Simple metric for a magnetized, spinning, deformed mass

V. S. Manko[1] and E. Ruiz[2]

[1]*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, A.P. 14-740, 07000 Ciudad de México, Mexico*

[2]*Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, 37008 Salamanca, Spain*

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We present and discuss a 4-parameter stationary axisymmetric solution of the Einstein-Maxwell equations, which is able to describe the exterior field of a rotating magnetized deformed mass. The solution arises as a system of two overlapping corotating magnetized nonequal black holes or hyperextreme disks, and we write it in a concise explicit form that is very suitable for concrete applications. An interesting peculiar feature of this electrovac solution is that it does not develop massless ring singularities outside the stationary limit surface, its first four electric multipole moments being equal to zero; it also has a nontrivial extreme limit, which we elaborate completely in terms of four polynomial factors.

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I. INTRODUCTION

During the last decade there has been considerable interest in the observational confirmation of the nature of the known black hole (BH) candidates as yet another possible test of general relativity [1,2]. It is clear that to be able to recognize a Kerr BH [3] by analyzing the data obtained in a real astronomical observation, it is also necessary to know the properties of the non-Kerr spacetimes, which could only slightly differ from those of a BH. Moreover, proposals for the study of the properties of black holes with electromagnetic radiation [4,5] make desirable the knowledge of simple models for compact magnetized objects permitting a clear physical interpretation. Such models may arise, in particular, from the extended 2-soliton electrovac solution with equatorial symmetry [6], determining a large 6-parameter family of the two-body configurations and permitting us to analytically approximate the exterior fields of compact astrophysical objects. Note that in order to form a configuration with reflection symmetry, a two-body system must be composed either of two separated identical corotating constituents or by two nonequal overlapping constituents with a common center and with a corotation or a counterrotation. While the former type of two-body system is more habitual for analysis and study, the systems of the latter type, as a rule, are ignored because their interpretation may look clumsy from the black hole point of view. At the same time, the use of overlapping black holes for modeling compact objects of physical interest seems very natural; for instance, a pair of superposed Schwarzschild black holes with a common center could be a good model of a static deformed mass.[1] Similarly, the double-Reissner-Nordström solution [9] in which the separation distance R is set equal to zero would describe a simple model of a charged static deformed mass; besides, as can be easily checked, in the $R = 0$ limit, this solution simplifies considerably and becomes a static specialization of the solution [6]. In the present paper, we will further explore the second type of equatorially symmetric configurations and consider a 4-parameter electrovac metric for a magnetized rotating deformed mass, which could also be regarded as representing two overlapping black hole constituents and which was identified with the help of a recent paper on the charged rotating masses [10]. The solution will be shown to have a simple form, with various distinctive features and interesting limits, which makes it very suitable for use in concrete applications.

Our paper is organized as follows. In the next section we present the 4-parameter electrovac solution, the corresponding complete metric, and the expression of the magnetic potential. Here we also analyze the sub- and hyperextreme cases of the solution and its multipole structure. In Sec. III we consider interesting specializations of the solution and work out the solution's extreme limit. Concluding remarks are given in Sec. IV.

II. THE 4-PARAMETER ELECTROVAC SOLUTION AND METRIC FUNCTIONS

Let us begin this section by noting that the general equatorially symmetric 2-soliton electrovac solution [6] (henceforth referred to as the MMR solution), obtained with the aid of Sibgatullin's integral method [11,12], is defined by the axis data

[1]As is well known [7,8], such a superposition with a common center does not lead to the Schwarzschild spacetime with the combined mass of the constituents.

$$e(z) = \frac{(z-m-ia)(z+ib)+k}{(z+m-ia)(z+ib)+k},$$
$$f(z) = \frac{qz+ic}{(z+m-ia)(z+ib)+k}, \quad (1)$$

where $e(z)$ and $f(z)$ are the expressions of the Ernst complex potentials $\mathcal{E}$ and Φ [13] on the upper part of the symmetry z axis; the six arbitrary real parameters entering (1) are m, a, b, k, q, and c.

The 4-parameter specialization of the MMR solution, which we are going to report in this paper, is determined by the following simple choice of parameters in (1):

$$b=a, \quad k=m_1m_2-\mu^2, \quad q=0, \quad c=m\mu, \quad (2)$$

the mass parameters m_1 and m_2 being such that $m_1+m_2=m$; the charge parameter q is set equal to zero because of its irrelevance for astrophysical applications, and hence the electromagnetic field in this solution is defined solely by the magnetic dipole parameter μ. Then, the axis data determining this particular case take the form

$$e(z)=\frac{(z-m_1)(z-m_2)-ia(m_1+m_2)+a^2-\mu^2}{(z+m_1)(z+m_2)+ia(m_1+m_2)+a^2-\mu^2},$$
$$f(z)=\frac{i\mu(m_1+m_2)}{(z+m_1)(z+m_2)+ia(m_1+m_2)+a^2-\mu^2}, \quad (3)$$

where the four arbitrary real parameters are m_1, m_2, a, and μ. The particular parameter choice (2) occurred to us when we noticed that the metric for two unequal counterrotating charged masses [10] becomes, in the limit $R=0$, a member of the MMR solution.

