is possible to obtain the expression for the periastron advance

$$\Delta\phi = \frac{\Delta\phi_{GR}}{(\delta+1)}\left(1 + \frac{2\delta G^2M^2}{3a^2c^4(1-e^2)^2} - \frac{2\pi\delta G^2M^2}{ac^4(1-e^2)\lambda} - \frac{3\delta GM}{ac^2(1-e^2)} - \frac{\delta G^2M^2}{6c^4(\delta+1)\lambda^2} + \frac{\delta GM}{3\lambda c^2}\right). \tag{59}$$

The Eq. (59) shows explicitly that it reduces to Eq. (58) for $\delta = 0$. Next, the amount of relativistic precession depends on

(i) the values M for the central mass,
(ii) tight orbits (small values of a),
(iii) large eccentricities e,
(iv) the Yukawa scale length λ,
(v) the Yukawa strength δ.

Therefore, the parameter space is larger than the one in the general relativistic case due to the presence of two extra parameters δ and λ that affect the precession. As already mentioned in Sec. IV, the Yukawa-correction can change the sign of the precession as found in semiclassical approaches [68–71,86].

A. Toy model stars around the Galactic center

Here we have particularized the periastron shift for a set of three toy model stars orbiting around the BH in the Galactic center. Let us remarks that being $\lambda \sim 10^3$ AU we cannot apply the Eq. (59) to the S-stars orbiting around the Galactic center for which one should solve the geodesic equations numerically. The BH mass is fixed to $M_{\mathrm{BH}} = 4.5 \times 10^6\ M_\odot$. The orbital parameters of the three models are summarized in Table I. In Fig. 5, we show the contribution of $f(R)$ gravity to the general relativistic periastron advance as a function of the strength of the Yukawa potential. Here, the scale length has been fixed to the confidence value $\lambda = 5000$ AU [68]. The figure shows that for $\delta > 0$ the contribution of the Yukawa-correction increases the periastron shift, while for $\delta < 0$ it decreases it. The shift in the periastron advance in $f(R)$ gravity with respect to GR can reach an order of magnitude of $\sim$10% for $\delta = \pm 0.1$, and it could be measurable with forthcoming observations of EHTC.

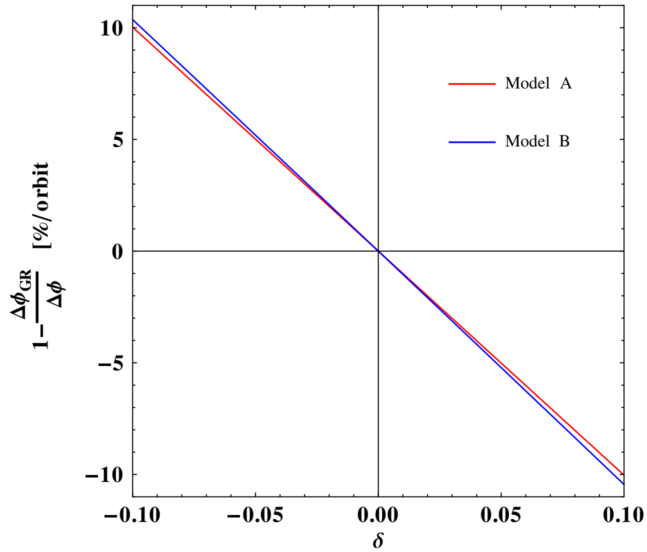

FIG. 5. The plot illustrate the change of the periastron advance with respect to the GR one as function of δ. We used Eq. (59) to compute analytically the periastron advance for a set of three toy model stars orbiting around the black hole at the Galactic center. The orbital parameters are given in Table I.

Finally, in Fig. 6, we demonstrate that the impact of the scale length is negligible, confirming the known degeneracy between $\delta - \lambda$ that cannot be constrained at the same time using the orbital motion [71].

