Self-force as a probe of global structure

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We calculate the self-force on an electric charge and electric dipole held at rest in a closed universe that results from joining two copies of Minkowski spacetime at a common boundary. Spacetime is strictly flat on each side of the boundary, but there is curvature at the surface layer required to join the two Minkowski spacetimes. We find that the self-force on the charge is always directed away from the surface layer. This is analogous to the case of an electric charge held at rest inside a spherical shell of matter, for which the self-force is also directed away from the shell. For the dipole, the direction of the self-force is a function of the dipole's position and orientation. Both self-forces become infinite when the charge or dipole is made to approach the surface layer. This study reveals that a self-force can arise even when the Riemann tensor vanishes at the position of the charge or dipole; in such cases the self-force is a manifestation of the global curvature of spacetime.

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I. INTRODUCTION AND SUMMARY

An electric charge held at rest in a curved spacetime creates an electric field that interacts with the spacetime curvature. In a flat spacetime the field lines would be isotropically distributed around the charge, and the field would exert no net force. In the curved spacetime, however, the isotropy is disturbed, and there is a self-force acting on the charge [1].

The prototypical example of a self-force in curved spacetime implicates an electric charge q held at a position r_0 outside a nonrotating black hole of mass M. This force was first calculated by Smith and Will [2], building up on earlier work by a number of researchers [3–7]. It is given by

$$\boldsymbol{F} = \frac{q^2 M}{r_0^3}\boldsymbol{n} \qquad \text{(charge outside a black hole)}, \qquad (1.1)$$

where $\boldsymbol{n}$ is a unit vector that points from the black hole to the charge.[1] The self-force points away from the black hole, meaning that the external force required to keep the particle in place is smaller when the particle is charged, compared to what it would be in the case of a neutral particle. This iconic result was later generalized to electric charges in the Reissner-Nordström spacetime [8,9], to scalar charges [10–12], and to higher-dimensional black holes [13–17].

Drivas and Gralla [18] have shown that the Smith-Will force of Eq. (1.1) is nearly universal, in the sense that its expression is nearly independent of the internal composition of the gravitating body. In other words, two bodies of the same mass but of distinct internal structure give rise to nearly identical self-forces, both given approximately by $q^2 M/r_0^3$. There is, however, a slight dependence on internal structure that produces a correction of order $q^2 M^3/r_0^5$, a factor $(M/r_0)^2$ smaller than the leading-order term; the correction depends on the internal composition of the gravitating body. The precise nature of these corrections was identified by Isoyama and Poisson [19], who concluded that, in principle, the self-force can be exploited as a probe of internal structure.

The approximate universality of the Smith-Will force suggests that a reliable estimate for the self-force acting on a static charge in any curved spacetime might be that it is given by q^2 times a measure of the local curvature, as provided by a typical component of the Riemann tensor in an orthonormal frame. In the case of a spacetime outside a body of mass M, the local curvature at position r_0 is measured by M/r_0^3, and the self-force is indeed given by $q^2 M/r_0^3$.

The rule of thumb might be adequate in many circumstances, but its limitations become apparent when the charge is placed in a region of spacetime in which the local curvature vanishes. An illuminating example, first examined by Unruh [20] and later revisited by Burko, Liu, and Soen [21], implicates an electric charge held at rest at a position r_0 inside a spherical shell of mass M and radius a. In this situation, the spacetime inside the shell is flat, and the local curvature at r_0 vanishes. But the spacetime is curved outside the shell, and the interaction between the electric field and this curvature still gives rise to a self-force. In this case the

[1] Throughout the paper we use a Cartesian language for vectors. In this case the spacetime is curved and the language is not entirely appropriate. The equation, however, is still valid. The vector $\boldsymbol{F}$ refers to an orthonormal basis attached to a static observer at position r_0, and $\boldsymbol{n}$ is the unit vector that points radially outward.

self-force provides a probe of the global curvature of the spacetime.

The precise expression for the force is complicated—it is presented as an infinite sum involving Legendre functions—but to leading order in an expansion in powers of M/a, it reduces to [21]

$$\boldsymbol{F} \simeq -\frac{q^2 M}{2ar_0^2}\left(\frac{r_0/a}{1-r_0^2/a^2} + \frac{1}{2}\ln\frac{1-r_0/a}{1+r_0/a}\right)\boldsymbol{n} \quad \text{(charge inside a shell, leading order in } M/a), \tag{1.2}$$

where $\boldsymbol{n}$ is a unit vector that points toward the charge from the center of the shell. Here the self-force points toward the center, away from the shell, and its scaling with M, a, and r_0 is not given by a simple expression. When r_0/a is small, the expression within brackets asymptotes to $\frac{2}{3}(r_0/a)^3$, and $\boldsymbol{F} \sim -\frac{1}{3}(q^2 M r_0/a^4)\boldsymbol{n}$. When r_0/a approaches unity, the expression within brackets is approximately equal to $[2(1-r_0/a)]^{-1}$, and the self-force becomes infinite in the limit. This example reveals that the scaling of the self-force is difficult to estimate when the charge is placed in a region of vanishing curvature. The rule of thumb proposed previously is clearly inadequate in such situations.

In this paper we examine an even more radical example of a self-force acting on an electric charge held at rest in a region of vanishing local curvature. We consider a static, spherically symmetric spacetime with a metric given by

$$ds^2 = -dt^2 + dr^2 + R^2(r)d\Omega^2, \tag{1.3}$$

where $d\Omega^2 \coloneqq d\theta^2 + \sin^2\theta d\phi^2$ and

$$R(r) \coloneqq \begin{cases} r & 0 \le r < a \\ 2a - r & a \le r \le 2a \end{cases}. \tag{1.4}$$

The spacetime represents a static universe with closed spatial sections, which extend from a first center at $r = 0$, at which $R(r) = 0$, to a second center at $r = 2a$, at which $R(r)$ also vanishes. The spacetime is strictly flat when $r < a$ and $r > a$, and it can be thought of as two copies of Minkowski spacetime joined together at $r = a$. The joint is achieved with a surface layer, and the Israel junction conditions [22] imply that the layer possesses a surface mass density $\sigma = (2\pi a)^{-1}$ and (negative) surface pressure $p = -(4\pi a)^{-1}$. The surface layer has an inertial mass $m = 4\pi a^2\sigma = 2a$, and it is the only place in spacetime where one can find curvature. The Riemann tensor is a Dirac distribution supported at $r = a$, and the Einstein field equations equate its Ricci piece to the distributional energy-momentum tensor provided by the surface layer. The spacetime is admittedly unrealistic, but it nevertheless gives rise to a striking (and simple) example of a self-force. In this case, the electric field interacts with curvature that is entirely confined to the surface layer.

