

Black hole shadow in an expanding universe with a cosmological constantVolker Perlick,^{1,*} Oleg Yu. Tsupko,^{2,†} and Gennady S. Bisnovaty-Kogan^{2,3,‡}¹ZARM, University of Bremen, 28359 Bremen, Germany²Space Research Institute of Russian Academy of Sciences, Profsoyuznaya 84/32, Moscow 117997, Russia³National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Kashirskoe Shosse 31, Moscow 115409, Russia

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We analytically investigate the influence of a cosmic expansion on the shadow of the Schwarzschild black hole. We suppose that the expansion is driven by a cosmological constant only and use the Kottler (or Schwarzschild–de Sitter) spacetime as a model for a Schwarzschild black hole embedded in a de Sitter universe. We calculate the angular radius of the shadow for an observer who is comoving with the cosmic expansion. It is found that the angular radius of the shadow shrinks to a nonzero finite value if the comoving observer approaches infinity.

DOI: [10.1103/PhysRevD.97.104062](https://doi.org/10.1103/PhysRevD.97.104062)**I. INTRODUCTION**

In recent years, strong evidence for the existence of supermassive black holes at the centers of most galaxies has been accumulated. According to theory, an observer should see such a black hole as a dark disk, known as the “shadow” of the black hole, in the sky against a backdrop of light sources. Attempts to actually observing the shadow of the black-hole candidates at the center of our own galaxy and at the center of M87 are under way, see the homepages of the Event Horizon Telescope (<http://eventhorizontelescope.org>) and of the BlackHoleCam (<http://blackholecam.org>).

For the simplest case of a nonrotating black hole, the shadow is a circular disk in the sky. If the black hole is uncharged, it is to be modeled by the Schwarzschild metric. For a static observer in the spacetime of a Schwarzschild black hole, the angular radius of the shadow was calculated in a seminal paper by Synge [1]. (Synge calculated what he called the “escape cone” of light which is just the complement in the sky of what we now call the shadow.) For a rotating black hole, the shadow is no longer circular but rather flattened on one side, as a consequence of the “dragging” of lightlike geodesics by the black hole. The shape of the shadow of a Kerr black hole for a stationary observer at a large distance was first calculated by Bardeen [2]. More generally, an analytical formula for the shape and the size of the shadow of a black hole of the Plebański-Demiański class, for an observer anywhere in the domain of outer communication, was derived by Grenzebach *et al.* [3,4]. In these two papers the observer’s four-velocity was assumed to be a linear combination of ∂_t and ∂_φ and in the

plane spanned by the two principal null directions; with this result at hand, the shadow can then be calculated for observers with any other four-velocities with the help of the standard aberration formula, see Grenzebach [5] for details. For the case of the Kerr metric, which is contained as a special case in the work by Grenzebach *et al.*, Tsupko [6] worked out an approximate formula that allows to extract the spin of the black hole from the shape of the shadow.

In all these works, the black hole is assumed to be eternal; i.e., the spacetime is assumed to be time independent. Then, of course, a static or stationary observer will see a time-independent shadow. Actually, we believe that we live in an expanding universe. This gives rise to the question of how the shadow depends on time. Also, in an expanding universe the dependence of the shadow on the momentary position of the observer will no longer be given by the formulas for a static or stationary black hole. Of course, for the black-hole candidates at the center of our own galaxy and at the centers of nearby galaxies the effect of the cosmological expansion is tiny. However, for galaxies at a larger distance the influence on the angular diameter of the shadow may be considerable. In any case, calculating this influence is an interesting question from a conceptual point of view. This is the purpose of the present paper. We restrict to the simplest model of a black hole in an expanding universe, viz. to the Kottler spacetime (also known as the Schwarzschild–de Sitter spacetime). This spacetime, which was found by Kottler [7] in 1918, describes a Schwarzschild-like (i.e., nonrotating and uncharged) black hole embedded in a de Sitter universe. More precisely, the Kottler metric depends on two parameters, m and Λ , both of which are assumed to be positive with $9\Lambda m^2 < 1$. It is a spherically symmetric solution of Einstein’s field equation for vacuum with a cosmological constant. Near the center the spacetime

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geometry is similar to a Schwarzschild black hole with mass parameter m , and far away from the center it is similar to a de Sitter universe with cosmological constant Λ . We admit that, according to the concordance model of cosmology, the de Sitter universe is a good model only for the late stage of our Universe, whereas for the present and earlier stages of our Universe the influence of matter cannot be neglected. Nonetheless, we believe that it is instructive to consider this model because it allows to determine the influence of the cosmological expansion on the shadow for the case that this expansion is driven by the cosmological constant only.

The Kottler metric admits a timelike Killing vector field. Observers whose worldlines are integral curves of this Killing vector field see a static (i.e., time-independent) spacetime geometry. We refer to them as to the static observers in the Kottler spacetime. When we consider the Kottler spacetime as a model for a black hole embedded in an expanding universe, we are not interested in these static observers, but rather in observers that are comoving with the cosmic expansion. However, the existence of the static observers gives us a useful tool for calculations: We may first consider the shadow as it is seen by a static observer. This was calculated for the Schwarzschild black hole without a cosmological constant by Synge [1], as was already mentioned above, and generalized to the case of a Kottler black hole by Stuchlík and Hledík [8]. From these results we can then calculate the angular radius of the shadow for an observer that is comoving with the cosmic expansion by applying the standard aberration formula.

In this paper, we want to concentrate on the influence of the cosmic expansion, as driven by the cosmological constant, on the shadow. Therefore, we simplify all other aspects as far as possible. In particular, we consider a black hole that is characterized by its mass only; i.e., it is nonspinning and carries no (electric, magnetic, gravitomagnetic,...) charges. It is certainly possible to consider, more generally, a Plebański-Demiański black hole, which may be spinning and carrying various kinds of charges, and to transform the above-mentioned results of Grenzebach *et al.* [3,4] with the help of the aberration formula to an observer that is comoving with the cosmic expansion. Then, however, it would be difficult to disentangle the influence of the various parameters on the result and to extract the effect of the Λ -driven expansion. Also, it would be possible to take the influence of a plasma onto the light rays into account. The shadow in a plasma for a static or stationary observer was calculated for nonrotating and rotating black holes by Perlick, Tsupko and Bisnovaty-Kogan [9,10], cf. [11]. Again, we will not do this because here we want to concentrate on the effect of the cosmic expansion driven by a cosmological constant.

As a starting point for our calculations we need the equation for lightlike geodesics in the Kottler spacetime, written in coordinates adapted to the static observers. It is well known that the set of solution curves of this differential equation is independent of Λ , see Islam [12]. It was widely

believed that, as a consequence, Λ has no influence on the lensing features. However, it was realized by Rindler and Ishak [13] that this is not true: Although the coordinate representation of the lightlike geodesics is unaffected by Λ , the cosmological constant does influence the lensing features because it changes the angle measurements. Therefore it should not come as a surprise that also the angular radius of the shadow does depend on Λ . When changing to the observers that are comoving with the cosmic expansion we have to apply the aberration formula. A detailed study of this formula in the Kottler spacetime was brought forward recently by Lebedev and Lake [14,15] and we will comment on the relation of our work to theirs in an appendix.

