Cosmological bouncing solutions in extended teleparallel gravity theories

Álvaro de la Cruz-Dombriz,[1,2] Gabriel Farrugia,[3,4] Jackson Levi Said,[3,4] and Diego Sáez-Chillón Gómez[5,6]

[1]*Cosmology and Gravity Group, University of Cape Town, 7701 Rondebosch, Cape Town, South Africa*
[2]*Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, Cape Town, South Africa*
[3]*Department of Physics, University of Malta, Msida MSD 2080, Malta*
[4]*Institute of Space Sciences and Astronomy, University of Malta, Msida MSD 2080, Malta*
[5]*Institut d'Estudis Espacials de Catalunya (IEEC), 08034 Barcelona, Spain*
[6]*Institut de Ciències de l'Espai (ICE, CSIC), Carrer de Can Magrans s/n, Campus UAB, 08193 Bellaterra (Barcelona), Spain*

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In the context of extended teleparallel gravity theories with a $3+1$-dimensional Gauss-Bonnet analog term, we address the possibility of these theories reproducing several well-known cosmological bouncing scenarios in a four-dimensional Friedmann-Lemaître-Robertson-Walker geometry. We study which types of gravitational Lagrangians are capable of reconstructing bouncing solutions provided by analytical expressions for symmetric, oscillatory, superbounce, matter bounce, and singular bounce. Some of the Lagrangians discovered are analytical at the origin, having both Minkowski and Schwarzschild vacuum solutions. All these results open up the possibility for such theories to be competitive candidates of extended theories of gravity in cosmological scales.

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I. INTRODUCTION

The appearance of cosmological bouncing scenarios has attracted much attention in recent years due to its power to avoid the unnaturalness of our Universe to be created from a big bang initial singularity. In such scenarios, the Universe contracts until reaching a minimal nonzero radius, bounces off, and then expands (cf. [1] and references therein for a recent thorough review on the subject), similarly to the so-called ekpyrotic scenario [2]. Apart from the possibility of preventing the initial cosmological singularity, the so-called big bounce cosmologies have been shown to provide competitive scenarios to the standard inflationary paradigm [3–7]; in some realizations, such as the so-called matter bounce scenario, they have been shown to generate a nearly scale-invariant power spectrum as in the usual inflationary models [8–19].

As such, bouncing solutions in the context of gravitational theories beyond the Einsteinian paradigm have also drawn some attention in recent literature. First, the idea of ekpyrotic/cyclic cosmologies was analyzed in the framework of $f(R)$ gravities in Ref. [20]. Related works on bounce cosmology reconstruction from scalar-tensor $f(R)$ theories can be found in [15,21]. Other recent proposals, such as unimodular $f(R)$ gravity, were studied in [22] where the authors studied well-known cosmological bouncing models and investigated which era of the whole bouncing model is responsible for the cosmological perturbations. Also, in Ref. [23] the authors investigated the superbounce and the loop quantum cosmological ekpyrosis bounce for $f(R)$, $f(G)$, and $f(T)$ gravity theories, showing the qualitative similarity of the different effective gravities realizing the two bouncing cosmologies mentioned above. Moreover, by performing a linear perturbation analysis, it was shown that the obtained solutions are conditionally or fully stable. In addition, in $f(T)$ extended teleparallel gravity, Ref. [24] focused on the simplest version of a matter bounce and studied the scalar and tensor modes of subsequent cosmological perturbations. Results showed that scalar metric perturbations lead to a background-dependent sound speed, which might be distinguishable from the Einsteinian prediction, and a scale-invariant primordial power spectrum, which is consistent with cosmological observations. Indeed, one can infer that extensions of teleparallel gravity reach a wide and rich family of solutions in the context of cosmology [25]. In addition, some alternative formulations of teleparallel gravity, where the Palatini approach is applied, show some interesting properties when dealing with the boundary terms in the Euclidean action [26].

In the present work, we investigate several well-established bouncing scenarios in the framework of extended teleparallel gravity theories with nonvanishing boundary terms, dubbed $f(T, T_G)$ theories, where an analog of the Gauss-Bonnet invariant is assumed in the framework of teleparallel gravity [27]. The existence of cosmological solutions has already been studied in such theories, where some reconstruction methods were implemented (see Ref. [28], and cf. [29] for a thorough review on

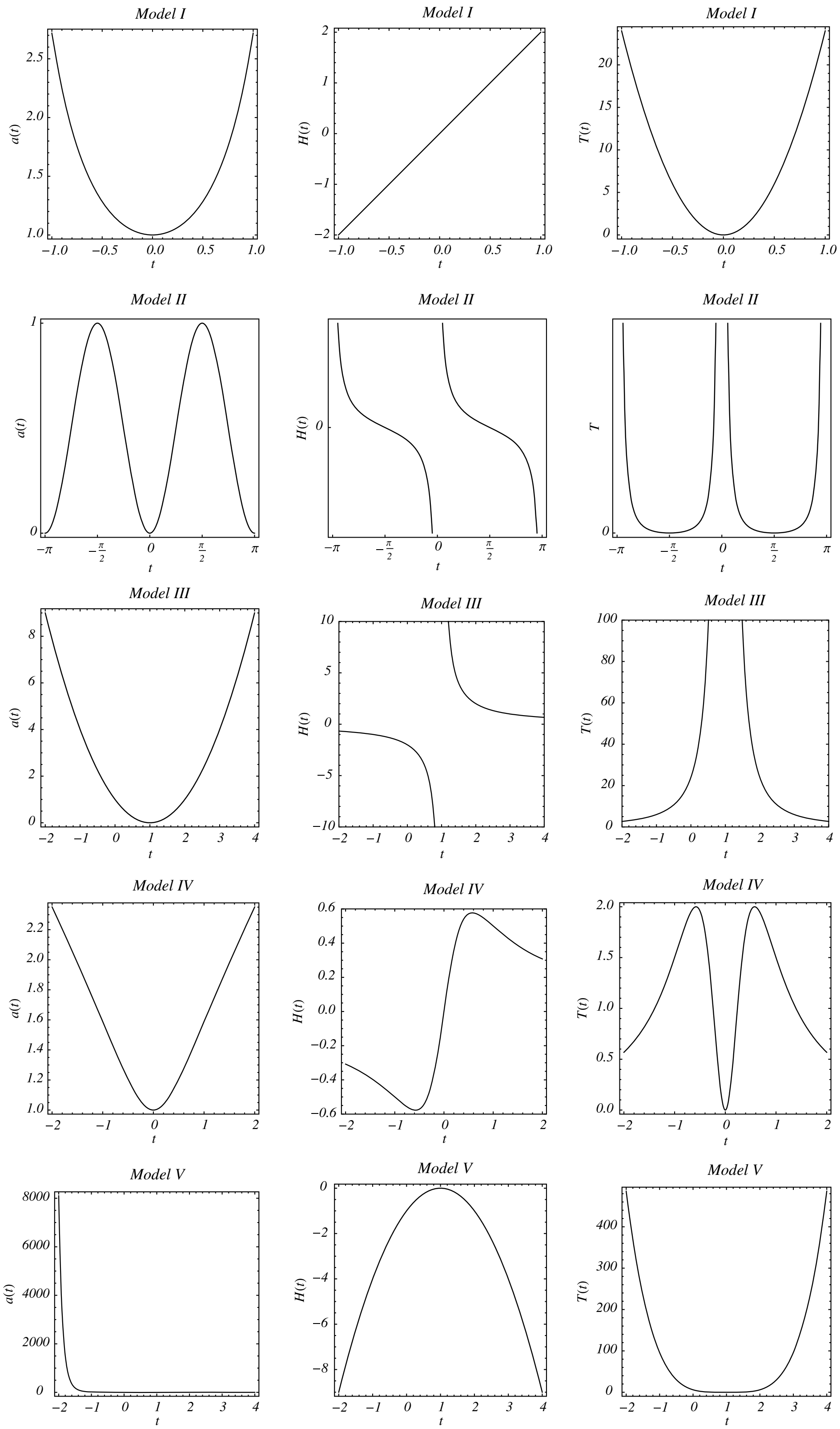

FIG. 1. A sample for each model analyzed in this paper, where the evolution of the scale factor, the Hubble parameter, and the torsion tensor are depicted for a particular set of free parameters of the models. The bouncing character of the solutions is clearly shown, as are the possible singularities that may occur.

the existence of cosmological solutions in such theories). Static spherically symmetric solutions and their relation to other extensions of Teleparallel General Relativity (TEGR) have also been analyzed [30]. Thus, we use the reconstruction method for $f(T,T_G)$ theories to realize such cosmological bouncing scenarios. In particular, we apply this method to bouncing cosmologies in spatially flat, four-dimensional, Friedmann-Lemaître-Robertson-Walker (FLRW) geometries to paradigmatic bouncing solutions, such as the symmetric bounce [15]; an oscillatory bouncing solution where the universe oscillates through a series of expansions and contractions [16–18]; a generic power-law bounce which has been studied, for instance, in the context of modified Gauss-Bonnet gravity [31] and loop quantum cosmology scenarios [32,33]; the superbounce [23,34,35]; the matter bounce scenario [8–19], also dubbed critical density bouncing, which naturally arises in loop quantum cosmology scenarios [36–40] and provides a viable alternative scenario to inflation compatible with Planck data; and finally, the so-called singular bounce [19,41–43] in which the Hubble radius is infinite as $t \to -\infty$ and gradually decreases until a minimal size, but near the bouncing point ($t = 0$), it increases and blows up at exactly the bouncing point. In this latter case, after the bouncing point, the Hubble radius eventually decreases gradually. This is different in comparison to other bouncing cosmologies, and this can be seen by directly comparing the behavior of the Hubble radius in Fig. 1.

For the sake of clarity, further technical details about each bouncing scenario are provided in upcoming sections. Moreover, in the bulk of the article, we show that these bouncing solutions can be obtained in universes filled with one standard fluid provided with a constant equation of state (EOS) and, when possible, in vacuum configurations. Thus, our results show that within this class of theories, bounce realizations do not rely on the existence of extra matter fields nor on the existence of fluids with an equation of state which violates the null energy condition, as is the case in other bouncing scenarios [44]. The types of gravitational actions analyzed in the paper are based on the idea of extending teleparallel gravity in such a way that the corresponding Lagrangians are constructed as separable (or multiplicative) additional terms, which perturbatively (depending on the extra parameters in the Lagrangians and the involved exponents) can be negligible in some scales but relevant in others (cosmological).

The paper is organized as follows: In Sec. II we briefly note the general features of the $f(T,T_G)$ gravity theories and the state of the art within this class of extended theories of gravity. There we provide the key equations to consider so the reconstruction mechanism can be performed. In the following sections, we briefly discuss the main features of the bouncing models to be studied and determine the $f(T,T_G)$ gravity theories capable of realizing such cosmologies. Thus, in Sec. III we discuss the reconstruction of the symmetric bounce. Then, Sec. IV addresses the same issue when the desired model to be reconstructed is a paradigmatic oscillatory bounce solution when parametrized as a squared sine function. Finally, Secs. V–VII are devoted to studying the possibility of reconstruction of superbounce, matter, and singular bounce solutions, respectively. We give our conclusions in Sec. VIII. At the end of the paper, the scale factor, the Hubble parameter, and the torsion scalar are depicted in Fig. 1 for a particular set of free parameters for the five bouncing models under consideration. The bouncing character of the solutions is clearly shown, as are the possible singularities that may occur.

Throughout the paper we use the following conventions: the Weitzenböck connection as defined in Sec. II will be denoted by $\tilde{\Gamma}^{\alpha}_{\mu\nu}$; D_μ shall represent the covariant derivative with respect to the usual Levi-Civita connection $\Gamma^{\alpha}_{\mu\nu}$; Greek indices such as $\mu, \nu, \ldots$ shall refer to spacetime indices, whereas Latin letters $a, b, c\ldots$ refer to the tetrad indices associated with the tangent space.

II. $f(T,T_G)$ THEORIES

Teleparallel gravities can be expressed by defining the mathematical objects known as vierbeins $e_a(x^\mu)$,

$$\mathrm{d}x^\mu = e_a{}^\mu \omega^a, \qquad \omega^a = e^a{}_\mu \mathrm{d}x^\mu, \tag{2.1}$$

which relate the spacetime of a manifold with its tangent space at every point x^μ,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}\omega^a\omega^b, \tag{2.2}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$ holds for the Minkowskian metric. In addition, the tetrads have the following properties:

$$e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu, \qquad e_a{}^\mu e^b{}_\mu = \delta^b_a. \tag{2.3}$$

The theory is constructed as a gauge theory of the translation group, leading to the so-called Weitzenböck connection, defined as

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = e_a{}^\alpha \partial_\nu e^a{}_\mu = -e^a{}_\mu \partial_\nu e_a{}^\alpha. \tag{2.4}$$

Whereas the Riemann tensor becomes null under this connection, the torsion does not vanish, so the torsion scalar is defined as

$$T = T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu} = \frac{1}{4} T^\lambda{}_{\mu\nu} T_\lambda{}^{\mu\nu} + \frac{1}{2} T^\lambda{}_{\mu\nu} T^{\nu\mu}{}_\lambda - T^\rho{}_{\mu\rho} T^{\nu\mu}{}_\nu, \tag{2.5}$$

where the torsion tensor is given by

$$T^\alpha{}_{\mu\nu} = \tilde{\Gamma}^{\alpha}_{\mu\nu} - \tilde{\Gamma}^{\alpha}_{\nu\mu} = e_a{}^\alpha(\partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu) \tag{2.6}$$

and

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha}T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha}T^{\beta\mu}{}_{\beta}). \tag{2.7}$$

Here the contorsion is given by the difference between the Weitzenböck and the Levi-Civita connection:

$$K^{\alpha}{}_{\mu\nu} = \tilde{\Gamma}^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\alpha}{}_{\nu} + T_{\nu}{}^{\alpha}{}_{\mu} - T^{\alpha}{}_{\mu\nu}). \tag{2.8}$$

Thus, the gravitational action for TEGR is solely given by the torsion scalar (2.5),

$$S_G = -\frac{1}{2\kappa^2}\int eT\mathrm{d}^4x, \tag{2.9}$$

where $\kappa^2 = 8\pi G_N$, with G_N the usual gravitational constant, and $e = \det(e^{a}{}_{\mu})$. This action is equivalent to the Einstein-Hilbert action since the relation of the torsion scalar and the Ricci curvature is given by

$$R = -T - 2D_{\mu}T^{\nu\mu}{}_{\nu}. \tag{2.10}$$

Here the last term is a total derivative and can be dropped out of the action. However, any nonlinear function of the torsion scalar will not be equivalent to $f(R)$ gravity as shown in Eq. (2.10).