In Sec. II we place an electric charge q at a position r_0 in the spacetime, keep it there by means of an external agent that balances the self-force, calculate the electric field, observe that it is affected by the global structure of the spacetime, and find that this field exerts a force on the charge. Taking $r_0 < a$ without loss of generality (because the spacetime is reflection-symmetric across $r = a$), we find that the self-force is given by

$$\boldsymbol{F} = -\frac{q^2 r_0}{2a^3}\frac{1}{1-(r_0/a)^2}\boldsymbol{n} \quad \text{(charge in the closed universe)}. \tag{1.5}$$

The minus sign indicates that the force is directed away from the surface layer. It scales as $q^2 r_0/a^3$ when $r_0/a \ll 1$, and it diverges in the limit $r_0 \to a$.

A technical complication arises because of the closed spatial sections of our spacetime. Gauss's law demands that the total charge be zero in a closed universe, and there must therefore be a second charge $-q$ in the spacetime. The second charge creates an additional force on the first charge, and this force must be distinguished from the self-force of Eq. (1.5). This can always be done, because the self-force depends only on the position r_0, while the interaction force between the two charges depends also on the position of the second charge. These details are presented in Sec. II.

The scaling of the self-force with $q^2 r_0/a^3$ and its blowup at $r_0 = a$ are reminiscent of the behavior of the self-force in the case of a charge inside a massive shell; see Eq. (1.2) and the discussion that follows. In this case we found the same blowup, and a rough scaling with $q^2 M r_0/a^4$. It is tempting to suggest that the $q^2 r_0/a^3$ scaling is recovered when M is of the same order of magnitude as a. This recovery is suggestive, but the suggestion suffers from the drawback that the surface layer of our closed universe has a vanishing gravitational mass; partial redemption may come from our earlier observation that its inertial mass is indeed given by $2a$. Another drawback comes from the fact that in the example of the massive shell, the scaling with $q^2 M r_0/a^4$ was identified in a self-force that was valid only to leading order in an expansion in powers of M/a. The self-force, however, was also calculated in the limit $a \to 2M$ in Ref. [21]. In this case it is given by

$$\boldsymbol{F} \simeq -\frac{q^2 r_0}{a^3}\frac{2 - r_0^2/a^2}{(1-r_0^2/a^2)^2}\boldsymbol{n} \quad \text{(charge inside a shell, } a \to 2M), \tag{1.6}$$

and its scaling with $q^2 r_0/a^3$ is confirmed. The blowup at $r = a$, however, is now stronger.

The spacetime introduced in this paper is sufficiently simple that it allows an easy investigation of self-forces in unusual and interesting circumstances. We take advantage of this simplicity and explore a new avenue by calculating the self-force acting on an electric dipole $\boldsymbol{p}$ held at rest at position $r_0 < a$ in the spacetime. To the best of our knowledge, there have been no studies of self-force on a dipole in curved spacetime, beyond the foundational work found in Ref. [23]. We hope that this initial study will motivate further work in this direction.

The calculation of the self-force on a dipole presents itself with a conundrum. The fact that the self-force on a charge q scales with q^2, and is therefore independent of the sign of the charge, suggests that the forces should simply add up when two opposite charges are brought together to form a dipole. On the other hand, we expect that a calculation carried out from first principles would reveal a self-force that scales as p^2, where $p \coloneqq |\boldsymbol{p}|$. Because $q = p/\epsilon$, where ϵ is the separation between the charges, the first suggestion would produce a force that diverges when $\epsilon \to 0$ with p fixed, while the second route would produce a finite self-force. The actual calculation of the self-force shows that it is finite, and the expectation that the individual self-forces add up is simply wrong.

In Sec. III we calculate the self-force acting on a point dipole $\boldsymbol{p}$ at rest at a position $r = r_0 < a$ in the closed universe. As in the case of the charge, the dipole is kept in place by an external agent that balances the self-force. We find that it is given by

$$\boldsymbol{F} = -\frac{p^2 r_0}{4a^5}\frac{1}{(1-r_0^2/a^2)^3}\boldsymbol{f} \qquad \text{(dipole in the closed universe)} \tag{1.7}$$

with

$$\boldsymbol{f} \coloneqq (1-r_0^2/a^2)(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})\hat{\boldsymbol{p}} + [3 - r_0^2/a^2 + 2(r_0^2/a^2)(\hat{\boldsymbol{p}}\cdot\boldsymbol{n})^2]\boldsymbol{n}, \tag{1.8}$$

where $\hat{\boldsymbol{p}} \coloneqq \boldsymbol{p}/p$ and $\boldsymbol{n}$ is a unit vector that points from $r = 0$ to the dipole. The directional structure of the self-force is rich. When $\boldsymbol{p}$ is aligned with $\boldsymbol{n}$, that is, when the dipole points in the radial direction, the force is directed along $-\boldsymbol{n}$. When $\boldsymbol{p}$ is orthogonal to $\boldsymbol{n}$, that is, when the dipole is transverse to the radial direction, the force is again directed along $-\boldsymbol{n}$. For a generic orientation of the dipole, the force is directed opposite to a linear combination of $\hat{\boldsymbol{p}}$ and $\boldsymbol{n}$. As in the case of a point charge, the self-force on a dipole diverges when $r_0 \to a$.

II. ELECTROMAGNETIC SELF-FORCE ON A POINT CHARGE

We wish to calculate the self-force acting on an electric charge q at rest at $r = r_0$ in the closed universe with the metric of Eq. (1.3). Because the spacetime is reflection-symmetric across $r = a$, there is no loss of generality in taking $r_0 < a$. Because the spatial sections are closed, Gauss's law demands that the total charge be zero, and therefore there must be a second charge $-q$ in the spacetime, which we put at $r = r_1$. We shall consider the cases $r_1 < a$ and $r_1 > a$. The total force acting on $+q$ is given by the sum of the self-force and the force exerted by $-q$.

