Integrability of geodesics in near-horizon extremal geometries: Case of Myers-Perry black holes in arbitrary dimensions

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We investigate dynamics of probe particles moving in the near-horizon limit of extremal Myers-Perry black holes in arbitrary dimensions. Employing ellipsoidal coordinates we show that this problem is integrable and separable, extending the results of the odd dimensional case discussed by Hakobyan *et al.* [Phys. Lett. B **772**, 586 (2017).]. We find the general solution of the Hamilton-Jacobi equations for these systems and present explicit expressions for the Liouville integrals and discuss Killing tensors and the associated constants of motion. We analyze special cases of the background near-horizon geometry were the system possesses more constants of motion and is hence superintegrable. Finally, we consider a near-horizon extremal vanishing horizon case which happens for Myers-Perry black holes in odd dimensions and show that geodesic equations on this geometry are also separable and work out its integrals of motion.

I. INTRODUCTION

Any dynamical system, particle or field dynamics alike, is classically described by equations of motion and some boundary conditions for the field theory case. The main task in analyzing the system is to solve the equations of motion, which are generically (partial) second order differential equations, and solving them is generically a formidable task. Symmetries, Noether theorem and constants of motion, are the usual tools facilitating tackling the problem. In this work we will focus on particle dynamics on certain d dimensional curved backgrounds; the question of field theories on such backgrounds are deferred to an upcoming publication.

In a dynamical system with N degrees of freedom and hence a $2N$ dimensional phase space, if the number of independent symmetries is equal to N, the system is called *integrable* and is usually solvable. If the system possesses $N + p$, $1 \leq p \leq N - 1$, independent symmetries (and hence functionally independent constants of motion), it is called *superintegrable* and the region it can probe in its $2N$ dimensional phase space is a compact $N - p$ dimensional surface; e.g. see [1–3].

For the question of particle dynamics on a general curved (usually a black hole) background in d dimensions, we are dealing with a $2d$ dimensional phase space. It is an established fact that isometries of the background, the Killing vectors, provide a set of constants of motion. Moreover, reparametrization invariance of the particle action implies that there is always a second rank Killing tensor whose conserved charge is the mass of particle. For backgrounds of interest, e.g. black holes or their near-horizon geometries, usually the number of Killing vectors plus one is less than d and one may wonder if the system is integrable.

The question of integrability of particle dynamics on black hole or near-horizon geometries have been extensively analyzed in the literature; e.g. see [4–15]. In particular, it has been shown that the problem is (super)integrable for a large class of black holes. The integrability is often associated with the existence of higher rank, usually second rank, Killing tensor fields [4] (see [16] for review).

Given an extremal black hole there are general theorems stating that in the near-horizon limit we obtain a usually smooth geometry with larger isometry group than the original extremal black hole [17]. It is hence an interesting question to explore if this symmetry enhancement yields further independent constants of motion and how it affects the (super)integrability of particle dynamics. This question, besides the academic interests, is also relevant to some of

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the observations related to black holes: It is now a well-accepted fact that there are fast rotating black holes in the sky which are well modeled by an extreme Kerr geometry [18] and the matter moving around these black holes in their accretion disks are essentially probing the near-horizon geometry [19].

The isometry group of generic stationary extremal black holes in the near-horizon region is shown to have an $SO(2,1) = SL(2,\mathbb{R})$ part [17,20]. Therefore, particle dynamics on the near-horizon extreme geometries possesses dynamical $0+1$ dimensional conformal symmetry, i.e. it defines a "conformal mechanics" [5,7,8]. This allows one to reduce the problem to the study of systems depending on latitudinal and azimuthal coordinates and their conjugate momenta with the effective Hamiltonian being Casimir of conformal algebra. Such associated systems have been investigated from various viewpoints in Ref. [21] where they were called "angular (or spherical) mechanics."

In this work, we continue our analysis of [22,23] and extend the analysis there to near-horizon extremal Myers-Perry [24] (NHEMP) black holes [20] in general odd and even dimensions. We discuss the separability of variables, constants of motion for "angular mechanics" associated with these systems, and how they are related to the second rank Killing tensors of the background. While the system is in general integrable, as we show, there are special cases where the system is superintegrable. Moreover, we discuss another interesting case, the extremal vanishing horizon (EVH) [25] Myer-Perry black holes [26] and show the integrability of geodesics in the near-horizon EVH Myers-Perry (NHEVH-MP) geometries.

The rest of this paper is organized as follows. In Sec. II we present the geometry of near-horizon extremal Myers-Perry black holes in generic even and odd dimensions, and construct the "angular mechanics" describing probe particle dynamics. In this section we set our notations and conventions. In Sec. III we analyze generic causal curve, massive or massless geodesic, in the NHEMP background. We show that this Hamiltonian system is separable in an ellipsoidal coordinate system, work out the constants of motion, and establish that the system is integrable. Moreover, we show how the Killing vectors and second rank Killing tensors are related to these constants of motion. In Sec. IV we analyze special cases where some of the rotation parameters of the background NHEMP are equal. In these cases we have some extra Killing vectors and tensors and the system is super-integrable. Section V contains the analysis of particle dynamics on the special class of extremal vanishing horizon (EVH) Myers-Perry black holes. We end this note with discussions and further comments.

II. NHEMP IN ARBITRARY DIMENSIONS; UNIFIED DESCRIPTION

The NHEMP metric in both odd and even dimensions in the Gaussian null coordinates was presented in [20].