Recently, the analog to the Gauss-Bonnet term with the Weitzenböck connection was found by using the above expression:

$$G = T_G + B_G, \tag{2.11}$$

where the Gauss-Bonnet invariant is defined as

$$G = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{2.12}$$

The second term in (2.11) is a total derivative, such that T_G can be expressed as follows [27]:

$$T_G = (K^{\alpha}{}_{\gamma\beta}K^{\gamma\lambda}{}_{\rho}K^{\mu}{}_{\epsilon\sigma}K^{\epsilon\nu}{}_{\varphi} - 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}K^{\gamma}{}_{\epsilon\sigma}K^{\epsilon\nu}{}_{\varphi} + 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}K^{\gamma\nu}{}_{\epsilon}K^{\epsilon}{}_{\sigma\varphi} + 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}K^{\gamma\nu}{}_{\sigma,\varphi})\delta^{\beta\rho\sigma\varphi}_{\alpha\lambda\mu\nu}. \tag{2.13}$$

Hence, any linear action on T_G leads to a total derivative, as in the metric case. Nevertheless, beyond the linear order the equivalence is broken. Here we are focusing on theories containing, in the action, such types of functions beyond the linear order on T_G,

$$S = S_G + S_m = \int e(f(T,T_G) + 2\kappa^2\mathcal{L}_m)\mathrm{d}^4x. \tag{2.14}$$

By assuming a spatially flat FLRW metric, T and T_G can be expressed in terms of the Hubble parameter as follows:

$$T = 6H^2; \qquad T_G = 24H^2(\dot{H} + H^2). \tag{2.15}$$

Note that T_G coincides with its GR counterpart, G, when assuming a spatial flatness. Then, the FLRW equations yield [28]

$$f - 12H^2 f_T - T_G f_{T_G} + 24H^3\dot{f}_{T_G} = 2\kappa^2\rho_m, \tag{2.16}$$

$$f - 4(3H^2 + \dot{H})f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H}T_G\dot{f}_{T_G} + 8H^2\ddot{f}_{T_G} = -2\kappa^2 p_m. \tag{2.17}$$

Here we have assumed the standard definition for the energy-momentum tensor $\mathcal{T}_{\mu}{}^{\nu} = \frac{e_a{}^{\nu}}{e}\frac{\delta\mathcal{L}_\mathrm{m}}{\delta e_a{}^{\mu}}$, together with the assumption of a perfect fluid. Combination of the previous equations leads to the usual conservation of the energy-momentum tensor. Thus, by using the above tools, we consider several types of bouncing solutions in the next sections, and some classes of Lagrangians are reconstructed.

III. BOUNCING COSMOLOGY I: EXPONENTIAL EVOLUTION

Let us start by considering a bouncing solution described by a scale factor with an exponential evolution,

$$a(t) = A\exp\left(\alpha\frac{t^2}{t_*{}^2}\right), \tag{3.1}$$

where t_* is some arbitrary time, and $A > 0$ and $\alpha > 0$ are constants. By evaluating the expression at $t = 0$, it can easily be concluded that $a(0) = A$. In such cases, H is given by

$$H = \frac{2\alpha t}{t_*{}^2}. \tag{3.2}$$

This means that a bounce is located at $t = 0$ since $H < 0$ for $t < 0$, $H = 0$ at $t = 0$ and $H > 0$ for $t > 0$. Consequently, T and T_G are given by

$$T = 6H^2 = \frac{24\alpha^2 t^2}{t_*{}^4}, \qquad T_G = \frac{8\alpha}{t_*{}^2}T + \frac{2T^2}{3}. \tag{3.3}$$

Furthermore, the scale factor can be solely expressed in terms of the torsion scalar T as

$$a(T) = a(0)\exp\left(\frac{Tt_*{}^2}{24\alpha}\right) = a(0)\exp\left(\alpha\frac{T}{T_*}\right), \tag{3.4}$$

where $T_* \equiv T(t = t_*) = 24\alpha^2/t_*{}^2$.

In order to solve the Friedmann equations for this model, some particular ansatz for the gravitational Lagrangian are considered. Before doing so, let us first simplify the stress-energy component of the field equations by setting $a(t_0) = 1$ at some arbitrary time $t_0 > 0$, such that

$$t_0{}^2 = -\frac{t_*{}^2}{\alpha}\ln A. \tag{3.5}$$

Since α is positive, the equation yields real values for the time t_0 if and only if $0 < A < 1$. Thus, as long as the value of A is restricted in the range $A \in (0, 1)$, one can define the parameters $T_0 \equiv T(t=t_0)$ and $\Omega_{w_i,0} \equiv \Omega_{w_i}(t=t_0)$, which may describe their present time values.

A. $f(T,T_G) = g(T) + h(T_G)$

By assuming a gravitational Lagrangian of the type $f(T,T_G) = g(T) + h(T_G)$, the Friedmann equation becomes

$$g + h - 2Tg_T - T_G h_{T_G} + 24H^3 h_{T_G T_G} \dot{T}_G = T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}. \quad (3.6)$$

Since the scale factor can be expressed in terms of T only, the differential equation (3.6) can be split into a pair of equations as follows:

$$g - 2Tg_T = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \exp\left[-\frac{(1+w_i)T{t_*}^2}{8\alpha}\right], \quad (3.7)$$

$$h - T_G h_{T_G} + 2h_{T_G T_G}\left({T_G}^2 - \frac{4T^4}{9}\right) = 0. \quad (3.8)$$

Then, the solution for g yields

$$g(T) = c_1\sqrt{T} + T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} [\sqrt{\pi} x_i \mathrm{erf}(x_i) + \mathrm{e}^{-{x_i}^2}], \quad (3.9)$$

where c_1 is a constant of integration (which corresponds to the DGP term) and $x_i \equiv \sqrt{\frac{T(1+w_i){t_*}^2}{8\alpha}}$.

In order to solve Eq. (3.8) for h, we rewrite the equation as follows:

$$h(x) - \frac{x\left(x^2 + 84\frac{\alpha}{{t_*}^2}x + 288\frac{\alpha^2}{{t_*}^4}\right)}{2\left(x + 12\frac{\alpha}{{t_*}^2}\right)^2} h'(x) + \frac{24\frac{\alpha}{{t_*}^2}x^2}{x + 12\frac{\alpha}{{t_*}^2}} h''(x) = 0, \quad (3.10)$$

where

$$x \equiv \sqrt{6T_G + 144\frac{\alpha^2}{{t_*}^4}} - 12\frac{\alpha}{{t_*}^2}. \quad (3.11)$$

Here we have used (3.3). The solution of Eq. (3.10) yields

$$h(x) = x\left(x + 24\frac{\alpha}{{t_*}^2}\right)c_1 + c_2 \exp\left(\frac{x{t_*}^2}{48\alpha}\right)\left[-6\sqrt{\frac{\alpha}{{t_*}^2}}x\left(x + 48\frac{\alpha}{{t_*}^2}\right) + \sqrt{3}x\left(x + 24\frac{\alpha}{{t_*}^2}\right)F\left(\frac{1}{4}\sqrt{\frac{x{t_*}^2}{3\alpha}}\right)\right], \quad (3.12)$$

where c_1 and c_2 are constants of integration and $F(z)$ is the Dawson integral which is defined as

$$F(z) \equiv \mathrm{e}^{-z^2}\int_0^z \mathrm{e}^{y^2}\mathrm{d}y. \quad (3.13)$$

The next step would be to check the existence of vacuum solutions, i.e., $f(0,0) = g(0) + h(0) = 0$. In this case, $h(T_G = x = 0)$ is equal to 0. Thus, we require $g(T=0) = 0$. However, the resulting limit is

$$g(0) = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)}, \quad (3.14)$$

which is trivially satisfied in vacuum, where $\Omega_{w_i,0} = 0$.

B. $f(T,T_G) = Tg(T_G)$

When considering T rescaling-type models, the Friedmann equations become

$$g - T_G g_{T_G} + \frac{4T^2}{3} g_{T_G} - 2g_{T_G T_G}\left({T_G}^2 - \frac{4T^4}{9}\right) = -\frac{T_0}{T}\sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \exp\left[-\frac{(1+w_i)T{t_*}^2}{8\alpha}\right]. \quad (3.15)$$

Similar to the previous case, the Friedmann equation has to be fully expressed in terms of T_G. By using the substitution $x \equiv \sqrt{6T_G + 144\frac{\alpha^2}{{t_*}^4}} - 12\frac{\alpha}{{t_*}^2}$, the resulting equation is given by

$$g(x) + \frac{x({t_*}^4x^2 + 36\alpha {t_*}^2 x - 288\alpha^2)}{2(12\alpha + {t_*}^2 x)^2} g'(x) - \frac{24\alpha x^2}{12\alpha + {t_*}^2 x} g''(x) = -2T_0 \sum_i \frac{\Omega_{w_i,0} A^{-3(1+w_i)}}{x} \exp\left[-\frac{(1+w_i)x{t_*}^2}{16\alpha}\right]. \quad (3.16)$$

The solution to this equation can be found by a power series, such that

$$g(x) = \sum_{n=0}^{\infty} a_n x^{n+r} + g_{\mathrm{part}}(x), \quad (3.17)$$

where $g_{\mathrm{part}}(x)$ corresponds to the particular solution of the inhomogeneous equation, while the exponent $r = 1$ and $r = -1/2$ for the homogeneous equation with the recurrence relation:

$$(n+r){t_*}^4 a_{n-2} + 12[4 + (n+r-1)(11 - 4n - 4r)]{t_*}^2 \alpha a_{n-1} - 288(n+r-1)(2n+2r+1)\alpha^2 a_n = 0, \quad (3.18)$$

with $a_{-2} = a_{-1} = 0$, which yields the following solutions for the homogeneous part of the equation:

$$g_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \frac{7c_1}{48 t_*^5}\left[12(t_*^5 x^2 + 84\alpha t_*^3 x - 864\alpha^2 t_*) + \sqrt{\frac{3\pi}{\alpha x}}(t_*^6 x^3 + 108\alpha t_*^4 x^2 + 20736\alpha^3) \times \mathrm{erf}\left(\sqrt{\frac{t_*^2 x}{48\alpha}}\right)\exp\left(\frac{t_*^2 x}{48\alpha}\right)\right], \tag{3.19}$$

$$g_2(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}} = \frac{t_*^6 x^3 + 108\alpha t_*^4 x^2 + 20736\alpha^3}{216\alpha^2 t_*^2 \sqrt{x}} \times \exp\left[\frac{t_*^2 x}{48\alpha}\right] c_1, \tag{3.20}$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\mathrm{d}t$ is the error function. Both solutions satisfy the vacuum constraint $g_i(0) = 0$. Finally, the particular solution can be found by using a Green function $G(x,s) = \frac{g_1(s)g_2(x) - g_2(s)g_1(x)}{W(s)}$, where $W(s)$ holds for the Wronskian. Nevertheless, it is not possible to find an analytical solution. However, the resulting particular solution at the $T \to 0$ limit (which corresponds to $x \to 0$) is defined since the integral would be equal to zero and since both $g_1(0) = g_2(0) = 0$ would imply $g_{\mathrm{part}}(0) = 0$.

C. $f(T,T_G) = T_G g(T)$

For a similar type of model where a rescaling of T_G is included, the Friedmann equation becomes

$$-\frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \exp\left[-\frac{(1+w_i)T t_*^2}{8\alpha}\right], \tag{3.21}$$

whose solution is given by

$$g(T) = c_1 + \sum_i \frac{3\Omega_{w_i,0} T_0 A^{-3(1+w_i)}}{8T^2}\left\{\left[1 - \frac{T(1+w_i)t_*^2}{8\alpha}\right] \times \exp\left[-\frac{T(1+w_i)t_*^2}{8\alpha}\right] - \left[\frac{T(1+w_i)t_*^2}{8\alpha}\right]^2 \mathrm{Ei}\left[-\frac{T(1+w_i)t_*^2}{8\alpha}\right]\right\}, \tag{3.22}$$

where c_1 is an integration constant and $\mathrm{Ei}(z)$ is the exponential integral function, $\mathrm{Ei}(z) \equiv -\int_{-z}^{\infty}\frac{\mathrm{e}^{-y}}{y}\mathrm{d}y$. The solution can be expressed in a more compact form by making use of the substitution variable $x_i \equiv -\frac{T(1+w_i)t_*^2}{8\alpha}$, which results in

$$g(x) = c_1 + \sum_i \frac{3T_0 \Omega_{w_i,0} A^{-3(1+w_i)}(1+w_i)^2 t_*^4}{512\alpha^2} \times \left[\frac{1+x_i}{x_i^2}\mathrm{e}^{x_i} - \mathrm{Ei}(x_i)\right]. \tag{3.23}$$

For vacuum solutions, we require the Lagrangian $f(T = x = 0) = 0$. In this case, we find

$$f(0) = -\frac{T_0}{8}\sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \times \left[1 + 3w_i + 3(1+w_i)\frac{\mathrm{e}^{x_i}}{x_i}\right]\Bigg|_{x_i \to 0}. \tag{3.24}$$

In the case of vacuum, the condition is satisfied, although the resulting Lagrangian would only be composed of the Gauss-Bonnet term, which effectively does not contribute to the field equations. On the other hand, for single fluids, the Lagrangian diverges even for the case $w_i = -1$. For a cosmological-constant-like fluid, x_i is already 0 by definition; hence, it requires more attention when taking the limit. The Lagrangian as $T \to 0$ in the presence of this fluid becomes

$$f(0) = \frac{\Omega_{-1,0} T_0}{4} + \frac{3\alpha\Omega_{-1,0} T_0}{T t_*^2}\Bigg|_{T \to 0}, \tag{3.25}$$

which diverges as $T \to 0$.