An attractive feature of the data in (3) is that the algebraic equation

$$e(z)+\bar{e}(z)+2f(z)\bar{f}(z)=0 \quad (4)$$

(the bars over symbols denote complex conjugation), which plays an important role in Sibgatullin's method, yields, for this case, four very simple roots α_n, namely,

$$\alpha_1=-\alpha_2=\sqrt{m_1^2-a^2+\mu^2}\equiv\sigma_1,$$
$$\alpha_3=-\alpha_4=\sqrt{m_2^2-a^2+\mu^2}\equiv\sigma_2, \quad (5)$$

which in turn define the following four functions of the Weyl-Papapetrou coordinates (ρ, z):

$$R_\pm=\sqrt{\rho^2+(z\pm\sigma_1)^2}, \quad r_\pm=\sqrt{\rho^2+(z\pm\sigma_2)^2}. \quad (6)$$

Formulas (3), (5), and (6) permit one to construct, by purely algebraic manipulations, the Ernst potentials $\mathcal{E}$ and Φ, as well as the corresponding metric functions f, γ, and ω of the line element

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\varphi^2]-f(dt-\omega d\varphi)^2, \quad (7)$$

using only the determinantal formulas of Ref. [14]. In our concrete case the desired expressions can also be worked out from the respective formulas of the MMR solution.[2] As a result, the potentials $\mathcal{E}$ and Φ of the 4-parameter solution can be shown to have the form

$$\mathcal{E}=\frac{A-B}{A+B}, \qquad \Phi=\frac{C}{A+B},$$
$$A=\sigma_1\sigma_2[(m_1^2+m_2^2)(R_++R_-)(r_++r_-)-4m_1m_2(R_+R_-+r_+r_-)]-(m_2^2\sigma_1^2+m_1^2\sigma_2^2)(R_+-R_-)(r_+-r_-)$$
$$+ia(m_1^2-m_2^2)[\sigma_1(R_++R_-)(r_+-r_-)-\sigma_2(R_+-R_-)(r_++r_-)],$$
$$B=-2(m_1^2-m_2^2)\{\sigma_1\sigma_2[m_2(R_++R_-)-m_1(r_++r_-)]+ia[m_2\sigma_2(R_+-R_-)-m_1\sigma_1(r_+-r_-)]\},$$
$$C=2i\mu(m_1^2-m_2^2)[m_1\sigma_2(R_+-R_-)-m_2\sigma_1(r_+-r_-)], \quad (8)$$

while for the metrical fields f, γ, and ω, one gets the following expressions:

[2] Unfortunately, the general formulas of Ref. [10] are not helpful for elaborating the $R=0$ limit because of the misprints in that paper, so Eqs. (8) and (9) have been worked out with the aid of our computer codes developed for the MMR solution.

$$
\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K_0\bar{K}_0 R_+ R_- r_+ r_-}, \qquad \omega = -\frac{\mathrm{Im}[G(\bar{A}+\bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \\
G &= -2zB + (m_1^2 - m_2^2)\{\sigma_1(2m_2^2 + \mu^2)(R_+ + R_-)(r_+ - r_-) - \sigma_2(2m_1^2 + \mu^2)(R_+ - R_-)(r_+ + r_-) \\
&\quad - 2ia(m_1^2 - m_2^2)(R_+ - R_-)(r_+ - r_-) \\
&\quad - 2\sigma_2[2m_2\sigma_1^2 - \mu^2(m_1 + m_2)](R_+ - R_-) + 2\sigma_1[2m_1\sigma_2^2 - \mu^2(m_1 + m_2)](r_+ - r_-) \\
&\quad - 4ia\sigma_1\sigma_2[m_2(R_+ + R_-) - m_1(r_+ + r_-)]\}, \\
I &= -i\mu(m_1 + m_2)\Big(\sigma_1\sigma_2[(m_1 + m_2)(R_+ + R_-)(r_+ + r_-) - 4m_1R_+R_- - 4m_2r_+r_-] \\
&\quad - (m_2\sigma_1^2 + m_1\sigma_2^2)(R_+ - R_-)(r_+ - r_-) + ia(m_1 - m_2)[\sigma_1(R_+ + R_-)(r_+ - r_-) \\
&\quad - \sigma_2(R_+ - R_-)(r_+ + r_-)] - 2(m_1 - m_2)\{\sigma_1\sigma_2[(2m_1 + m_2)(R_+ + R_-) \\
&\quad - (m_1 + 2m_2)(r_+ + r_-) + 2m_1^2 - 2m_2^2] + ia(m_1 + m_2)[\sigma_2(R_+ - R_-) - \sigma_1(r_+ - r_-)]\}\Big), \\
K_0 &= 4\sigma_1\sigma_2(m_1 - m_2)^2.
\end{aligned} \tag{9}
$$