The NHEMP is a (generically) smooth solution to vacuum Einstein equations in odd $d = (2N+1)$ and $d = (2N+2)$ dimensions; in general it is specified by N number of rotation parameters a_i (or N angular momenta J_i) and has $SL(2,\mathbb{R}) \times U(1)^N$ isometry. In Boyer-Lindquist coordinates NHEMP metric has the form

$$ds^2 = \frac{F_H}{b}\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma}(r_H^2 + a_I^2)d\mu_I^2 + \gamma_{ij}D\varphi^i D\varphi^j,$$

$$D\varphi^i \equiv d\varphi^i + \frac{B^i}{b}r d\tau, \tag{1}$$

where $N_\sigma = [\frac{d}{2}] = N + \sigma$, i.e. $\sigma = 0$ for the odd and $\sigma = 1$ for the even dimensions cases, r_H is a black hole radius which satisfies the equation

$$\sum_{I=1}^{N_\sigma} \frac{r_H^2}{r_H^2 + a_I^2} = \frac{1+2\sigma}{1+\sigma}, \quad \text{with} \quad a_{N+1} = 0 \tag{2}$$

and,[1]

$$F_H = 1 - \sum_{i=1}^N \frac{a_i^2 \mu_i^2}{r_H^2 + a_i^2},$$

$$b = \frac{1}{r_H^2}\left(\sum_{i=1}^N \frac{\sigma r_H^2}{r_H^2 + a_i^2} + 4\sum_{i<j}^N \frac{r_H^2}{r_H^2 + a_i^2}\frac{r_H^2}{r_H^2 + a_j^2}\right),$$

$$B^i = \frac{2r_H a_i}{(r_H^2 + a_i^2)^2}, \tag{3}$$

$$\gamma_{ij} = (r_H^2 + a_i^2)\mu_i^2 \delta_{ij} + \frac{1}{F_H}a_i\mu_i^2 a_j\mu_j^2, \qquad \sum_{I=1}^{N_\sigma}\mu_I^2 = 1. \tag{4}$$

In our notations lowercase Latin indices i, j run from 1 to N and uppercase Latin indices I, J run over 1 to N_σ and r_H satisfies.

For the case when all a_i take generic nonzero values[2] it is convenient to introduce new parameters m_i

$$m_i = \frac{r_H^2 + a_i^2}{r_H^2} > 1, \quad m_{N+1} = 1 \text{ and } \sum_{I=1}^{N_\sigma}\frac{1}{m_I} = \frac{1+2\sigma}{1+\sigma}, \tag{5}$$

and rescaled coordinates x_I,

$$x_I = \sqrt{m_I}\mu_I : \quad \sum_{I=1}^{N_\sigma}\frac{x_I^2}{m_I} = 1. \tag{6}$$

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[1]There seems to be a minor typo in the expressions for NHEMP metrics given in [20], which we have corrected here.
[2]The case when one of the a_i is zero is the EVH case we will discuss separately in Sec. V.

In these terms the near-horizon metrics read

$$\frac{ds^2}{r_H^2} = A(x)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I$$

$$+ \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i x_j D\varphi^i D\varphi^j,$$

$$D\varphi^i \equiv d\varphi^i + k^i r d\tau, \tag{7}$$

where

$$A(x) = \frac{\sum_{I=1}^{N_\sigma} x_I^2/m_I^2}{\frac{\sigma}{1+\sigma} + 4\sum_{i<j}^{N} \frac{1}{m_i}\frac{1}{m_j}},$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_I^{N_\sigma} x_I^2/m_I^2} \frac{\sqrt{m_i-1}\,x_i}{m_i} \frac{\sqrt{m_j-1}\,x_j}{m_j},$$

$$k^i = \frac{2\sqrt{m_i-1}}{m_i^2\left(\frac{\sigma}{1+\sigma} + 4\sum_{k<l}^{N}\frac{1}{m_k}\frac{1}{m_l}\right)}, \tag{8}$$

with

$$\sum_{I=1}^{N_\sigma} \frac{x_I^2}{m_I} = 1, \qquad \sum_{I=1}^{N_\sigma} \frac{1}{m_I} = \frac{1+2\sigma}{1+\sigma}. \tag{9}$$

With this unified description at hand, we are ready to describe probe particle dynamics.

A. Probe-particle dynamics

The metric (7) has $SL(2,\mathbb{R})$ isometry group and hence the particle dynamics on this background exhibits dynamical conformal symmetry; we are dealing with a "conformal mechanics" problem [5,7,8]. Let us denote the three generators of this $sl(2,\mathbb{R})$ algebra by H, D, K, and its Casimir by $\mathcal{I}$:

$$\{H,D\} = H, \qquad \{H,K\} = 2D,$$
$$\{D,K\} = K, \qquad \mathcal{I} = HK - D^2. \tag{10}$$

The mass-shell equation for a particle of mass m_0 moving in the background metric

$$m_0^2 = -\sum_{A,B=1}^{2N+1+\sigma} g^{AB} p_A p_B \tag{11}$$

leads to the following expression:

$$m_0^2 r_H^2 = \frac{1}{A}\left[\left(\frac{p_0}{r} - \sum_{i=1}^{N} k_i p_{\varphi_i}\right)^2 - (rp_r)^2\right] - \sum_{a,b=1}^{N_\sigma-1} h^{ab} p_a p_b$$

$$- \sum_{i,j=1}^{N} \tilde{\gamma}^{ij} \frac{p_{\varphi_i}}{x_i} \frac{p_{\varphi_j}}{x_j}, \tag{12}$$

where

$$h^{ab} = \delta^{ab} - \frac{1}{\sum_{I=1}^{N_\sigma} x_I^2/m_I^2} \frac{x_a}{m_a} \frac{x_b}{m_b},$$

$$\tilde{\gamma}^{ij} = \delta^{ij} - x_i \frac{\sqrt{m_i-1}}{m_i} x_j \frac{\sqrt{m_j-1}}{m_j},$$

$$a,b = 1,\cdots N_\sigma - 1, i,j = 1,...,N. \tag{13}$$

Using (12), as in [22], we can construct the Hamiltonian $H = p_0$ and the other generators of the conformal algebra