The spacetime comes with a timelike Killing vector t^α, with nonvanishing component $t^t = 1$. The vector is covariantly constant, so that $\nabla_\alpha t_\beta = 0$.

A. Maxwell's equations

Maxwell's equations are $\nabla_{[\alpha}F_{\beta\gamma]} = 0$ and $\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha$, where $F_{\alpha\beta}$ is the electromagnetic field tensor, and

$$j^\alpha(x) = q\int u_0^\alpha \delta(x, z_0)d\tau_0 - q\int u_1^\alpha \delta(x, z_1)d\tau_1 \tag{2.1}$$

is the current density, with $z_0^\alpha(\tau_0)$ describing the world line of the charge $+q$, while $z_1^\alpha(\tau_1)$ describes the world line of the charge $-q$; τ_0 and τ_1 are the respective proper times, $u_0^\alpha \coloneqq dz_0^\alpha/d\tau_0$ and $u_1^\alpha \coloneqq dz_1^\alpha/d\tau_1$ are the respective velocity vectors, and $\delta(x, z)$ is a scalarized Dirac distribution. The force exerted on $+q$ is formally given by $F^\alpha = qF^\alpha{}_\beta u_0^\beta$. This expression must be regularized to account for the singularity of the field tensor on the charge's world line.

For two charges at rest in the spacetime, we have that $u_0^\alpha = u_1^\alpha = t^\alpha$, and the current density is $j^\alpha = \rho t^\alpha$ with

$$\rho = \frac{q}{R_0^2}\delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) - \frac{q}{R_1^2}\delta(r - r_1)\delta(\cos\theta - \cos\theta_1)\delta(\phi - \phi_1), \tag{2.2}$$

where $R_0 \coloneqq R(r_0) = r_0$, $R_1 \coloneqq R(r_1)$, (θ_0, ϕ_0) are the polar angles of the charge $+q$, and (θ_1, ϕ_1) are those of the charge $-q$.

We introduce a vector potential A_α and express the field tensor as $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. With $A_\alpha = \Phi t_\alpha$ and Φ time-independent, Maxwell's equations become

$$\Box\Phi = -4\pi\rho, \tag{2.3}$$

where $\Box \coloneqq g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. The force acting on $+q$ can then be expressed as

$$F_\alpha = qE_\alpha = -q\nabla_\alpha\Phi, \tag{2.4}$$

where $E_\alpha = -\nabla_\alpha\Phi$ is the electric field, related to the field tensor by $F_{\alpha\beta} = t_\alpha E_\beta - E_\alpha t_\beta$.

Incorporating the metric of Eq. (1.3), Eq. (2.3) reduces to

$$\partial_{rr}\Phi + \frac{2R'}{R}\partial_r\Phi + \frac{1}{R^2}D^2\Phi = -4\pi\rho, \tag{2.5}$$

where $R' \coloneqq dR/dr$ and $D^2 \coloneqq \partial_{\theta\theta} + (\cos\theta/\sin\theta)\partial_\theta + (\sin\theta)^{-2}\partial_{\phi\phi}$ is the Laplacian operator on the unit two-sphere. The potential Φ is expanded in spherical harmonics,

$$\Phi(r,\theta,\phi) = \sum_{\ell m}\Phi_{\ell m}(r)Y_{\ell m}(\theta,\phi), \tag{2.6}$$

and we make use of the completeness relation

$$\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') = \sum_{\ell m} Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi), \tag{2.7}$$

as well as the eigenvalue equation $D^2 Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$. Equation (2.5) becomes

$$\Phi''_{\ell m} + \frac{2R'}{R}\Phi'_{\ell m} - \frac{\ell(\ell+1)}{R^2}\Phi_{\ell m} = -\frac{4\pi q}{r_0^2}Y^*_{\ell m}(\theta_0,\phi_0)\delta(r-r_0) + \frac{4\pi q}{R_1^2}Y^*_{\ell m}(\theta_1,\phi_1)\delta(r-r_1). \tag{2.8}$$

For simplicity we place the two charges on the same radial line, so that $(\theta_1,\phi_1) = (\theta_0,\phi_0)$, and for convenience we align the polar axis with this line, so that $\theta_0 = \theta_1 = 0$. Noting that

$$Y^*_{\ell m}(0,\phi') = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m0}, \qquad Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta), \tag{2.9}$$

we find that Φ is axisymmetric and admits an expansion in Legendre polynomials,

$$\Phi(r,\theta) = \sum_\ell \Phi_\ell(r)P_\ell(\cos\theta), \tag{2.10}$$

with radial functions that satisfy

$$\Phi''_\ell + \frac{2R'}{R}\Phi'_\ell - \frac{\ell(\ell+1)}{R^2}\Phi_\ell = -(2\ell+1)\frac{q}{r_0^2}\delta(r-r_0) + (2\ell+1)\frac{q}{R_1^2}\delta(r-r_1). \tag{2.11}$$

These are related by $\Phi_\ell = [(2\ell+1)/(4\pi)]^{1/2}\Phi_{\ell 0}$ to the radial functions that appear in the original expansion of Eq. (2.6).

When $r \neq r_0$ and $r \neq r_1$, Eq. (2.11) admits the linearly independent solutions $\Phi_\ell = \{R^\ell, R^{-(\ell+1)}\}$. The solution must be regular at $r = 0$ and $r = 2a$, and to account for the delta function at $r = r_0$, it must satisfy the junction conditions

$$[\Phi_\ell]_{r_0} = 0, \qquad [\Phi'_\ell]_{r_0} = -(2\ell+1)\frac{q}{r_0^2}, \tag{2.12}$$

with $[f]_{r_0} := f(r=r_0^+) - f(r=r_0^-)$ denoting the jump of f across $r = r_0$. Similarly, the junction conditions

$$[\Phi_\ell]_{r_1} = 0, \qquad [\Phi'_\ell]_{r_1} = (2\ell+1)\frac{q}{R_1^2} \tag{2.13}$$

account for the delta function at $r = r_1$. We assume that the surface layer is electrically inert, so that both Φ_ℓ and Φ'_ℓ are continuous at $r = a$.