D. $f(T,T_G) = -T + T_G g(T)$

If we consider a model expressed as a correction to the teleparallel action with a rescaling of T_G, the Friedmann equation becomes

$$T - \frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \exp\left[-\frac{(1+w_i)T t_*^2}{8\alpha}\right], \tag{3.26}$$

whose solution is identical to the previous model with an extra particular solution of the form

$$g_{\mathrm{part}}(T) = -\frac{3}{4T}. \tag{3.27}$$

The contribution to the Lagrangian in this case is given by

$$f_{\mathrm{part}}(T, T_G) = -\frac{3T_G}{4T} = -\frac{6\alpha}{t_*^2} - \frac{T}{2}, \tag{3.28}$$

which in the $T \to 0$ limit reduces to a nonzero constant. As in the previous case, there is no trivial vacuum solution.

E. $f(T,T_G) = -T + \mu(\frac{T}{T_0})^\beta(\frac{T_G}{T_{G,0}})^\gamma$

For TEGR with a power-law model, the Friedmann equation becomes

$$\mu\left(\frac{T}{T_0}\right)^{\beta+\gamma}\left(\frac{12\alpha+T{t_*}^2}{12\alpha+T_0{t_*}^2}\right)^\gamma\left[1-2\beta-\gamma+\frac{24\alpha\beta\gamma}{12\alpha+T{t_*}^2}\right.$$
$$\left.+\frac{48\gamma(\gamma-1)\alpha(6\alpha+T{t_*}^2)}{(12\alpha+T{t_*}^2)^2}\right]+T=T_0\sum_i\Omega_{w_i,0}a^{-3(1+w_i)}, \tag{3.29}$$

where μ, β, and γ are constants. By evaluating the expression at current times, the value of μ is found to be

$$\mu=\frac{T_0(-1+\sum_i\Omega_{w_i,0})}{1-2\beta-\gamma+\frac{24\alpha\beta\gamma}{12\alpha+T_0{t_*}^2}+\frac{48\gamma(\gamma-1)\alpha(6\alpha+T_0{t_*}^2)}{(12\alpha+T_0{t_*}^2)^2}}$$
$$\equiv\frac{T_0(-1+\sum_i\Omega_{w_i,0})}{\nu}, \tag{3.30}$$

where ν is defined as the denominator. The expression is true provided that $\nu\neq 0$. To obtain vacuum solutions, the following condition must be satisfied:

$$\beta+\gamma>0. \tag{3.31}$$

Since when $t=0$, $T=0$, another condition has to be obeyed:

$$\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}=0. \tag{3.32}$$

This condition is satisfied in the case of vacuum. However, this condition cannot be satisfied if fluids exist since $\Omega_{w_i,0}, A>0$. Therefore, we only consider the former. From the definition of μ, the Friedmann equation simplifies to

$$\left(\frac{T}{T_0}\right)^{\beta+\gamma}\left(\frac{12\alpha+T{t_*}^2}{12\alpha+T_0{t_*}^2}\right)^\gamma\left[1-2\beta-\gamma+\frac{24\alpha\beta\gamma}{12\alpha+T{t_*}^2}\right.$$
$$\left.+\frac{48\gamma(\gamma-1)\alpha(6\alpha+T{t_*}^2)}{(12\alpha+T{t_*}^2)^2}\right]=\nu\frac{T}{T_0}. \tag{3.33}$$

In order to determine which values of β and γ satisfy this equation, the equation must hold at all times. The equation trivially holds when $t=0$; however, this must also hold for arbitrary time. Thus, the time-dependent (torsion scalar) terms must cancel. The only possible solution is $\gamma=0$, $\beta=1$, which sets $\nu=-1$ and consequently $\mu=T_0$. However, this implies that $f(T,T_G)=0$, which is not physical. Thus, a power-law solution with a TEGR contribution cannot describe this bouncing cosmology.

IV. BOUNCING MODEL II: OSCILLATORY MODEL

The second bouncing model we are considering here is described by an oscillatory scale factor:

$$a(t)=A\sin^2\left(B\frac{t}{t_*}\right), \tag{4.1}$$

where $t_*>0$ is some reference time, and $A>0$ and $B>0$ are dimensionless constants. Here, the restrictions for t_* and B can be relaxed to simply be nonzero. The choice here helps us define the subsequent parameters and ease the analysis for determining which models obey the necessary conditions. For such a model, the Hubble parameter is

$$H=\frac{2B}{t_*}\cot\left(B\frac{t}{t_*}\right). \tag{4.2}$$

This oscillatory model produces two different bounces. For times $t=\frac{n\pi t_*}{B}$, $n\in\mathbb{Z}$, the model describes the time when the universe reaches a crunch ($a=0$, $H\to-\infty$) and is reborn with a big bang ($a=0$, $H\to\infty$). This corresponds to a superbounce. On the other hand, for times $t=\frac{(2n+1)\pi t_*}{2B}$, $n\in\mathbb{Z}$, the universe reaches its maximum size with no further expansion ($a=A, H=0$). This also corresponds to a bounce since H transitions from positive to negative. In this case, T and T_G are

$$T=6H^2=\frac{24B^2}{{t_*}^2}\cot^2\left(B\frac{t}{t_*}\right),\qquad T_G=4T\left(\frac{T}{12}-\frac{2B^2}{{t_*}^2}\right).$$

Using these definitions, the scale factor can be expressed in terms of the torsion scalar to be

$$a(T)=\frac{A}{1+\frac{T{t_*}^2}{24B^2}}. \tag{4.4}$$

Before solving for the gravitational actions considered above, we first assume the existence of some time $t_0>0$ at which the scale factor is 1,

$$\frac{1}{A}=\sin^2\left(B\frac{t_0}{t_*}\right). \tag{4.5}$$

Since $0\leq\sin^2(x)\leq 1$, $A>1$ is required. If we set our first big bang to be at $t=0$, for instance, and the first maximum of the expansion at $t=\frac{\pi t_*}{2B}$, then the present time would lie at $t_0=\frac{t_*\sin^{-1}(\frac{1}{A})}{B}$. With this time defined, the remaining present-time parameters—such as the $\Omega_{w_i,0}$ density parameters, current times torsion scalar $T_0\equiv T(t=t_0)=\frac{24(A^2-1)B^2}{{t_*}^2}$, and so on—can be defined.

A. $f(T,T_G)=g(T)+h(T_G)$

For this type of model, the Friedmann equation in

$$g+h-2Tg_T-T_G h_{T_G}-\frac{4T(2T^2-3T_G)(T{t_*}^2-12B^2)}{9{t_*}^2}h_{T_GT_G} = T_0\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}\left(1+\frac{T{t_*}^2}{24B^2}\right)^{3(1+w_i)}. \tag{4.6}$$

Before solving the Ordinary differential equation (ODE), we point out that since the T and T_G are related through a quadratic expression (4.3), the torsion scalar can be expressed in terms of T_G as

$$T=\frac{12B^2}{{t_*}^2}\left(1-\sqrt{1+\frac{T_G{t_*}^4}{48B^4}}\right), \tag{4.7}$$

where the plus solution is neglected since it is inconsistent at maximum size periods (i.e., when $T=T_G=0$). In doing so, the ODE can be separated into two ODEs, for g and for h. This is only possible provided their respective ODEs result in a constant. It turns out that, similar to other bouncing models, this constant drops out of the Lagrangian, so it is neglected from the solutions. The resulting ODEs to solve are the following:

$$g-2Tg_T=T_0\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}\left(1+\frac{T{t_*}^2}{24B^2}\right)^{3(1+w_i)}, \tag{4.8}$$

$$h-T_G h_{T_G}-\frac{4T(2T^2-3T_G)(T{t_*}^2-12B^2)}{9{t_*}^2}h_{T_GT_G}=0, \tag{4.9}$$

where T is expressed in terms of T_G in the ODE for h.

The solution for $g(T)$ is given by

$$\begin{aligned}g(T)=c_1\sqrt{T}-T_0\sum_i\frac{\Omega_{w_i,0}A^{-3(1+w_i)}}{69120B^6}\Bigg\{&T^3{t_*}^6{}_2F_1\left(\frac{5}{2},-3w_i;\frac{7}{2};-\frac{T{t_*}^2}{24B^2}\right)\\&+120B^2T{t_*}^2\left[72B^2{}_2F_1\left(\frac{1}{2},-3w_i;\frac{3}{2};-\frac{T{t_*}^2}{24B^2}\right)+T{t_*}^2{}_2F_1\left(\frac{3}{2},-3w_i;\frac{5}{2};-\frac{T{t_*}^2}{24B^2}\right)\right]\\&-69120B^6{}_2F_1\left(-\frac{1}{2},-3w_i;\frac{1}{2};-\frac{T{t_*}^2}{24B^2}\right)\Bigg\},\end{aligned} \tag{4.10}$$

where c_1 is an integration constant corresponding to the DGP contribution in the Lagrangian. The solution for $h(T_G)$ leads to

$$h(x)=x(x-2)c_1+\left[2(8-3x)\sqrt{x}-3\sqrt{2}x(x-2)\tan^{-1}\left(\sqrt{\frac{x}{2}}\right)\right]c_2, \tag{4.11}$$

where $c_{1,2}$ are integration constants and $x\equiv 1-\sqrt{1+\frac{T_G{t_*}^4}{48B^4}}$. In the vacuum limit $T_G\to 0$ (or equivalently, $x\to 0$), $h(0)=0$, whereas in the limit $T\to 0$, the resulting function leads to $g(0)=T_0\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}$. Hence, $f(0,0)=0$ is only possible in the absence of fluids, $\Omega_{w_i,0}=0$, as usual.

B. $f(T,T_G)=Tg(T_G)$

For a TEGR rescaling model, the Friedmann equation is given by

$$g+\left(\frac{4T^2}{3}-T_G\right)g_{T_G}+\frac{4T(2T^2-3T_G)(T{t_*}^2-12B^2)}{9{t_*}^2}g_{T_GT_G} = -\frac{T_0}{T}\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}\left(1+\frac{T{t_*}^2}{24B^2}\right)^{3(1+w_i)}. \tag{4.12}$$

Using Eq. (4.7), the ODE can be expressed fully in terms of T_G and hence can be solved for g. To simplify the ODE, we make a change of variables by introducing the variable $x\equiv 1-\sqrt{1+\frac{T_G{t_*}^4}{48B^4}}$. This results in

$$g(x)+\frac{x(x^2-5x-2)}{2(x-1)^2}g'(x)+\frac{x^2(x+2)}{x-1}g''(x) = -T_0\sum_i\Omega_{w_i,0}\frac{{t_*}^2 2^{-3w_i-5}}{3B^2x}\left(\frac{A}{x+2}\right)^{-3(1+w_i)}. \tag{4.13}$$

The homogeneous solution can be expressed by a power series, leading to

$$g(x)=\sum_{n=0}^{\infty}a_nx^n, \tag{4.14}$$

where the following recurrence relation is obtained:

$$2(n-1)^2 a_n + n a_{n-2} + (n-5)(2n-1)a_{n-1} = 4n(n+1)a_{n+1}, \tag{4.15}$$

with $a_{-2} = a_{-1} = 0$. A general solution to the recurrence relation cannot be found. Nonetheless, the first few terms of the series are found to be

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{3}{8}a_1,$$
$$a_4 = 0, \quad a_5 = \frac{21}{640}a_1, \quad a_6 = -\frac{11}{1600}a_1. \tag{4.16}$$

Thus, the first solution to the homogeneous equation is

$$g_1(x) = \sum_{n=0}^{\infty} a_n x^n = a_1\left(x - \frac{3x^3}{8} + \frac{21x^5}{640} - \frac{11x^6}{1600} + \cdots\right), \tag{4.17}$$

where a_1 takes the role of the integration constant. In order to find the second solution, one can use Abel's identity, although only in certain intervals [45]. By using Abel's identity, we obtain

$$g_2(x) = C g_1(x) \int^x \frac{(1-\eta)}{\sqrt{\eta}(2+\eta){g_1}^2(\eta)} \mathrm{d}\eta, \tag{4.18}$$

where C is an integration constant. However, the above homogeneous solution is only applicable for $x \in (0,1) \cup (1,\infty)$, as $g_1(x)$ and its derivative are not continuous at $x = 0$ and $x = 1$, respectively. Finally, since the power series is not expressed in terms of some analytical function, integrating over an infinite series is intractable. We also point out that, in the vacuum limit, $g_1(0) = 0$ although nothing can be inferred about $g_2(0)$.