$$H = r\left(\sqrt{L(x_a, p_a, p_{\varphi_i}) + (rp_r)^2} + \sum_{i=1}^{N} k_i p_{\varphi_i}\right), \tag{14}$$

$$D = rp_r,$$

$$K = \frac{1}{r}\left(\sqrt{L(x_a, p_a, p_{\varphi_i}) + (rp_r)^2} - \sum_{i=1}^{N} k_i p_{\varphi_i}\right), \tag{15}$$

where

$$L(x_a, p_a, p_{\varphi_i}) = A\left(m_0 r_H^2 + \sum_{a,b=1}^{N_\sigma-1} h^{ab} p_a p_b \right.$$

$$\left. + \sum_{i,j=1}^{N} \tilde{\gamma}^{ij} \frac{p_{\varphi_i}}{x_i} \frac{p_{\varphi_j}}{x_j}\right),$$

and the momenta p_a, p_{φ_i}, p_r are conjugate to x_a, φ_i, r with the canonical Poisson brackets

$$\{p_a, x_b\} = \delta_{ab}, \qquad \{p_{\varphi_i}, \varphi_j\} = \delta_{ij}, \qquad \{p_r, r\} = 1. \tag{16}$$

Thus, the Casimir element of the conformal algebra reads

$$\mathcal{I} = A\left[\sum_{a,b=1}^{N_\sigma-1} h^{ab} p_a p_b + \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} + g_0\right] - \mathcal{I}_0 \tag{17}$$

where

$$g_0 = -\left(\sum_{i=1}^{N} \frac{\sqrt{m_i-1}\,p_{\varphi_i}}{m_i}\right)^2 + m_0^2 r_H^2, \qquad \mathcal{I}_0 = \left(\sum_i^{N} k_i p_{\varphi_i}\right)^2. \tag{18}$$

In an appropriately chosen frame H can be written in formally nonrelativistic form [7,8]

$$H = \frac{p_R^2}{2} + \frac{2\mathcal{I}}{R^2}, \tag{19}$$

where $R = \sqrt{2K}$, $p_R = \frac{2D}{\sqrt{2K}}$ are the effective "radius" and its canonical conjugate "radial momentum." As we will show below the Casimir $\mathcal{I}$ encodes all the essential information about the system of particles on these

backgrounds. The Casimir $\mathcal{I}$ (17) is at most quadratic in momenta canonically conjugate to the remaining angular variables and it can conveniently be viewed as the Hamiltonian of a reduced "angular/spherical mechanics" [21] describing motion of particle on some curved background. Note that the "time parameter" conjugate to $\mathcal{I}$ is different than the time parameter τ shown in metric (7) whose conjugate variable is $H = p_0$. See [27] for more detailed discussions.

Since the azimuthal angular variables φ^i are cyclic, corresponding conjugate momenta p_{φ_i} are constants of motion. We then remain with a reduced $(N_\sigma - 1)$-dimensional system described by Hamiltonian (17) and x_a variables and their conjugate momenta.

III. FULLY NONISOTROPIC CASE

To show that the angular/spherical mechanics system is integrable, we show that it is separable in the ellipsoidal coordinates when we are dealing with cases where all parameters m_i are nonequal. The ellipsoidal coordinates λ_I for odd and even dimensions are then defined as

$$x_I^2 = (m_I - \lambda_I) \prod_{J=1, J \neq I}^{N_\sigma} \frac{m_I - \lambda_J}{m_I - m_J},$$

$$\lambda_{N_\sigma} < m_{N_\sigma} < \ldots < \lambda_2 < m_2 < \lambda_1 < m_1. \tag{20}$$

To resolve the condition $\sum_{I=1}^{N_\sigma} \frac{x_I^2}{m_I} = 1$ we choose $\lambda_{N_\sigma} = 0$ and hence there are $N_\sigma - 1$ independent λ_I variables, which will be denoted by λ_a.

In these coordinates the angular Hamiltonian $\mathcal{I}$ (shifted by a constant and appropriately rescaled) reads

$$\tilde{\mathcal{I}} = \lambda_1 \ldots \lambda_{N_\sigma - 1} \left[-\sum_a^{N_\sigma - 1} \frac{4 \prod_{I=1}^{N_\sigma} (m_I - \lambda_a) \pi_a^2}{\lambda_a \prod_{b=1, a \neq b}^{N_\sigma - 1} (\lambda_b - \lambda_a)} \right.$$

$$\left. + \sum_{i=1}^{N_\sigma} \frac{g_I^2}{\prod_{a=1}^{N_\sigma - 1} (m_I - \lambda_a)} + g_0 \right], \tag{21}$$

where

$$\tilde{\mathcal{I}} \equiv (\mathcal{I} + \mathcal{I}_0) \left(\frac{\sigma}{1 + \sigma} + 4 \sum_{k<l}^{N} \frac{1}{m_k} \frac{1}{m_l} \right) \prod_{i=1}^{N} m_i,$$

$$\mathcal{I}_0 = \left(\sum_i^N k_i p_{\varphi_i} \right)^2, \tag{22}$$

with

$$g_I^2 = \frac{p_{\varphi_I}^2}{m_I} \prod_{J=1, J \neq I}^{N_\sigma} (m_I - m_J),$$

$$g_{N+1} = p_{\varphi_{N+1}} \equiv 0, \tag{23}$$

and $\{\pi_a, \lambda_b\} = \delta_{ab}$, $\{p_{\varphi_i}, \varphi_j\} = \delta_{ij}$.