B. Constraining Yukawa potential with pulsars in binary systems

Binary systems composed by double pulsars or by a pulsar and a companion star provide a excellent laboratory

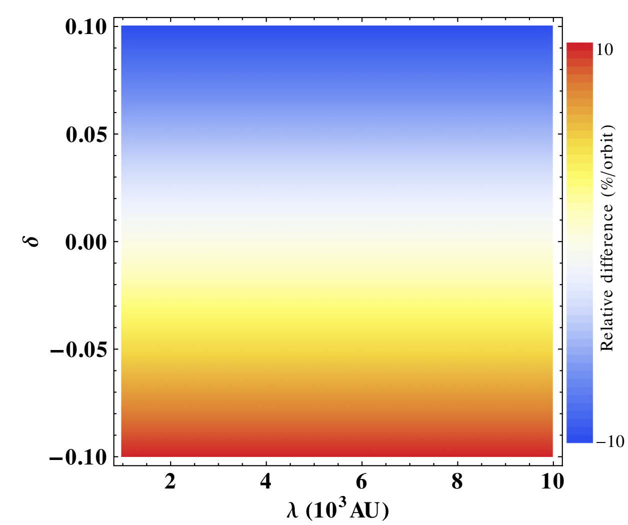

FIG. 6. We illustrated the dependence of the periastron advance in Eq. (59) from both the strength and the scale length of the Yukawa potential. The plot is particularized for the toy model A in Table I.

to study alternative theories of gravity. It is well known that pulsars act as very precise clocks. Monitoring one such a clock allows us to measure the time of arrival (TAO) of pulses at the telescopes and to obtain the pulse profile. In case the pulsar is part of a binary system, the pulse profile shows a periodic variation in the arrival time. This variation is related to the orbital motion around the center of mass of the binary system, and it needs to be modeled. Binary systems can be described in terms of the Keplerian parameters: the orbital period P_b, the projected semi-major axis $a_p \sin i$, the eccentricity of the orbit e, the periastron, ω, and the time of the transition at periastron T_0. Nevertheless, when considering close binary systems relativistic effects due to the strong field regime must be introduced. It is customary to parametrize the timing model using the post-Keplerian (PK) parameters: the time variation of the orbital period $\dot{P}_b$, the advance of the periastron $\dot{\omega}$, the time delay γ, and other two parameters, r and s, related to the Shapiro delay due to the gravitational field of the companion star. Although GR is capable of describing those systems, alternative theories of gravity can be probed using specific generalizations of the PK parameters. The main difference is that, in GR, the two masses are the only free parameters. Therefore, observing two PK parameters leads to estimating the masses uniquely. Clearly, precise measurements of the all PK parameters will provide an accurate estimation of the masses. Nevertheless, in $f(R)$-gravity this is not true. The two masses are not the only free parameters, one also has the parameters of the gravitational potential (δ, λ) or alternatively, their expression in terms of the Taylor coefficients (f'_0, f''_0), and they are degenerate with the masses. The only way to break this degeneracy is to fix the masses [44,45]. Therefore, calculating more PK parameters in alternative theories of gravity will give a powerful tool to estimate the masses of the two stars and, at same time, to constrain/rule our theory.

The theoretical expression for the periastron advance in the case of binary systems is obviously dependent on the pulsar mass m_p and on the mass of the companion star m_c. To generalize the periastron advance in Eq. (59) to the case of a binary system we have to use Kepler's law and the fact that the total mass in Eq. (59) can be recast as $M = m_c + m_p$. Thus, Eq. (59) becomes

$$\begin{aligned}\dot{\omega} = \frac{\dot{\omega}_{GR}}{(\delta+1)}\Bigg[1 &+ \frac{2\delta}{(1-e^2)^2}\left(\frac{2\pi}{P_b}\right)^{4/3}\frac{G^{4/3}}{c^4}(m_p+m_c)^{4/3} \\ &- \frac{2\delta}{(1-e^2)\lambda}\left(\frac{2\pi}{P_b}\right)^{2/3}\frac{G^{5/3}}{c^4}(m_p+m_c)^{2/3} \\ &- \frac{2\delta}{(1-e^2)}\left(\frac{2\pi}{P_b}\right)^{2/3}\frac{G^{2/3}}{c^2}(m_p+m_c)^{2/3} \\ &- \frac{\delta}{2\lambda^2}\frac{G^2}{c^4}(m_p+m_c)^2 + \frac{\delta}{\lambda}\frac{G}{c^2}(m_p+m_c)\Bigg], \end{aligned} \tag{60}$$

where the masses m_p and m_c are expressed in solar masses, and we have defined

$$\dot{\omega}_{GR} = \left(\frac{2\pi}{P_b}\right)^{5/3}\frac{G^{2/3}}{c^2}\frac{(m_p+m_c)^{2/3}}{(1-e^2)}. \tag{61}$$

The previous equation can be further simplified using the constant $T_\odot = GM_\odot/c^3 = 4.925490947\ \mu s$, and can be expressed in term of f'_0 and f''_0. The previous equation, together with the equation of the time variation of the orbital period in [44] provides a very powerful tool to test $f(R)$-gravity with current observations from the Parkes Pulsar Timing Array (PPTA) and, in particular, with next-generation facilities such as the Square-Kilometre-Array (SKA) [91–93].