C. $f(T, T_G) = T_G g(T)$

For a Teleparallel Gauss-Bonnet (TEGB) rescaling, the Friedmann equation is given by

$$-\frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \left(1 + \frac{T{t_*}^2}{24B^2}\right)^{3(1+w_i)}. \tag{4.19}$$

To simplify this equation, we introduce a change of variables defined by $x \equiv 1 + \frac{T{t_*}^2}{24B^2}$, which leads to

$$(x-1)^3 g_x = \sum_i \xi_{w_i} x^{3(1+w_i)}, \tag{4.20}$$

where $\xi_{w_i} \equiv -\frac{3}{4}\left[\frac{{t_*}^2}{24B^2}\right]^2 T_0 \Omega_{w_i,0} A^{-3(1+w_i)}$. Depending on the value of w_i, we have different particular solutions. Because the sum is finite, the sum of the particular solutions corresponding to each w_i will be the general solution.

1. Case 1: $w \neq n/3$, $n \in \mathbb{Z}$, $n \geq -1$

For this set of values, the solution is given by

$$g(x) = \frac{\xi_{w_i}}{2} x^{1+3w_i} \left\{ -\frac{1}{3w_i(x-1)^3} \left[(x-1)(2-x+3w_i x) \times {}_2F_1\left(1,1;1-3w_i;\frac{1}{1-x}\right) + x(4+3w_i-5x-3w_i x) \times {}_2F_1\left(1,2;1-3w;\frac{1}{1-x}\right)\right] + \frac{2}{1+3w_i} \right\}. \tag{4.21}$$

For every w_i, the Lagrangian diverges in the vacuum limit.

2. Case 2: $w = n/3$, $n \in \mathbb{Z}$, $n \geq -1$

For the remaining set of values, we solve the ODE as follows:

$$(x-1)^3 g_x = \xi_{w_i} x^{3+n}, \tag{4.22}$$

where the summation is suppressed for simplicity. Next, we define the variable $y \equiv x - 1$ to transform the ODE into

$$g_y = \xi_{w_i} \frac{(y+1)^{3+n}}{y^3}. \tag{4.23}$$

Since $n \in \mathbb{Z}$, $n \geq -1$, by the binomial theorem, the binomial term can be expanded as

$$(y+1)^{3+n} = \sum_{k=0}^{3+n} \binom{n}{k} y^k. \tag{4.24}$$

Therefore, the resulting solution is given by

$$g(y) = \xi_{w_i} \sum_{k=0}^{3+n} \binom{n}{k} \int y^{k-3} \mathrm{d}y. \tag{4.25}$$

For these values, the Lagrangian diverges in the vacuum limit.

D. $f(T, T_G) = -T + T_G g(T)$

For a TEGB rescaling with TEGR, the Friedmann equation is given by

$$T - \frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}, \tag{4.26}$$

whose solution is

$$g(T) = c_1 - \frac{3}{4T} + h(T), \tag{4.27}$$

where c_1 is a constant of integration corresponding to the Gauss-Bonnet contribution in the Lagrangian and $h(T)$ is the solution found in the previous model. In the vacuum limit, the Lagrangian is

$$f(0,0)=\frac{6B^2}{t_*{}^2}+T_Gh(T)|_{T,T_G\to 0}. \tag{4.28}$$

Following the discussions in the previous section, the last term is finite only in vacuum, leading to $T_Gh(T)|_{T,T_G\to 0}=0$. However, since B, $t_*>0$, the Lagrangian does not satisfy the vacuum condition. Therefore, this model cannot describe the oscillating cosmology while obeying the vacuum condition.

E. $f(T,T_G)=-T+\mu(\frac{T}{T_0})^\beta(\frac{T_G}{T_{G,0}})^\gamma$

For a power-law model with a TEGR contribution, the Friedmann equation becomes

$$\begin{aligned}T&+\mu\left(\frac{T}{T_0}\right)^\beta\left(\frac{T_G}{T_{G,0}}\right)^\gamma\left[1-2\beta-\gamma+\beta\gamma\left(2-\frac{4T^2}{3T_G}\right)\right.\\&\left.-\frac{\gamma(\gamma-1)}{9t_*{}^2}\frac{4T(2T^2-3T_G)(Tt_*{}^2-12B^2)}{T_G{}^2}\right]\\&=T_0\sum_i\Omega_{w_i,0}a^{-3(1+w_i)}.\end{aligned} \tag{4.29}$$

The Lagrangian satisfies the vacuum condition as long as $\beta+\gamma>0$. At times when $T=T_G=0$ (which occurs at the maximum universe size), the Friedmann equation yields the following condition:

$$0=\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}. \tag{4.30}$$

However, this is possible only in vacuum. Then, the Friedmann equation can be evaluated at current times to evaluate μ,

$$\begin{aligned}\mu&=\frac{-T_0}{1-2\beta-\gamma+\beta\gamma\left(2-\frac{4T_0{}^2}{3T_{G,0}}\right)-\frac{\gamma(\gamma-1)}{9t_*{}^2}\frac{4T_0(2T_0{}^2-3T_{G,0})(Tt_*{}^2-12B^2)}{T_{G,0}{}^2}}\\&\equiv-\frac{T_0}{\nu},\end{aligned} \tag{4.31}$$

where $\nu\neq 0$ is defined as the denominator. This simplifies the Friedmann equation to

$$\begin{aligned}T&-\frac{T_0}{\nu}\left(\frac{T}{T_0}\right)^\beta\left(\frac{T_G}{T_{G,0}}\right)^\gamma\left[1-2\beta-\gamma+\beta\gamma\left(2-\frac{4T^2}{3T_G}\right)\right.\\&\left.-\frac{\gamma(\gamma-1)}{9t_*{}^2}\frac{4T(2T^2-3T_G)(Tt_*{}^2-12B^2)}{T_G{}^2}\right]=0.\end{aligned} \tag{4.32}$$

This equation has to be satisfied at all times. Trivially, this is satisfied when $T=T_G=0$ and at $t=t_0$, so other time instances are assumed. This allows for a rearranging of the equation to

$$\begin{aligned}&\left(\frac{T}{T_0}\right)^{\beta-1}\left(\frac{T_G}{T_{G,0}}\right)^\gamma\left[1-2\beta-\gamma+\beta\gamma\left(2-\frac{4T^2}{3T_G}\right)\right.\\&\left.-\frac{\gamma(\gamma-1)}{9t_*{}^2}\frac{4T(2T^2-3T_G)(Tt_*{}^2-12B^2)}{T_G{}^2}\right]=\nu.\end{aligned} \tag{4.33}$$

Since ν is a constant, all time-dependent (or, equivalently, the torsional and TEGB terms) must vanish. This is possible for the following cases: $\beta=-1$, $\gamma=1$ and $\beta=1$, $\gamma=0$. In the former case, although it leads to a nontrivial Lagrangian, the vacuum condition is not satisfied. On the other hand, the latter is the TEGR result, which leads to $\nu=-1$ and, consequently, a zero Lagrangian which is nonphysical. Thus, there is no Lagrangian which describes the oscillating cosmology while obeying the vacuum condition.

In a Lagrangian composed of the TEGR term with DGP and Gauss-Bonnet terms, the resulting Friedmann equation is given by

$$T=T_0\sum_i\Omega_{w_i,0}a^{-3(1+w_i)}. \tag{4.34}$$

However, evaluating at times when the universe size is maximum (i.e., $T=T_G=0$) yields the previous restriction on the omega parameters:

$$0=\sum_i\Omega_{w_i,0}A^{-3(1+w_i)}, \tag{4.35}$$

which is only possible in vacuum. If this is assumed, this sets $T=0$ at all times, which is clearly not the case. Thus, this Lagrangian composition cannot describe the oscillating cosmology.

V. BOUNCING MODEL III: POWER-LAW MODEL

For this section, we consider a scale factor of the form

$$a(t)=\left(\frac{t_s-t}{t_0}\right)^{2/c^2}, \tag{5.1}$$

where t_s represents the time at which the bounce occurs, $t_0>0$ is an arbitrary time parameter which defines the scale factor to be 1 when $t=t_s+t_0$, and c is a constant. In this case, we have the following expressions

$$H=-\frac{2}{c^2}\frac{1}{t_s-t},\qquad T=6H^2,\qquad T_G=\frac{2T^2}{3}\left(1-\frac{c^2}{2}\right). \tag{5.2}$$

Furthermore, the scale factor can be solely expressed in terms of the torsion scalar as

$$a(T)=\left(\frac{24}{Tc^4t_0{}^2}\right)^{1/c^2}. \tag{5.3}$$

Before continuing further, we make note of the following. We define the quantities $t_*\equiv t-t_s$ and $\alpha\equiv 2/c^2$. Thus, the scale factor becomes $a(t_*)=(t_*/t_0)^\alpha$, while the Hubble parameter, torsion, and teleparallel Gauss-Bonnet quantities become

$$H = \frac{\alpha}{t_*}, \qquad T = 6H^2 = 6\frac{\alpha^2}{t_*{}^2}, \qquad T_G = \frac{2T^2}{3}\left(1 - \frac{1}{\alpha}\right). \tag{5.4}$$

Note that at $t_* = t_0$, $T_0 \equiv T(t_* = t_0) = 6\alpha^2/t_0{}^2$. This simplifies the expression for the scale factor to

$$a(T) = \left(\frac{T_0}{T}\right)^{\alpha/2}. \tag{5.5}$$

This transformation effectively simplifies the model to a standard power-law model encountered in single-fluid-dominated universes with the difference being that multiple fluids are considered. In fact, the Friedmann equation remains unchanged since the time-dependent differentiations remain unchanged: $\dot{T} \equiv dT/dt = dT/dt_*$ and $\dot{T}_G \equiv dT_G/dt = dT_G/dt_*$. Hence, the resulting Friedmann equation is

$$f - 2Tf_T - T_G f_{T_G} - \frac{4T^3}{3\alpha} f_{TT_G} - \frac{8T^2 T_G}{3\alpha} f_{T_G T_G} = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}. \tag{5.6}$$

Let us now find the corresponding Lagrangians for this type of cosmology.

A. $f(T,T_G) = g(T) + h(T_G)$

For an additive-type model, with two functions g and h of the torsion scalar and TEGB term, respectively, the Friedmann equation simplifies to

$$g + h - 2Tg_T - T_G h_{T_G} - \frac{8T^2}{3\alpha} T_G h_{T_G T_G} = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}. \tag{5.7}$$

Note that when $\alpha = 1$ sets $T_G = 0$, one has to be careful in solving the Friedmann equation in this scenario. Thus, we solve the Friedmann equation for the cases when $\alpha = 1$ and $\alpha \neq 1$ separately.

For $\alpha = 1$, the function h results in a constant, say, $h(T_G) = h(0) = \mu$.[1] However, nothing can be inferred on the behavior of its derivatives, becoming degenerate. However, we can analyze the case when the derivatives are constant, i.e., $h'(0) = \beta$ and $h''(0) = \gamma$ for some constants β and γ. Here, the resulting Friedmann equation is

$$g - 2Tg_T = -\mu + T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}, \tag{5.8}$$

whose solution is given by

$$g(T) = c_1\sqrt{T} - \mu + B_j T \sqrt{\frac{T_0}{T}} \ln\left(\frac{T_0}{T}\right) + \sum_i A_i \left(\frac{T_0}{T}\right)^{-3\alpha(1+w_i)/2}, \tag{5.9}$$

for some integration constant c_1, whose term corresponds to the DGP term, with $A_i \equiv \frac{\Omega_{w_i,0} T_0}{1-3\alpha(1+w_i)}$ having $1-3\alpha(1+w_i) \neq 0 \,\forall\, i$ and $B_j \equiv \frac{\Omega_{w_j,0}}{2}$ obeying $1 - 3\alpha(1 + w_j) = 0 \,\exists\, j$. Next, we demand the vacuum condition $f(0,0) = g(0) + h(0) = 0$. This can be satisfied for various scenarios, for instance, in vacuum ($B_j = A_i = 0\ \forall\ i, j$), for a single fluid obeying the B_j condition, for fluids having EOS $w_i > -1\ \forall\ i$, and so on. Examples of functions obeying these sets of conditions include $h(T_G) = \sum_{n=1}^{i} \eta_n \exp(\xi_n T_G{}^n)$ for $i < \infty$ and constants η_n and ξ_n and $h(T_G) = \xi + \sum_{n=1}^{i} \eta_n T_G{}^n$ for $i < \infty$, where ξ and η_n are constants.

Lastly, another solution can be obtained for the case when $h'(0) = \beta$ and $h''(0) \to \infty$ with $T_G h''(T_G)|_{T_G \to 0} = \gamma$. In this case, the Friedmann equation reduces to

$$g - 2Tg_T = -\mu + \frac{8T^2}{3}\gamma + T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}. \tag{5.10}$$

The resulting solution is

$$g(T) = c_1\sqrt{T} - \frac{8\gamma T^2}{9} - \mu + B_j T \sqrt{\frac{T_0}{T}} \ln\left(\frac{T_0}{T}\right) + \sum_i A_i \left(\frac{T_0}{T}\right)^{-3\alpha(1+w_i)/2}, \tag{5.11}$$

where c_1, A_i, and B_j have the same definitions and conditions as the previous case. The only difference lies in the extra contribution of $-8\gamma T^2/9$ in the Lagrangian. Since in the $T \to 0$ limit this reduces to 0, the same vacuum conditions obtained previously can be applied. An example of a function with these properties is the function $h(T_G)$ such that $h''(T_G) = \sin(\alpha/T_G)$, for some constant $\alpha > 0$.

In principle, other solutions can be obtained under different conditions, say, $h'(0) \to \infty$ with $T_G h'(T_G)|_{T_G \to 0} = \beta$ and $h''(0) \to \infty$ with $T_G h''(T_G)|_{T_G \to 0} = \gamma$. However, since functions obeying these properties have not been found, these were not considered in the analysis.

