Metric anisotropies and emergent anisotropic hydrodynamics

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Expansion of a locally equilibrated fluid is considered in an anisotropic space-time given by the Bianchi type-I metric. Starting from the isotropic equilibrium phase-space distribution function in the local rest frame, we obtain expressions for components of the energy-momentum tensor and conserved current, such as number density, energy density, and pressure components. In the case of an axissymmetric Bianchi type-I metric, we show that they are identical to those obtained within the setup of anisotropic hydrodynamics. We further consider the case in which the Bianchi type-I metric is a vacuum solution of the Einstein equation: the Kasner metric. For the axissymmetric Kasner metric, we discuss the implications of our results in the context of anisotropic hydrodynamics.

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I. INTRODUCTION

The success of relativistic hydrodynamics in explaining the space-time evolution of strongly interacting hot and dense matter, produced in relativistic heavy-ion collisions, has initiated new developments in the theoretical formulation of relativistic viscous hydrodynamics [1,2]. In recent years, there have been significant advances in our understanding of the theory of relativistic hydrodynamics and its application to high-energy heavy-ion collisions at the Relativistic Heavy-Ion Collider and the LHC [3–5]. The formulation of dissipative hydrodynamic equations is achieved by obtaining the long-wavelength, low-frequency limit of the underlying microscopic dynamics of a system [6–11]. The traditional derivation of dissipative hydrodynamics from kinetic theory relies on a linearization around an equilibrium distribution function that is isotropic in momentum space [12–18]. This amounts to expansion of the underlying microscopic kinetic theory in terms of the inverse Reynolds number and Knudsen number around local equilibrium [19–24]. This type of expansion may not be accurate in situations in which deviations from the local equilibrium and/or space-time gradients are large.

Recent studies have shown that a phase of QCD called the quark gluon plasma (QGP), which is created in relativistic heavy-ion collisions, is not isotropic in momentum space. For instance, at very early times, large pressure anisotropies are created in the center of the fireball for viscosities consistent with experimental observations. Moreover, the level of plasma anisotropy increases as one moves away from the center of the fireball to the peripheral regions of the plasma where the temperature is low [25,26]. Such large pressure anisotropies indicate large viscous corrections to the distribution function, which is contradictory to the near-equilibrium assumption of the formulation of viscous hydrodynamic equations. Furthermore, application of traditional linearized viscous hydrodynamics leads to regions of phase space in which the single-particle phase-space distribution function may be negative, which may in turn lead to negative longitudinal pressure [27–29]. Depending on whether one considers early times or colder regions of the plasma, the size of these unphysical regions increases. It is important to note that the single-particle phase-space distribution function is used to calculate observable plasma signatures, such as dilepton and photon production/flow, quarkonium suppression, and hadronic spectrum through freeze-out. Therefore, inaccuracies in the distribution function can potentially lead to incorrect estimation of these observables [30–33].

Because of the aforementioned problems in traditional dissipative hydrodynamics, there was motivation to create an alternative framework that could more accurately capture the far-from-equilibrium dynamics of highly momentum-space anisotropic systems. The framework of anisotropic hydrodynamics has proven to be quite successful in this context [34–43]; see Ref. [44] for a comprehensive review. Anisotropic hydrodynamics is a nonperturbative approximation of relativistic dissipative hydrodynamics that takes into account the large momentum-space anisotropies generated in relativistic heavy-ion collisions. The motivation for the formulation of anisotropic hydrodynamics is to create a framework that is better suited to deal with such large anisotropies and accurately describes several interesting features such as the early-time dynamics of the QGP, dynamics near the transverse edges of the fireball, and the possibility of a large shear viscosity–to–entropy density ratio η/s.[1] As a consequence, it allows one to extend the regime of applicability of dissipative hydrodynamics to systems that can be quite far from isotropic local thermal equilibrium.

In this paper, we present an alternate derivation of anisotropic hydrodynamic equations by considering the expansion of a locally equilibrated fluid in an anisotropic space-time given by the Bianchi type-I metric. Assuming the isotropic phase-space distribution function at the initial time in the local rest frame, we obtain expressions for components of the energy-momentum tensor and conserved current, such as the number density, energy density, and pressure components. We show that these expressions are identical to those obtained within the setup of anisotropic hydrodynamics when one considers the axissymmetric Bianchi type-I metric.

We further consider the case in which the Bianchi type-I metric is a solution of the Einstein equation: the Kasner metric. The Kasner metric describes a curved space-time in general. However, it has been shown that the Kasner space-time can be treated as a well-controlled approximation of a local rest frame of an anisotropically expanding fluid in Minkowski space-time [47]. Therefore, the Kasner space-time provides a useful framework for studying the anisotropic expansion because of the simplification of the hydrodynamic equations [48]. For the axissymmetric Kasner metric, we further discuss the implications of our results in the context of anisotropic hydrodynamics.