B. Case $r_1 = 0$ or $r_1 = 2a$

The simplest situation has the charge $-q$ at either $r_1 = 0$ or $r_1 = 2a$, the two centers of the spacetime. In this situation the charge $-q$ creates a monopole field, and its attraction on $+q$ is simple to describe. With this attraction accounted for, what is left over is the self-force acting on the original charge.

When $\ell \neq 0$, the solution to Eq. (2.11) is

$$\Phi_\ell(0 < r < r_0) = q\frac{r^\ell}{r_0^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}}, \tag{2.14a}$$

$$\Phi_\ell(r_0 < r < a) = q\frac{r_0^\ell}{r^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}}, \tag{2.14b}$$

$$\Phi_\ell(a < r < 2a) = q\frac{(r_0 R)^\ell}{a^{2\ell+1}} + \frac{q}{2\ell}\frac{(r_0 R)^\ell}{a^{2\ell+1}}, \tag{2.14c}$$

where $R = 2a - r$. When $\ell = 0$, the solution is defined up to an overall additive constant. When $r_1 = 0$ we choose

$$\Phi_0(0 < r < r_0) = -q/r + q/r_0, \tag{2.15a}$$

$$\Phi_0(r_0 < r < a) = 0, \tag{2.15b}$$

$$\Phi_0(a < r < 2a) = 0, \tag{2.15c}$$

and note that in this situation, the solution cannot be regular at $r = 0$. When $r_1 = 2a$, we choose instead

$$\Phi_0(0 < r < r_0) = q/r_0, \tag{2.16a}$$

$$\Phi_0(r_0 < r < a) = q/r, \tag{2.16b}$$

$$\Phi_0(a < r < 2a) = 2q/a - q/R, \tag{2.16c}$$

and note that this solution cannot be regular at $r = 2a$.

The complete potential in the region $0 \le r \le a$ is given by

$$\Phi = \Phi^{\rm S} + \Phi^{\rm R} + \Phi^{\rm int}, \tag{2.17}$$

where

$$\Phi^{\rm S} = q\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\theta) \tag{2.18}$$

with $r_< := \min(r, r_0)$ and $r_> := \max(r, r_0)$,

$$\Phi^{\rm R} = \frac{q}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{(r_0 r)^{\ell}}{a^{2\ell+1}}P_\ell(\cos\theta), \tag{2.19}$$

and

$$\Phi^{\rm int} = \begin{cases} -q/r & -q \text{ at } r_1 = 0 \\ 0 & -q \text{ at } r_1 = 2a \end{cases}. \tag{2.20}$$

The monopole potential $\Phi^{\rm int}$ describes the interaction between the two charges. When the charge $-q$ is at $r_1 = 0$, so that $r_1 < r_0$, the interaction potential is given by $-q/r$, and the charge $+q$ feels the force created by $-q$, given by $-q^2/r_0^2$; this is the expected attraction described by the usual Coulomb law. When, on the other hand, the charge $-q$ is at $r_1 = 2a$, so that $r_1 > r_0$, the force vanishes because $-q$ represents a spherical distribution of charge external to the sphere $r = r_0$. The potential $\Phi^{\rm S}$ is recognized as the potential that would be created by the charge $+q$ if it were situated in a globally flat spacetime. This potential is singular at $r = r_0$, but it produces an isotropic electric field around the charge, and this field does not contribute to the self-force acting on the charge. This potential can be identified with the Detweiler-Whiting *singular potential* [24]. The remaining contribution to Φ is $\Phi^{\rm R}$, which is smooth at $r = r_0$ and is entirely responsible for the self-force; this is identified with the Detweiler-Whiting *regular potential*.

The mode sums for $\Phi^{\rm S}$ and $\Phi^{\rm R}$ can be evaluated; the details are provided in the Appendix. For the singular potential we have the familiar expression

$$\Phi^{\rm S} = \frac{q}{\sqrt{r^2 - 2r_0 r\cos\theta + r_0^2}}, \tag{2.21}$$

and for the regular potential we have

$$\Phi^{\rm R} = -\frac{q}{2a}\ln\left(\frac{a^2 - r_0 r\cos\theta + \sqrt{r_0^2 r^2 - 2a^2 r_0 r\cos\theta + a^4}}{2a^2}\right). \tag{2.22}$$

The regular potential produces the electric field $E_a^{\rm R} := -\partial_a\Phi^{\rm R}$, and the radial component evaluated at $\theta = 0$ is given by

$$E_r^{\rm R}(r, \theta = 0) = -\frac{q r_0}{2a^3}\frac{1}{1 - r_0 r/a^2}. \tag{2.23}$$

This expression is valid for $0 \le r \le a$ and $r_0 < a$, and the apparent singularity at $r = a^2/r_0$ is situated beyond $r = a$, where another (nonsingular) form of solution takes over. The self-force acting on the charge $+q$ is $F_r^{\rm self} = qE_r^{\rm R}(r = r_0, \theta = 0)$, or

$$F_r^{\rm self} = -\frac{q^2 r_0}{2a^3}\frac{1}{1 - (r_0/a)^2}, \tag{2.24}$$

as was first displayed in Eq. (1.5). The expression applies when $r_0 < a$, and it becomes singular in the limit $r_0 \to a$.

C. Case $r_1 < a$

Next we place the charge $-q$ at an arbitrary position r_1 inside the surface layer, so that $r_1 < a$. For concreteness we present the calculation assuming that $r_1 > r_0$; the case $r_1 < r_0$ is very similar, and there is no need to describe it in detail.

When $\ell \neq 0$, the solution to Eq. (2.11) is given by

$$\Phi_\ell(0 < r < r_0) = q\frac{r^\ell}{r_0^{\ell+1}} - q\frac{r^\ell}{r_1^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(r_1 r)^\ell}{a^{2\ell+1}}, \tag{2.25a}$$

$$\Phi_\ell(r_0 < r < r_1) = q\frac{r_0^\ell}{r^{\ell+1}} - q\frac{r^\ell}{r_1^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(r_1 r)^\ell}{a^{2\ell+1}}, \tag{2.25b}$$

$$\Phi_\ell(r_1 < r < a) = q\frac{r_0^\ell}{r^{\ell+1}} - q\frac{r_1^\ell}{r^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(r_1 r)^\ell}{a^{2\ell+1}}, \tag{2.25c}$$