The paper is organized as follows. In Sec. II, we calculate the shadow in the Kottler spacetime for a static observer. The results are not new, but we have to repeat them here because we want to use them later. Section III contains the main results of this paper: Here we calculate the shadow in the Kottler spacetime as it is seen by an observer that is comoving with the cosmic expansion. An approximation for these results is given in Sec. IV for the case that the observer is far away from the black hole. We conclude with a discussion of our results in Sec. V. In the Appendix, we point out how our work is related to the above-mentioned work by Lebedev and Lake. Throughout the paper, we use Einstein's summation convention for greek indices taking values 0,1,2,3. Our choice of signature is $(-, +, +, +)$.

II. SHADOW IN THE KOTTLER SPACETIME AS SEEN BY A STATIC OBSERVER

The Kottler metric is the unique spherically symmetric solution to Einstein's vacuum field equation with a cosmological constant. In its standard form, it reads

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad (1)$$

where

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \quad d\Omega^2 = \sin^2\theta d\varphi^2 + d\theta^2. \quad (2)$$

m is the mass parameter,

$$m = \frac{GM}{c^2} \quad (3)$$

where M is the mass of the central object and Λ is the cosmological constant. (As usual, G is Newton's gravitational constant and c is the vacuum speed of light). We assume throughout that

$$0 < \Lambda < \frac{1}{9m^2}. \quad (4)$$

Then the Kottler metric has two event horizons, given by the zeros of the function $f(r)$, an inner one at a radius r_{HI}

and an outer one at a radius r_{H2} where $2m < r_{H1} < 3m < r_{H2} < \infty$. The region between the two horizons is called the “domain of outer communication” because any two observers in this region may communicate with each other without being hindered by a horizon. In this region, the function $f(r)$ is positive; i.e., the vector field ∂_t is timelike. As a consequence, the integral curves of the vector field ∂_t may be interpreted as the worldlines of observers. Since ∂_t is a Killing vector field, these observers see a time-independent universe. As mentioned already in the introduction, we will refer to them as to the static observers in the Kottler spacetime. For the following, it is crucial that the static observers exist only in the domain of outer communication.

The horizon at $r = r_{H1}$ consists of a future inner horizon that separates the domain of outer communication from a black-hole region and of a past inner horizon that separates it from a white-hole region. (For literature on white holes, see e.g. [16–18].) Similarly, the horizon at $r = r_{H2}$ consists of a future outer horizon and a past outer horizon. In this paper, we are interested in the shadow of the black hole. It is constructed under the assumption that there are light sources only in the domain of outer communication. As the light emitted from such a light source can never reach one of the two past horizons, the regions beyond the past horizons will be of no relevance for us. We will be concerned only with the domain of outer communication, tagged *I* in Fig. 1, and to the regions beyond the future

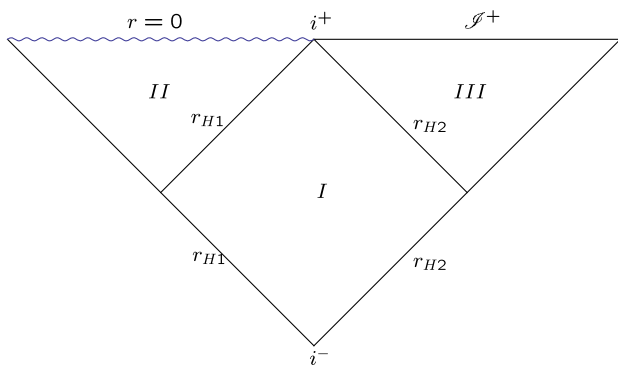


FIG. 1. Carter-Penrose diagram of the Kottler spacetime. The picture shows only the part of spacetime that is of relevance to us: The domain of outer communication *I*, the black-hole region *II* and the region beyond the (future) cosmological horizon *III*. A signal (i.e., a future-oriented causal worldline) that starts somewhere in the domain of outer communication may do one of three things: (i) It may stay inside *I* forever, approaching future timelike infinity i^+ ; examples are the circular lightlike geodesics at $r = 3m$. (ii) It may cross the black-hole horizon and end up in the singularity at $r = 0$; examples are the ingoing radial lightlike geodesics. (iii) It may cross the cosmological horizon and go to future null infinity \mathcal{S}^+ ; examples are the outgoing radial lightlike geodesics.—The Carter-Penrose diagram of the (maximal) Kottler spacetime was first determined by Gibbons and Hawking [19].

horizons, tagged *II* and *III* in Fig. 1. We will refer to the future inner horizon as the black-hole horizon and to the future outer horizon as the (future) cosmological horizon.

Before introducing moving observers in the next section, we will now calculate the shadow as it is momentarily seen by a static observer at a spacetime point $(t_O, r_O, \vartheta_O = \pi/2, \varphi_O = 0)$ in the domain of outer communication. Because of the symmetry, it is no restriction to place the observer in the equatorial plane and it suffices to consider lightlike geodesics in the equatorial plane. Geodesics in the equatorial plane derive from the Lagrangian

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} \left(-f(r)c^2\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\varphi}^2 \right). \quad (5)$$

The t and φ components of the Euler-Lagrange equation give us two constants of motion,

$$E = f(r)c^2\dot{t}, \quad L = r^2\dot{\varphi}. \quad (6)$$

For lightlike geodesics, we have

$$-f(r)c^2\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\varphi}^2 = 0. \quad (7)$$

Solving for $\dot{r}^2/\dot{\varphi}^2 = (dr/d\varphi)^2$ and inserting (6) yields the orbit equation for lightlike geodesics,

$$\left(\frac{dr}{d\varphi} \right)^2 = r^4 \left(\frac{E^2}{c^2L^2} + \frac{\Lambda}{3} - \frac{1}{r^2} + \frac{2m}{r^3} \right). \quad (8)$$

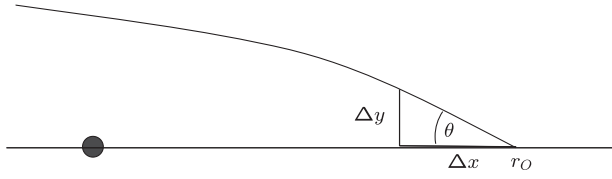
We see that Λ can be absorbed into a new constant of motion $C = E^2/(c^2L^2) + \Lambda/3$, i.e., that the set of all lightlike geodesics is independent of Λ in the chosen coordinate representation. This, however, does not mean that Λ has no influence on the lensing features because angle measurements do depend on Λ , see Rindler and Ishak [13].