It should also be noted that the electric A_t and magnetic A_φ components of the electromagnetic 4-potential defined by the solution (8) have the form

$$
A_t = -\mathrm{Re}\left(\frac{C}{A+B}\right), \qquad A_\varphi = \mathrm{Im}\left(\frac{I - zC}{A+B}\right), \tag{10}
$$

and these formulas complement the description of our particular 4-parameter electrovac spacetime.

Turning now to the discussion of the properties of the solution (8), we first mention that the form of σ_1 and σ_2 defined in (5) clearly shows which type of constituents may form the two-body configurations described by this solution. In the subextreme case, the quantities σ_1 and σ_2 are real valued, which means that both m_1 and m_2 must fulfill the inequalities $m_1^2 > a^2 - \mu^2$ and $m_2^2 > a^2 - \mu^2$. Similarly, in the hyperextreme case both σ_1 and σ_2 take pure imaginary values, which means that $m_1^2 < a^2 - \mu^2$ and $m_2^2 < a^2 - \mu^2$. Since $\sigma_1 \neq \sigma_2$ generically, we can suppose, say, that $m_1 > m_2$; then the mixed subextreme-hyperextreme case arises when $m_2^2 < a^2 - \mu^2 < m_1^2$. These three basic types of two-body configurations described by the solution (8) are depicted in Fig. 1. The fact that the constituents are overlapping can be most easily seen by setting the rotational parameter a equal to zero in the above formulas and observing that in this case the solution reduces to the $R = 0$ limit of the asymmetric dihole spacetime considered in [15].

The calculation of the first five Beig-Simon relativistic multipole moments [16], with the aid of the Hoenselaers-Perjés procedure [17] rectified by Sotiriou and Apostolatos [18], yields for the solution (8) the expressions

$$
\begin{aligned}
&P_0 = m_1 + m_2, \quad P_1 = ia(m_1 + m_2), \quad P_2 = -(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \\
&P_3 = -ia(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \quad P_4 = (m_1 + m_2)(m_1m_2 + a^2 - \mu^2)^2 + \frac{1}{70}(m_1 + m_2)^3(10m_1m_2 - 7\mu^2), \\
&Q_0 = 0, \quad Q_1 = i\mu(m_1 + m_2), \quad Q_2 = 0, \quad Q_3 = -i\mu(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \quad Q_4 = -\frac{1}{10}a\mu(m_1 + m_2)^3,
\end{aligned} \tag{11}
$$

where it follows that the parameters m_1 and m_2 can be associated with the individual masses of the first and second constituents, respectively. Moreover, the expression of the total angular momentum P_1 indicates that the constituents are corotating with the same angular momentum per unit mass ratio: $j_1/m_1 = j_2/m_2 = a$, with j_1 and j_2 being angular momenta of the first and second constituents, respectively. A surprising feature of the electromagnetic moments Q_n is that the first four electric multipoles (these are represented by the real parts of the respective Q_n) are zeros, the first nonzero electric moment being the hexadecapole one. Note also that by setting $\mu = m_1 = 0$ (or $\mu = m_2 = 0$) in (11), one immediately obtains the multipoles P_n of the Kerr solution [3], so the latter important solution is contained in a simple way in our general formulas (8) and (9).