The level surface of angular Hamiltonian (21), $\tilde{\mathcal{I}} = \mathcal{E}$, can be conveniently represented through

$$\sum_{a=1}^{N_\sigma - 1} \frac{R_a - \mathcal{E}}{\lambda_a \prod_{b=1, a \neq b}^{N_\sigma - 1} (\lambda_b - \lambda_a)} = 0, \tag{24}$$

where[3]

$$R_a \equiv -4 \prod_{I=1}^{N_\sigma} (m_I - \lambda_a) \pi_a^2 + (-1)^{N_\sigma} \sum_{I=1}^{N_\sigma} \frac{g_I^2 \lambda_a}{m_I - \lambda_a} - g_0(-\lambda_a)^{N_\sigma - 1}, \tag{25}$$

and we used the identities

$$\frac{1}{\prod_{a=1}^{N_\sigma - 1} (\lambda_a - \kappa)} = \sum_{a=1}^{N_\sigma - 1} \frac{1}{\prod_{b=1; a \neq b}^{N_\sigma - 1} (\lambda_b - \lambda_a)} \frac{1}{\lambda_a - \kappa},$$

$$\frac{1}{\lambda_1 \ldots \lambda_{N_\sigma - 1}} = \sum_{a=1}^{N_\sigma - 1} \frac{1}{\prod_{b=1; b \neq a}^{N_\sigma - 1} (\lambda_b - \lambda_a)} \frac{1}{\lambda_a}. \tag{26}$$

We can rewrite the expression (24) in a more useful form, recalling the identities,

$$\sum_{a=1}^{N_\sigma - 1} \frac{\lambda_a^\alpha}{\prod_{\substack{b=1 \\ b \neq a}}^{N_\sigma - 1} (\lambda_a - \lambda_b)} = \delta_{\alpha, N_\sigma - 2} \quad \alpha = 0, \ldots, N_\sigma - 2. \tag{27}$$

Multiplying both sides of (27) by arbitrary constants ν_α and adding to (24), we get

$$\sum_{a=1}^{N_\sigma - 1} \frac{R_a(\pi, \lambda) - \sum_{c=1}^{N_\sigma - 1} \nu_{c-1} \lambda_a^{c-1}}{\lambda_a \prod_{b=1, a \neq b}^{N_\sigma - 1} (\lambda_b - \lambda_a)} = 0, \qquad \nu_0 = \mathcal{E}. \tag{28}$$

Equipped with the above we can solve the Hamilton-Jacobi equations

$$\mathcal{E} \left(\lambda_a, \frac{\partial S_{\text{gen}}}{\partial \lambda_a} \right) = \nu_0, \tag{29}$$

and obtain the generating function S_{gen} depending on $N_\sigma - 1$ integration constants (i.e. the general solution of Hamilton-Jacobi equation). To this end we substitute in (28)

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[3]Note that $R_a \lambda_a \to R_a$ and $\nu_a \to F_{a+1}$ replacements have been assumed in the current paper compared to [22].

$$\pi_a = \frac{\partial S_{\text{gen}}}{\partial \lambda_a}, \tag{30}$$

and choose the ansatz

$$S_{\text{gen}}(\lambda_1, \ldots, \lambda_{N_\sigma-1}) = \sum_{a=1}^{N_\sigma-1} S(\lambda_a). \tag{31}$$

This reduces the Hamilton-Jacobi equation to a set of $N_\sigma - 1$ ordinary differential equations

$$R\left(\lambda_a, \frac{dS(\lambda_a)}{d\lambda_a}\right) - \sum_{b=1}^{N_\sigma-1} \nu_{b-1}\lambda_a^{b-1} = 0, \tag{32}$$

or in an explicit form,

$$-4\left(\frac{dS(\lambda_a)}{d\lambda_a}\right)^2 \prod_{I=1}^{N_\sigma} (m_I - \lambda_a) + (-1)^{N_\sigma} \sum_{I=1}^{N_\sigma} \frac{g_I^2 \lambda_a}{m_I - \lambda_a}$$

$$- g_0(-\lambda_a)^{N_\sigma-1} - \sum_{b=1}^{N_\sigma-1} \nu_{b-1}\lambda_a^{b-1} = 0. \tag{33}$$

Hence, the analytic solution to the Hamilton-Jacobi equation is given through the generating function (31) with

$$S(\lambda, \nu_a) = \frac{1}{2} \frac{d\lambda}{\sqrt{\prod_{I=1}^{N_\sigma} (m_I - \lambda)}} \sqrt{(-1)^{N_\sigma}\left[\sum_{I=1}^{N_\sigma} \frac{g_I^2 m_I}{m_I - \lambda} + g_0\lambda^{N_\sigma-1} - \sum_{i=1}^{N} g_i^2\right] - \sum_{b=1}^{N_\sigma-1} \nu_{b-1}\lambda^{b-1}}. \tag{34}$$

Then, differentiating with respect to constants ν_a, we can get the explicit solutions of the equations of motion

$$\tau = \frac{\partial S_{\text{gen}}}{\partial \nu_0} \equiv \frac{\partial S_{\text{gen}}}{\partial \mathcal{E}}, \qquad c_a = \frac{\partial S_{\text{gen}}}{\partial \nu_a}. \tag{35}$$

To include the dynamics of azimuthal coordinates φ_i we have to consider the generating function $S_{\text{tot}} = S_{\text{gen}} + \sum_{i=1}^{N} p_{\varphi_i}\varphi_i$, where we take into account functional dependence of g_0, g_i from p_{φ_i}. This yields the solutions for azimuthal coordinates

$$\varphi_i = -\frac{\partial S_{\text{gen}}}{\partial p_{\varphi_i}}. \tag{36}$$

Thus, we get the solutions of the angular sector of generic NHEMP with nonequal nonvanishing rotational parameters.