By evaluating the equations $dr/d\varphi = 0$ and $d^2r/d\varphi^2 = 0$ we find that there is a circular lightlike geodesic at radius $r = 3m$ and that the constants of motion for this circular light ray satisfy

$$\frac{E^2}{c^2L^2} = \frac{1}{27m^2} - \frac{\Lambda}{3}. \quad (9)$$

This circular light ray is unstable in the sense that a slight perturbation of the initial direction in the equatorial plane gives a light ray that moves away from the circle at $r = 3m$ and crosses one of the two horizons. If we take all three spatial dimensions into account, we find that there is such an unstable circular light ray in any plane through the origin. These circular light rays fill the photon sphere at $r = 3m$.

For constructing the shadow, we consider all light rays that go from the position of the static observer at $(t_O, r_O, \vartheta_O = \pi/2, \varphi_O = 0)$ into the past. They leave the


 FIG. 2. Definition of the angle θ .

observer at an angle θ with respect to the radial line that satisfies

$$\tan \theta = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}, \quad (10)$$

see Fig. 2. From the Kottler metric (1) we read that Δx and Δy satisfy, in the desired limit,

$$\tan \theta = \left. \frac{rd\varphi}{(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)^{-1/2} dr} \right|_{r=r_O}. \quad (11)$$

Expressing $dr/d\varphi$ with the help of the orbit equation (8) results in

$$\tan^2 \theta = \frac{r_O - 2m - \frac{\Lambda}{3}r_O^3}{(\frac{E^2}{c^2 L^2} + \frac{\Lambda}{3})r_O^3 - r_O + 2m}. \quad (12)$$

By elementary trigonometry,

$$\sin^2 \theta = \frac{1 - \frac{2m}{r_O} - \frac{\Lambda}{3}r_O^2}{\frac{E^2}{c^2 L^2} r_O^2}. \quad (13)$$

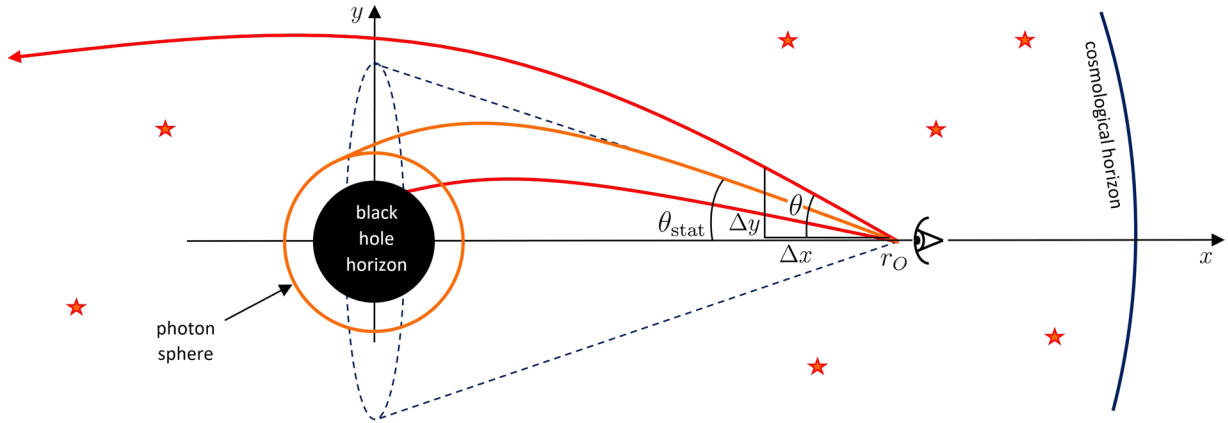


FIG. 3. Formation of the shadow as seen by a static observer in the Kottler spacetime. The Kottler metric has a black hole event horizon at r_{H1} and a cosmological event horizon at r_{H2} . The observer is at radial coordinate r_O . Without loss of generality, we consider light rays in the equatorial plane and we assume that the observer is located on the x -axis. If the observer “emits light rays into the past,” some of them go towards the horizon at r_{H1} while others, after approaching the black hole, go towards the horizon at r_{H2} . The borderline cases between these two classes are light rays which asymptotically spiral towards the photon sphere at $r = 3m$ which is filled with unstable circular light orbits. In the case of light sources distributed everywhere in the domain of outer communication but not between the black hole and the observer, the cone bounded by light rays that spiral towards the photon sphere will be empty, so the observer will see the shadow as a black disk of angular radius θ_{stat} . We have extended the tangents to the initial directions of these light rays in the coordinate picture by straight dashed lines up to the plane $x = 0$. This dashed cone has no coordinate-independent meaning, but it shows that application of the naive Euclidean formula $\tan \theta_{\text{stat}} = 3m/r_O$ gives an angular radius of the shadow that is smaller than the correct one. Also note that the Euclidean formula is independent of Λ whereas the correct one, given by (14), is not.

The shadow is constructed in the following way, see Fig. 3. We assume that there are light sources everywhere in the domain of outer communication but not between the observer and the black hole. Each point in the observer’s sky corresponds to a light ray issuing from the observer position into the past. We assign darkness (respectively brightness) to those directions which correspond to light rays that go to the horizon at r_{H1} (respectively to the horizon at r_{H2}). The boundary of the shadow corresponds to light rays that spiral asymptotically towards circular lightlike geodesics at $r = 3m$. Therefore, the angular radius of the shadow is found by equating E^2/L^2 to the constant of motion that corresponds to the circular light ray at $r = 3m$. Substituting from (9) into (13) yields the angular radius θ_{stat} of the shadow as it is seen by a static observer,

$$\sin^2 \theta_{\text{stat}} = \frac{1 - \frac{2m}{r_O} - \frac{\Lambda}{3}r_O^2}{(\frac{1}{27m^2} - \frac{\Lambda}{3})r_O^2}. \quad (14)$$

θ_{stat} varies from 0 (bright sky) to π (dark sky) when the observer position r_O varies from r_{H2} to r_{H1} . For $r_O = 3m$, we have $\theta_{\text{stat}} = \pi/2$, i.e., half of the sky is dark; see Fig. 4.

Eq. (14) is equivalent to a result found by Stuchlík and Hledík [8]. For $\Lambda \rightarrow 0$, (14) reduces of course to the formula for the shadow of a Schwarzschild black hole which was first calculated by Synge [1]. The word “shadow” is used neither by Synge nor by Stuchlík and Hledík. They calculated what they called the “escape cone” of light which is the complement of the shadow.

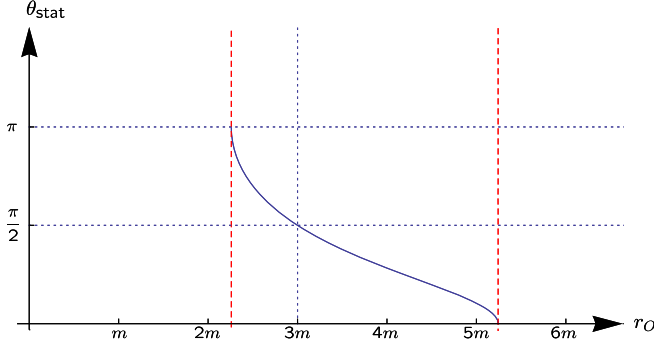


FIG. 4. Angular radius θ_{stat} of the shadow plotted against the observer position r_O . The picture is for $\sqrt{\Lambda/3} = H_0/c = 0.15 m^{-1}$. The dashed (red) lines mark the horizons at $r = r_{\text{H1}}$ and $r = r_{\text{H2}}$.