II. METRIC

The most general anisotropic Bianchi type-I metric is [49–51]

$$ds^2 = dt^2 - g_{ij}dx^i dx^j. \quad (1)$$

When there is no *a priori* preferred direction, the metric simply takes a diagonal form given as

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2. \quad (2)$$

The quantities $A(t)$, $B(t)$, and $C(t)$ are scale factors for the expansion along x, y, and z axes. The metric tensor is diagonal and is given by

$$g_{\mu\nu} = \mathrm{diag}[1, -A^2(t), -B^2(t), -C^2(t)], \quad (3)$$

and the inverse of the metric tensor is given by

$$g^{\mu\nu} = \mathrm{diag}\left[1, -\frac{1}{A^2(t)}, -\frac{1}{B^2(t)}, -\frac{1}{C^2(t)}\right]. \quad (4)$$

Later, we will specialize to the axissymmetric case in which we will have $A(t) = B(t)$. We will also consider the case in which the Bianchi type-I metric is a solution of the Einstein equation: the Kasner metric.

III. STRESS-ENERGY TENSOR FOR A GAS IN THERMAL EQUILIBRIUM

First, we will investigate the form of stress energy for a gas of strongly interacting massless particles in thermal equilibrium at a time $t = t_0$. This is possible if the characteristic interaction time is much shorter than the dynamic expansion time of the system.

The stress-energy tensor is defined as

$$T^{\mu\nu} = \int \sqrt{-g}\frac{d^3p}{p^0} p^\mu p^\nu f(x_\mu, p_\mu) = \int \sqrt{-g}\frac{dp^1 dp^2 dp^3}{p^0} p^\mu p^\nu f(x_\mu, p_\mu), \quad (5)$$

where $f(x_\mu, p_\mu)$ is the scalar distribution function in the relativistic phase space and g is the determinant of metric tensor $g_{\mu\nu}$ and is equal to

$$g = -A^2B^2C^2 = -V^2, \quad (6)$$

where V is the physical volume occupied by the particles. Similarly, one can define the number density to be

$$n = \int \sqrt{-g} d^3p f(x_\mu, p_\mu). \quad (7)$$

For ultrarelativistic particles of which the masses can be ignored, we know that

$$g^{\mu\nu}p_\mu p_\nu = m^2 = 0, \quad (8)$$

where m is mass. Hence, we can write Eq. (8) as

$$E_0 = p_0|_{t=t_0} = \left[\left(\frac{p_1}{A(t_0)}\right)^2 + \left(\frac{p_2}{B(t_0)}\right)^2 + \left(\frac{p_3}{C(t_0)}\right)^2\right]^{1/2}, \quad (9)$$

where we have denoted E_0 as the energy of the particles at time $t = t_0$. Since we have thermal equilibrium at time $t = t_0$, we can recast our momenta in spherical polar coordinates (p_0, θ, ϕ) to extract the components of stress-energy tensor ($T_{\mu\nu}$) using Eq. (5).

The physical components of 4-momenta that a local homogeneous observer reads are defined as

[1] Although phenomenological analyses of experimental data suggests that the average value of η/s of QGP is small, lattice QCD predicts relatively large values at high temperature [45]. Also see Ref. [46] for phenomenological implications of temperature-dependent η/s.

$$P_\mu = (g^{\mu\mu})^{1/2} p_\mu \tag{10}$$

such that $P_\mu P^\mu = m^2$ and no sum is implied in Eq. (10). Thus, in spherical polar coordinates, one finds

$$\begin{aligned} P_1 &= \frac{p_1}{A(t_0)} = p_0 \sin\theta \sin\phi \\ P_2 &= \frac{p_2}{B(t_0)} = p_0 \sin\theta \cos\phi \\ P_3 &= \frac{p_3}{C(t_0)} = p_0 \cos\theta. \end{aligned} \tag{11}$$

One can readily verify that the above system of equations satisfies Eq. (9). We can calculate the transformation Jacobian of Eq. (5) to be

$$dp^1 dp^2 dp^3 = \frac{1}{V} p_0^2 dp_0 d\Omega. \tag{12}$$

We can choose the scalar distribution function $f(x,p)$ as

$$f(x,p) = g_0 \frac{1}{e^{E_0/T_0} + r}, \tag{13}$$

where g_0 is the degeneracy factor; T_0 is temperature at time $t = t_0$; and $r = 0, +1$, and -1 for Boltzmann, Fermi-Dirac, and Bose-Einstein distribution functions, respectively. Note that we have assumed Boltzmann's constant and Planck's constant to be unity, i.e., $k = \hbar = 1$.