For $\alpha \neq 1$, the Friedmann equation can be expressed fully in terms of T and T_G as follows:

[1]Note that, in principle, $h(0)$ could be divergent. However, in order to satisfy the vacuum condition, this would require that $g(0)$ also diverges and would need to cancel exactly. Thus, for simplicity, we consider only the finite case.

$$g + h - 2Tg_T - T_G h_{T_G} - \frac{4T_G{}^2}{\alpha - 1} h_{T_G T_G} = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}, \tag{5.12}$$

which can be split into the following system of equations:

$$g - 2Tg_T - T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2} = \lambda, \tag{5.13}$$

$$h - T_G h_{T_G} - \frac{4T_G{}^2}{\alpha - 1} h_{T_G T_G} = -\lambda. \tag{5.14}$$

Here, λ is a constant. Hence, the following solutions are obtained,

$$g(T) = \lambda + c_1\sqrt{T} + B_j T \sqrt{\frac{T_0}{T}} \ln\left(\frac{T_0}{T}\right) + \sum_i A_i \left(\frac{T_0}{T}\right)^{-3\alpha(1+w_i)/2}, \tag{5.15}$$

$$h(T_G) = -\lambda + T_G c_2 + T_G{}^{\frac{1-\alpha}{4}} c_3, \tag{5.16}$$

where $A_i \equiv \frac{\Omega_{w_i,0} T_0}{1-3\alpha(1+w_i)}$, with $1 - 3\alpha(1 + w_i) \neq 0\ \forall\ i$, $B_j \equiv \frac{\Omega_{w_j,0}}{2}$ obeying $1 - 3\alpha(1 + w_j) = 0 \exists j$, and $c_{1,2,3}$ are integration constants. The c_1 term corresponds to the DGP term, while c_2 corresponds to the Gauss-Bonnet term. We also remark that the contribution of λ is fictitious since the total contribution of λ to the Lagrangian f is zero.

In order to keep vacuum solutions where $g(0) = h(0) = 0$, the following conditions must be satisfied:

$$\alpha(1 + w_i) > 0 \Rightarrow w_i > -1 \quad \forall\ i, \tag{5.17}$$

$$\alpha < 1. \tag{5.18}$$

The first condition is obtained provided that a fluid obeying the A_i condition exists; otherwise the condition is not applicable in vacuum. On the other hand, the second condition holds provided that $c_3 \neq 0$. Otherwise, for cases for which $\alpha \geq 1$, c_3 can be set to zero, and we can obtain nontrivial solutions from the $g(T)$ contribution.

B. $f(T,T_G) = Tg(T_G)$

For a rescaling of the T model, the resulting Friedmann equation is

$$g + \left(T_G + \frac{4T^2}{3\alpha}\right) g_{T_G} + \frac{8T^2}{3\alpha} T_G g_{T_G T_G} = -\sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{\frac{-3(1+w_i)\alpha+2}{2}}. \tag{5.19}$$

Similar to the previous case, the equation yields different solutions depending on the values of α, i.e., between $\alpha = 1$ and $\alpha \neq 1$.

For $\alpha = 1$, $T_G = 0$, such that the function $g(T_G)$ results in a constant, namely, $g(T_G) = g(0) = \mu$.[2] Note that this automatically satisfies the vacuum condition $f(0, 0) = 0$.

For $\alpha \neq 1$, the Friedmann equation can be expressed fully in terms of T_G as

$$g + \frac{\alpha + 1}{\alpha - 1} T_G g_{T_G} + \frac{4T_G{}^2}{\alpha - 1} g_{T_G T_G} = -\sum_i \Omega_{w_i,0} \left(\frac{T_{G,0}}{T_G}\right)^{\frac{-3(1+w_i)\alpha+2}{4}}, \tag{5.20}$$

which yields a solution of the form

$$g(T_G) = c_1 T_G{}^{m_-} + c_2 T_G{}^{m_+} - \sum_i A_i \left(\frac{T_{G,0}}{T_G}\right)^{\frac{-3(1+w_i)\alpha+2}{4}}, \tag{5.21}$$

where

$$m_\pm \equiv \frac{1}{8}\left(3 - \alpha \pm \sqrt{\alpha^2 - 22\alpha + 25}\right), \tag{5.22}$$

$$A_i \equiv \frac{4(\alpha - 1)\Omega_{w_i,0}}{3\alpha^2(3w_i{}^2 + 7w_i + 4) - \alpha(21w_i + 19) + 6}, \tag{5.23}$$

provided that the denominator of A_i is nonzero $\forall i$, which is satisfied as long as

$$w_i \neq \frac{7 - 7\alpha - \sqrt{\alpha^2 - 22\alpha + 25}}{6\alpha}. \tag{5.24}$$

It is important to distinguish the different solutions stemming from the c_1 and c_2 contributions. This is done by examining the square-root term. The following subcases are obtained:

(i) $\alpha^2 - 22\alpha + 25 > 0$: When the square root is real, this gives two distinct power-law solutions. Here, the ranges of values of α obeying the condition are $0 < \alpha < 11 - 4\sqrt{6}$ and $\alpha > 11 + 4\sqrt{6}$. In this case, the vacuum condition is satisfied as long as $0 < \alpha < 11 - 4\sqrt{6}$; otherwise the integration constants are set to zero.

(ii) $\alpha^2 - 22\alpha + 25 = 0$: In this case, $m_+ = m_-$, effectively combining the two solutions into one,

[2]Similar to the additive case, $h(0)$ could diverge. Even though this satisfies the vacuum condition, one needs to satisfy the resulting Friedmann equation. Since here we are only interested in illustrating some possible solutions, this case is not considered, for simplicity.

$g(T_G) \propto T_G{}^{\frac{3-\alpha}{8}}$. The values of α giving rise to this particular case are $\alpha = 11 \pm 4\sqrt{6}$. In this case, the vacuum condition for this homogeneous solution is satisfied only for $\alpha = 11 - 4\sqrt{6}$ unless the constant of integration is zero for the other value.

(iii) $\alpha^2 - 22\alpha + 25 < 0$: When the square root becomes complex, the homogeneous solution has to be reexpressed using the relation

$$a^{b+ic} = a^b[\cos(c \ln a) + i \sin(c \ln a)]. \qquad (5.25)$$

For simplicity, we define $i\beta \equiv \sqrt{\alpha^2 - 22\alpha + 25}$. This leads to the following homogeneous solution:

$$g_{\rm hom}(T_G) = c_1 T_G{}^{\frac{3-\alpha}{8}} \cos\left(\frac{\beta}{8} \ln T_G\right) + c_2 T_G{}^{\frac{3-\alpha}{8}} \sin\left(\frac{\beta}{8} \ln T_G\right), \qquad (5.26)$$

where the constants of integration c_1 and c_2 have been redefined. Equivalently, the homogeneous solution can be expressed as

$$g_{\rm hom}(T_G) = c_3 T_G{}^{\frac{3-\alpha}{8}} \cos\left(c_4 + \frac{\beta}{8} \ln T_G\right), \qquad (5.27)$$

where $c_3 \equiv \sqrt{c_1{}^2 + c_2{}^2}$ and $c_4 = -\arctan(c_1/c_2)$. In this case, α lies in the range $11 - 4\sqrt{6} < \alpha < 11 + 4\sqrt{6}$. For $11 - 4\sqrt{6} < \alpha < 7$, the vacuum condition is satisfied, while for $7 \leq \alpha < 11 + 4\sqrt{6}$, the latter is satisfied when $c_3 = 0$; i.e., there would be no contribution from the homogeneous solution for this particular range of values.

On the other hand, the particular solution satisfies the vacuum condition as long as $w_i > -1$, $\forall i$.

C. $f(T,T_G) = T_G g(T)$

For this model, the Friedmann equation becomes

$$-\frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}, \qquad (5.28)$$

whose solution is given as

$$g(T) = c_1 + B_j \ln\left(\frac{T_0}{T}\right) + \sum_i \frac{A_i}{T^2} \left(\frac{T_0}{T}\right)^{-3\alpha(1+w_i)/2}, \qquad (5.29)$$

where $A_i \equiv \frac{3\Omega_{w_i,0} T_0}{2[4-3\alpha(1+w_i)]}$, with $4 - 3\alpha(1+w_i) \neq 0 \ \forall\ i$, $B_j \equiv \frac{3\Omega_{w_j,0}}{4T_0}$ obeying $4 - 3\alpha(1+w_j) = 0 \exists j$, and c_1 a constant of integration. The latter corresponds to the Gauss-Bonnet term in the Lagrangian, while the others are the nontrivial solutions. Trivially, the vacuum solution is also a solution since $A_i = B_j = 0 \ \forall i$ and $f(0,0) = 0$, although this leaves the Lagrangian to be the Gauss-Bonnet term, which does not contribute to the Friedmann equation and hence cannot be a source to the bounce. Thus, a fluid must exist. In this case, the vacuum condition is satisfied provided that any fluid obeying the A_i condition satisfies

$$1 + \frac{3\alpha}{2}(1 + w_i) > 0 \Rightarrow w_i > \frac{-2 - 3\alpha}{3\alpha} \quad \forall i. \qquad (5.30)$$

D. $f(T,T_G) = -T + T_G g(T)$

In this case, we enforce the presence of the TEGR term. This yields the following Friedmann equation:

$$T - \frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}, \qquad (5.31)$$

which yields the same solutions found in the previous section with an extra particular solution of the form

$$g_{\rm part}(T) = -\frac{3}{4T}. \qquad (5.32)$$

This introduces an extra contribution in the Lagrangian of the form $f_{\rm part}(T, T_G) = -\frac{3T_G}{4T}$. The vacuum conditions are identical to those found in the previous model since the new contributions reduce to zero in the $T \to 0$ limit.

E. $f(T,T_G) = -T + \mu(\frac{T}{T_0})^\beta (\frac{T_G}{T_{G,0}})^\gamma$

For this model, the Friedmann equation becomes

$$T + \mu \left(\frac{T}{T_0}\right)^\beta \left(\frac{T_G}{T_{G,0}}\right)^\gamma \left\{1 - 2\beta - \gamma - \frac{2\beta\gamma}{\alpha - 1} - \frac{4\gamma(\gamma-1)}{\alpha - 1}\right\} = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}. \qquad (5.33)$$

The constant μ can be found by evaluating the expression at $t_* = t_0$, resulting in

$$\mu = \frac{T_0(-1 + \sum_i \Omega_{w_i,0})}{1 - 2\beta - \gamma - \frac{2\beta\gamma}{\alpha-1} - \frac{4\gamma(\gamma-1)}{\alpha-1}}, \qquad (5.34)$$

provided that the denominator is nonzero. This simplifies the Friedmann equation to

$$\frac{T}{T_0} + \left(\frac{T}{T_0}\right)^\beta \left(\frac{T_G}{T_{G,0}}\right)^\gamma \left(-1 + \sum_i \Omega_{w_i,0}\right) = \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)\alpha/2}. \qquad (5.35)$$

At this point, we consider two distinct scenarios, $\alpha = 1$ and $\alpha \neq 1$. In the former case, $T_G = 0$ at all times. Thus, the ratio of $T_G/T_{G,0}$ is not properly defined in this instance. Nonetheless, since T_0 and $T_{G,0}$ are constants, one can alternatively define a Lagrangian of the form $f(T,T_G) = -T + \nu T^\beta T_G{}^\gamma$, for some constant ν. The Lagrangian is defined provided that $\gamma > 0$ (and by the vacuum condition, provided that $\beta \geq 0$). In this case, the field equation reduces to

$$T + \nu T^\beta T_G{}^\gamma \left[1 - 2\beta - \gamma - \frac{4T^2}{3T_G}\gamma(\beta + 2\gamma - 2)\right] = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)/2}. \tag{5.36}$$

For the field equation to give physical results, one needs to further restrict the parameters β and γ. The following cases are generated. If either $\gamma > 1$ or $\beta = 2 - 2\gamma$ (and since $\beta \geq 0$ and $\gamma > 0$, this restricts to $0 < \gamma \leq 1$), the equation simplifies to

$$1 = \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{\frac{-3(1+w_i)+2}{2}}. \tag{5.37}$$

Since the lhs is a constant, the Friedmann equation is satisfied only when there exists a single fluid with EOS $w = -1/3$. Lastly, if $\gamma = 1$, the Friedmann equation simplifies to

$$T - \frac{4}{3}\beta\nu T^{\beta+2} = T_0 \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{-3(1+w_i)/2}. \tag{5.38}$$

By evaluating the expression at $t_* = t_0$, the value of ν can be found, which is

$$\nu = \frac{1 - \sum_i \Omega_{w_i,0}}{\frac{4}{3}\beta T_0{}^{\beta+1}}, \tag{5.39}$$

which is defined when $\beta > 0$. Assuming this is the case, the Friedmann equation can be expressed as

$$1 = \left(1 - \sum_i \Omega_{w_i,0}\right) \left(\frac{T}{T_0}\right)^{\beta+1} + \sum_i \Omega_{w_i,0} \left(\frac{T_0}{T}\right)^{\frac{-3(1+w_i)+2}{2}}. \tag{5.40}$$

Since the lhs is a constant, the time (torsional)-dependent components must cancel. Irrespective of whether or not vacuum or fluids exist, the condition $\beta = -1$ must be satisfied, which originates from the first term on the lhs. However, this does not obey the vacuum condition $f(0,0) = 0$ since it requires $\beta \geq 0$. Now, if we consider $\beta = 0$, this would correspond to a Gauss-Bonnet contribution. However, from Eq. (5.38), this is only possible provided that a fluid exists with EOS $w = -1/3$. In fact, the result agrees with the case when $\beta = 2 - 2\gamma$ since when $\gamma = 1$, $\beta = 0$.