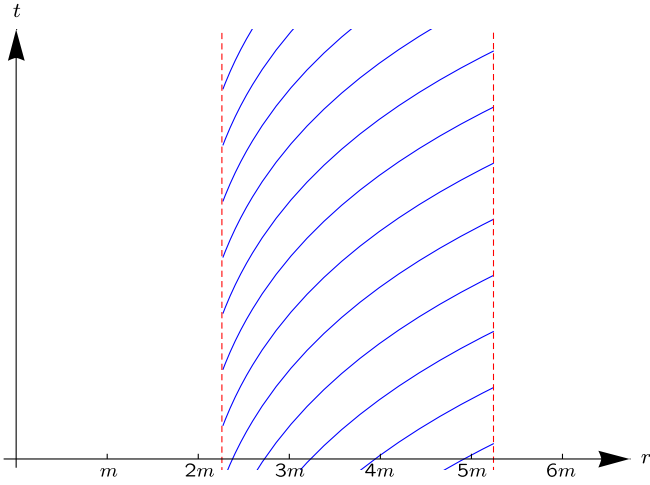


FIG. 5. Worldlines of the comoving observers in the $r-t$ coordinate system. As in Fig. 4, we have chosen $\sqrt{\Lambda/3} = H_0/c = 0.15 m^{-1}$. The worldlines of the comoving observers are shown here in the region between the two horizons which are, again, marked by dashed (red) lines. This corresponds to the region *I* in Fig. 1. If extended beyond the cosmological horizon, the worldlines of the comoving observers fill the regions *I* and *III* in Fig. 1 and terminate at \mathcal{S}^+ .

III. SHADOW IN THE KOTTLER SPACETIME AS SEEN BY AN OBSERVER COMOVING WITH THE EXPANDING UNIVERSE

We will now turn to the shadow as it is seen by an observer who is comoving with the cosmic expansion. To that end we introduce on the Kottler spacetime a new coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta} = \vartheta, \tilde{\varphi} = \varphi)$ which is related to the old coordinate system by

$$r = \tilde{r} e^{H_0 \tilde{t}} \left(1 + \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^2, \quad (15)$$

$$t = \tilde{t} + \int_{w_0}^{\tilde{r} e^{H_0 \tilde{t}}} \frac{H_0 \left(1 + \frac{m}{2w} \right)^6 w dw}{c^2 \left(1 - \frac{m}{2w} \right)^2 - H_0^2 w^2 \left(1 + \frac{m}{2w} \right)^6} \quad (16)$$

where

$$H_0 = \sqrt{\frac{\Lambda}{3}} c \quad (17)$$

and w_0 is an integration constant that has to be chosen appropriately. If we differentiate (15) and (16), we find the relation between the coordinate differentials,

$$dr = e^{H_0 \tilde{t}} \left(1 - \frac{m^2}{4\tilde{r}^2} e^{-2H_0 \tilde{t}} \right) (d\tilde{r} + \tilde{r} H_0 d\tilde{t}), \quad (18)$$

$$cdt = \frac{\left(1 - \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^2 c d\tilde{t} + \frac{H_0}{c} \tilde{r} e^{2H_0 \tilde{t}} \left(1 + \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^6 d\tilde{r}}{\left(1 - \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^2 - \frac{H_0^2}{c^2} \tilde{r}^2 e^{2H_0 \tilde{t}} \left(1 + \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^6} d\tilde{r}. \quad (19)$$

Inserting these expressions into (1) gives us the Kottler metric in the new coordinates,

$$\tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = - \left(1 - \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^2 \left(1 + \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^{-2} c^2 d\tilde{t}^2 + e^{2H_0 \tilde{t}} \left(1 + \frac{m}{2\tilde{r}} e^{-H_0 \tilde{t}} \right)^4 (d\tilde{r} + \tilde{r} d\Omega^2). \quad (20)$$

In this coordinate system, observers on \tilde{t} lines see an exponentially expanding universe with a (time-independent) Hubble constant H_0 . We call them the comoving observers, where ‘‘comoving’’ refers to the cosmic expansion, see Fig. 5. The twiddled coordinates are known as the McVittie coordinates, referring to the 1933 work by McVittie [20] on a more general class of spacetimes, although for the Kottler metric, Robertson [21] had used these coordinates already in 1928. For $H_0 \rightarrow 0$, the Kottler spacetime in the Robertson-McVittie representation (20) reduces to the Schwarzschild spacetime in isotropic coordinates while for $m \rightarrow 0$ it reduces to the steady-state universe, i.e., to one half of the de Sitter spacetime in Robertson-Walker coordinates adapted to a spatially flat slicing.

If solved for the differentials of the twiddled coordinates, (18) and (19) can be expressed as

$$d\tilde{t} = dt - \frac{H_0 r dr}{c^2 \sqrt{1 - \frac{2m}{r}} \left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2} \right)}, \quad (21)$$

$$\frac{d\tilde{r}}{\tilde{r}} = \frac{\sqrt{1 - \frac{2m}{r}} dr}{r \left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2} \right)} - H_0 dt. \quad (22)$$

This transformation can be equivalently rewritten in terms of the Gaussian basis vector fields as

$$\frac{\partial}{\partial \tilde{t}} = \frac{\left(1 - \frac{2m}{r} \right)}{\left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2} \right)} \frac{\partial}{\partial t} + H_0 r \sqrt{1 - \frac{2m}{r}} \frac{\partial}{\partial r}, \quad (23)$$

$$\tilde{r} \frac{\partial}{\partial \tilde{r}} = \frac{H_0 r^2}{c^2 \left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}\right)} \frac{\partial}{\partial t} + r \sqrt{1 - \frac{2m}{r}} \frac{\partial}{\partial r}. \quad (24)$$

We want to find the angular radius θ_{comov} of the shadow as it is seen by a comoving observer. We have calculated in (14) the angular radius θ_{stat} of the shadow for a static observer. The angle θ_{comov} we are looking for is related to θ_{stat} by the standard aberration formula

$$\sin^2 \theta_{\text{comov}} = \left(1 - \frac{v^2}{c^2}\right) \frac{\sin^2 \theta_{\text{stat}}}{\left(1 - \frac{v}{c} \cos \theta_{\text{stat}}\right)^2} \quad (25)$$

where v is the three-velocity of the comoving observer with respect to the static observer at the same observation event. Here we have to be careful when expressing $\cos \theta_{\text{stat}}$ with the help of our formula (14) for $\sin^2 \theta_{\text{stat}}$: We know from the preceding section that θ_{stat} lies between $\pi/2$ and π for $r_{H1} < r_O < 3m$ and that it lies between 0 and $\pi/2$ for $3m < r_O < r_{H2}$. Therefore, we rewrite (25) as

$$\sin^2 \theta_{\text{comov}} = \left(1 - \frac{v^2}{c^2}\right) \frac{\sin^2 \theta_{\text{stat}}}{\left(1 \pm \frac{v}{c} \sqrt{1 - \sin^2 \theta_{\text{stat}}}\right)^2} \quad (26)$$

where we have to choose the upper sign in the domain $r_{H1} < r_O < 3m$ and the lower sign in the domain $3m < r_O < r_{H2}$.