Inserting Eqs. (11)–(13) into Eqs. (5) and (7) and doing the angular integration, one easily obtains the equilibrium relations

$$\begin{aligned} n &= g_1 (T_0)^3, \\ \varepsilon &= T_{00} = g_2 (T_0)^4, \\ \mathcal{P} &= T_{11} = T_{22} = T_{33} = \frac{1}{3}\varepsilon, \end{aligned} \tag{14}$$

where we have absorbed some constants appearing after integration into the redefined degeneracy factors g_1 and g_2. Thus, we have established that the form of the stress-energy tensor completely agrees with that of a gas in thermal equilibrium with its surrounding having temperature $T = T_0$. In the next section, we will derive the evolution of stress energy when the gas gets completely decoupled from its surrounding.

IV. COLLISIONLESS STRESS-ENERGY TENSOR

We idealize the decoupling of the gas from its surrounding happens at time $t = t_0$, such that after time t_0 the gas experiences a collisionless adiabatic expansion or contraction as specified by the metric of Eq. (2). Also, Liouville's theorem guarantees that the distribution function $f(x,p)$ of Eq. (13) remains constant throughout the phase space for all time during the evolution. This in turn implies that the energy E and the temperature T, at a given time t, are redshifted by the same amount, i.e.,

$$\frac{E}{E_0} = \frac{T}{T_0} = z, \tag{15}$$

where z is the usual redshift factor.

The evolution of the stress-energy tensor depends only on the function z, which needs to be determined. From Eq. (8), the energy E for the particles evolving by the metric given in Eq. (2) at time t is

$$E_0 = \left[\left(\frac{p_1}{A(t_0)}\right)^2 + \left(\frac{p_2}{B(t_0)}\right)^2 + \left(\frac{p_3}{C(t_0)}\right)^2\right]^{1/2}. \tag{16}$$

Since, the 3-momenta p_i are constants of motion, i.e., $dp_i/d\tau = 0$ (as shown in Appendix), the contravariant components of Eq. (16) are

$$E_0 = \left[\left(\frac{A^2(t)p^1}{A(t_0)}\right)^2 + \left(\frac{B^2(t)p^2}{B(t_0)}\right)^2 + \left(\frac{C^2(t)p^3}{C(t_0)}\right)^2\right]^{1/2}. \tag{17}$$

Now, using Eqs. (10) and (11) in Eq. (17), we can easily find the redshift factor z to be

$$z = \left[\left(\frac{A(t)\sin\theta\sin\phi}{A(t_0)}\right)^2 + \left(\frac{B(t)\sin\theta\cos\phi}{B(t_0)}\right)^2 + \left(\frac{C(t)\cos\theta}{C(t_0)}\right)^2\right]^{-1/2}. \tag{18}$$

From Eq. (18), we see that the characteristic temperature is dependent on the direction of the motion of the particles.

Using Eqs. (15) and (18), we can calculate the components of the stress-energy tensor at a later time $t > t_0$ from Eq. (14). We proceed in the same way as before, except the angular integration is altered. We obtain

$$n = \frac{n_0}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta z^3 d\theta \tag{19}$$

$$\varepsilon = T_{00} = \frac{\varepsilon_0}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta z^4 d\theta, \tag{20}$$

$$\mathcal{P}_x = T_{11} = \frac{3(\mathcal{P}_x)_0}{4\pi}\int_0^{2\pi} \cos^2\phi d\phi \int_0^\pi \sin^3\theta z^4 d\theta, \tag{21}$$

$$\mathcal{P}_y = T_{22} = \frac{3(\mathcal{P}_y)_0}{4\pi}\int_0^{2\pi} \sin^2\phi d\phi \int_0^\pi \sin^3\theta z^4 d\theta, \tag{22}$$

$$\mathcal{P}_z = T_{33} = \frac{3(\mathcal{P}_z)_0}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta\cos^2\theta z^4 d\theta. \tag{23}$$

If we assume an axissymmetric case, i.e., $\frac{A(t)}{A(t_0)} = \frac{B(t)}{B(t_0)} = \xi_1$ and $\frac{C(t)}{C(t_0)} = \xi_3$, in which case the system of Eqs. (19)–(22) reduces to a more tractable form,

$$n = \frac{n_0}{2}\int_{-1}^{1} (\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2)^{-3/2} d\lambda, \tag{24}$$

$$\varepsilon = \frac{\varepsilon_0}{2}\int_{-1}^{1} (\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2)^{-2} d\lambda, \tag{25}$$

$$\mathcal{P}_\perp = \frac{3(P_\perp)_0}{4}\int_{-1}^{1} (1-\lambda^2)(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2)^{-2} d\lambda, \tag{26}$$

$$\mathcal{P}_\parallel = \frac{3(P_\parallel)_0}{2}\int_{-1}^{1} \lambda^2(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2)^{-2} d\lambda, \tag{27}$$

where $\lambda = \cos\theta$ and we have defined $\mathcal{P}_x = \mathcal{P}_y = \mathcal{P}_\perp$ and $\mathcal{P}_z = \mathcal{P}_\parallel$.