For the case when $\alpha \neq 1$, the Friedmann equation (5.35) can be expressed in terms of time as

$$\left(\frac{t_0}{t_*}\right)^2 + \left(\frac{t_0}{t_*}\right)^{2\beta+4\gamma} \left(-1 + \sum_i \Omega_{w_i,0}\right) = \sum_i \Omega_{w_i,0} \left(\frac{t_0}{t_*}\right)^{3(1+w_i)\alpha}. \tag{5.41}$$

The expression is satisfied for all times when the powers of t_* cancel, leading to the following conditions:

$$\beta + 2\gamma = 1, \tag{5.42}$$

$$3(1 + w_i)\alpha = 2, \quad \forall i. \tag{5.43}$$

The first condition restricts the powers of β and γ, while the second restricts the possible choice of fluids depending on the value of α. In the case of vacuum, the second condition is not present. One can easily conclude that, in a non-vacuum universe, since all fluids must satisfy the second condition, the only possibility is that only one fluid is present (i.e., two fluids with different EOS parameters are not achievable). This reduces the problem to a standard single-fluid-dominated universe (unless vacuum is considered). Furthermore, since $\alpha > 0$, the range of EOS parameter values is restricted within $w > -1$.

Lastly, given that the denominator of μ has to be nonzero, we get an extra condition,

$$\gamma \neq \frac{\alpha - 1}{3\alpha - 1}, \tag{5.44}$$

while the vacuum solution condition demands $\beta + 2\gamma > 0$, which is ensured by the first condition.

VI. BOUNCING MODEL IV: CRITICAL DENSITY

For this bouncing model, the scale factor takes the form

$$a(t) = A\left(\frac{3}{2}\rho_{\mathrm{cr}} t^2 + 1\right)^{1/3}, \tag{6.1}$$

where ρ_{cr} is the critical density and $A > 0$ is a dimensionless constant, which is the value of the scale factor at $t = 0$, i.e., $A = a(0)$. In this case, we find

$$H = \frac{2t\rho_{\mathrm{cr}}}{2 + 3t^2\rho_{\mathrm{cr}}}, \qquad T = 6H^2 = 6\left(\frac{2t\rho_{\mathrm{cr}}}{2 + 3t^2\rho_{\mathrm{cr}}}\right)^2, \qquad T_G = \frac{T^2}{3}\left(\frac{2}{t^2\rho_{\mathrm{cr}}} - 1\right). \tag{6.2}$$

Here, the bounce occurs at $t = 0$ since $H(t < 0) < 0$, $H(t = 0) = 0$, and $H(t > 0) > 0$. Let us first express the scale factor and T_G solely in terms of T. This can be achieved by expressing the time parameter t in terms of H. From the definition of H, we have

$$3t^2\rho_{\mathrm{cr}} H - 2t\rho_{\mathrm{cr}} + 2H = 0, \tag{6.3}$$

which is a quadratic in t whose solution is

$$3Ht = 1 - \sqrt{1 - \frac{6H^2}{\rho_{\rm cr}}}. \tag{6.4}$$

The correct sign was obtained by evaluating the expression at $t = 0$ since for $t = 0$, $H = 0$, thus leaving the negative sign as the physical solution. Therefore, the scale factor can be expressed in terms of T as

$$a(T) = A\left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{1/3}, \tag{6.5}$$

while the TEGB term is given by

$$T_G = -\frac{4T^2}{3} + 2T\rho_{\rm cr}\left(1 + \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right). \tag{6.6}$$

We also remark that the square root is always real. From the definition of H, one can easily find that the maximum value is achieved at the maximum turning point(s), which occurs at $t_{\rm max} = \pm\sqrt{\frac{2}{3\rho_{\rm cr}}}$, with $H_{\rm max} = \pm\sqrt{\frac{\rho_{\rm cr}}{6}}$. Thus, the maximum value for the torsion scalar is $T_{\rm max} = \rho$. Consequently, this leads to $0 \le T/\rho_{\rm cr} \le 1$. In addition, in order to simplify the field equations and express them to be compared to observational data, we define the current time $t_0 > 0$ where $a(t_0) = 1$,

$${t_0}^2 = \frac{2}{3\rho_{\rm cr}}\left(\frac{1}{A^3} - 1\right). \tag{6.7}$$

Since $\rho_{\rm cr} > 0$, this equation holds provided that $A < 1$, which will be assumed from here on. Then, the parameters $T_0 \equiv T(t = t_0) = 4A^3(1 - A^3)\rho_{\rm cr}$ and $\Omega_{w_i,0} \equiv \Omega_{w_i}(t = t_0)$ provide their values at the current time.

A. $f(T,T_G) = g(T) + h(T_G)$

For this type of model, the Friedmann equation becomes

$$\begin{aligned} &g + h - 2Tg_T - T_G h_{T_G} \\ &\quad + \frac{2T}{9}(-20T^3 + 12\rho T^2 - 51TT_G + 36\rho T_G)h_{T_G T_G} \\ &= T_0\sum_i \Omega_{w_i,0}A^{-3(1+w_i)}\left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{-(1+w_i)}. \end{aligned} \tag{6.8}$$

By using the above expressions for T and T_G, the following relation is found:

$$\begin{aligned} T = {}& \frac{3\rho_{\rm cr}}{16} - \frac{1}{32}\sqrt{96T_G\left(\frac{9\rho_{\rm cr}}{\sqrt{x}} - 8\right) + 27\rho^2\left(\frac{\rho_{\rm cr}}{\sqrt{x}} + 4\right) - 256x} \\ & + \frac{\sqrt{x}}{2}, \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} x \equiv {}& \frac{9{\rho_{\rm cr}}^2}{64} + \frac{1}{8}\sqrt[3]{-512{T_G}^3 + 1161{\rho_{\rm cr}}^2{T_G}^2 + 9\sqrt{3}\sqrt{-3072{\rho_{\rm cr}}^2{T_G}^5 + 4523{\rho_{\rm cr}}^4{T_G}^4 + 192{\rho_{\rm cr}}^6{T_G}^3}} \\ & + \frac{16{T_G}^2 - 9{\rho_{\rm cr}}^2 T_G}{2\sqrt[3]{-512{T_G}^3 + 1161{\rho_{\rm cr}}^2{T_G}^2 + 9\sqrt{3}\sqrt{-3072{\rho_{\rm cr}}^2{T_G}^5 + 4523{\rho_{\rm cr}}^4{T_G}^4 + 192{\rho_{\rm cr}}^6{T_G}^3}}} - T_G. \end{aligned} \tag{6.10}$$

Thus, Eq. (6.8) can be split in the following system of equations:

$$g - 2Tg_T = T_0\sum_i \Omega_{w_i,0}A^{-3(1+w_i)}\left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{-(1+w_i)}, \tag{6.11}$$

$$h - T_G h_{T_G} + \frac{2T}{9}(-20T^3 + 12\rho_{\rm cr}T^2 - 51TT_G + 36\rho_{\rm cr}T_G)h_{T_G T_G} = 0, \tag{6.12}$$

whose solution for $g(T)$ yields

$$\begin{aligned} g(T) = {}& c_1\sqrt{T} - \frac{\Omega_{0,0}\sqrt{T}T_0}{4A^3}\left[\frac{2\sqrt{T}}{\rho_{\rm cr}\left(\sqrt{1 - \frac{T}{\rho_{\rm cr}}} - 1\right)} + \frac{2\tan^{-1}\left(\sqrt{\frac{\rho_{\rm cr}}{T} - 1}\right)}{\sqrt{\rho_{\rm cr}}}\right] + \sum_i \frac{\Omega_{w_i,0}T_0}{2w_i}A^{-3(1+w_i)}\left(\sqrt{1 - \frac{T}{\rho_{\rm cr}}} + 1\right) \\ & \times\left[(1 + w_i)\,{}_2F_1\left(-\frac{1}{2}, w_i; \frac{1}{2}; 1 - \frac{2}{\sqrt{1 - \frac{T}{\rho_{\rm cr}}} + 1}\right) - \left(\frac{2}{\sqrt{1 - \frac{T}{\rho_{\rm cr}}} + 1}\right)^{-w_i}\right], \end{aligned} \tag{6.13}$$

where c_1 is an integration constant corresponding to the DGP term. Note that in the case of dust ($w = 0$), it has a distinct solution due to the divergence present in the summation. In this case, the vacuum condition implies

$$g(0) = \sum_j \Omega_{w_j,0} T_0 A^{-3(1+w_j)}, \tag{6.14}$$

where the summation includes the matter fluid.

The solution for $h(T_G)$ is more difficult to obtain analytically, as Eq. (6.12), together with the expression (6.12), requires numerical resources. Moreover, vacuum $f(0,0) = g(0) + h(0) = 0$ is only achieved in the absence of matter fluids for $g(T)$, while the absence of an analytical solution for $h(T_G)$ prevents us from going further with this analysis.

B. $f(T,T_G) = Tg(T_G)$

For a T rescaling model for some function $g(T_G)$, the Friedmann equation simplifies to

$$\begin{aligned} g + &\left(\frac{4T^2}{3} - T_G\right) g_{T_G} \\ &- \frac{2T}{9} g_{T_G T_G}(-20T^3 + 12\rho_{\rm cr}T^2 - 51TT_G + 36\rho_{\rm cr}T_G) \\ &= -\frac{T_0}{T}\sum_i \Omega_{w_i,0} A^{-3(1+w_i)}\left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{-(1+w_i)}. \end{aligned} \tag{6.15}$$

Let us rewrite this equation by defining the variable $x \equiv \sqrt{1 - \frac{T}{\rho_{\rm cr}}}$, which yields

$$\begin{aligned} &-3(x^2-1)^2[x(2x+1)(7x-2)-3]^2 g''(x) \\ &\quad + (x^2-1)(332x^7 + 778x^6 - 1036x^5 - 1013x^4 \\ &\quad + 164x^3 + 388x^2 - 36x - 9)g'(x) \\ &\quad + (8x^2 + x - 3)[x(2x+1)(7x-2)-3]^2 g(x) \\ &\quad = \sum_i \frac{\xi_i}{(x^2-1)}(8x^2 + x - 3)[x(2x+1)(7x-2)-3]^2 \\ &\quad \times (1+x)^{1+w_i}, \end{aligned} \tag{6.16}$$

where $\xi_i \equiv \frac{\Omega_{w_i,0}T_0}{\rho_{\rm cr}} 2^{-1-w_i} A^{-3(1+w_i)}$.

The solution for the homogeneous part of Eq. (6.16) is given by

$$\begin{aligned} g(x) = c_1&\left(1 - \frac{1}{2}x^2 + \frac{15}{648}x^4 + \frac{1851}{2430}x^5 + \cdots\right) \\ &+ c_2\left(x + \frac{a_1}{6}x^2 - \frac{769a_1}{648}x^4 + \frac{1706}{2430}x^5 + \cdots\right). \end{aligned} \tag{6.17}$$

For the vacuum condition, we require $f(0,0) = 0$. In this case, after multiplying the homogeneous solution by the torsion scalar, the condition is satisfied. Nevertheless, the general solution cannot be found analytically since the rhs of Eq. (6.16) is not necessarily a polynomial, depending on w_i. Furthermore, using the Wronskian and Green's function method is not feasible either since neither homogeneous solution is expressed analytically in terms of some known function. Nonetheless, the homogeneous solutions correspond to the vacuum solution, which satisfies the vacuum condition.

C. $f(T,T_G) = T_G g(T)$

For a T_G rescaling model, the Friedmann equation is given by

$$\begin{aligned} -\frac{4T^3}{3} g_T = T_0 &\sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \\ &\times \left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{-(1+w_i)}. \end{aligned} \tag{6.18}$$

By defining the variable $x \equiv 1 + \sqrt{1 - \frac{T}{\rho_{\rm cr}}}$, the equation becomes

$$(2-x)^3 g_x = (1-x)\sum_i \xi_{w_i} x^{w_i - 2}, \tag{6.19}$$

where $\xi_{w_i} \equiv -\frac{3T_0}{2\rho_{\rm cr}{}^2}\Omega_{w_i,0} 2^{-(1+w_i)} A^{-3(1+w_i)}$. The general solution is given by

$$\begin{aligned} g(x) = c_1 + \frac{1}{16}\xi_w x^w &\left[\frac{2}{(w-1)x} + \frac{1}{w} \right. \\ &+ \frac{4(\frac{x}{x-2})^{-w}{}_2F_1(2-w,-w;3-w;-\frac{2}{x-2})}{(w-2)(x-2)^2} \\ &\left. - \frac{(\frac{x}{x-2})^{-w}{}_2F_1(-w,-w;1-w;-\frac{2}{x-2})}{w}\right], \end{aligned} \tag{6.20}$$

which diverges for dust $w = 0$. For the case of a pressureless fluid, the solution reduces to

$$g(x) = c_1 + \frac{\xi_0}{16}\left[-\frac{2}{(x-2)^2} - \frac{2}{x} + \ln\left(\frac{x}{2-x}\right)\right]. \tag{6.21}$$

Nevertheless, such Lagrangians diverge in vacuum, where $T = T_G = 0$. However, by assuming more than a single fluid, the general solution leads to the sum of the solutions (6.20) for each EOS w, and vacuum may be achieved by the cancelation of the divergences. Particularly, by assuming an arbitrary number of fluids, the following condition is found:

$$0 = \sum_i a_i \xi_{w_i}, \tag{6.22}$$

where $a_i > 0$ are unknown coefficients corresponding to each EOS. However, as $a_i\xi_{w_i} < 0\ \forall i$, the solution does not describe the bouncing cosmology while obeying the vacuum condition.