The three-velocity v has to be calculated from the special-relativistic equation

$$g_{\mu\nu} U_{\text{stat}}^\mu U_{\text{comov}}^\nu = \frac{-c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (27)$$

where $U_{\text{stat}}^\mu \partial/\partial x^\mu$ is the four-velocity vector of the static observer and $U_{\text{comov}}^\mu \partial/\partial x^\mu$ is the four-velocity vector of the comoving observer. The former is proportional to $\partial/\partial t$ while the latter is proportional to $\partial/\partial \tilde{t}$,

$$U_{\text{stat}}^\mu \frac{\partial}{\partial x^\mu} = N_{\text{stat}} \frac{\partial}{\partial t}, \quad (28)$$

$$\begin{aligned} U_{\text{comov}}^\mu \frac{\partial}{\partial x^\mu} &= N_{\text{comov}} \frac{\partial}{\partial \tilde{t}} \\ &= N_{\text{comov}} \left(\frac{\left(1 - \frac{2m}{r}\right)}{\left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}\right)} \frac{\partial}{\partial t} + H_0 r \sqrt{1 - \frac{2m}{r}} \frac{\partial}{\partial r} \right), \end{aligned} \quad (29)$$

where in the last equality we have used (23). The factors N_{stat} and N_{comov} follow from the normalization condition,

$$\begin{aligned} -c^2 &= g_{\mu\nu} U_{\text{stat}}^\mu U_{\text{stat}}^\nu \\ &= -c^2 N_{\text{stat}}^2 \left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}\right), \end{aligned} \quad (30)$$

$$-c^2 = g_{\mu\nu} U_{\text{comov}}^\mu U_{\text{comov}}^\nu = -c^2 N_{\text{comov}}^2 \left(1 - \frac{2m}{r}\right); \quad (31)$$

hence, (28) and (29) yield

$$U_{\text{stat}}^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}}} \frac{\partial}{\partial t}, \quad (32)$$

$$U_{\text{comov}}^\mu \frac{\partial}{\partial x^\mu} = \frac{\sqrt{1 - \frac{2m}{r}}}{\left(1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}\right)} \frac{\partial}{\partial t} + H_0 r \frac{\partial}{\partial r}. \quad (33)$$

Inserting these expressions for U_{stat}^μ and U_{comov}^ν into (27) results in

$$1 - \frac{v^2}{c^2} = \frac{1 - \frac{2m}{r} - \frac{H_0^2 r^2}{c^2}}{1 - \frac{2m}{r}} \quad (34)$$

which is equivalent to

$$v = \frac{H_0 r}{\sqrt{1 - \frac{2m}{r}}}. \quad (35)$$

From (34) we read that v tends to c if one of the two horizons is approached; this is clear because on the horizons the worldlines of the static observers become lightlike. Between the two horizons, v is decreasing from c to a local minimum at the photon sphere and then increasing again to c , see Fig. 6.

We can now calculate θ_{comov} by inserting (14) and (35) with $r = r_O$ into (26). After some elementary algebra we find

$$\begin{aligned} \sin \theta_{\text{comov}} &= \frac{\sqrt{27}m}{r_O} \sqrt{1 - \frac{2m}{r_O}} \sqrt{1 - \frac{27H_0^2 m^2}{c^2}} \\ &\mp \frac{\sqrt{27}mH_0}{c} \sqrt{1 - \frac{27m^2}{r_O^2}} \left(1 - \frac{2m}{r_O}\right). \end{aligned} \quad (36)$$

This equation makes sense for all momentary observer positions r_O with $r_{H1} < r_O < \infty$, although for the derivation it was assumed that $r_{H1} < r_O < r_{H2}$. This reflects the

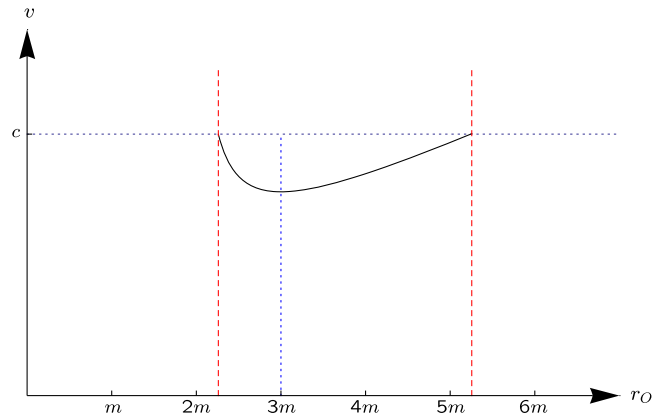


FIG. 6. Three-velocity v of a comoving observer relative to a static observer at the same event, plotted as a function of the radius coordinate r_O . As in the preceding pictures, we have chosen $\sqrt{\Lambda/3} = H_0/c = 0.15 \text{ m}^{-1}$ and the dashed (red) lines mark the horizons.

fact that the worldlines of the comoving observers may be analytically extended beyond the cosmological horizon. In (36), we have to choose the upper sign in the domain $r_{H1} < r_O < 3m$ and the lower sign in the domain $3m < r_O < \infty$; for $r_O = 3m$ the term with the \mp sign is equal to zero. (36) gives us the angular radius of the shadow as it is seen by a comoving observer on his way from the inner horizon through the outer horizon to infinity. Recall that a comoving observer has a constant twiddled radius coordinate, $\tilde{r}_O = \text{constant}$; hence, when we express r_O in terms of \tilde{r}_O and \tilde{t}_O with the help of (15) we get from (36) the angle θ_{comov} as a function of the time coordinate \tilde{t}_O .