$$\Phi_\ell(a < r < 2a) = q\frac{(r_0 R)^\ell}{a^{2\ell+1}} - q\frac{(r_1 R)^\ell}{a^{2\ell+1}} + \frac{q}{2\ell}\frac{(r_0 R)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(r_1 R)^\ell}{a^{2\ell+1}}, \tag{2.25d}$$

where $R = 2a - r$. For $\ell = 0$ we have

$$\Phi_0(0 < r < r_0) = q/r_0 - q/r_1, \tag{2.26a}$$

$$\Phi_0(r_0 < r < r_1) = q/r - q/r_1, \tag{2.26b}$$

$$\Phi_0(r_1 < r < a) = 0, \tag{2.26c}$$

$$\Phi_0(a < r < 2a) = 0. \tag{2.26d}$$

The complete potential in the region $0 \le r < r_1$ is given by

$$\Phi = \Phi^{\rm S} + \Phi^{\rm R} + \Phi^{\rm int}, \tag{2.27}$$

with $\Phi^{\rm S}$ given by Eqs. (2.18) and (2.21), $\Phi^{\rm R}$ given by Eqs. (2.19) and (2.22), and

$$\Phi^{\rm int} = -q\sum_{\ell=0}^{\infty}\frac{r^\ell}{r_1^{\ell+1}}P_\ell(\cos\theta) - \frac{q}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{(r_1 r)^\ell}{a^{2\ell+1}}P_\ell(\cos\theta). \tag{2.28}$$

A calculation carried out for $r_1 < r_0$ would return the same expression for Φ, except that the first sum for $\Phi^{\rm int}$ would have $r_1^\ell/r^{\ell+1}$ in front of the Legendre polynomials; the general expression is $r_<^\ell/r_>^{\ell+1}$, with $r_< := \min(r, r_1)$ and $r_> := \max(r, r_1)$. The potentials keep their interpretations: $\Phi^{\rm S}$ is the singular potential, $\Phi^{\rm R}$ is the regular potential responsible for the self-force, and $\Phi^{\rm int}$ describes the interaction between charges.

The interaction potential can be evaluated explicitly (see the Appendix for details). We have

$$\Phi^{\rm int} = \frac{-q}{\sqrt{r^2 - 2r_1 r\cos\theta + r_1^2}} + \frac{q}{2a}\ln\left(\frac{a^2 - r_1 r\cos\theta + \sqrt{r_1^2 r^2 - 2a^2 r_1 r\cos\theta + a^4}}{2a^2}\right). \tag{2.29}$$

The potential produces the electric field $E_a^{\rm int} = -\partial_a\Phi^{\rm int}$, and the force acting on the charge $+q$ is $F_r^{\rm int} = qE_r^{\rm int}(r = r_0, \theta = 0)$, or

$$F_r^{\rm int} = \pm\frac{q^2}{(r_1 - r_0)^2} + \frac{q^2 r_1}{2a^3}\frac{1}{1 - r_0 r_1/a^2}, \tag{2.30}$$

with the positive sign applying when $r_1 > r_0$, and the negative sign applying when $r_1 < r_0$. Equation (2.30) reduces to $F_r^{\rm int} = -q^2/r_0^2$ when $r_1 = 0$, in agreement with our results in Sec. II B. The first term in the interaction force is the usual expression of Coulomb's law; the force is directed toward the $-q$ charge at $r = r_1$. The second term is a modification to Coulomb's law contributed by the global curvature of the spacetime; it is directed toward the surface layer.

D. Case $r_1 > a$

In this section we place the charge $-q$ at an arbitrary position r_1 outside the surface layer, so that $r_1 > a$. When $\ell \neq 0$, the solution to Eq. (2.11) is given by

$$\Phi_\ell(0<r<r_0) = q\frac{r^\ell}{r_0^{\ell+1}} - q\frac{(R_1 r)^\ell}{a^{2\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(R_1 r)^\ell}{a^{2\ell+1}}, \tag{2.31a}$$

$$\Phi_\ell(r_0<r<a) = q\frac{r_0^\ell}{r^{\ell+1}} - q\frac{(R_1 r)^\ell}{a^{2\ell+1}} + \frac{q}{2\ell}\frac{(r_0 r)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(R_1 r)^\ell}{a^{2\ell+1}}, \tag{2.31b}$$

$$\Phi_\ell(a<r<r_1) = q\frac{(r_0 R)^\ell}{a^{2\ell+1}} - q\frac{R_1^\ell}{R^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 R)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(R_1 R)^\ell}{a^{2\ell+1}}, \tag{2.31c}$$

$$\Phi_\ell(r_1 < r < 2a) = q\frac{(r_0 R)^\ell}{a^{2\ell+1}} - q\frac{R^\ell}{R_1^{\ell+1}} + \frac{q}{2\ell}\frac{(r_0 R)^\ell}{a^{2\ell+1}} - \frac{q}{2\ell}\frac{(R_1 R)^\ell}{a^{2\ell+1}}, \tag{2.31d}$$

where $R = 2a - r$ and $R_1 = 2a - r_1$. For $\ell = 0$ we have

$$\Phi_0(0 < r < r_0) = q/r_0 - q/a, \tag{2.32a}$$

$$\Phi_0(r_0 < r < a) = q/r - q/a, \tag{2.32b}$$

$$\Phi_0(a < r < r_1) = -q/R + q/a, \tag{2.32c}$$

$$\Phi_0(r_1 < r < 2a) = -q/R_1 + q/a. \tag{2.32d}$$

The complete potential in the region $0 \le r \le a$ is given by

$$\Phi = \Phi^{\rm S} + \Phi^{\rm R} + \Phi^{\rm int}, \tag{2.33}$$

with $\Phi^{\rm S}$ given by Eqs. (2.18) and (2.21), $\Phi^{\rm R}$ given by Eqs. (2.19) and (2.22), and $\Phi^{\rm int}$ now given by

$$\Phi^{\rm int} = -q\sum_{\ell=0}^{\infty}\frac{(R_1 r)^\ell}{a^{2\ell+1}}P_\ell(\cos\theta) - \frac{q}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{(R_1 r)^\ell}{a^{2\ell+1}}P_\ell(\cos\theta). \tag{2.34}$$