A. Constants of motion

The expressions for commuting constants of motion F_a can be found from (32), by expressing constants ν_a in terms of λ_a, $\pi_a = \partial S_{\text{gen}}/\partial \lambda_a$:

$$\sum_{b=1}^{N_\sigma-1} F_b \lambda_a^{b-1} = R_a(\pi_a, \lambda_a) \Leftrightarrow \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N_\sigma-2} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N_\sigma-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N_\sigma-1} & \lambda_{N_\sigma-1}^2 & \cdots & \lambda_{N_\sigma-1}^{N_\sigma-2} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N_\sigma-1} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_{N_\sigma-1} \end{pmatrix}, \tag{37}$$

where $R_a(\lambda_a, \pi_a)$ are given by (25). Integrals of motion are the solutions to this equation and may be expressed via the inverse Vandermonde matrix, explicitly,

$$F_\alpha = (-1)^{\alpha-1} \sum_{a=1}^{N_\sigma-1} R_a \frac{A_{N_\sigma-\alpha-1}^{\neq a}}{\prod_{\substack{b=1 \\ b\neq a}}^{N_\sigma-1}(\lambda_b - \lambda_a)}, \qquad \alpha = 1, \ldots, N_\sigma - 2, \qquad F_{N_\sigma-1} = \sum_{a=1}^{N_\sigma-1} \frac{R_a}{\prod_{\substack{b=1 \\ b\neq a}}^{N_\sigma-1}(\lambda_a - \lambda_b)}, \tag{38}$$

where

$$A_\alpha^{\neq a} \equiv \sum_{\substack{1\leq k_1 < \ldots < k_\alpha \\ k_1,\ldots,k_\alpha \neq a}}^{N_\sigma-1} \lambda_{k_1}\ldots\lambda_{k_\alpha}. \tag{39}$$

After tedious transformations one can rewrite these expressions in x_a, φ_i coordinates,

$$F_a = (-1)^a \sum_{b,c=1}^{N_\sigma-1} K_{(a)}^{bc}(x)p_b p_c - \sum_{i,j=1}^{N} L_{(a)}^{ij} p_{\varphi_i} p_{\varphi_j} + (-1)^{a-1} A_{N_\sigma-a} m_0^2 r_H^2, \tag{40}$$

where

$$K_{(a)}^{bc} = \left(\sum_{\alpha=0}^{N_\sigma-a-1} (-1)^{N_\sigma+\alpha-a} A_\alpha m_b^{N_\sigma-\alpha-a} + x_b^2 \sum_{\alpha=1}^{N_\sigma-a-1} (-1)^\alpha M_{N_\sigma-\alpha-a-1}^{\neq b} m_b^\alpha \right) \delta^{bc} + M_{N_\sigma-a-1}^{\neq b,c} x_b x_c \tag{41}$$

$$L_{(a)}^{ij} = \left((1-\delta_a^1) \sum_{\alpha=1}^{N_\sigma-a} (-1)^{N_\sigma+\alpha} A_{\alpha-1} m_i^{N_\sigma-a-\alpha+1} - \delta_a^1 A_{N_\sigma-1} \right) \frac{\delta^{ij}}{x_i^2} + (-1)^{a-1} A_{N_\sigma-a} \frac{\sqrt{m_i-1}}{m_i} \frac{\sqrt{m_j-1}}{m_j} \tag{42}$$

with

$$A_a(x_i, m_j) \equiv \sum_{1 \leq k_1 < \dots < k_a}^{N_\sigma-1} \lambda_{k_1} \dots \lambda_{k_a} = -\sum_{i=1}^{N_\sigma} x_i^2 M_{a-1}^{\neq i} + \sum_{1 \leq k_1 < \dots < k_a}^{N_\sigma} m_{k_1} \dots m_{k_a}, \quad a = 1, \dots, N_\sigma-1, \tag{43}$$

and

$$M_i^{\neq a_1,\dots,a_j} \equiv \sum_{\substack{1 \leq k_1 < \dots < k_i \\ k_1, \dots, k_i \neq a_1, \dots, a_j}}^{N_\sigma} m_{k_1} \dots m_{k_i}, \quad j = 0, \dots, N_\sigma-1, \quad i = 1, \dots, N_\sigma-j. \tag{44}$$

It is also assumed that

$$A_0 \equiv 1, \qquad M_0^{\neq a_1,\dots,a_j} \equiv 1. \tag{45}$$

One can check that in odd dimensions in the special cases of F_{N-1}, F_{N-2} and F_{N-3}, the above reduce to the corresponding integrals of motion given in [23]. One can also check that simply requiring the rotation parameters to be equal in these expressions, one does not recover all the integrals of the special case of $a_i = a, \forall i$ NHEMP. In such special cases all of the first integrals of the spherical mechanics of the generic (nonequal a_i) case transform into the Hamiltonian of the spherical mechanics of the equal a_i case. So, to obtain the Liouville integrals in the isotropic case we need to develop a more sophisticated contraction procedure.

We also note that the above expressions for the constants of motion were found in the ellipsoidal coordinates introduced for the special case of nonequal rotational parameters a_i. However, we then wrote them in the initial coordinates, and they hold for generic nonzero values of the rotation parameters a_i. We will analyze the special cases where some of the a_i or m_i are equal in Sec. IV and when one of them is vanishing in Sec. V.

B. Killing tensors

In the previous subsection we presented the constants of motion in the form of demonstrating their explicit dependence on the momenta p_a, p_{φ_i}. To represent (40) through the respective N_σ second rank Killing tensors, one can replace the last term proportional to m_0^2 from the mass-shell equations (11) and (12). Note also that the F_a, $a = 1, \dots, N_\sigma-1$, provides $N_\sigma-1$ one constants of motion. We can then add F_{N_σ} to this collection, which is proportional to the mass with the corresponding second rank killing tensor being the inverse metric, i.e.

$$F_{N_\sigma} = (-1)^{a-1} \left(r_H^2 \sum_{A,B=1}^{2N+1+\sigma} g^{AB} p_A p_B \right.$$
$$\left. - \left(\sum_{i=1}^{N} \frac{\sqrt{m_i-1} p_{\varphi_i}}{m_i} \right)^2 \right), \tag{46}$$

where we assumed $M_{-1}^{\neq b,c} = 0$.