Integrating Eqs. (24)–(27), we get

$$n = \frac{n_0}{\xi_1^3 \xi^{1/2}}, \tag{28}$$

$$\varepsilon = \frac{\varepsilon_0 \xi^{-1}}{2\xi_1^4}\left(1 + \frac{\xi \arctan\sqrt{\xi-1}}{\sqrt{\xi-1}}\right) = \frac{\varepsilon_0 \mathcal{R}(\xi)}{\xi_1^4}, \tag{29}$$

$$\begin{aligned}\mathcal{P}_\perp &= \frac{3(\mathcal{P}_\perp)_0}{4\xi_1^4}\left(\frac{1}{\xi-1} + \frac{(\xi-2)\arctan\sqrt{\xi-1}}{(\xi-1)^{3/2}}\right)\\ &= \frac{3(\mathcal{P}_\perp)_0}{2\xi_1^4}\left(\frac{1+\xi(\xi-2)\mathcal{R}(\xi)}{\xi(\xi-1)}\right),\end{aligned} \tag{30}$$

$$\begin{aligned}\mathcal{P}_\parallel &= \frac{3(\mathcal{P}_\parallel)_0}{2\xi_1^4}\left(\frac{1}{\xi(1-\xi)} + \frac{\arctan\sqrt{\xi-1}}{(\xi-1)^{3/2}}\right)\\ &= \frac{3(\mathcal{P}_\parallel)_0}{\xi_1^4}\left(\frac{\xi\mathcal{R}(\xi)-1}{\xi(\xi-1)}\right),\end{aligned} \tag{31}$$

where $\xi = \frac{\xi_3^2}{\xi_1^2}$ and $\mathcal{R}(\xi) = \frac{1}{2\xi}(1 + \frac{\xi \arctan\sqrt{\xi-1}}{\sqrt{\xi-1}})$ for $\xi > 1$, while we substitute ξ as $\frac{1}{\xi}$ for $\xi < 1$.

V. COLLISIONLESS BOLTZMANN EQUATION

The expression for components of the energy-momentum tensor and conserved current, given in Eqs. (28)–(31), for a system of anisotropically expanding collisionless plasma, can also be obtained by considering the Boltzmann equation in the free-streaming case. The collisionless Boltzmann equation for the Bianchi type-I metric is given as

$$p^\mu \partial_\mu f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = 0. \tag{32}$$

Taking into account that f cannot depend on the position x_i, because of the homogeneity of space, the collisionless Boltzmann equation thus becomes

$$p^0 \partial_0 f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = 0, \tag{33}$$

$$p^0 \partial_0 f - (2\Gamma^\mu_{0j} p^0 + \Gamma^\mu_{ij} p^i) p^j \frac{\partial f}{\partial p^\mu} = 0. \tag{34}$$

The nonzero components of the Christoffel symbols are $\Gamma^x_{0x} = \frac{\dot{A}}{A}$, $\Gamma^y_{0y} = \frac{\dot{B}}{B}$, and $\Gamma^z_{0z} = \frac{\dot{C}}{C}$. Substituting these values of Christoffel symbols into the above equation gives us

$$\partial_0 f - 2\left(\frac{\dot{A}}{A} p^x \frac{\partial f}{\partial p^x} + \frac{\dot{B}}{B} p^y \frac{\partial f}{\partial p^y} + \frac{\dot{C}}{C} p^z \frac{\partial f}{\partial p^z}\right) = 0. \tag{35}$$

Solving the above equation by the method of characteristics yields $f = f(A^2(t)p^x, B^2(t)p^y, C^2(t)p^z) = f(A(t)P^x, B(t)P^y, C(t)P^z)$, where we have used Eq. (10) and redefined momenta in terms of physical momenta [52].

From the above equation, we see that for a collisionless system the momentum dependence of the distribution function should be of the form $f = f(A^2(t)p^x, B^2(t)p^y, C^2(t)p^z)$. Using this form of functional dependence in the equilibrium distribution function, given in Eq. (13), and taking the appropriate moments again leads to the same expressions for the components of the energy-momentum tensor and conserved current, Eqs. (28)–(31), of a system of anisotropically expanding collisionless plasma [53]. This, however, is not surprising because in the previous section we used the fact that the particle momenta, p_i, are constants of motion; i.e., particles do not suffer any collision and are free streaming. Therefore, the solution of the collisionless Boltzmann equation should also lead to the same expressions for the components of the energy-momentum tensor and conserved current, as demonstrated here. We note that anisotropic expansion through the metric, as considered here, naturally leads to the Romatshke-Strickland form of the distribution function [54].