The singularities of the solution (8) in the cases of both gravitational and electromagnetic fields are defined by the roots of the equation $A + B = 0$. While it is well known that in the pure vacuum limit ($\mu = 0$) the singularities are

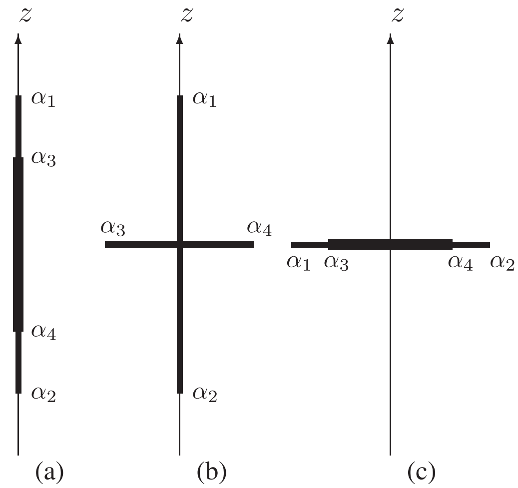

FIG. 1. Three different types of systems with overlapping constituents: (a) subextreme configuration, (b) subextreme-extreme configuration, and (c) hyperextreme configuration.

located on the stationary limit surface ($f = 0$), it is most remarkable that even in the case of nonvanishing μ the singularities of this solution lie on that surface too, as we have checked numerically for a wide range of values of the parameters; this is due to the specific property of the solution (8) according to which its singularities are also zeros of the function C from the electromagnetic potential Φ. In the subextreme case, for instance, the massless ring singularities off the symmetry axis do not appear if m_1 and m_2 are positive quantities and the stationary limit surface consists of two disconnected regions [see Fig. 2(a)]. However, the positiveness of m_1 and m_2 does not prevent the appearance of the ring singularity outside the z axis when the two regions constituting the stationary limit surface overlap [Fig. 3(a)], and this singularity seems to correspond to the geometrical singularity determining the shape of the latter surface. Additionally, in Figs. 2(b) and 3(b) we have plotted the magnetic lines of force for the same particular parameter choices that were used for plotting Figs. 2(a) and 3(a), respectively.

We conclude this section by giving useful and concise formulas for the nonzero ρ and z components of the electric and magnetic fields in terms of the Ernst complex potential Φ:

$$E_\rho + iB_z = \sqrt{f}e^{-2\gamma}[\mathrm{Re}(\Phi_{,\rho}) + i\mathrm{Im}(\Phi_{,z})],$$
$$E_z + iB_\rho = \sqrt{f}e^{-2\gamma}[\mathrm{Re}(\Phi_{,z}) + i\mathrm{Im}(\Phi_{,\rho})], \qquad (12)$$

which are obtainable from the formulas $E_\mu = F_{\mu\nu}u^\nu$, $B_\mu = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}u^\nu$, where $F_{\mu\nu}$ is the electromagnetic field tensor and $u^\nu = (1/\sqrt{f}, 0, 0, 0)$ [19], and could be used in concrete applications; it may be noted that their form is simpler than the one employed, for instance, in Ref. [20]. Let us also mention that the nonzero component of the Poynting vector in the stationary axisymmetric case is defined by a simple formula [21,22],

$$S^\varphi = \frac{\sqrt{f}e^{-2\gamma}}{4\pi\rho}\mathrm{Im}(\bar{\Phi}_{,\rho}\Phi_{,z}), \qquad (13)$$

where it follows, in particular, that an application of the duality rotation transformation $\Phi \to e^{i\alpha}\Phi$, $\alpha = \mathrm{const}$, to our solution would lead to a spacetime representing, in the limit $a = 0$, an electrovac *static* solution for a mass endowed with both electric and magnetic dipole moments.

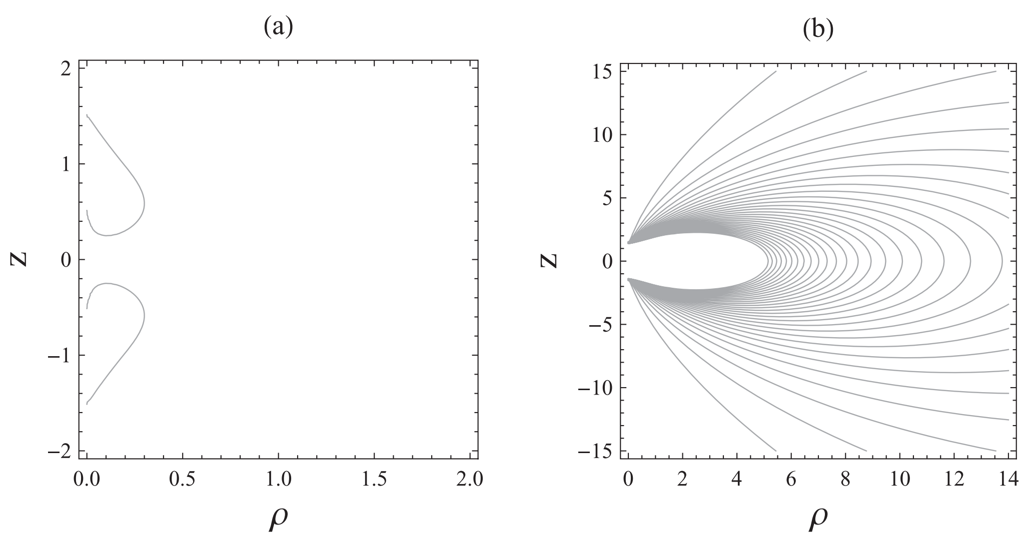

FIG. 2. Plots of the solution's (a) stationary limit surface and (b) magnetic lines of force for the particular parameter choice $m_1 = 1.5$, $m_2 = 0.5$, $a = 0.125$, $\mu = 0.25$.