If one of the horizons is approached,

$$\frac{1}{r_O} \sqrt{1 - \frac{2m}{r_O}} \rightarrow \frac{H_0}{c}. \quad (37)$$

For the inner horizon, we have to use the upper sign in (36). Then (37) yields

$$\sin \theta_{\text{comov}} \rightarrow 0 \quad \text{for } r_O \rightarrow r_{H1}. \quad (38)$$

The angle θ_{comov} itself goes to π . For the outer horizon, however, we have to use the lower sign in (36). Then (37) yields

$$\sin \theta_{\text{comov}} \rightarrow 2\sqrt{27} \frac{H_0 m}{c} \sqrt{1 - \frac{27H_0^2 m^2}{c^2}} \quad \text{for } r_O \rightarrow r_{H2}. \quad (39)$$

Moreover, from (36) with the lower sign we read that

$$\sin \theta_{\text{comov}} \rightarrow \sqrt{27} \frac{H_0 m}{c} \quad \text{for } r_O \rightarrow \infty. \quad (40)$$

When the comoving observer starts at the inner horizon, the shadow covers the entire sky, $\theta_{\text{comov}} = \pi$. On his way out to infinity, the shadow monotonically shrinks to a finite value given by (40), see Fig. 7. Nothing particular happens when the observer crosses the outer horizon. Note that the (future) cosmological horizon is an event horizon for all observers who stay forever in the domain of outer communication, in particular for the static observers, but *not* for the comoving observers. This can be clearly seen from Fig. 1: Even after crossing this horizon a comoving observer can receive light signals from region *I*.

According to Eq. (40), the angular radius θ_{comov} of the shadow of very distant black holes is determined by the cosmological constant and of course, by the mass of the black hole. With a value of $\Lambda \approx 1.1 \times 10^{-46} \text{ km}^{-2}$, which is in agreement with present day observations, (17) gives us a Hubble time of $H_0^{-1} \approx 5 \times 10^{17} \text{ s}$. Upon inserting this value into (40) we find for a supermassive black hole of 10^{10} Solar masses in the limit $r_O \rightarrow \infty$ an angular radius of $\theta_{\text{comov}} \approx 0.1$ microarcseconds. Present-day VLBI technology allows to resolve angles of a few dozen

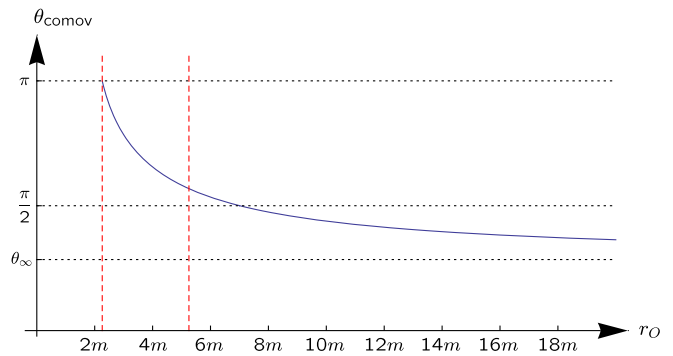


FIG. 7. Angular radius θ_{comov} of the shadow plotted against the observer position r_O . As before, we have chosen $\sqrt{\Lambda/3} = H_0/c = 0.15 \text{ m}^{-1}$ and the dashed (red) lines mark the horizons.

microarcseconds, so a resolution of 0.1 microarcseconds cannot be achieved at the moment but it could come into reach within one or two decades. Also, the existence of black holes with masses of more than 10^{10} Solar masses, for which the shadow would be bigger, cannot be ruled out. Note, however, that this line of argument does not necessarily imply that the shadows of very distant black holes will become observable with VLBI instruments in a few years' time. Firstly, we have to keep in mind that our calculation was done in a universe where the cosmic expansion is driven by the cosmological constant only. In a realistic model of the Universe, taking the matter content into account, the Hubble "constant" is a function of time; the chosen value of the Hubble time, $H_0^{-1} \approx 5 \times 10^{17} \text{ s}$ is a reasonably good approximation for the present time (and an even better approximation for later times, when the cosmological constant dominates even more over matter), but at earlier times the Hubble time had different values. So one would have to repeat our calculation in a universe with a time-dependent Hubble "constant" to see how the matter content influences our result. Secondly, for the observability of the shadow it is necessary not only that the angular radius of the shadow is big enough but also that there are sufficiently bright light sources that can serve as a backdrop against which the shadow can be observed. This requires calculating, for a realistic model of our Universe, the influence of the spacetime geometry on the surface brightness of distant light sources.

IV. SHADOW FOR OBSERVERS AT LARGE DISTANCES

In the preceding sections, we have calculated the shadow for any possible observer position, i.e. $r_{H1} < r_O < r_{H2}$ for static observers and $r_{H1} < r_O < \infty$ for comoving observers. In this section, we want to derive approximate formulas for the case that the observer is far away from the black hole, $r_O \gg m$. Physically this means that over a large part of a light ray to the observer the effect of the cosmic expansion dominates over the gravitational attraction

by the black hole. Clearly, for a static observer the condition $r_O \gg m$ can be satisfied only if $r_{H2} \gg m$. No such restriction is necessary for comoving observers. Therefore, we will consider the cases of static and comoving observers separately.

A. Static observer

As a preliminary note, we want to discuss an important difference between the black-hole shadow in Schwarzschild and Kottler spacetimes that arises from the fact that the former is asymptotically flat whereas the latter is not. In the case of the Schwarzschild metric, the angular radius of the shadow (as seen by a static observer) can be written as

$$\text{(Schwarzschild)} \quad \sin^2 \theta_{\text{stat}} = \frac{(1 - \frac{2m}{r_O}) b_{\text{cr}}^2}{r_O^2}, \quad (41)$$

where b_{cr} is the critical value of the impact parameter $b = cL/E$ corresponding to photons on unstable circular orbits filling the photon sphere. In the Schwarzschild metric, the radius of the photon sphere equals $3m$ and

$$\text{(Schwarzschild)} \quad b_{\text{cr}} = 3\sqrt{3}m; \quad (42)$$

see (9) with $\Lambda = 0$.

With increasing distance r_O , both the sine of the angular radius of the shadow and the angular radius itself tend to zero. This is because the denominator of the fraction in (41) increases while the factor in brackets in the numerator tends to unity. Therefore, for large distances the angular size of the shadow can be written as

$$\text{(Schwarzschild)} \quad \sin^2 \theta_{\text{stat}} \approx \frac{b_{\text{cr}}^2}{r_O^2}, \quad r_O \gg m. \quad (43)$$

This approach reduces the determination of the angular size of the shadow at large distances to the calculation of the critical value of the impact parameter: knowing the critical impact parameter, one gets an approximate value for $\sin \theta_{\text{stat}}$ after dividing by r_O . Bardeen [2] has used this approach for the more general case of the Kerr metric. In this case, the shadow is not circular; its shape for distant observers is determined by two impact parameters. Accordingly, the angular radii of the shadow can be approximately found by dividing these impact parameters by the (Boyer-Lindquist) radius coordinate r_O of the observer.

This method works for metrics that are asymptotically flat at infinity. The Kottler spacetime, however, is not asymptotically flat; the metric coefficient $f(r)$ does not tend to unity for large r . In this metric, the angular radius of the shadow (as seen by a static observer) can be written as

$$\text{(Kottler)} \quad \sin^2 \theta_{\text{stat}} = \frac{(1 - \frac{2m}{r_O} - \frac{\Lambda}{3} r_O^2) b_{\text{cr}}^2}{r_O^2}, \quad (44)$$

where the critical value of the impact parameter $b = cL/E$ is given by (9),

$$\text{(Kottler)} \quad b_{\text{cr}} = \frac{3\sqrt{3}m}{(1 - 9\Lambda m^2)^{1/2}}. \quad (45)$$

This value of the critical impact parameter for the Kottler metric is well known, see e.g. [22,23].