To get the expression for Killing tensors, we should simply replace the momenta by the respective vector fields, $p_A \to \frac{\partial}{\partial x^A}$. That is, in the coordinates (x_a, φ_a) where the constants of motion (40) are written, one should replace

$$p_a \to \frac{\partial}{\partial x^a}, \qquad p_{\varphi_i} \to \frac{\partial}{\partial \varphi^i}, \qquad p_r \to \frac{\partial}{\partial r}, \qquad p_0 \to \frac{\partial}{\partial \tau}.$$

In ellipsoidal coordinates the above presented $N_\sigma - 2$ Killing tensors read

$$K_a = \sum_\alpha A_\alpha^{\neq a} h^\alpha (\partial_{\lambda_\alpha})^2 + \sum_I \sum_\alpha \frac{A_\alpha^{\neq a} \prod_{J \neq I} (m_J - m_I)}{m_I (m_I - \lambda_a) \prod_b'(\lambda_b - \lambda_a)}$$
$$\times (\partial_{\varphi_I})^2 + \frac{A^a}{A(\lambda)} \left(-\frac{1}{r^2} (\partial_\tau)^2 + r^2 (\partial_r)^2 \right). \tag{47}$$

Thus, we have $N + 1$ mutually commuting Killing vectors $\partial/\partial \varphi_i$, ∂/∂_τ and N_σ Killing tensors, summing up to $d = N_\sigma + N + 1$ and hence the system is integrable. One may check that our expressions for the Killing tensors match with those that appeared in [14,28] after taking the near-horizon limit. We note that the two extra Killing vectors of the $SL(2, \mathbb{R})$ part of the isometry which appear in the near-horizon limit and in the coordinates of (7) take the form

$$r\frac{\partial}{\partial r} - \tau\frac{\partial}{\partial \tau}, \quad \left(\tau^2 + \frac{1}{r^2}\right)\frac{\partial}{\partial \tau} - 2\tau r\frac{\partial}{\partial r} - \frac{2}{r}\sum_{i=1}^{N}\frac{\partial}{\partial \varphi_i}, \quad (48)$$

and do not yield new independent constants of motion.

IV. ISOTROPIC AND PARTIALLY ISOTROPIC CASES

When some of the $a_i \neq 0$'s are equal, the geometry (1) exhibits a bigger isometry group than $SL(2,\mathbb{R}) \times U(1)^N$; depending on the number of equal a_i's the $U(1)^N$ part is enhanced to a rank N subgroup of $U(N)$. This larger isometry group brings a larger number of Killing vectors and tensors and one hence expects the particle dynamics for these cases to become a superintegrable system. This is what we will explore in this section and construct the corresponding conserved charges.

A. The fully isotropic, equal m_i case

When all of the rotational parameters coincide, the Hamiltonian of the probe particle reduces to the system on sphere and admits separation of variables in spherical coordinates [8]. It can be checked that in this case, the Hamiltonian of the reduced mechanics derived from (17) transforms into the corresponding mechanics with equal parameters derived in [8] for both odd and even dimensional cases. Notice that in this limit the difference between even and odd cases becomes visible:

(i) In the odd case, $\sigma = 0$, the isotropic limit corresponds to the choice $m_i = N$, $i = 1, ..., N$. As a result, the angular Hamiltonian (17) which we will denote by $\mathcal{I}_N$ takes the form

$$\mathcal{I}_N = \sum_{a,b=1}^{N-1}(N\delta_{ab} - x_a x_b)p_a p_b + N\sum_{i=1}^{N}\frac{p_{\varphi_i}^2}{x_i^2},$$

$$\sum_{i=1}^{N}x_i^2 = N. \quad (49)$$

For the fixed p_{φ_i} configuration space of this system is an $(N-1)$-dimensional sphere, and the Hamiltonian defines specific generalization of the Higgs oscillator, which is also known as a Rossochatius system [29].

(ii) In the even case, $\sigma = 1$, one has $m_i = 2N$ when $i = 1, ..., N$ and $m_{N+1} = 1$; i.e. we cannot choose all parameters m_I be equal. As a result, the angular Hamiltonian (17) reads

$$\mathcal{I}_N = \sum_{i,j=1}^{N}(\eta^2\delta_{ij} - x_i x_j)p_i p_j + \sum_{i=1}^{N}\frac{\eta^2 p_{\varphi_i}^2}{x_i^2} + \omega\sum_{i=1}^{N}x_i^2, \quad (50)$$

where

$$\eta^2 = 4N^2 - (2N-1)\sum_{i=1}^{N}x_i^2,$$

$$\omega = \left(1 - \frac{1}{2N}\right)^2\sum_{i,j=1}^{N}p_{\varphi_i}p_{\varphi_j} - m_0^2(2N-1). \quad (51)$$

In the case of even dimension, configuration space fails to be a sphere (even with fixed p_{φ_i}).