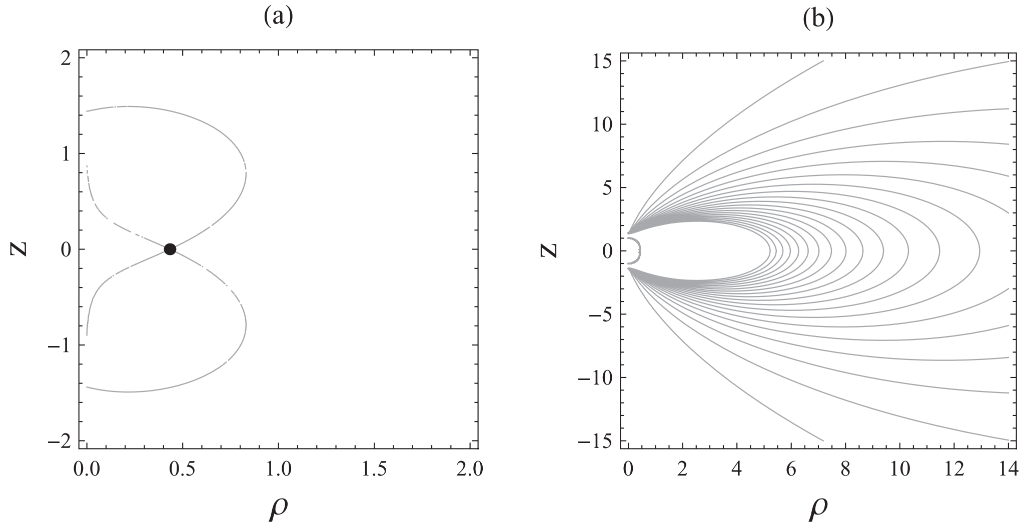

FIG. 3. Plots of the solution's (a) stationary limit surface and (b) magnetic lines of force for the particular parameter choice $m_1 = 1.5$, $m_2 = 1$, $a = 0.5$, $\mu = 0.25$. The ring singularity is located at $\rho \approx 0.433$, $z = 0$.

III. PHYSICALLY INTERESTING LIMITS OF THE SOLUTION

The 4-parameter solution (8) has various physically interesting limits, which we will briefly consider below.

A. Magnetostatic limit

When $a = 0$, the solution describes a static deformed mass endowed with a magnetic dipole moment, and it coincides, as was already mentioned earlier, with the $R = 0$ specialization of the asymmetric dihole solution considered in [15]. In this case $\sigma_1 = \sqrt{m_1^2 + \mu^2}$, $\sigma_2 = \sqrt{m_2^2 + \mu^2}$, which means that only the subextreme type of overlapping constituents is possible [see Fig. 1(a)]. An interesting particular case of this magnetostatic solution is the magnetized Schwarzschild metric, which arises by further setting to zero one of the parameters m_1 or m_2, and its explicit form is the following ($|C|^2 \equiv C\bar{C}$):

$$
\begin{aligned}
&\mathcal{E} = \frac{A-B}{A+B}, \quad \Phi = \frac{C}{A+B}, \quad A_\varphi = -\mu + \frac{2\mu I}{A+B}, \quad f = \frac{A^2 - B^2 + |C|^2}{(A+B)^2}, \quad e^{2\gamma} = \frac{A^2 - B^2 + |C|^2}{16\sigma^2 R_+ R_- r_+ r_-}, \\
&A = \sigma(R_+ + R_-)(r_+ + r_-) - \mu(R_+ - R_-)(r_+ - r_-), \quad B = 2m\sigma(r_+ + r_-), \quad C = 2im\mu(R_+ - R_-), \\
&I = 2\sigma(R_+ + m)(R_- + m) - mz(R_+ - R_-), \quad R_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \mu)^2}, \quad \sigma = \sqrt{m^2 + \mu^2}.
\end{aligned} \tag{14}
$$

By putting $\mu = 0$ in (14), one gets the Schwarzschild solution.