VI. KASNER METRIC

We shall restrict ourselves here even further to the classical vacuum solutions of the Einstein equation and consider only the subclass of Bianchi type-I metrics in which the expansion factors take the Kasner form [55,56]

$$ds^2 = dt^2 - t^{2a}dx^2 - t^{2b}dy^2 - t^{2c}dz^2, \tag{36}$$

where a, b, and c are three parameters that are related to each other by the equations

$$a + b + c = 1 \tag{37}$$

$$a^2 + b^2 + c^2 = 1. \tag{38}$$

The above constraints are obtained by requiring that the metric given in Eq. (36) is a vacuum solution of the Einstein equation. However, as was shown in Ref. [55], the fluid satisfies the above relations if we impose conformal invariance. In general, the Kasner metric describes a curved space-time. However, it was shown that the Kasner space-time can be treated as an approximation of a local rest frame of an anisotropically expanding fluid in Minkowski space-time [47]. Hence, the hydrodynamic equations for anisotropic expansion take a simple form in Kasner space-time [48].

Since the particle current must be conserved, the number density n of particles that is measured by a comoving observer satisfies the continuity equation

$$\frac{dn}{dt} + \Gamma^i_{i0} n = 0. \tag{39}$$

The nonvanishing components of the Christoffel symbols for the Kasner metric are

$$\Gamma^1_{10} = \frac{a}{t}, \qquad \Gamma^2_{20} = \frac{b}{t}, \qquad \Gamma^3_{30} = \frac{c}{t}. \tag{40}$$

Using Eq. (40) in Eq. (39), we have

$$\frac{dn}{dt} + (a+b+c)\frac{n}{t} = 0 \tag{41}$$

$$\frac{dn}{dt} + \frac{n}{t} = 0, \tag{42}$$

where in the second equation we have used Eq. (37). On integrating Eq. (42), we have

$$n = \frac{n_0 t_0}{t}. \tag{43}$$

It is interesting to note that the above equation holds for all Kasner-type expansion. As demonstrated in the following, the Milne metric turns out to be a special case of the Kasner metric.

From Eqs. (37) and (38), we see that out of the three parameters a, b, and c only one is independent. If we impose an additional constraint of azimuthal symmetry, we have only two possibilities for (a, b, c):

$$\text{Case I}:\ (0,0,1) \quad \text{Case II}:\ \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right). \tag{44}$$

The first one of course reduces to the usual Milne coordinates by a coordinate transformation $t \sinh z = y$ and $t \cosh z = \tau$, where y and τ are rapidity and proper time. The second one is a new finding in the context of azimuthally symmetric anisotropic hydrodynamics. It is important to note that for case I all the components of the Riemann curvature tensor vanishes and hence can be obtained by a general coordinate transformation of the Minkowski metric. On the other hand, case II has curvature and therefore cannot be obtained by a general coordinate transformation of the Minkowski metric, which is flat.

Imposing the condition in Eq. (44) on the variable ξ gives us

$$\text{Case I}:\ \xi = \frac{t^2}{t_0^2} \quad \text{Case II}:\ \xi = \frac{t_0^2}{t^2}. \tag{45}$$

Case I refers refers to longitudinal expansion, while case II denotes transverse expansion. We note that case I corresponds to the usual free-streaming solution in Bjorken expansion, which has been obtained in the past by other authors [57–59]. It is also interesting to note that a system that is Bjorken expanding in Minkowski space-time is static in the Milne coordinate system, which is case I of Eqs. (44) and (45).

VII. RESULTS AND DISCUSSION

In this section, we investigate the evolution of the stress-energy tensor of the fireball that has experienced a relative longitudinal contraction or expansion at time t along the z axis such that $t > t_0$, where t_0 is the time of isotropization.

For a violent longitudinal expansion, $\xi > 1$, and the form of ξ in Eq. (45) is that of case I in Fig. 1. By Eqs. (28)–(31), this induces a stress-energy tensor of the form

$$\mathcal{P}_\parallel = 0 \quad \mathcal{P}_\perp = \frac{1}{2}\varepsilon \tag{46}$$

as the anisotropic longitudinal expansion increases, i.e., in the limit $1/\xi \to 0$. Throughout the evolution starting from initial isotropization at time t_0 to the final asymptotic limit, we have $\mathcal{P}_\parallel < \mathcal{P}_\perp$ as dictated by Eqs. (28)–(31).

The scenario of case II happens when there is a violent transverse expansion i.e., $\xi < 1$, which leads to stress energy of the limiting form

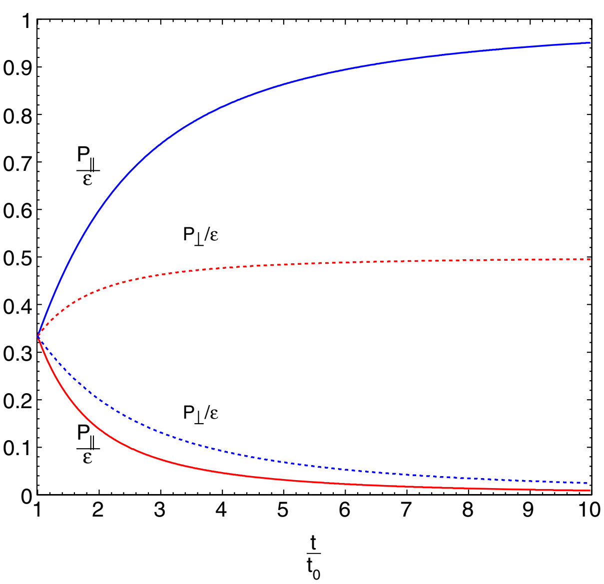

FIG. 1. Evolution of longitudinal and transverse pressures, scaled by the energy density, for case I (red) and case II (blue) as described in the text.