The interaction potential is given explicitly by (see the Appendix)

$$\Phi^{\rm int} = \frac{-qa}{\sqrt{R_1^2 r^2 - 2a^2 R_1 r\cos\theta + a^4}} + \frac{q}{2a}\ln\left(\frac{a^2 - R_1 r\cos\theta + \sqrt{R_1^2 r^2 - 2a^2 R_1 r\cos\theta + a^4}}{2a^2}\right), \tag{2.35}$$

and it produces the electric field $E_a^{\rm int} = -\partial_a\Phi^{\rm int}$. The force acting on the charge $+q$ is $F_r^{\rm int} = qE_r^{\rm int}(r = r_0, \theta = 0)$, or

$$F_r^{\rm int} = \frac{q^2 R_1}{2a^3}\frac{3 - r_0 R_1/a^2}{(1 - r_0 R_1/a^2)^2}. \tag{2.36}$$

The force vanishes when $R_1 = 0$, or $r_1 = 2a$, in agreement with our results in Sec. II B. With the charges situated on opposite sides of the surface layer, the interaction force bears little resemblance to the usual Coulomb force, and it is always directed toward the surface layer.

III. ELECTROMAGNETIC SELF-FORCE ON A POINT DIPOLE

In this section we consider a point electric dipole $\boldsymbol{p}$ situated at $r = r_0 < a$ (inside the surface layer), and we calculate the self-force on this dipole.

A. Point dipole

Because the dipole is at rest in a local patch of Minkowski spacetime, it is convenient to describe its local physics in a Newtonian language involving Cartesian vectors such as $\boldsymbol{p}$; relativistic aspects reveal themselves only when we integrate Maxwell's equations for the electrostatic potential Φ. We shall use both the original spherical coordinates (r, θ, ϕ) and the associated Cartesian coordinates (x, y, z), but express all vectors and tensors in Cartesian coordinates. We recall that the vector basis attached to the spherical coordinates is given by

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}, \tag{3.1a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}}, \tag{3.1b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}, \tag{3.1c}$$

in terms of the Cartesian basis vectors. All vectors have a unit length.

The charge density of a point dipole is given by

$$\rho(\boldsymbol{x}) = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{r}_0), \tag{3.2}$$

where $\delta(\boldsymbol{x}-\boldsymbol{r}_0)$ is a three-dimensional delta function, and $\boldsymbol{r}_0 = (0, 0, r_0)$ is the dipole's position vector. The potential Φ created by the dipole satisfies Eq. (2.3), or its explicit expression of Eq. (2.5). The force exerted on the dipole is given formally by

$$F_a = p^b\partial_b E_a = -p^b\partial_{ab}\Phi, \tag{3.3}$$

in which $E_a := -\partial_a\Phi$ is the electric field, and the potential is evaluated at $\boldsymbol{x} = \boldsymbol{r}_0$ after differentiation. This expression must be regularized before a meaningful result is obtained for the self-force.

B. Dipole in the z direction

We first calculate the potential for a dipole aligned with the z axis. We have $\boldsymbol{p} = p\hat{\boldsymbol{z}}$, so that $p \equiv p_z$. We wish to perform the calculation in spherical coordinates, and to handle the coordinate singularity on the polar axis—ϕ is not defined there—we first place the dipole at (r_0, θ_0, ϕ_0) and eventually take the limit $\theta_0 \to 0$; the limit is independent of ϕ_0. We also put the dipole in the direction of $\hat{\boldsymbol{r}}_0$, the radial unit vector evaluated at $(\theta = \theta_0, \phi = \phi_0)$; in the limit the dipole becomes aligned with the z axis.

Working momentarily in Cartesian coordinates, the dipole's charge density is given by

$$\rho = -p\hat{\boldsymbol{r}}_0\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{r}_0) = +p\hat{\boldsymbol{r}}_0\cdot\boldsymbol{\nabla}_0\delta(\boldsymbol{x}-\boldsymbol{r}_0), \tag{3.4}$$

where $\boldsymbol{\nabla}_0$ is the gradient operator associated with the variables contained in $\boldsymbol{r}_0$. Switching now to the spherical coordinates, we have

$$\rho = p\frac{\partial}{\partial r_0}\left[\frac{\delta(r-r_0)}{r^2}\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)\right], \tag{3.5}$$

or

$$\rho = -p\left[\frac{\delta'(r-r_0)}{r_0^2} + 2\frac{\delta(r-r_0)}{r_0^3}\right]\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0), \tag{3.6}$$

with a prime indicating differentiation with respect to r; we made use of the distributional identity $f(r)\delta'(r-r_0) = f(r_0)\delta'(r-r_0) - f'(r_0)\delta(r-r_0)$. The same expression for ρ can be obtained from the two-charge model of Eq. (2.2) by letting $r_1 = r_0 - \delta r$, $\theta_1 = \theta_0$, $\phi_1 = \phi_0$, and taking the limit $\delta r \to 0$ with $p := q\delta r$ kept fixed.

The calculation of Φ proceeds as in Sec. II A. In the limit $\theta_0 \to 0$ with ϕ_0 arbitrary, the potential is axisymmetric and admits the decomposition

$$\Phi(r,\theta) = \sum_\ell \Phi_\ell(r)P_\ell(\cos\theta), \tag{3.7}$$

with radial functions that satisfy

$$\Phi_\ell'' + \frac{2R'}{R}\Phi_\ell' - \frac{\ell(\ell+1)}{R^2}\Phi_\ell = (2\ell+1)p\left[\frac{\delta'(r-r_0)}{r_0^2} + 2\frac{\delta(r-r_0)}{r_0^3}\right]. \tag{3.8}$$

When $r \neq r_0$, the differential equation admits the linearly independent solutions $\Phi_\ell = \{R^\ell, R^{-(\ell+1)}\}$. The solution must be regular at $r = 0$ and $r = 2a$, and to account for the singularity at $r = r_0$, it must satisfy the junction conditions

$$[\Phi_\ell]_{r_0} = (2\ell+1)\frac{p}{r_0^2}, \qquad [\Phi_\ell']_{r_0} = 0. \tag{3.9}$$

We again assume that the surface layer is electrically inert, so that both Φ_ℓ and Φ_ℓ' are continuous at $r = a$.