VI. CONCLUSION AND REMARKS

In this paper we have investigated the impact of the Yukawa-like gravitational potential on the periastron shift of an orbiting body. The Yukawa-like correction to the Newtonian potential is a very well established result of many different alternative theories of gravity. Here, we have particularized our calculation to the framework of $f(R)$-gravity where the gravitational potential assumes the functional form given in Eq. (14). Thus, the modifications due to the $f(R)$ gravity is encoded in two parameters: the strength δ and the scale length λ of the Yukawa-term.

First, we have computed the geodesic equations and we have solved them numerically to visually show the presence of stable orbits and the orbital precession of a test particle moving around a massive body. Second, we have computed an analytic formula for the periastron shift in the limit that the orbital radius is much lower that the scale length λ. Since the most suitable candidates to test the theory are binary systems composed by a SMBH and an orbiting star, we have computed the periastron advance particularizing the Eq. (59) for three toy models of stars orbiting around the Galactic center. We have illustrated our results in the Fig. 5 fixing $\lambda = 5000AU$. Let us remark that our results are showing the capability of the periastron shift to constrain the Yukawa strength once the scale length is fixed. Then, we have generalized the expression of the periastron advance for a binary systems composed by two neutron stars or pulsars with comparable masses. Finally, the results showed above will represent a fundamental tool to be used with forthcoming observations of pulsars near the Galactic center.

We have considered idealized systems, where the internal structure of the two masses and others effects that can affect their motion (like as tidal effect, dusts, etc.) have not been taken into account. Nevertheless, even in a realistic system, the internal structure of the stars is decoupled from the orbital motion not producing relevant difference in the precession. Moreover, we have particularized our plots for pulsars near to the SMBH at Galactic center. However, one should have in mind that finding pulsars near the SMBH is difficult due to the relatively high density of free electrons in the gas around the Galactic center. Radio waves scatter off of these electrons, smearing out the sharp pulses from a pulsar in a phenomenon known as interstellar dispersion. Because of searches for pulsars rely on detecting periodic bursts, if the pulses are smeared out over the entire pulse period, a pulsar becomes essentially undetectable. More stable radio pulsars in the region would allow astronomers to sample more areas of the accretion disk and to make accurate measurements of the curvature of space-time [94].

Also, estimates of the pulsar population around Sgr A* range from the hundreds to the thousands [95]. To find these pulsars and overcome the high dispersion of pulses near the galactic center, astronomers will use further searches in high frequency X-rays as well as computer-intensive attempts to "dedisperse" observations by testing different estimates of the density of free electrons between earth and the pulsar at each observed point. Thus, we will soon have many more pulsars to map out the area around the SMBH. Forthcoming observations of the EHTC may provide a measure of the periastron shift, and other pulsar's observables such as the time dependence of the orbital period and the time delay, for these pulsars. Therefore they will provide the ultimate test for GR and alternative theories of gravity.

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[1] C. M. Will, *Theory and Experiment in Gravitational Physics*, 2nd ed. (Cambridge University Press, Cambridge, England, 1993).

[2] I. H. Stairs, Living Rev. Relativity **6**, 5 (2003).

[3] C. W. F. Everitt, D. B. DeBra, B. W. Parkinson, J. P. Turneaure, and J. W. Conklin, Phys. Rev. Lett. **106**, 221101 (2011).

[4] S. Perlmutter, S. Gabi, G. Goldhaber, A. Goobar, D. E. Groom *et al.*, Astrophys. J. **483**, 565 (1997).

[5] A. G. Riess, L.-G. Strolger, J. Tonry, S. Casertano, H. C. Ferguson *et al.*, Astrophys. J. **607**, 665 (2004).

[6] P. Astier, J. Guy, N. Regnault, R. Pain, E. Aubourg *et al.*, Astron. Astrophys. **447**, 31 (2006).

[7] N. Suzuki, D. Rubin, C. Lidman, G. Aldering, R. Amanullah *et al.*, Astrophys. J. **746**, 85 (2012).

[8] A. C. Pope, T. Matsubara, A. S. Szalay, M. R. Blanton, D. J. Eisenstein *et al.*, Astrophys. J. **607**, 655 (2004).