B. Pure vacuum limit

In the absence of the magnetic dipole parameter μ, the solution (8) reduces to the $R = 0$ special case of the metric for two unequal counterrotating black holes [23]. Our alternative derivation of the solution [23] permitted us to work out a simple representation for the $R = 0$ case, which we give below:

$$
\begin{aligned}
&f = \frac{A\bar{A} - B\bar{B}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{\mathcal{K}_0\bar{\mathcal{K}}_0 R_+ R_- r_+ r_-}, \qquad \omega = -\frac{2\mathrm{Im}[G(\bar{A}+\bar{B})]}{A\bar{A} - B\bar{B}}, \\
&A = (\sigma_1 + \sigma_2)^2 (R_+ - R_-)(r_- - r_+) - 4\sigma_1\sigma_2 (R_+ - r_-)(R_- - r_+), \qquad B = 2(m_1^2 - m_2^2)[\sigma_2(R_- - R_+) + \sigma_1(r_+ - r_-)], \\
&G = -zB + (m_1^2 - m_2^2)[\sigma_1(R_+ + R_-)(r_+ - r_-) - \sigma_2(R_+ - R_-)(r_+ + r_-) - 2\sigma_1\sigma_2(R_+ + R_- - r_+ - r_-)], \\
&R_\pm = \frac{m_1 \mp \sigma_1 - ia}{m_1 \mp \sigma_1 + ia}\sqrt{\rho^2 + (z \pm \sigma_1)^2}, \qquad r_\pm = \frac{m_2 \mp \sigma_2 - ia}{m_2 \mp \sigma_2 + ia}\sqrt{\rho^2 + (z \pm \sigma_2)^2}, \qquad \sigma_1 = \sqrt{m_1^2 - a^2}, \\
&\sigma_2 = \sqrt{m_2^2 - a^2}, \qquad K_0 = 4\sigma_1\sigma_2(m_1 - m_2)^2/(m_1 m_2).
\end{aligned} \tag{15}
$$

Note that the functions $R_\pm$ and $r_\pm$ are defined here in a slightly different way than in (6). Of course, the above formulas (15) are fully equivalent to the $\mu = 0$ limit of the solution in (8) and (9).

We would like to remark that it was precisely Ref. [23] that motivated us to write this paper after we incidentally discovered a misprint in the formula (38) of [23] and then took notice of a rather unusual property of that formula whose correct form is

$$J_2 = -\frac{J_1 M_2}{M_1}\left(\frac{R + M_1 - M_2}{R - M_1 + M_2}\right), \tag{16}$$

where M_i and J_i are, respectively, the Komar [24] masses and angular momenta of the black hole constituents, while R is the separation distance. Indeed, as it follows from (16), for all $R > |M_1 - M_2|$ the two Kerr black holes are counterrotating; nevertheless, for $0 \le R < |M_1 - M_2|$ the black holes become corotating, as the expression in parentheses on the right-hand side of (16) then takes negative values. Obviously, the authors of [23] were only interested in the configurations with $R > M_1 + M_2$, when the counterrotating black holes are separated by a massless strut, so they discarded other possibilities as unphysical or uninteresting. However, in our opinion, it is the $R = 0$ case that is probably most interesting from the physical point of view because this is the only case of unequal constituents with equatorial symmetry, and also because it might represent a legitimate end (or intermediate) state of two merging Kerr black holes. It is worth pointing out that the change from counterrotation to corotation does not occur in the case of equal black holes [$M_1 = M_2$ in (16)], so the intrinsic inequality of black holes is necessary for the formation of the final configuration of corotating Kerr black holes described by (15). The change of the total angular momentum of the system between its final ($R = 0$) state and the initial state of infinitely separated sources ($R = \infty$) is given by the simple formula

$$\Delta J = 2J_1 M_2/M_1, \tag{17}$$

and it would be tempting to speculate that this change of the total angular momentum, which might be attributed to the extremely strong frame-dragging effects inside a larger black hole, could have a relation to the production of relativistic jets in the centers of galaxies. It would also be worth noting, in conclusion of this subsection, that although it was long conjectured [25] that the collision of two Kerr black holes leads to the formation of another Kerr black hole of larger mass (the gravitational radiation being an important part of such a process), the solution (15) might suggest that this is not necessarily the case for the head-on collisions of nonequal Kerr black holes when the gravitational radiation is absent due to axial symmetry.