The solution that satisfies all these requirements is

$$\Phi_\ell(0<r<r_0) = -(\ell+1)p\frac{r^\ell}{r_0^{\ell+2}} + \frac{1}{2}p\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}, \tag{3.10a}$$

$$\Phi_\ell(r_0<r<a) = \ell p\frac{r_0^{\ell-1}}{r^{\ell+1}} + \frac{1}{2}p\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}, \tag{3.10b}$$

$$\Phi_\ell(a < r < 2a) = \frac{1}{2}(2\ell+1)p\frac{r_0^{\ell-1}R^\ell}{a^{2\ell+1}}. \tag{3.10c}$$

The complete potential in the region $0 < r < a$ is then

$$\Phi = \Phi^{\rm S} + \Phi^{\rm R} \tag{3.11}$$

with

$$\Phi^{\rm S} = p\frac{\partial}{\partial r_0}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta) \tag{3.12}$$

and

$$\Phi^{\rm R} = \frac{1}{2}p\sum_{\ell=0}^{\infty}\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}P_\ell(\cos\theta), \tag{3.13}$$

where $r_< := \min(r, r_0)$ and $r_> := \max(r, r_0)$.

Comparison with Eqs. (2.18) and (2.21) allows us to evaluate the sums, and we arrive at

$$\Phi^{\rm S} = \frac{p(r\cos\theta - r_0)}{(r^2 - 2r_0 r\cos\theta + r_0^2)^{3/2}} \tag{3.14}$$

and

$$\Phi^{\rm R} = \frac{pa}{2r_0 s^2}, \tag{3.15}$$

where

$$s^2 := (r_0^2 r^2 - 2a^2 r_0 r\cos\theta + a^4)^{1/2}. \tag{3.16}$$

Equation (3.14) is the familiar expression for the potential of a point dipole aligned with the z axis, when the dipole is placed in a globally flat spacetime; this potential diverges at the dipole's position, and it can be identified with the singular Detweiler-Whiting potential [24]. The potential of Eq. (3.15) is smooth at the dipole's position, and it can be identified with the regular Detweiler-Whiting potential. The self-force acting on the dipole will come entirely from $\Phi^{\rm R}$.

C. Dipole in the x direction

Next we take the dipole to be aligned with the x axis, so that $\boldsymbol{p} = p\hat{\boldsymbol{x}}$ and $p \equiv p_x$. To set up the calculation we place the dipole at (r_0, θ_0, ϕ_0) and align it with $\hat{\boldsymbol{\theta}}_0$, the angular unit vector evaluated at the dipole's position. Taking the limit $\theta_0 \to 0$ with $\phi_0 = 0$ will take the dipole to the z axis and align it with the x direction.

The charge density is given by

$$\begin{aligned}\rho &= -p\hat{\boldsymbol{\theta}}_0\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{r}_0) = p\hat{\boldsymbol{\theta}}_0\cdot\boldsymbol{\nabla}_0\delta(\boldsymbol{x}-\boldsymbol{r}_0)\\ &= \frac{p}{r_0}\frac{\partial}{\partial\theta_0}\left[\frac{\delta(r-r_0)}{r^2}\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)\right],\end{aligned} \tag{3.17}$$

or

$$\rho = \frac{p}{r_0^3}\delta(r-r_0)\partial_{\theta_0}\delta(\cos\theta - \cos\theta_0)\delta(\phi-\phi_0). \tag{3.18}$$

The same expression can be obtained from the two-charge model of Eq. (2.2) by letting $r_1 = r_0$, $\theta_1 = \theta_0 - \delta\theta$, $\phi_1 = \phi_0$, and taking the limit $\delta\theta \to 0$ keeping $p := qr_0\delta\theta$ fixed.

The potential is expanded in spherical harmonics as in Eq. (2.6), and the completeness relation of Eq. (2.7) is differentiated with respect to θ_0. The identity

$$\begin{aligned}\partial_\theta Y_{\ell m} &= \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}Y_{\ell,m+1}e^{-i\phi}\\ &\quad -\frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}Y_{\ell,m-1}e^{i\phi}\end{aligned} \tag{3.19}$$

allows us to express $\partial_{\theta_0}Y_{\ell m}(\theta_0, \phi_0)$ in terms of spherical harmonics, and to take the limit $\theta_0 \to 0$ with $\phi_0 = 0$; we obtain

$$\lim_{\theta_0\to 0}\frac{\partial}{\partial\theta_0}Y_{\ell m}(\theta_0, 0) = \frac{1}{2}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\ell(\ell+1)}(\delta_{m,-1} - \delta_{m,1}). \tag{3.20}$$

This relation implies that $\Phi_{\ell,-1} = -\Phi_{\ell,1}$. With

$$Y_{\ell,\pm 1} = \mp\sqrt{\frac{2\ell+1}{4\pi}}\frac{1}{\sqrt{\ell(\ell+1)}}P_\ell^1(\cos\theta)e^{\pm i\phi}, \tag{3.21}$$

where P_ℓ^1 is an associated Legendre function, and the definition

$$\Phi_\ell := -2\sqrt{\frac{2\ell+1}{4\pi}}\frac{1}{\sqrt{\ell(\ell+1)}}\Phi_{\ell,1}, \tag{3.22}$$

we find that the potential admits the decomposition

$$\Phi(r,\theta,\phi) = \sum_{\ell=1}^{\infty}\Phi_\ell(r)P_\ell^1(\cos\theta)\cos\phi \tag{3.23}$$

with radial functions that satisfy

$$\Phi_\ell'' + \frac{2R'}{R}\Phi_\ell' - \frac{\ell(\ell+1)}{R^2}\Phi_\ell = -(2\ell+1)\frac{p}{r_0^3}\delta(r-r_0). \tag{3.24}$$

The solution to the differential equation is

$$\Phi_\ell(0 < r < r_0) = p\frac{r^\ell}{r_0^{\ell+2}} + \frac{p}{2\ell}\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}, \tag{3.25a}$$

$$\Phi_\ell(r_0 < r < a) = p\frac{r_0^{\ell-1}}{r^{\ell+1}} + \frac{p}{2\ell}\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}, \tag{3.25b}$$

$$\Phi_\ell(a < r < 2a) = p\frac{r_0^{\ell-1}R^\ell}{a^{2\ell+1}} + \frac{p}{2\ell}\frac{r_0^{\ell-1}R^\ell}{a^{2\ell+1}}, \quad (3.25c)$$

and the potential in the region $0 < r < a$ is

$$\Phi = \Phi^S + \Phi^R \quad (3.26)$$

with

$$\Phi^S = \frac{p}{r_0}\sum_{\ell=1}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell^1(\cos\theta)\cos\phi \quad (3.27)$$

and

$$\Phi^R = \frac{1}{2}p\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{r_0^{\ell-1}r^\ell}{a^{2\ell+1}}P_\ell^1(\cos\theta)\cos\phi. \quad (3.28)$$