[9] W. J. Percival, C. M. Baugh, J. Bland-Hawthorn, T. Bridges, R. Cannon *et al.*, Mon. Not. R. Astron. Soc. **327**, 1297 (2001).

[10] M. Tegmark, M. R. Blanton, M. A. Strauss, F. Hoyle, D. Schlegel *et al.*, Astrophys. J. **606**, 702 (2004).

[11] G. Hinshaw, D. Larson, E. Komatsu, D. N. Spergel, C. L. Bennett *et al.*, Astrophys. J. Suppl. Ser. **208**, 19 (2013).

[12] C. Blake, E. A. Kazin, F. Beutler, T. M. Davis, D. Parkinson *et al.*, Mon. Not. R. Astron. Soc. **418**, 1707 (2011).

[13] Planck Collaboration, Astron. Astrophys. **594**, A13 (2016).

[14] I. de Martino, R. Génova-Santos, F. Atrio-Barandela, H. Ebeling, A. Kashlinsky, D. Kocevski, and C. J. A. P. Martins, Astrophys. J. **808**, 128 (2015).

[15] I. de Martino, C. J. A. P. Martins, H. Ebeling, and D. Kocevski, Phys. Rev. D **94**, 083008 (2016).

[16] C. W. Misner, in *Relativity*, edited by M. Carmeli, L. Fickler, and L. Witten (Plenum, San Francisco, 1970).

[17] C. J. Isham, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon Press, Oxford, 1981); C. Itzykson and J. C. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[18] J. L. Feng, Annu. Rev. Astron. Astrophys. **48**, 495 (2010).

[19] G. Bertone, D. Hooper, and J. Silk, Phys. Rep. **405**, 279 (2005).

[20] A. Capolupo, Adv. High Energy Phys. **2016**, 8089142 (2010).

[21] H.-Y. Schive, T. Chiueh, and T. Broadhurst, Nat. Phys. **10**, 496 (2014).

[22] A. Capolupo, Adv. High Energy Phys. **2018**, 9840351 (2018).

[23] I. De Martino, T. Broadhurst, S.-H. H. Tye, T. Chiueh, H.-Y. Schive, and R. Lazkoz, Phys. Rev. Lett. **119**, 221103 (2017).

[24] I. De Martino, T. Broadhurst, S.-H. H. Tye, T. Chiueh, H.-Y. Schive, and R. Lazkoz, Galaxies **6**, 10 (2018).

[25] I. Lopes and G. Panotopoulos, Phys. Rev. D **97**, 024030 (2018).

[26] G. Panotopoulos and I. Lopes, Phys. Rev. D **97**, 024025 (2018).

[27] I. de Martino, M. De Laurentis, and S. Capozziello, Universe **1**, 123 (2015).