For $\Lambda \neq 0$, the dependence of the shadow size on r_O is very different from the Schwarzschild case. With increasing r_O , the denominator of the fraction in (44) increases, while the factor in brackets in the numerator tends to zero if r_O approaches its maximal value r_{H2} . Therefore for the Kottler spacetime the determination of the angular size of the shadow at large distances does not reduce to the calculation of the critical value of the impact parameter:

$$\text{(Kottler)} \quad \sin^2 \theta_{\text{stat}} \not\approx \frac{b_{\text{cr}}^2}{r_O^2}, \quad r_O \gg m. \quad (46)$$

Note that in the above argument we implicitly assume that Λ is sufficiently small such that $r_{H2} \gg m$ because otherwise the condition $r_O \gg m$ could not hold for a static observer.

Let us now approximate formula (14) for static observers at large distances, $r_O \gg m$. As this requires $r_{H2} \gg m$, the equation for the outer horizon

$$1 - \frac{2m}{r_{H2}} - \frac{\Lambda}{3} r_{H2}^2 = 0 \quad (47)$$

can be approximated by

$$1 - \frac{\Lambda}{3} r_{H2}^2 \approx 0, \quad r_{H2}^2 \approx \frac{3}{\Lambda}. \quad (48)$$

Combining (48) with the condition that $r_{H2} \gg m$, we obtain a restriction on the value of Λ :

$$\Lambda m^2 \ll 1. \quad (49)$$

With $r_O \gg m$ and (49), Eq. (14) for the angular size of the shadow for static observers can be simplified to

$$\sin^2 \theta_{\text{stat}} \approx \frac{27m^2}{r_O^2} \left(1 - \frac{\Lambda}{3} r_O^2\right) \quad \text{for } r_O \gg m. \quad (50)$$

B. Comoving observer

In the case of comoving observers, the condition $r_O \gg m$ does not require any restriction on r_{H2} because such observers can exist both inside and outside the cosmological horizon.

For $r_O \gg m$, Eq. (36) for the angular size of the shadow for comoving observers is simplified to

$$\sin \theta_{\text{comov}} \approx \frac{\sqrt{27}m}{r_O} \left(\sqrt{1 - \frac{27H_0^2 m^2}{c^2}} + \frac{H_0 r_O}{c} \right). \quad (51)$$

Here we have to choose the + sign in (36) because the condition $r_O \gg m$ implies that $r_O > 3m$. For $r_O \rightarrow \infty$, we recover, of course, (40).

If we want to apply the approximation formula (51) for comoving observers near r_{H2} , we need to assume that $r_{H2} \gg m$. As we already know, this requires (48) and (49) which read, in terms of H_0 ,

$$r_{H2} \approx \frac{c}{H_0}, \quad \frac{H_0^2 m^2}{c^2} \ll 1. \quad (52)$$

Then we obtain from (51) the approximate formula

$$\sin \theta_{\text{comov}} \approx 2\sqrt{27} \frac{H_0 m}{c} \quad \text{for } r_O \approx r_{H2} \gg m. \quad (53)$$

V. CONCLUSIONS

In this paper, we have calculated the angular radius of the shadow for an observer that is comoving with the cosmic expansion in Kottler (Schwarzschild–de Sitter) spacetime. As far as we know, the shadow for a comoving observer in an expanding universe was not calculated before. The resulting expression is presented in formula (36).

Quite generally, the cosmic expansion has a magnifying effect on the shadow. This is in agreement with the well-known fact that the image of an object is magnified by aberration if the observer moves away from the object. Moreover, it is found that the shadow shrinks to a finite value if the comoving observer approaches infinity, see formula (40). As a consequence, even the most distant black holes have a shadow whose angular radius is bigger than the bound given by (40).

The magnification effect caused by a cosmological constant of $\Lambda \approx 10^{-46} \text{ km}^{-2}$ is rather strong: for a black hole of 10^{10} Solar masses we found that even in the limit $r_O \rightarrow \infty$ the angular radius of the shadow is not smaller than $\theta_{\text{comov}} \approx 0.1$ microarcseconds. This is only two orders of magnitude beyond the resolvability of present-day VLBI technology. However, there are two caveats. Firstly, our

calculations were done in the Kottler spacetime in which the cosmic expansion is driven by the cosmological constant only. It has to be checked how our results are to be modified in a more realistic spacetime model, taking the matter content of the Universe into account. Secondly, the shadow can be observed only if there is a backdrop of sufficiently bright light sources against which the shadow can be seen as a dark disk. Therefore, when doing the calculations in a realistic model of our Universe one would also have to estimate the influence of the spacetime geometry on the surface brightness of light sources.

Note that a comoving observer in the Kottler spacetime can exist behind the cosmological event horizon, in contrast to a static observer, and that he can see the shadow until he ends up at future null infinity. Simplified approximative formulas for distant observers, both static and comoving, are presented in Sec. IV.

In an Appendix, we demonstrate that our results for the angular size of the shadow can be also obtained by using formulas for the deflection angle in Kottler spacetime derived by Lebedev and Lake [14,15].

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APPENDIX: DERIVATION OF THE ANGULAR SIZE OF THE SHADOW USING RESULTS OF LEBEDEV AND LAKE

Here we show how to obtain formulas (26), (36) and (51) using results from Lebedev and Lake [14] (cf. [15]) on the deflection of light in the Kottler (Schwarzschild–de Sitter) spacetime.

(i) Formula (128) from [14] is

$$\cos(\alpha_{\text{radial}}) = \frac{\sqrt{\frac{f(r_0)}{r_0^2} - \frac{f(r)}{r^2}} + \left(\sqrt{\frac{f(r_0)}{r_0^2}} + \sqrt{\frac{f(r_0)}{r_0^2} - \frac{f(r)}{r^2}} \right) \left(\frac{U^{r2}}{f(r)} - \frac{U^r}{\sqrt{f(r)}} \sqrt{1 + \frac{U^{r2}}{f(r)}} \right)}{\left(\sqrt{\frac{f(r_0)}{r_0^2}} \sqrt{1 + \frac{U^{r2}}{f(r)}} - \sqrt{\frac{f(r_0)}{r_0^2} - \frac{f(r)}{r^2}} \frac{U^r}{\sqrt{f(r)}} \right) \left(\sqrt{1 + \frac{U^{r2}}{f(r)}} - \frac{U^r}{\sqrt{f(r)}} \right)}. \quad (A1)$$

Here α_{radial} is the angle, as measured by a radially moving observer in the Kottler spacetime, between a radial light ray and a light ray with r_0 as radial coordinate of the point of closest approach. The observer's radial

coordinate is r and the observer's four-velocity is $U = (U^t, U^r, 0, 0)$. In this appendix, we follow Lebedev and Lake and choose units such that $c = 1$. Then the function $f(r)$ is

$$f(r) = 1 - \frac{2m}{r} - H_0^2 r^2. \quad (\text{A2})$$

Note that in our notation the observer's radial coordinate is denoted r_O which should not be confused with the r_0 of Lebedev and Lake.