$$\mathcal{P}_{\parallel} = \varepsilon \quad \mathcal{P}_{\perp} = 0 \tag{47}$$

as the anisotropic transverse expansion increases, i.e., in the limit $\xi \to 0$. Equations (28)–(31) imply that expansion along the transverse direction is accompanied by a simultaneous longitudinal contraction that eventually builds up an enormous pressure along the z direction while the slow expansion along the transverse direction continues, as evident in Fig. 1.

In both cases, we observe that the system never reaches the isotropic state. This is due to the fact that we have considered free streaming, i.e., noninteracting evolution. Note that this is in contrast to dissipative hydrodynamics in which the evolution drives the system toward equilibrium.

VIII. SUMMARY AND OUTLOOK

In this paper, we have considered the free streaming of a locally equilibrated fluid in an anisotropic space-time given by the Bianchi type-I metric. We obtained expressions for components of the energy-momentum tensor and conserved current, such as energy density, pressure components, and number density, for an asymptotic observer. In the case of an axissymmetric Bianchi type-I metric, we showed that they are identical to that obtained within the setup of anisotropic hydrodynamics. We further considered the case in which the Bianchi type-I metric is a solution of the Einstein equation: the Kasner metric. For the axissymmetric Kasner metric, we discussed the implications of our results in the context of anisotropic hydrodynamics.

The framework presented in this paper may also find applications in the context of cosmology. In the standard cosmological model, it is assumed that the space-time is isotropic about every point in space and time. However, after the discovery of temperature anisotropies of the cosmic microwave background (CMB), we now know that the Universe is isotropic up to small perturbations. If the CMB temperature were isotropic about every point in space-time, then the Universe can be described by an exact Friedmann-Lemaître model [60]. However, since the CMB radiation is not exactly isotropic, it can be described by a perturbed Friedmann-Robertson-Walker metric that can be obtained as a special case of the Bianchi type-I metric. Since the framework of anisotropic hydrodynamics is well studied, one may apply similar techniques in cosmological models.

Looking forward, it will be interesting to consider an interacting medium within the present setup by considering all possible corrections to the energy-momentum tensor up to a particular order in gradients. Alternatively, one can consider an evolving medium through interactions, i.e., in the presence of a nonvanishing collision kernel in the Boltzmann equation. This will lead to viscous corrections in the local distribution function. One can also study the evolution of dissipative quantities within this setup. We leave these questions for future studies.

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APPENDIX: GEODESIC EQUATION AND FREE STREAMING

In this Appendix, we show that after the decoupling time t_0 as the particles stream freely through space-time the momenta p_μ of particles are constants of the motion along their phase-space trajectories. Consider the general geodesic equation for the Bianchi type-I metric

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0, \tag{A1}$$

where $\Gamma^\mu_{\rho\sigma}$ are the usual Christoffel symbols, p^μ is the 4-momentum of the particle, and τ is the proper time. For convenience, we consider only the x component of Eq. (A1), and other components could be derived straightforwardly. For the x component, the nonvanishing components of $\Gamma^x_{\rho\sigma}$ are $\Gamma^x_{0x} = \Gamma^x_{x0} = \frac{\dot{A}}{A}$. Substituting this into the geodesic equation, Eq. (A1), we have

$$\frac{dp^x}{d\tau} + 2\frac{\dot{A}}{A} p^0 p^x = 0 \tag{A2}$$

$$\frac{dp^x}{d\tau} + \frac{2p^x}{A}\frac{dA}{d\tau} = 0, \tag{A3}$$

where we used the identity $\dot{A}p^0 = \frac{dA}{dt}\frac{dt}{d\tau} = \frac{dA}{d\tau}$. We can rewrite Eq. (A3) as

$$\frac{d}{d\tau}(A^2 p^x) = \frac{dp_x}{d\tau} = 0, \tag{A4}$$

which implies $p_x = \text{const.}$ is a constant of motion. A similar relation holds for other components of p_μ.