The sum for Φ^S can be evaluated by noting that $P_\ell^1(\cos\theta) = -(d/d\theta)P_\ell(\cos\theta)$, and we obtain

$$\Phi^S = \frac{pr\sin\theta\cos\phi}{(r^2 - 2r_0r\cos\theta + r_0^2)^{3/2}}. \quad (3.29)$$

This is the familiar expression for the potential of a point dipole in a globally flat spacetime, when the dipole is aligned with the x axis; this potential diverges at the dipole's position, and it can be identified with the singular Detweiler-Whiting potential [24]. The sum for Φ^R can also be evaluated by comparing with Eqs. (2.19) and (2.22). After differentiation with respect to θ we arrive at

$$\Phi^R = \frac{p}{2a}\frac{(a^2+s^2)r\sin\theta\cos\phi}{s^2(a^2 - r_0r\cos\theta + s^2)}, \quad (3.30)$$

where s^2 is defined by Eq. (3.16). This potential is smooth at the dipole's position, and it can be identified with the regular Detweiler-Whiting potential. The self-force acting on the dipole will originate entirely from Φ^R.

D. Dipole in the y direction

The calculation of Φ for a dipole aligned with the y axis proceeds as in Sec. III C. The steps are identical, except for the fact that the limit $\theta_0 \to 0$ is now taken with $\phi_0 = \pi/2$ to produce the correct orientation for the dipole. The end result for Φ^R is

$$\Phi^R = \frac{p}{2a}\frac{(a^2+s^2)r\sin\theta\sin\phi}{s^2(a^2 - r_0r\cos\theta + s^2)}, \quad (3.31)$$

which can be obtained directly from Eq. (3.30) by replacing the factor of $\cos\phi$ with $\sin\phi$.

E. Self-force on a dipole

The complete Φ^R for an arbitrarily aligned dipole can be obtained by combining Eqs. (3.15), (3.30), and (3.31). We get

$$\Phi^R = \frac{(a^2+s^2)(p_xr\sin\theta\cos\phi + p_yr\sin\theta\sin\phi)}{2as^2(a^2 - r_0r\cos\theta + s^2)} + \frac{ap_z}{2r_0s^2}, \quad (3.32)$$

with s^2 defined by Eq. (3.16). The self-force acting on the dipole is calculated from Eq. (3.3), which we write as

$$F_a^{\rm self} = -p^b\partial_{ab}\Phi^R. \quad (3.33)$$

To calculate the components of the force we express the potential in Cartesian coordinates, take the derivatives, and evaluate the result at $x = y = 0$ and $z = r_0$. We obtain

$$F_x^{\rm self} = -\frac{r_0p_xp_z}{4a^5(1 - r_0^2/a^2)^2}, \quad (3.34a)$$

$$F_y^{\rm self} = -\frac{r_0p_yp_z}{4a^5(1 - r_0^2/a^2)^2}, \quad (3.34b)$$

$$F_z^{\rm self} = -\frac{r_0[(3 - r_0^2/a^2)(p_x^2 + p_y^2) + 4p_z^2]}{4a^5(1 - r_0^2/a^2)^3}. \quad (3.34c)$$

This can be put in the vectorial form of Eq. (1.7), with $\boldsymbol{n} := \boldsymbol{r}_0/r_0 = \hat{\boldsymbol{z}}$.

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APPENDIX: DERIVATION OF EQS. (2.21), (2.22), (2.29), AND (2.35)

Equation (2.21) follows directly from Eq. (2.18) and the identity

$$(1 - 2tx + t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x), \quad (A1)$$

which holds for $|t| < 1$.

To obtain Eq. (2.22), we note that Eq. (2.19) can be expressed as

$$\Phi^R = \frac{qa}{2}\int dr_0\left[\frac{1}{r_0^2}\sum_{\ell=1}^{\infty}\frac{r^\ell}{(a^2/r_0)^{\ell+1}}P_\ell(\cos\theta)\right], \quad (A2)$$

or

$$\Phi^R = \frac{qa}{2}\int dr_0\left[\frac{1}{r_0^2}\sum_{\ell=0}^{\infty}\frac{r^\ell}{(a^2/r_0)^{\ell+1}}P_\ell(\cos\theta) - \frac{1}{a^2r_0}\right]. \quad (A3)$$

Making use of Eq. (A1), the sum over Legendre polynomials evaluates to

$$\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{(a^2/r_0)^{\ell+1}}P_{\ell}(\cos\theta)=\frac{r_0}{\sqrt{r_0^2r^2-2a^2r_0r\cos\theta+a^4}}, \tag{A4}$$

and the potential becomes

$$\Phi^{\rm R}=\frac{qa}{2}\int dr_0\left[\frac{1}{r_0\sqrt{r_0^2r^2-2a^2r_0r\cos\theta+a^4}}-\frac{1}{a^2r_0}\right]. \tag{A5}$$

Direct integration gives rise to Eq. (2.22) after adjusting the constant of integration so that $\Phi^{\rm R}(r=0,\theta)=0$, as implied by Eq. (2.19).

The steps required to arrive at Eq. (2.29) from Eq. (2.28) are identical to those described previously, with the changes $q\to -q$ and $r_0\to r_1$. To go from Eq. (2.34) to Eq. (2.35), we let $q' := -qa/R_1$ and $R_1' := a^2/R_1$ in the first sum, which becomes

$$q'\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{(R_1')^{\ell+1}}P_{\ell}(\cos\theta)=\frac{q'}{\sqrt{r^2-2R_1'r\cos\theta+R_1'^2}}, \tag{A6}$$

and which gives rise to the first term in Eq. (2.35). It is interesting to note that the expressions for q' and R_1' are the same ones that arise for the image charge in the problem of a point charge outside a grounded, spherical conductor. The second sum in Eq. (2.34) is of the same form as Eq. (2.19) with $r_0\to R_1$, and it leads directly to the second term in Eq. (2.35).

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