To rederive the formula for the sine of the angular radius of the shadow, $\sin \theta_{\text{comov}}$, we have to choose the minimal coordinate distance as $r_0 = 3m$, the observer's position as $r = r_O$, and the observer's four-velocity as $U^r = H_0 r_O$, see (33). With these substitutions α_{radial} in (A1) gives us θ_{comov} .

To rewrite (A1) in a more compact way, we use the equation

$$U^t = \frac{1}{\sqrt{f(r_O)}} \sqrt{1 + \frac{U^{r^2}}{f(r_O)}}, \quad (\text{A3})$$

and we introduce the notation

$$w_1 \equiv \sqrt{1 - \frac{h^2(3m)}{h^2(r_O)}}, \quad w_t \equiv \sqrt{f(r_O)} U^t, \\ w_r \equiv \frac{U^r}{\sqrt{f(r_O)}}. \quad (\text{A4})$$

Here the function $h(r)$ is defined by

$$h^2(r) = \frac{r^2}{f(r)} = \frac{r^2}{1 - \frac{2m}{r} - H_0^2 r^2}, \quad (\text{A5})$$

similar to our previous work [9].

The quantities w_1, w_t, w_r are introduced for convenience only and have no specific physical meaning. In particular, they are not the covariant components of any four-vector. Note that the expression $h^2(3m)/h^2(r_O)$ coincides with $\sin^2 \theta_{\text{stat}}$ from formula (14). With this notation the expression (A1) takes the following form (compare with Eq. (129) of [14]):

$$\cos \theta_{\text{comov}} = \frac{w_1 + (1 + w_1)w_r(w_r - w_t)}{(w_t - w_1 w_r)(w_t - w_r)}. \quad (\text{A6})$$

From $U^\mu U_\mu = -1$, we find that $w_t^2 - w_r^2 = 1$, hence

$$(w_t - w_1 w_r)(w_t - w_r) = 1 + (1 + w_1)w_r(w_r - w_t). \quad (\text{A7})$$

This allows us to rewrite (A6) as

$$\cos \theta_{\text{comov}} = \frac{w_1 + z_w}{1 + z_w}, \quad z_w \equiv (1 + w_1)w_r(w_r - w_t). \quad (\text{A8})$$

As a consequence,

$$\sin^2 \theta_{\text{comov}} = 1 - \frac{(w_1 + z_w)^2}{(1 + z_w)^2} = \frac{1 + 2z_w - w_1^2 - 2w_1 z_w}{(1 + z_w)^2} \\ = \frac{(1 - w_1^2)(w_t - w_r)^2}{(1 + z_w)^2} = \frac{1 - w_1^2}{(w_t - w_1 w_r)^2}. \quad (\text{A9})$$

Note that the numerator $1 - w_1^2$ coincides with $\sin^2 \theta_{\text{stat}}$ from formula (14).

From these results, we can reobtain a formula for the shadow in the form of (26) in the following way. We substitute $U^r = H_0 r_O$ into (A3) and (A4) and obtain:

$$w_t = \frac{1}{\sqrt{1 - v^2}}, \quad w_r = \frac{v}{\sqrt{1 - v^2}}. \quad (\text{A10})$$

Here we have introduced for compactness the variable v in the same way as in (34) and (35). With these expressions, we can transform formula (A9) to (26) with v given by (35).

Lebedev and Lake assume that the radial coordinate of the observer is bigger than the radial coordinate of the point of the closest approach of the light ray. In our problem, this means that $r_O > 3m$. Therefore we get from their approach Eq. (26) only with the minus sign in the denominator. If $r_{H1} < r_O < 3m$, we have to use Eq. (26) with the plus sign because $\cos \theta_{\text{stat}} < 0$ in this case.

(ii) If we want to obtain a formula for the shadow in the form of (36), we can perform the following transformation:

$$\sin \theta_{\text{comov}} = \frac{\sin \theta_{\text{stat}}}{w_t \pm w_1 w_r} = \frac{\sin \theta_{\text{stat}}(w_t \mp w_1 w_r)}{w_t^2 - w_1^2 w_r^2} \\ = \frac{\sin \theta_{\text{stat}}(w_t \mp w_1 w_r)}{1 + w_r^2 \sin^2 \theta_{\text{stat}}}. \quad (\text{A11})$$

By substituting $U^r = H_0 r_O$ into (A3) and (A4) we recover (36).

(iii) Our approximative formula (51) for the size of the shadow as seen by a distant observer can also be derived using formula (132) from [14]:

$$\cos(\alpha_{\text{comoving}}) = \frac{\sqrt{\frac{f(r_O)}{r_0^2} - \frac{f_{m=0}(r)}{r^2}} - \sqrt{\frac{f(r_O)}{r_0^2}} \sqrt{\frac{\Lambda}{3}} r}{\sqrt{\frac{f(r_O)}{r_0^2}} - \sqrt{\frac{f(r_O)}{r_0^2} - \frac{f_{m=0}(r)}{r^2}} \sqrt{\frac{\Lambda}{3}} r}, \quad (\text{A12})$$

where

$$f_{m=0}(r) = 1 - \frac{\Lambda}{3} r^2. \quad (\text{A13})$$

Then the four-velocity of a comoving observer in static coordinates is

$$U_{\text{comoving}}^\mu = \left(\frac{1}{f_{m=0}(r)}, \sqrt{\frac{\Lambda}{3}} r, 0, 0 \right). \quad (\text{A14})$$

Substituting $r_0 = 3m$, $r = r_O$ and $\sqrt{\frac{\Lambda}{3}} = H_0$ we rewrite (A12) in our notation as

$$\cos \theta_{\text{comov}} = \frac{w_1 - H_0 r_O}{1 - w_1 H_0 r_O}, \quad (\text{A15})$$

where

$$w_1 = \sqrt{1 - \frac{9m^2 f_{m=0}(r_O)}{r_O^2 f(3m)}}. \quad (\text{A16})$$

By applying the transformation

$$\begin{aligned} \sin \theta_{\text{comov}} &= \frac{\sqrt{1 - w_1^2} \sqrt{1 - H_0^2 r_O^2}}{1 - w_1 H_0 r_O} \\ &= \frac{\sqrt{1 - w_1^2} \sqrt{1 - H_0^2 r_O^2} (1 + w_1 H_0 r_O)}{1 - w_1^2 H_0^2 r_O^2} \end{aligned} \quad (\text{A17})$$

and simplifying $\sin \theta_{\text{comov}}$ with $r_O \gg m$, we recover (51).

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