[1] S. Jeon and U. Heinz, Introduction to hydrodynamics, Int. J. Mod. Phys. E **24**, 1530010 (2015).
[2] W. Florkowski, M. P. Heller, and M. Spalinski, New theories of relativistic hydrodynamics in the LHC era, Rep. Prog. Phys. **81**, 046001 (2018).
[3] U. Heinz and R. Snellings, Collective flow and viscosity in relativistic heavy-ion collisions, Annu. Rev. Nucl. Part. Sci. **63**, 123 (2013).
[4] P. Braun-Munzinger, V. Koch, T. Schfer, and J. Stachel, Properties of hot and dense matter from relativistic heavy ion collisions, Phys. Rep. **621**, 76 (2016).
[5] A. Jaiswal and V. Roy, Relativistic hydrodynamics in heavy-ion collisions: general aspects and recent developments, Adv. High Energy Phys. **2016**, 9623034 (2016).
[6] C. Eckart, The thermodynamics of irreversible processes. 1. The simple fluid, Phys. Rev. **58**, 267 (1940).
[7] H. Grad, On the kinetic theory of rarefied gases, Commun. Pure Appl. Math. **2**, 331 (1949).
[8] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, England, 1970).
[9] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, Oxford, 1987).
[10] P. Romatschke, New developments in relativistic viscous hydrodynamics, Int. J. Mod. Phys. E **19**, 1 (2010).
[11] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, Relativistic viscous hydrodynamics, conformal invariance, and holography, J. High Energy Phys. 04 (2008) 100.
[12] W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, Ann. Phys. (N.Y.) **118**, 341 (1979).
[13] A. Muronga, Causal theories of dissipative relativistic fluid dynamics for nuclear collisions, Phys. Rev. C **69**, 034903 (2004).
[14] M. A. York and G. D. Moore, Second order hydrodynamic coefficients from kinetic theory, Phys. Rev. D **79**, 054011 (2009).
[15] G. S. Denicol, T. Koide, and D. H. Rischke, Dissipative Relativistic Fluid Dynamics: A New Way to Derive the Equations of Motion from Kinetic Theory, Phys. Rev. Lett. **105**, 162501 (2010).
[16] A. Jaiswal, R. S. Bhalerao, and S. Pal, New relativistic dissipative fluid dynamics from kinetic theory, Phys. Lett. B **720**, 347 (2013).
[17] A. Jaiswal, R. S. Bhalerao, and S. Pal, Complete relativistic second-order dissipative hydrodynamics from the entropy principle, Phys. Rev. C **87**, 021901 (2013).
[18] A. Jaiswal, Relativistic dissipative hydrodynamics from kinetic theory with relaxation time approximation, Phys. Rev. C **87**, 051901 (2013).
[19] A. Jaiswal, Relativistic third-order dissipative fluid dynamics from kinetic theory, Phys. Rev. C **88**, 021903 (2013).
[20] R. S. Bhalerao, A. Jaiswal, S. Pal, and V. Sreekanth, Relativistic viscous hydrodynamics for heavy-ion collisions: A comparison between the Chapman-Enskog and Grad methods, Phys. Rev. C **89**, 054903 (2014).
[21] A. Jaiswal, R. Ryblewski, and M. Strickland, Transport coefficients for bulk viscous evolution in the relaxation time approximation, Phys. Rev. C **90**, 044908 (2014).
[22] W. Florkowski, A. Jaiswal, E. Maksymiuk, R. Ryblewski, and M. Strickland, Relativistic quantum transport coefficients for second-order viscous hydrodynamics, Phys. Rev. C **91**, 054907 (2015).
[23] B. Betz, D. Henkel, and D. H. Rischke, From kinetic theory to dissipative fluid dynamics, Prog. Part. Nucl. Phys. **62**, 556 (2009).
[24] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Derivation of transient relativistic fluid dynamics from the Boltzmann equation, Phys. Rev. D **85**, 114047 (2012).
[25] H. Song, Causal viscous hydrodynamics for relativistic heavy ion collisions, arXiv:0908.3656.
[26] M. Martinez, R. Ryblewski, and M. Strickland, Boost-invariant $(2+1)$-dimensional anisotropic hydrodynamics, Phys. Rev. C **85**, 064913 (2012).
[27] G. Torrieri and I. Mishustin, Instability of Boost-invariant hydrodynamics with a QCD inspired bulk viscosity, Phys. Rev. C **78**, 021901 (2008).
[28] M. Martinez and M. Strickland, Constraining relativistic viscous hydrodynamical evolution, Phys. Rev. C **79**, 044903 (2009).
[29] M. Habich and P. Romatschke, Onset of cavitation in the quark-gluon plasma, J. High Energy Phys. 12 (2014) 054.
[30] J. R. Bhatt, H. Mishra, and V. Sreekanth, Thermal photons in QGP and non-ideal effects, J. High Energy Phys. 11 (2010) 106.
[31] J. R. Bhatt, H. Mishra, and V. Sreekanth, Cavitation and thermal dilepton production in QGP, Nucl. Phys. **A875**, 181 (2012).
[32] J. R. Bhatt, H. Mishra, and V. Sreekanth, Shear viscosity, cavitation and hydrodynamics at LHC, Phys. Lett. B **704**, 486 (2011).
[33] R. S. Bhalerao, A. Jaiswal, S. Pal, and V. Sreekanth, Particle production in relativistic heavy-ion collisions: A consistent hydrodynamic approach, Phys. Rev. C **88**, 044911 (2013).
[34] M. Martinez and M. Strickland, Dissipative dynamics of highly anisotropic systems, Nucl. Phys. **A848**, 183 (2010).
[35] W. Florkowski and R. Ryblewski, Highly-anisotropic and strongly-dissipative hydrodynamics for early stages of relativistic heavy-ion collisions, Phys. Rev. C **83**, 034907 (2011).
[36] M. Martinez and M. Strickland, Non-boost-invariant anisotropic dynamics, Nucl. Phys. **A856**, 68 (2011).
[37] R. Ryblewski and W. Florkowski, Highly-anisotropic and strongly-dissipative hydrodynamics with transverse expansion, Eur. Phys. J. C **71**, 1761 (2011).
[38] M. Nopoush, M. Strickland, R. Ryblewski, D. Bazow, U. Heinz, and M. Martinez, Leading-order anisotropic hydrodynamics for central collisions, Phys. Rev. C **92**, 044912 (2015).
[39] R. Ryblewski and W. Florkowski, Highly-anisotropic hydrodynamics in $3+1$ space-time dimensions, Phys. Rev. C **85**, 064901 (2012).
[40] D. Bazow, U. W. Heinz, and M. Strickland, Second-order $(2+1)$-dimensional anisotropic hydrodynamics, Phys. Rev. C **90**, 054910 (2014).
[41] L. Tinti and W. Florkowski, Projection method and new formulation of leading-order anisotropic hydrodynamics, Phys. Rev. C **89**, 034907 (2014).