What is important is that both systems admit separation of variables in spherical coordinates. Namely, by recursively introducing spherical coordinates

$$x_{N_\sigma} = \sqrt{N_\sigma}\cos\theta_{N_\sigma-1},$$
$$x_a = \sqrt{N_\sigma}\tilde{x}_a \sin\theta_{N_\sigma-1},$$

$$\sum_{a=1}^{N_\sigma-1}\tilde{x}_a^2 = 1, \quad (52)$$

we get the following recurrent formulas for the constants of motion

$$\sigma = 0: \mathcal{I}_{\text{odd}} = F_{N-1}, \qquad F_a = p_{\theta_a}^2 + \frac{p_{\varphi_{a+1}}^2}{\cos^2\theta_a} + \frac{F_{a-1}}{\sin^2\theta_a}, \qquad F_0 = p_{\varphi_1}^2, \quad (53)$$

$$\sigma = 1: \mathcal{I}_{\text{even}} = 2Np_{\theta_N}^2 + \nu\sin^2\theta_N + (2N\cot^2\theta_N + 1)F_{N-1}. \quad (54)$$

It is clear that $F_1, ..., F_{N_\sigma-1}$ defines a complete set of Liouville constants of motion and the $\sigma = 1$ system contains $\sigma = 0$ as a subsystem. Moreover, the Rosochatius system (angular Hamiltonian for $\sigma = 0$ case with fixed p_{φ_i}) is superintegrable: it has $N - 2$ additional functionally independent constants of motion defined by the expression

$$I_{a,a-1} = (p_{\theta_{a-2}}\sin\theta_{a-2}\cot\theta_{a-1} - p_{\theta_{a-1}}\cos\theta_{a-2})^2$$
$$+ \left(p_{\varphi_{a-1}}\frac{\cot\theta_{a-1}}{\cos\theta_{a-2}} + p_{\varphi_a}\cos\theta_{a-2}\tan\theta_{a-1}\right)^2. \quad (55)$$

When p_{φ_i} are not fixed, the system is an $(N_\sigma - 1 + N)$-dimensional one. In that case, from its action-angle

formulation [8] one can observe, it remains maximally superintegrable for $\sigma = 0$, i.e. possesses $4N - 3$ constants of motion: Besides $2N - 3$ constants of motion given by (53) and (55), and the N commuting integrals p_{φ_i} (associated with axial Killing vectors), there are N additional constants of motion with quadratic term mixing p_{θ_a} and p_{φ_i}, i.e. N second rank Killing tensors in the $\partial_{\theta_a} \partial_{\varphi_i}$ direction. When $\sigma = 1$, the system is $2N$ dimensional, and has $4N - 2$ integrals, i.e., lacks one integral from being maximally superintegrable.

From these constants of motion one can readily read the associated Killing vectors and second rank Killing tensors. Hence, the isotropic system has $N + 1$ mutually commuting Killing vectors and $d - 3 = 2N + \sigma - 2$ Killing tensors, and additional N noncommuting second rank Killing tensors.

For more detailed analysis of the isotropic case, see [8]. Here we present it mainly to set the conventions we use in the study of an "intermediate case," when only some of the rotation parameters are equal to each other.

B. Partially isotropic case in odd dimensions

Let us start with the simpler odd dimensional system, $\sigma = 0$, with $p = N - l$ nonequal rotation parameters and l equal ones:

$$m_1 \neq m_2 \neq \ldots \neq m_p \neq m_{p+1},$$
$$m_{p+1} = m_{p+2} = \ldots = m_N \equiv \kappa. \qquad (56)$$

Starting from the metric (7) we will construct the Hamiltonian for the reduced mechanics by introducing spherical and ellipsoidal coordinates. Spherical coordinates $\{y, \theta_i\}$, $i = 1 \ldots l - 1$ will be introduced for the l latitudinal coordinates $x_{p+1}, \ldots, x_N$ corresponding to the equal rotational parameters

$$x_{p+1} = y \prod_{i=1}^{l-1} \sin\theta_i,$$

$$x_{p+a} = y \cos\theta_{a-1} \prod_{i=a}^{l-1} \sin\theta_i,$$

$$x_{p+l} = y \cos\theta_{l-1}, \quad a = 2, \ldots, l - 1. \qquad (57)$$

Hence,

$$\sum_{a=1}^{l} \frac{x_{p+a}^2}{m_{p+a}} = \frac{y^2}{\kappa}, \qquad \sum_{a=1}^{l} (dx_{p+a})^2 = (dy)^2 + y^2 d\Omega_{l-1},$$

$$(58)$$

with $d\Omega_{l-1}$ being the metric on the $(l - 1)$-dimensional sphere: $d\Omega_{l-1} = d\theta_{l-1}^2 + \sin^2\theta_{l-1} d\Omega_{l-2}$.

Performing the coordinate transformation (57) in (1), it is seen that the radial coordinate y of the spherical subsystem behaves very much like the other latitudinal coordinates of nonequal rotational parameters. Therefore, we will treat y and $x_1 \ldots x_p$ in the same way:

$$y_a = (x_1, \ldots, x_p, y), \quad \tilde{m}_a = (m_1, \ldots, m_p, m_{p+1}):$$

$$\sum_{a=1}^{p+1} \frac{1}{\tilde{m}_a} = 1, \qquad \sum_{a=1}^{p} \frac{y_a^2}{\tilde{m}_a} + \frac{y^2}{\tilde{m}_{p+1}} = 1, \qquad (59)$$

in terms of which the metric takes the form

$$\frac{ds^2}{r_H^2} = A(y)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + dy_{p+1}^2 + y_{p+1}^2 d\Omega_{l-1}$$

$$+ \sum_{a=1}^{p} (dy_a)^2 + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i(y) x_j(y) D\varphi^i D\varphi^j, \qquad (60)$$

with

$$A(y) = \frac{\sum_{a=1}^{p+1} y_a^2 / \tilde{m}_a^2}{4 \sum_{a<b}^{p+1} \frac{1}{\tilde{m}_a} \frac{1}{\tilde{m}_b}}, \qquad \sum_{a=1}^{p+1} \frac{y_a^2}{\tilde{m}_a} = 1. \qquad (61)$$