D. $f(T,T_G) = -T + T_G g(T)$

For a T_G rescaling with a TEGR contribution, the Friedmann equation becomes

$$T - \frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} A^{-3(1+w_i)} \times \left[\frac{2\rho_{\rm cr}}{T}\left(1 - \sqrt{1 - \frac{T}{\rho_{\rm cr}}}\right)\right]^{-(1+w_i)}. \tag{6.23}$$

In this case, the solution is similar to the previous model with an extra particular solution of the form $g_{\rm part} = -3/4T$. Thus, the Lagrangian is given by

$$f(T,T_G) = -T - \frac{3T_G}{4T} + T_G h(T), \tag{6.24}$$

where $h(T)$ represents the previous model's solution. To satisfy the vacuum condition, we again require $f(0,0) = 0$. However, as indicated in the previous model, $T_G h(T)|_{T,T_G \to 0}$ yields finite results only in vacuum. This leads to $f(0,0) = -3\rho_{\rm cr} < 0$. Therefore, this model does not satisfy the vacuum condition.

E. $f(T,T_G) = -T + \mu(\frac{T}{T_0})^\beta (\frac{T_G}{T_{G,0}})^\gamma$

For a power-law model, the Friedmann equation becomes

$$T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)} = T + \mu\left(\frac{T}{T_0}\right)^\beta \left(\frac{T_G}{T_{G,0}}\right)^\gamma \Bigg[1 - 2\beta - \gamma + \beta\gamma\left(2 - \frac{4T^2}{3T_G}\right) + \gamma(\gamma-1)\frac{2T}{9{T_G}^2}(-20T^3 + 12\rho_{\rm cr}T^2 - 51TT_G + 36\rho_{\rm cr}T_G)\Bigg]. \tag{6.25}$$

For this model, the vacuum condition $f(0,0) = 0$ is satisfied as long as $\beta + \gamma > 0$. Evaluating the Friedmann equation at $t = 0$ yields the following condition,

$$0 = \sum_i \Omega_{w_i,0} A^{-3(1+w_i)}. \tag{6.26}$$

However, since both parameters are positive, this is not achievable unless vacuum is considered. Thus, the latter is assumed. By evaluating the Friedmann equation at $t = t_0$, the constant μ can be determined to be

$$\mu = \frac{-T_0}{1 - 2\beta - \gamma + \beta\gamma\left(2 - \frac{4{T_0}^2}{3T_{G,0}}\right) + \frac{2\gamma(\gamma-1)T_0}{9{T_{G,0}}^2}(-20{T_0}^3 + 12\rho_{\rm cr}{T_0}^2 - 51T_0T_{G,0} + 36\rho_{\rm cr}T_{G,0})} \equiv -\frac{T_0}{\nu}, \tag{6.27}$$

where $\nu \neq 0$ is defined as denominator. This simplifies the Friedmann equation to

$$T = \frac{T_0}{\nu}\left(\frac{T}{T_0}\right)^\beta\left(\frac{T_G}{T_{G,0}}\right)^\gamma\left[1 - 2\beta - \gamma + \beta\gamma\left(2 - \frac{4T^2}{3T_G}\right) + \gamma(\gamma-1)\frac{2T}{9{T_G}^2}(-20T^3 + 12\rho_{\rm cr}T^2 - 51TT_G + 36\rho_{\rm cr}T_G)\right]. \tag{6.28}$$

Since we require the equation to hold at all times, assuming $T \neq 0$, the Friedmann equation can be rearranged to be in the form $\nu = g(T)$, for some function g. Thus, since the lhs is a constant, the rhs must also be a constant, meaning that the function must be independent of T. This is true in two cases, $\beta = -1$, $\gamma = 1$ and $\beta = 1$, $\gamma = 0$. The former, albeit leading to a nontrivial Lagrangian, does not satisfy the vacuum condition. On the other hand, the second case corresponds to a TEGR rescaling with $\nu = -1$. However, this leads to a zero Lagrangian, which is nonphysical. Therefore, this case is also neglected.

We conclude this section by examining the TEGR with DGP and Gauss-Bonnet terms since the latter two do not contribute to the Friedmann equation. In this case, the equation becomes

$$T = T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}. \tag{6.29}$$

At time $t = 0$, the same condition is obtained, which is only true when vacuum is considered. However, this would imply that $T = 0$ at all times, which is a contradiction. Thus, this implies that the TEGR term cannot describe the bouncing cosmology. Therefore, no Lagrangian has been found which satisfies the vacuum condition.

VII. BOUNCING MODEL V: EXPONENTIAL MODEL II

The last bouncing model is similar to the first one, but it may include a future singularity, similar to the power-law model studied above:

$$a(t) = A \exp\left[\frac{f_0}{\alpha+1}(t - t_s)^{\alpha+1}\right], \tag{7.1}$$

where $A > 0$ is a dimensionless constant which corresponds to the scale factor at the bouncing point time t_s;

i.e., $A = a(t_s)$, $f_0 > 0$ is some arbitrary constant having time dimensions $[T]^{-\alpha-1}$, and α is a constant. In this case, the Hubble parameter and, consequently, the torsion scalar and TEGB term are given by

$$H = f_0(t - t_s)^\alpha, \qquad T = 6H^2,$$
$$T_G = 4T\left[\frac{T}{6} + f_0\alpha\left(\frac{T}{6f_0{}^2}\right)^{\frac{\alpha-1}{2\alpha}}\right]. \tag{7.2}$$

Furthermore, the scale factor can be solely expressed in terms of the torsion scalar as

$$a(T) = A\exp\left[\frac{f_0}{\alpha+1}\left(\frac{T}{6f_0{}^2}\right)^{\frac{\alpha+1}{2\alpha}}\right]. \tag{7.3}$$

A type IV singularity (see Ref. [46]) may occur in this bouncing cosmology when

$$\alpha = \frac{2n+1}{2m+1}, \tag{7.4}$$

where n, $m \in \mathbb{N}$ and $\alpha > 1$. Before reconstructing the corresponding Lagrangians, we make note that by introducing the new time variable $t_* \equiv t - t_s$, the scale factor and Hubble parameter become

$$a(t_*) = A\exp\left[\frac{f_0}{\alpha+1}t_*{}^{\alpha+1}\right], \qquad H = f_0 t_*{}^\alpha. \tag{7.5}$$

This effectively simplifies the Hubble parameter to a standard power-law relation in the time variable t_*. Lastly, we define an instant of time $t_* = t_0 > 0$ at which $a(t_0) = 1$ to simplify the Friedmann equation's calculations. The time is given by

$$t_0{}^{\alpha+1} = -\frac{\alpha+1}{f_0}\ln A. \tag{7.6}$$

Since we demand that $t_0 > 0$ and α, $f_0 > 0$, we require $0 < A < 1$. In what follows, this will be assumed. By defining this time, we define the torsion scalar at this instant as follows:

$$T_0 \equiv T(t_* = t_0) = 6f_0{}^2 t_0{}^{2\alpha}. \tag{7.7}$$

By doing so, the scale factor simplifies to

$$a(T) = A^{1-(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}}, \tag{7.8}$$

where we have used Eq. (7.6). Furthermore, the TEGB term can be reexpressed into a simpler form as follows:

$$T_G = 4T\left[\frac{T}{6} + \alpha f_0 t_0{}^{\alpha-1}\left(\frac{T}{T_0}\right)^{\frac{\alpha-1}{2\alpha}}\right]. \tag{7.9}$$

However, working with this scale factor may introduce difficulties when reconstructing the corresponding gravitational actions. Instead, we make use of Eq. (7.6), such that the scale factor can be expressed as

$$a(T) = \exp\left\{-\frac{f_0 t_0{}^{\alpha+1}}{\alpha+1}\left[1 - \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}. \tag{7.10}$$

A. $f(T,T_G) = g(T) + h(T_G)$

For a separable additional model for T and T_G, the Friedmann equation reduces to

$$g + h - 2Tg_T - T_G h_{T_G} - \frac{(2T^2 - 3T_G)[2(\alpha+1)T^2 + 3(3\alpha-1)T_G]}{9\alpha}h_{T_G T_G} = T_0\sum_i \Omega_{w_i,0}\exp\left\{\frac{3f_0 t_0{}^{\alpha+1}(1+w_i)}{\alpha+1}\left[1 - \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}. \tag{7.11}$$

This equation cannot be split as in previous cases due to the coefficient of $h_{T_G T_G}$. There may exist an invertible relation for T in terms of T_G, such that $T = p(T_G)$, but there is not a general one for any arbitrary α. Indeed, given the form of α in Eq. (7.4) with $\alpha > 1$, the form of T_G is given as

$$T_G = \mu T^2 + \nu T^{\frac{3n-m+1}{2n+1}}, \tag{7.12}$$

where μ and ν are the corresponding coefficients of Eq. (7.9). It is clear that due to the last term, the equation is, in general, not invertible. Nonetheless, in some particular cases, the equation is invertible. For the sake of generality, we assume that T is invertible and that some function $p(T_G)$ exists. In other words, the Friedmann equation now becomes

$$g + h - 2Tg_T - T_G h_{T_G} - q(T_G)h_{T_G T_G} = T_0\sum_i \Omega_{w_i,0}\exp\left\{\frac{3f_0 t_0{}^{\alpha+1}(1+w_i)}{\alpha+1}\left[1 - \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}, \tag{7.13}$$

where $q(T_G)$ is a function of the TEGB term only representing the coefficient of $h_{T_G T_G}$, which is now possible due to the demand that $T = p(T_G)$. Now, the equation can be separated, with each side of the equation in terms of T and T_G independently, leading to the same procedure used in Sec. V A. In fact, the constant which is generated can be set to zero, as it will not contribute to the Lagrangian. Thus, the system of differential equations leads to

$$g-2Tg_T = T_0\sum_i \Omega_{w_i,0} \times \exp\left\{\frac{3f_0 {t_0}^{\alpha+1}(1+w_i)}{\alpha+1}\left[1-\left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}, \tag{7.14}$$

$$h - T_G h_{T_G} - q(T_G)h_{T_G T_G} = 0. \tag{7.15}$$

The solution for $g(T)$ is given by

$$g(T) = c_1\sqrt{T} + \sum_i \sum_{n=0}^{\infty} \frac{\Omega_{w_i,0}T_0}{n!}\left(\frac{3f_0 {t_0}^{\alpha+1}(1+w_i)}{\alpha+1}\right)^n \times {}_2F_1\left[-n,-\frac{\alpha}{\alpha+1};\frac{1}{\alpha+1};\left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right], \tag{7.16}$$

where c_1 is an integration constant whose term corresponds to the DGP term and ${}_2F_1(a,b,c;z)$ is Gauss' hypergeometric function. Note that since $\alpha > 1$, the hypergeometric function is always defined. When $T = 0$, the solution reduces to

$$g(0) = \sum_i \Omega_{w_i,0}T_0 \exp\left[\frac{3f_0(1+w_i){t_0}^{\alpha+1}}{\alpha+1}\right]. \tag{7.17}$$

As discussed at the beginning of this section, the form of $q(T_G)$ is unknown or nonexistent depending on the value of α. The exponent of the last term in Eq. (7.12) lies in the range $(1, 3/2)$, which makes it difficult to obtain an invertible condition. Nonetheless, the equation generates two independent homogeneous solutions since it is a linear homogeneous type, say, $u_1(T_G)$ and $u_2(T_G)$. Thus, the solution for h can always be expressed as

$$h(T_G) = c_1u_1(T_G) + c_2u_2(T_G), \tag{7.18}$$

for some arbitrary integration constants $c_{1,2}$. In fact, it is easy to verify that one of the solutions is the Gauss-Bonnet contribution T_G. In other words, the solution is

$$h(T_G) = c_1u_1(T_G) + c_2T_G. \tag{7.19}$$

Now, independently of the form of u_1, we can reach the following conclusions. If the function $u_1(0) = 0$, then this gives a nontrivial solution with $h(0) = 0$. This demands that $g(0) = 0$ for the vacuum condition to be satisfied, which is possible only in the absence of matter. On the other hand, if this results in a constant, it still defines a nontrivial solution; however, $h(0)$ can be nonzero depending on the integration constant. If the integration constant is set to zero, then $g(0) = 0$, which is only possible in vacuum. On the other hand, if $h(0)$ is equal to some constant $\mu \neq 0$, then $g(0) = -h(0) = -\mu$. Furthermore, since $g(0) > 0$ in these cases, this restricts to $\mu < 0$. Lastly, if the function diverges at $T_G = 0$, the singularity can be removed by setting the integration constant to zero. Again, this sets $h(0) = 0$; thus, we need $g(0) = 0$ for vacuum solutions to occur, which is again only satisfied in vacuum.

B. $f(T,T_G) = Tg(T_G)$

For a rescaling of the T model, the resulting Friedmann equation is given by

$$g + g_{T_G}\left(-T_G + \frac{4T^2}{3}\right) + \frac{(2T^2-3T_G)[2(\alpha+1)T^2 + 3(3\alpha-1)T_G]}{9\alpha}g_{T_GT_G} = -\frac{T_0}{T}\sum_i \Omega_{w_i,0}\exp\left\{\frac{3f_0{t_0}^{\alpha+1}(1+w_i)}{\alpha+1} \times \left[1-\left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}. \tag{7.20}$$

Similar to the previous case, a problem arises due to the invertibility issue of the torsion scalar as a function of the TEGB term. Nonetheless, we can extract and analyze some behaviors of the solution even in the absence of its explicit form. Let us express Eq. (7.20) in terms of T_G:

$$g + p(T_G)g_{T_G} + q(T_G)g_{T_GT_G} = h(T_G), \tag{7.21}$$

where p, q, and h are unknown functions of T_G. Thus, the complete solution would be given by

$$g(T_G) = c_1u_1(T_G) + c_2u_2(T_G) + \int^{T_G} G(T_G,s)h(s)\mathrm{d}s, \tag{7.22}$$

where $G(T_G,s)$ is the Green function of Eq. (7.21), while $u_{1,2}(T_G)$ are the solutions of the homogeneous part of Eq. (7.21). Finally, the vacuum condition is satisfied; i.e., $T = T_G = 0$ implies $f(0,0) = 0$, as long as the solution (7.22) is finite at $T_G = 0$.