C. Magnetized Kerr solution

By choosing $m_1 = m$, $m_2 = 0$ in (8), we get a 3-parameter variant of the magnetized Kerr spacetime of the form

$$\begin{aligned}
&\mathcal{E} = \frac{A-B}{A+B}, \quad \Phi = \frac{C}{A+B}, \quad A_\varphi = \mathrm{Im}\left(\frac{I - zC}{A+B}\right), \quad f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})},\\
&e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K_0\bar{K}_0 R_+ R_- r_+ r_-}, \quad \omega = -\frac{\mathrm{Im}[G(\bar{A}+\bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}},\\
&\quad A = \sigma_1\sigma_2(R_+ + R_-)(r_+ + r_-) - \sigma_2^2(R_+ - R_-)(r_+ - r_-) + ia[\sigma_1(R_+ + R_-)(r_+ - r_-) - \sigma_2(R_+ - R_-)(r_+ + r_-)],\\
&\quad B = 2m\sigma_1[\sigma_2(r_+ + r_-) + ia(r_+ - r_-)], \quad C = 2im\mu\sigma_2(R_+ - R_-),\\
&\quad G = -2zB + \sigma_1\mu^2(R_+ + R_-)(r_+ - r_-) - \sigma_2(2m^2 + \mu^2)(R_+ - R_-)(r_+ + r_-)\\
&\qquad - 2im^2a(R_+ - R_-)(r_+ - r_-) + 2\sigma_2 m\mu^2(R_+ - R_-) + 2m\sigma_1(\mu^2 - 2a^2)(r_+ - r_-) + 4ima\sigma_1\sigma_2(r_+ + r_-),\\
&\quad I = -i\mu\{\sigma_1\sigma_2(R_+ + R_-)(r_+ + r_-) - \sigma_2^2(R_+ - R_-)(r_+ - r_-) + ia[\sigma_1(R_+ + R_-)(r_+ - r_-)\\
&\qquad - \sigma_2(R_+ - R_-)(r_+ + r_-)] - 2\sigma_1\sigma_2[2(R_+ + m)(R_- + m) - m(r_+ + r_-)] - 2ima[\sigma_2(R_+ - R_-) - \sigma_1(r_+ - r_-)]\},\\
&R_\pm = \sqrt{\rho^2 + (z \pm \sigma_1)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \sigma_2)^2}, \quad \sigma_1 = \sqrt{m^2 - a^2 + \mu^2}, \quad \sigma_2 = \sqrt{\mu^2 - a^2}, \quad K_0 = 4\sigma_1\sigma_2.
\end{aligned} \tag{18}$$

This solution is different from the known generalization of the Kerr solution obtained by one of us more than two decades ago [26]. The difference is clearly seen if one considers the axis data of the solution (18), namely,

$$e(z) = \frac{(z - m - ia)(z + ia) - \mu^2}{(z + m - ia)(z + ia) - \mu^2}, \qquad f(z) = \frac{im\mu}{(z + m - ia)(z + ia) - \mu^2}, \tag{19}$$

where the magnetic dipole parameter μ enters the expressions of both $e(z)$ and $f(z)$, unlike the solution [26] whose $e(z)$ coincides with the axis data of the Kerr metric [3] and hence is free of the magnetic parameter. However, the presence of μ on the right-hand side of $e(z)$ in (19) is quite acceptable as it is well known that the magnetic field is able to distort the stars [27], thus affecting the structure of their gravitational multipoles. In this respect it is worth noting that the structure of the mass-quadrupole moment P_2 both in the 4-parameter solution (8) and in its special case (18) is congruent with the characteristic deformation in the recently considered models of magnetized neutron stars [28,29] where the magnetic field induces a prolate contribution. We also mention that our solutions confirm the effect found by Bocquet *et al.* [30] consisting in the increase of the maximum rotational velocity of a neutron star due to the magnetic field—in the case of the solution (18) this effect shows itself, for instance, as a positive contribution of the magnetic dipole μ to the quantity σ_1, so the latter σ_1, for a given mass m, takes the extremal zero value at a larger absolute value of the rotational parameter a than in the absence of μ. It is also important to point out that, as was mentioned earlier in Sec. II, the magnetic field in the solution (8) does not give rise to singularities outside the stationary limit surface, which makes the solution very suitable for the analytical approximation of the exterior field of magnetized neutron stars.