[28] S. Capozziello, M. De Laurentis, M. Francaviglia, and S. Mercadante, Found. Phys. **39**, 1161 (2009).
[29] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011).
[30] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rep. **692**, 1 (2017).
[31] S. Nojiri and S. D. Odintsov, eConf C **0602061**, 06 (2006); Int. J. Geom. Methods Mod. Phys. **04**, 115 (2007).
[32] S. Capozziello and M. De Laurentis, Phys. Rep. **509**, 167 (2011).
[33] The LIGO Scientific Collaboration and The Virgo Collaboration *et al.*, *Nature (London)* 551, 88 (2017).
[34] M. De Laurentis, O. Porth, L. Bovard, B. Ahmedov, and A. Abdujabbarov, Phys. Rev. D **94**, 124038, 124038 (2016).
[35] S. Bellucci, S. Capozziello, M. De Laurentis, and V. Faraoni, Phys. Rev. D **79**, 104004 (2009).
[36] S. Capozziello, M. De Laurentis, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **95**, 083524 (2017).
[37] C. Bogdanos, S. Capozziello, M. De Laurentis, and S. Nesseris, Astropart. Phys. **34**, 236 (2010).
[38] J. M. Ezquiaga and M. Zumalacárregui, arXiv:1710.05901.
[39] J. Sakstein and B. Jain, Phys. Rev. Lett. **119**, 251303 (2017).
[40] L. Lombriser and A. Taylor, J. Cosmol. Astropart. Phys. 03 (2016) 031; L. Lombriser and N. A. Lima Phys. Lett. B **765**, 382 (2017).
[41] S. Capozziello and M. De Laurentis, Ann. Phys. (Berlin) **524**, 545 (2012).
[42] D.R. Lorimer and M. Kramer, *Handbook of Pulsar Astronomy* (Cambridge University Press, Cambridge, England, 2004); A. G. Lyne and F. Graham-Smith, *Pulsar Astronomy* (Cambridge University Press, Cambridge, England, 1998); R. N. Manchester and J. H. Taylor, *Pulsars* (W. H. Freeman and Company, San Francisco, 1977).
[43] M. De Laurentis, R. De Rosa, F. Garufi, and L. Milano, Mon. Not. R. Astron. Soc. **424**, 2371 (2012).
[44] M. De Laurentis and I. de Martino, Mon. Not. R. Astron. Soc. **431**, 741 (2013).
[45] M. De Laurentis and I. de Martino, Int. J. Geom. Methods Mod. Phys. **12**1550040 (2015).
[46] C. P. L. Berry and J. R. Gair, Phys. Rev. D **83**, 104022 (2011).
[47] S. Lee, arXiv:1711.09038.
[48] T. Liu, X. Zhang, and W. Zhao, Phys. Lett. B **777**, 286 (2018).
[49] E. Berti, E. Barausse, V. Cardoso, L. Gualtieri, P. Pani *et al.*, Classical Quantum Gravity **32**, 243001 (2015).
[50] J. Antoniadis, Astrophys. Space Sci. Proc. **40**, 1 (2015).
[51] P. C. C. Freire, N. Wex, G. Esposito-Farèse, J. P. W. Verbiest, M. Bailes, B. A. Jacoby, M. Kramer, I. H. Stairs, J. Antoniadis, and G. H. Janssen, Mon. Not. R. Astron. Soc. **423**, 3328 (2012).
[52] L. Shao, N. Sennett, A. Buonanno, M. Kramer, and N. Wex, Phys. Rev. X **7**, 041025 (2017).
[53] K. Liu, N. Wex, M. Kramer, J. M. Cordes, and T. J. W. Lazio, Astrophys. J. **747**, 1 (2012).
[54] C. Goddi, H. Falcke, M. Kramer, L. Rezzolla *et al.*, Mod. Phys. D **26**, 02 (2017).
[55] R. Angelil, P. Saha, and D. Merritt, Astrophys. J. **720**, 1303 (2010).
[56] Akiyama *et al.*, Astrophys. J. **807**, 150 (2015).
[57] L. Huang, M. Cai, Z.-Q. Shen, and F. Yuan, Mon. Not. R. Astron. Soc. **379**, 833 (2007).
[58] S. S. Doeleman *et al.*, Nature (London) **455**, 78 (2008).
[59] S. S. Doeleman, arXiv 0906.3899; Astro2010: The Astronomy and Astrophysics Decadal Survey, Science White Papers, no. 68.
[60] S. S. Doeleman *et al.*, Science **338**, 355 (2012).
[61] H. Falcke, F. Melia, and E. Agol, Astrophys. J. Lett. **528**, L13 (2000).
[62] V. L. Fish, S. S. Doeleman, C. Beaudoin, R. Blundell, D. E. Bolin, G. C. Bower, R. Chamberlin, R. Freund, P. Friberg, M. A. Gurwell *et al.*, Astrophys. J. Lett. **727**, L36 (2011).
[63] V. L. Fish, M. D. Johnson, S. S. Doeleman, A. E. Broderick, D. Psaltis, R.-S. Lu, K. Akiyama, W. Alef, J. C. Algaba, K. Asada *et al.*, Astrophys. J. **820**, 90 (2016).
[64] M. D. Johnson, V. L. Fish, S. S. Doeleman *et al.*, Science **350**, 1242 (2015).
[65] Z. Younsi and A. Grenzebach, in *Proceedings, 14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Rela- tivistic Field Theories (MG14) (In 4 Volumes): Rome, Italy, 2015* (2017), Vol. 4, p. 3525, DOI: 10.1142/10614.
[66] A. A. Abdujabbarov, L. Rezzolla, and B. J. Ahmedov, Mon. Not. R. Astron. Soc. **454**, 2423 (2015).
[67] G. Hobbs, in *Proceedings Science with Parkes @ 50 Years Young*, edited by R. Braun (2012).
[68] A. F. Zakharov, P. Jovanović, D. Borka, and V. Borka Jovanović, J. Cosmol. Astropart. Phys. 05 (2016) 045.
[69] D. Borka, P. Jovanović, V. Borka Jovanović, and A. F. Zakharov, Phys. Rev. D **85**, 124004 (2012).
[70] D. Borka, P. Jovanović, V. Borka Jovanović, and A. F. Zakharov, J. Cosmol. Astropart. Phys. 11 (2013) 050.
[71] I. de Martino, R. Lazkoz, and M. de Laurentis, preceding paper, Phys. Rev. D **97**, 104067 (2018).
[72] L. Iorio, Astrophys. J. **137**, 3 (2009); Int. J. Mod. Phys. D **15**, 473 (2006); Classical Quantum Gravity **22**, 5271 (2005); High Energy Phys. 2012 (**2012**) 73.; Phys. Lett. A **298**, 315 (2002); Planet. Space Sci. **55**, 1290 (2007).
[73] J. Khoury and A. Weltman, Phys. Rev. Lett. **93**, 171104 (2004).
[74] J. Khoury and M. Wyman, Phys. Rev. D **80**, 064023 (2009).
[75] A. de Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[76] G. Magnano and L. Sokolowski, Phys. Rev. D **50**, 5039 (1994).
[77] V. Faraoni and E. Gunzig, Int. J. Theor. Phys. **38**, 217 (1999).
[78] S. Capozziello, A. Stabile, and A. Troisi, Classical Quantum Gravity **25**, 085004 (2008).
[79] S. Capozziello, A. Stabile, and A. Troisi, Classical Quantum Gravity **24**, 2153 (2007).
[80] S. Weimberg, *Gravitational and Cosmology: Principles and Applications* (John Wiley & Sons, New York, 1972).
[81] I. de Martino, M. De Laurentis, F. Atrio-Barandela, and S. Capozziello, Mon. Not. R. Astron. Soc. **442**, 921, (2014).
[82] I. de Martino, Phys. Rev. D **93**, 124043 (2016).
[83] V. F. Cardone and S. Capozziello, Mon. Not. R. Astron. Soc. **414**, 1301 (2011).
[84] N. R. Napolitano, S. Capozziello, A. J. Romanowsky, M. Capaccioli, and C. Tortora, Astrophys. J. **748**, 87 (2012).