[42] M. Nopoush, R. Ryblewski, and M. Strickland, Bulk viscous evolution within anisotropic hydrodynamics, Phys. Rev. C **90**, 014908 (2014).
[43] E. Molnar, H. Niemi, and D. H. Rischke, Derivation of anisotropic dissipative fluid dynamics from the Boltzmann equation, Phys. Rev. D **93**, 114025 (2016).
[44] M. Strickland, Anisotropic hydrodynamics: Three lectures, Acta Phys. Pol. B **45**, 2355 (2014).
[45] A. Nakamura and S. Sakai, Transport Coefficients of Gluon Plasma, Phys. Rev. Lett. **94**, 072305 (2005).
[46] H. Niemi, G. S. Denicol, P. Huovinen, E. Molnar, and D. H. Rischke, Influence of a temperature-dependent shear viscosity on the azimuthal asymmetries of transverse momentum spectra in ultrarelativistic heavy-ion collisions, Phys. Rev. C **86**, 014909 (2012).
[47] S. J. Sin, S. Nakamura, and S. P. Kim, Elliptic Flow, Kasner universe and holographic dual of RHIC fireball, J. High Energy Phys. 12 (2006) 075.
[48] H. Culetu, Bjorken expansion in the isotropic Kasner spacetime, Eur. Phys. J. Plus **126**, 126 (2011).
[49] C. W. Misner, The isotropy of the universe, Astrophys. J. **151**, 431 (1968).
[50] K. C. Jacobs, Ph.D. dissertation, California Institute of Technology, 1969, https://thesis.library.caltech.edu/4107/.
[51] K. S. Thorne, Primordial element formation, primordial magnetic fields, and the isotropy of the universe, Astrophys. J. **148**, 51 (1967).
[52] P. Romatschke, Collective flow without hydrodynamics: simulation results for relativistic ion collisions, Eur. Phys. J. C **75**, 429 (2015).
[53] L. Keegan, A. Kurkela, P. Romatschke, W. van der Schee, and Y. Zhu, Weak and strong coupling equilibration in nonabelian gauge theories, J. High Energy Phys. 04 (2016) 031.
[54] P. Romatschke and M. Strickland, Collective modes of an anisotropic quark gluon plasma, Phys. Rev. D **68**, 036004 (2003).
[55] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology, Adv. Phys. **19**, 525 (1970).
[56] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970).
[57] J. I. Kapusta, L. D. McLerran, and D. K. Srivastava, Rates for dilepton production at RHIC and LHC between J / psi and upsilon are big, Phys. Lett. B **283**, 145 (1992).
[58] M. Martinez and M. Strickland, Pre-equilibrium dilepton production from an anisotropic quark-gluon plasma, Phys. Rev. C **78**, 034917 (2008).
[59] W. Florkowski, Anisotropic fluid dynamics in the early stage of relativistic heavy-ion collisions, Phys. Lett. B **668**, 32 (2008).
[60] J. Ehlers, P. Geren, and R. K. Sachs, Isotropic solutions of the Einstein-Liouville equations, J. Math. Phys. **9**, 1344 (1968).