D. Extreme limit

The extreme limit of the solution (8) corresponds to the case of equal overlapping constituents, when $M_1 = M_2$ and $\sigma_1 = \sigma_2$, and the application of the L'Hôpital rule to formulas (8) and (9) is then required. By introducing the spheroidal coordinates x and y via the relations

$$x = \frac{1}{2\sigma}(r_+ + r_-), \qquad y = \frac{1}{2\sigma}(r_+ - r_-),$$
$$r_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \qquad \sigma = \sqrt{m^2 - a^2 + \mu^2}, \tag{20}$$

it is possible to write down the resulting expressions in terms of four polynomials: λ, ν, κ, and χ. Thus, for the potentials $\mathcal{E}$, Φ, and A_φ we get

$$\mathcal{E} = \frac{A - B}{A + B}, \qquad \Phi = \frac{C}{A + B}, \qquad A_\varphi = \mathrm{Im}\left(\frac{I}{A + B}\right),$$
$$A = \lambda^2 + 2m^2[\lambda + 2ia\sigma xy(1 - y^2)],$$
$$B = 2m[(\sigma x + iay)\lambda + 2im^2ay(1 - y^2)],$$
$$C = 2i\mu my\lambda,$$
$$I = \frac{i}{2}\mu(1 - y^2)[\kappa + 4m^2(\sigma x + m - iay)^2], \tag{21}$$

while the metrical fields f, γ, and ω are defined by the expressions

$$f = \frac{N}{D}, \qquad e^{2\gamma} = \frac{N}{\sigma^8(x^2 - y^2)^4}, \qquad \omega = \frac{(y^2 - 1)W}{N},$$
$$N = \lambda^4 - \sigma^2(x^2 - 1)(1 - y^2)\nu^2,$$
$$D = N + \lambda^2\kappa + (1 - y^2)\nu\chi, \qquad W = \sigma^2(x^2 - 1)\nu\kappa + \lambda^2\chi,$$
$$\lambda = \sigma^2(x^2 - y^2) - m^2(1 - y^2), \qquad \nu = 4m^2ay^2,$$
$$\kappa = 4m[\sigma^2(\sigma x + 2m)(x^2 - y^2) + m^2\sigma x(y^2 + 1) + m(2m^2 + \mu^2)y^2],$$
$$\chi = 4ma[\sigma^2(\sigma x + 2m)(x^2 - y^2) + m^2\sigma x(1 - y^2)]. \tag{22}$$

Note that in the literature on exact solutions the polynomials λ, ν, κ, and χ have previously been used exclusively in applications to the metric functions f, γ, and ω [31,32], so our paper actually pioneers the use of these polynomials for getting a concise form of the Ernst potentials $\mathcal{E}$ and Φ too. We also note that the magnetic potential A_φ can be written alternatively in the form

$$A_\varphi = \frac{2\mu(y^2 - 1)F}{D},$$
$$F = \lambda\left[\frac{1}{4}\kappa + m^2(\sigma x + m)^2 - m^2a^2y^2\right][\lambda + 2m(\sigma x + m)] - 4m^3a^2y^2(\sigma x + m)[\lambda + 2m(\sigma x + m)(1 - y^2)]. \tag{23}$$

Remarkably, the vacuum ($\mu = 0$) limit of the solution (21) differs from the well-known Tomimatsu-Sato [33] and Kinnersley-Chitre [34] solutions. This was surprising since our initial intention was only to see how this limit is contained in the Kinnersley-Chitre 5-parameter family of solutions. In view of the potential interest the new vacuum solution might represent, below we write it out explicitly:

$$\xi = \frac{1 - \mathcal{E}}{1 + \mathcal{E}} = \frac{2m[(\sigma x + iay)\lambda + 2im^2ay(1 - y^2)]}{\lambda^2 + 2m^2[\lambda + 2ia\sigma xy(1 - y^2)]},$$
$$\lambda = m^2(x^2 - 1) - a^2(x^2 - y^2), \qquad \sigma = \sqrt{m^2 - a^2}. \tag{24}$$

The corresponding metric functions are easily obtainable from (22).

IV. CONCLUDING REMARKS

We believe that the 4-parameter electrovacuum solution considered in the present paper, as well as some of its particular limits, provides interesting new opportunities for modeling the exterior gravitational and electromagnetic fields of rotating bodies and enlarges our knowledge about possible final states of two interacting black holes. The solution has a clear physical interpretation since it arises within a legitimate binary system of counterrotating nonequal black holes, and the corotation of its constituents, though unexpected at first glance but still natural, should be attributed to strong dragging effects that involve a smaller black hole in corotation with the larger one. We have shown that the overlapping constituents in the case of two Kerr black holes have a larger total angular momentum than at infinite separation, the increase being defined by formula (17), and the latter formula also holds in the presence of the electromagnetic field. In future research it would be interesting to clarify whether the aforementioned change of the angular momentum could be related to the mechanisms that are responsible for the production of relativistic jets at the galactic nuclei [35–37].

As a final remark we would like to mention that the 4-parameter solution considered in this paper can be trivially generalized to include an additional parameter of electric dipole moment ε by the substitution $i\mu \to \varepsilon + i\mu$, $\mu^2 \to \varepsilon^2 + \mu^2$, but we have not found any physical justification to do it here.

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