[85] S. Capozziello, E. De Filippis, and V.Salzano, Mon. Not. R. Astron. Soc. **394**, 947 (2009).

[86] A. F. Zakharov, P. Jovanović, D. Borka, V. Borka Jovanović, J. Cosmol. Astropart. Phys. 04 (2018) 050.

[87] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, International Series of Monographs on Physics Vol. 69 (Clarendon Press/Oxford University Press, Oxford/New York, 1983), p. 663.

[88] A. M. Ghez, S. Salim, N. N. Weinberg, J. R. Lu, T. Do, J. K. Dunn, K. Matthews, M. Morris, S. Yelda, E. E. Becklin, T. Kremenek, M. Milosavljevic, and J. Naiman, Astrophys. J. **689**, 1044 (2008).

[89] F. Eisenhauer, R. Schoedel, R. Genzel, T. Ott, M. Tecza, R. Abuter, A. Eckart, and T. Alexander, Astrophys. J. **597**, L121 (2003).

[90] M. De Laurentis, Z. Younsi, O. Porth, Y. Mizuno, and L. Rezzolla, Phys. Rev. D **97**, 104024 (2018).

[91] M. Kramer, D. C. Backer, J. M. Cordes, T. J. W. Lazio, B. W. Stappers, and S. Johnston, in *Strong-Field Gravity Tests Using Pulsars and Black Holes*, edited by C. Carilli and S. Rawlings, Science with the Square-Kilometer-Array (SKA) (Elsevier Science, New York, 2004).

[92] J. M. C. M. Kramer, T. J. W. Lazio, and B. W. Stappers, in *Pulsars as Tools for Fundamental Physics & Astrophysics*, edited by C. Carilli and S. Rawlings, Science with the Square-Kilometer-Array (SKA) (Elsevier Science Publishers, New York, 2004).

[93] C. Carilli and S. Rawlings, *Science with the Square-Kilometer-Array (SKA)* (Elsevier Science Publishers, New York, 2004).

[94] R. P. Eatough *et al.*, Nature (London) **501**, 391 (2013).

[95] J. Chennamangalam and D. R. Lorimer, Mon. Not. R. Astron. Soc. Lett. **440**, L86 (2014).