C. $f(T,T_G) = T_Gg(T)$

For a TEGB rescaling model, the resulting equation yields

$$-\frac{4T^3}{3}g_T = T_0\sum_i \Omega_{w_i,0} \times \exp\left\{\frac{3f_0{t_0}^{\alpha+1}(1+w_i)}{\alpha+1}\left[1-\left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}. \tag{7.23}$$

The first solution of this equation is given by

$$g_1(T) = c_1 + \sum_i \sum_{n=0}^{\infty} \frac{3\Omega_{w_i,0}T_0}{8T^2 n!} \left[\frac{3f_0 t_0^{\alpha+1}(1+w_i)}{\alpha+1}\right]^n {}_2F_1\left[-n, -\frac{4\alpha}{\alpha+1}; 1-\frac{4\alpha}{\alpha+1}; \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right], \tag{7.24}$$

where c_1 is an integration constant, which corresponds to the Gauss-Bonnet contribution in the Lagrangian, and ${}_2F_1(a, b; c; z)$ is Gauss' hypergeometric function. The solution exists and is defined provided that the third argument in the hypergeometric function $c \equiv 1 - \frac{4\alpha}{\alpha+1} \notin \mathbb{Z}^- \cup \{0\}$. For the values of α considered for the type IV singularity in Eq. (7.4) with $\alpha > 1$, the only allowed value is $\alpha = 3$, which results in $c = -2$. This leads to the second solution

$$g_2(T) = c_1 + \sum_i \frac{3\Omega_{w_i,0}T_0}{16T^2} \exp\left(\frac{3f_0 t_0^4(1+w_i)}{4}\right)[\mathrm{e}^{x_i}(x_i^2 + x_i + 2) - x_i^3 \mathrm{Ei}(x_i)], \tag{7.25}$$

where $x_i \equiv -\frac{3f_0 t_0^4(1+w_i)}{4} \frac{T^{2/3}}{T_0^{2/3}}$ and $\mathrm{Ei}(z)$ is the exponential integral. Whether both solutions satisfy $f(0,0) = 0$ can be checked by evaluating the solutions in vacuum:

$$f(0,0) = \sum_i \frac{\Omega_{w_i,0}T_0}{4} \exp\left[\frac{3f_0 t_0^{\alpha+1}(1+w_i)}{\alpha+1}\right] + \sum_i \sum_{n=0}^{\infty} \frac{3\alpha f_0 \Omega_{w_i,0} T_0 t_0^{\alpha-1}}{2Tn!} \left(\frac{T}{T_0}\right)^{\frac{\alpha-1}{2\alpha}} \left[\frac{3f_0 t_0^{\alpha+1}(1+w_i)}{\alpha+1}\right]^n \times {}_2F_1\left[-n, -\frac{4\alpha}{\alpha+1}; 1-\frac{4\alpha}{\alpha+1}; \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right], \tag{7.26}$$

which gives a singularity in the second summation due to the $\alpha > 1$ condition. Trivially, the condition is satisfied when vacuum is considered, although this results in a Lagrangian with only the Gauss-Bonnet term which is nonphysical. On the other hand, the singularity can be removed only when all the coefficients sum to zero, i.e.,

$$0 = \sum_i \sum_{n=0}^{\infty} \frac{3\alpha f_0 \Omega_{w_i,0} T_0 t_0^{\alpha-1}}{2n!} \left[\frac{3f_0(1+w_i)t_0^{\alpha+1}}{\alpha+1}\right]^n = \sum_i \frac{3\alpha f_0 \Omega_{w_i,0} T_0 t_0^{\alpha-1}}{2} \exp\left[\frac{3f_0 t_0^{\alpha+1}(1+w_i)}{\alpha+1}\right]. \tag{7.27}$$

However, since every contribution is positive, the condition cannot be satisfied.

On the other hand, for the second solution, one finds

$$f(0,0) = \sum_i \frac{\Omega_{w_i,0}}{2} \exp\left(\frac{3f_0 t_0^4(1+w_i)}{4}\right) \left\{-\frac{T_0(1+9w_i)}{16} + 9f_0 t_0^2 \left(\frac{T_0}{T}\right)^{2/3} \exp\left(-\frac{3}{4} f_0 t_0^4 (1+w_i) \frac{T^{2/3}}{T_0^{2/3}}\right)\right\}\Bigg|_{T\to 0}, \tag{7.28}$$

which has a singularity in the exponential term provided that vacuum is not considered (in this case, the solution trivially holds, although the Lagrangian would only be provided by the Gauss-Bonnet term which is nonphysical). The singularity in the exponential term can be removed only if the coefficients sum to 0, i.e.,

$$\sum_i f_0 t_0^2 \Omega_{w_i,0} = 0. \tag{7.29}$$

However, since f_0, t_0, $\Omega_{w_i,0} > 0$, this condition cannot be satisfied, leading to the vacuum solution as the only solution which satisfies the vacuum condition, as usual.

D. $f(T,T_G) = -T + T_G g(T)$

For models with a TEGB rescaling and a TEGR contribution, the resulting equation is

$$T - \frac{4T^3}{3} g_T = T_0 \sum_i \Omega_{w_i,0} \exp\left\{\frac{3f_0 t_0^{\alpha+1}(1+w_i)}{\alpha+1}\left[1 - \left(\frac{T}{T_0}\right)^{\frac{\alpha+1}{2\alpha}}\right]\right\}. \tag{7.30}$$

Here, the solutions are identical to the previous case with an extra particular solution

$$g_{\mathrm{part}}(T) = -\frac{3}{4T}. \tag{7.31}$$

To check for vacuum solutions, we demand the condition $f(0,0) = 0$. Since the results in the previous section show that only vacuum can yield finite results in the T, $T_G \to 0$ limit, the resulting Lagrangian which must be checked for the vacuum condition is

$$f(T,T_G) = -T - \frac{3T_G}{4T} + c_1 T_G, \tag{7.32}$$

where c_1 is a constant of integration. In this case, the limit does satisfy the vacuum condition and hence can describe the bouncing cosmology.

E. $f(T,T_G) = -T + \mu(\frac{T}{T_0})^\beta(\frac{T_G}{T_{G,0}})^\gamma$

For a power-law model in both T and T_G, the Friedmann equation reduces to

$$T + \mu\left(\frac{T}{T_0}\right)^{\beta+\gamma\frac{3\alpha-1}{2\alpha}}\left[\frac{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0}\right]^\gamma \left\{1 - 2\beta - \gamma + \frac{12\beta\gamma\alpha f_0 {t_0}^{\alpha-1}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}} + \frac{12\alpha\gamma(\gamma-1) f_0 {t_0}^{\alpha-1}\left[3(3\alpha-1) f_0 {t_0}^{\alpha-1} + 2T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]}{\left[6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]^2}\right\} = T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}. \tag{7.33}$$

For this model, vacuum solutions are obtained provided that

$$\beta + \frac{(3\alpha-1)\gamma}{2\alpha} > 0. \tag{7.34}$$

The value of μ is obtained by evaluating the expression at the current time, yielding

$$\mu = \frac{-T_0 + T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}}{1 - 2\beta - \gamma + \frac{12\beta\gamma\alpha f_0 {t_0}^{\alpha-1}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0} + \frac{12\alpha\gamma(\gamma-1) f_0 {t_0}^{\alpha-1}[3(3\alpha-1) f_0 {t_0}^{\alpha-1} + 2T_0]}{(6\alpha f_0 {t_0}^{\alpha-1} + T_0)^2}} \equiv \frac{1}{\nu}\left(-T_0 + T_0 \sum_i \Omega_{w_i,0} a^{-3(1+w_i)}\right), \tag{7.35}$$

where ν is defined by the denominator provided that it is nonzero. Note that the DGP ($\beta = 1/2$, $\gamma = 0$) and Gauss-Bonnet ($\beta = 0$, $\gamma = 1$) contributions give $\nu = 0$ and hence are excluded for the subsequent analysis. The special case when these are considered is discussed at the end of the section. Furthermore, by evaluating the expression at the bouncing time $t = t_s$ (or equivalently, $t_* = 0$) results in the following condition:

$$\sum_i \Omega_{w_i,0} A^{-3(1+w_i)} = 0. \tag{7.36}$$

This condition can only be satisfied in the absence of any type of matter, i.e., $\Omega_{w_i,0} = 0$. Let us assume such a case. The Friedmann equation is simplified as follows:

$$\frac{T_0}{\nu}\left(\frac{T}{T_0}\right)^{\beta+\gamma\frac{3\alpha-1}{2\alpha}}\left[\frac{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0}\right]^\gamma \left\{1 - 2\beta - \gamma + \frac{12\beta\gamma\alpha f_0 {t_0}^{\alpha-1}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}} + \frac{12\alpha\gamma(\gamma-1) f_0 {t_0}^{\alpha-1}\left[3(3\alpha-1) f_0 {t_0}^{\alpha-1} + 2T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]}{\left[6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]^2}\right\} = T. \tag{7.37}$$

By assuming $T \neq 0$ (which already trivially satisfies the relation), we obtain

$$\left(\frac{T}{T_0}\right)^{\beta+\gamma\frac{3\alpha-1}{2\alpha}-1}\left[\frac{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0}\right]^\gamma \left\{1 - 2\beta - \gamma + \frac{12\beta\gamma\alpha f_0 {t_0}^{\alpha-1}}{6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}} + \frac{12\alpha\gamma(\gamma-1) f_0 {t_0}^{\alpha-1}\left[3(3\alpha-1) f_0 {t_0}^{\alpha-1} + 2T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]}{\left[6\alpha f_0 {t_0}^{\alpha-1} + T_0(\frac{T}{T_0})^{\frac{\alpha+1}{2\alpha}}\right]^2}\right\} = \nu. \tag{7.38}$$

Since the lhs is constant, all the torsion terms on the rhs must vanish and yield a constant. This is possible only if $\beta = 1$ and $\gamma = 0$. This sets $\nu = -1$, so the Lagrangian is zero, which is not physical.

VIII. CONCLUSIONS

Bouncing cosmologies have become a reliable alternative to the inflationary paradigm, especially because of the absence of initial conditions to start the cosmological evolution and also because of the absence of an initial singularity within some models. In general, such a scenario results in a universe that expands and then slows down and contracts again, a similar framework to the so-called ekpyrotic universes. Here, we have investigated the possibility of reproducing some bouncing cosmologies in the framework of a class of extended teleparallel theories, where the gravitational action includes functions of the torsion scalar and an analog of the Gauss-Bonnet invariant. To do so, we have considered some particular forms of the Lagrangian according to some physical properties.

Then, several bouncing cosmologies have been considered, including some singular bouncing solutions, and the corresponding Lagrangian is reconstructed. The existence of vacuum (null torsion) solutions has also been analyzed since it guarantees that such Lagrangians will indeed contain both Minkowski and Schwarzschild solutions, a fundamental requirement for the viability of any theory of gravity. Let us now summarize the solutions explored throughout the paper. First, we have considered a class of exponential laws for the scale factor, free of singularities, where the scale factor decreases and reaches a minimum, avoiding the occurrence of big bang-like singularity, and then increases. The Hubble parameter is then described by a linear function of the cosmic time, as shown in the first row of Fig. 1. Even though this is not a realistic example, it represents quite well the idea of a bounce in the universe expansion. By considering several forms of the gravitational action, the corresponding function of the torsion scalar and the Gauss-Bonnet invariant is reconstructed. As shown in Sec. III, the analytical expression for the gravitational Lagrangian is difficult to obtain, but in general, the action fulfills the requirement of vacuum solutions. In addition, an oscillating bouncing universe is considered. Such an example is unusual for the whole cosmological history but contains a singularity, a big bang/crunch singularity, such that the scale factor goes to zero and then the universe starts in a big bang again. Nevertheless, note that such a singularity may be alleviated by imposing a minimum value larger than zero on the scale factor. The reconstructed Lagrangians corresponding to this oscillating solution are provided in Sec. IV, although, in general, the Lagrangians do not behave well in vacuum, where some of the reconstructed functions diverge. Then, a similar solution in terms of the occurrence of a big bang/crunch singularity is also given in the form of a power-law solution in Sec. V. This case makes the gravitational action simpler for some of the classes of Lagrangians explored in the paper. In addition, vacuum solutions are better achieved for the power-law solution than in the previous case. Another important bouncing solution widely explored in the literature is the so-called critical density solution, which is free of singularities and very similar to the exponential case even though it exhibits a more complex—and realistic—evolution of the Hubble parameter. Nevertheless, the reconstruction of the corresponding Lagrangians turns out to be more difficult than in the previous cases, and only some analytical expressions are obtained, as shown in Sec. VI. Finally, we have explored an extension of the first model, the exponential case, with the presence of a possible future singularity. The corresponding discussion about the gravitational Lagrangians is given in Sec. VII, but in general, the action becomes very complex and the analysis of vacuum solutions is not possible.

Hence, we have explored a wide range of bouncing solutions in the framework of $f(T, T_G)$ actions, such that the corresponding Lagrangians can be reconstructed. Here, we have thus provided some techniques and tools for the analysis of these types of Lagrangians when analyzing such cosmological solutions. Thus, we have shown the viability of some Lagrangians to reproduce the corresponding bouncing solution and the possibility of containing other important physical features to be considered a viable alternative to teleparallel gravity.

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