The Hamiltonian of the corresponding spherical mechanics then reads

$$\mathcal{I} = A\left[\sum_{a,b=1}^{p} h^{ab} p_a p_b + \sum_{a=1}^{p+1} \frac{g_a^2}{y_a^2} + g_0\right],$$

with $g_a^2 = (p_{\varphi_1}^2, \ldots, p_{\varphi_p}^2, \mathcal{I}_{p+1})$,

$$h^{ab} = \delta^{ab} - \frac{1}{\sum_{a=1}^{p+1} y_a^2 / \tilde{m}_a^2} \frac{y_a}{\tilde{m}_a} \frac{y_b}{\tilde{m}_b}, \qquad (62)$$

and $\mathcal{I}_{p+1}$ is defined as by (49) in the $(p + 1)$-dimensional space. The above describes a lower-dimensional version of (17), where all rotational parameters are nonequal and we can analyze it as we did for the general case in the previous section. That is, we introduce on the $(p + 1)$-dimensional ellipsoidal coordinates

$$y_a^2 = \frac{\prod_{b=1}^{p+1} (\tilde{m}_a - \lambda_b)}{\prod_{b=1; b\neq a}^{p+1} (\tilde{m}_a - \tilde{m}_b)}, \qquad (63)$$

and take $\lambda_{p+1} = 0$ for resolving the constraint (61) given by the second expression. This coordinate transformation is similar to the one introduced in [30] in the context of general Kerr-NUT-AdS geometries. The rest of the analysis goes through as in [22] and as in Sec. III.

The partially isotropic case discussed here, as we see, interpolates between the generic case of Sec. III ($p = N - 1$) and the fully isotropic case ($p = 0$) of Sec. IV A: It decouples to the Hamiltonians of type (21)

and (49). The case $l = 1$ corresponds to the system with nonequal parameters, and the spherical subsystem is trivial ($\mathcal{I}_{p+1} = p_{\varphi_{p+1}}^2$). For $l \geq 2$ the $(l-1)$-dimensional spherical subsystem is not trivial anymore and has $2(l-1)-1$ constants of motion. Thus the reduced $(N-1)$-dimensional angular system has $p + 2l - 3 = N - 1 + l - 2$ constants of motion, i.e. the number of extra constants of motion compared to the generic case is $l - 2$, with $l > 2$. It becomes maximally superintegrable only for $l = N$, i.e. when all rotational parameters are equal.

This discussion can be easily extended to the case of even dimensions ($\sigma = 1$). Here we will have an additional latitudinal coordinate ($p + l = N + 1$) and a rotational parameter with a fixed value ($m_{N+1} = 1$). One should note that m_{N+1} cannot be equal to any other rotational parameter, so it is one of the p nonequal parameters. In the limiting case when $l = 1$ and all rotational parameters are different, we have an integrable system with $p = N$ configuration space degrees of freedom, as expected. Since m_{N+1} cannot be equal to the others, p cannot be equal to 0 and the even dimensional system cannot be maximally superintegrable. In the limit when all rotational parameters are equal except m_{N+1} ($p = 1$), the system will lack one integral of motion to be maximally superintegrable.

C. General case

Having discussed the some equal m_i's but the rest nonequal case, we now turn to the most general case when there are s sets (blocks) of equal rotation parameters each containing l_i members. As before we assume that there are p rotation parameters which are not equal to the others, so that $p + \sum_{i=1}^s l_i = N_\sigma$. Note that in our conventions $l_i \geq 2$. We introduce an upper index which, written on a parameter or a function, denotes the number of the block under consideration. So, for example $m_a^{(i)}$ will denote all the equal rotational parameters in the ith set of rotation parameters and $x_a^{(i)}$ will denote their corresponding latitudinal coordinates and

$$\{m_a^{(i)}\} = m_{p+l_1+\cdots+l_{i-1}+a} \equiv \kappa^{(i)} \quad i = 1, \ldots, s, \quad a = 1, \ldots, l_i.$$
(64)

The list of all rotational parameters can be written as

$$\{m_\alpha\} = m_1, m_2, \ldots, m_p, \quad \{m_a^{(1)}\}, \{m_a^{(2)}\}, \ldots, \{m_a^{(s)}\},$$
$$\alpha = 1, \ldots, N,$$
$$m_1 \neq m_2 \neq \ldots \neq m_p, \quad \{m_a^{(i)}\} = \kappa^{(i)}$$
with $\kappa^{(i)} \neq \kappa^{(j)}, \quad p + l_1 + \cdots + l_s = N.$
(65)

Let us start with the odd ($\sigma = 0$) case and the metric (7). We can construct the Hamiltonian for the reduced

mechanics by introducing spherical and ellipsoidal coordinates. Different spherical coordinates will be introduced separately for each set of latitudinal coordinates corresponding to different sets of equal rotational parameters,

$$x_1^{(i)} = r_i \prod_{\alpha=1}^{l_i-1} \sin \theta_\alpha^{(i)}$$

$$x_k^{(i)} = r_i \cos \theta_{k-1}^{(i)} \prod_{a=k}^{l_i-1} \sin \theta_a^{(i)}$$

$$x_{l_i}^{(i)} = r_i \cos \theta_{l_i-1}^{(i)}, \quad k = 2, \ldots, l_i - 1.$$
(66)

One should note that these spherical coordinates satisfy the relations

$$\sum_{a=1}^{l_i} (x_a^{(i)})^2 = r_i^2 \quad \text{and} \quad \sum_{a=1}^{l_i} (dx_a^{(i)})^2 = dr_i^2 + r_i^2 d\Omega_{l_i-1}^{(i)},$$
(67)

where $d\Omega_n^{(i)} = (d\theta_n^{(i)})^2 + \sin^2\theta_n^{(i)} d\Omega_{n-1}^{(i)}$ denotes the metric on unit n-dimensional sphere. For the rest of the latitudinal coordinates $x_1 \ldots x_p$ corresponding to nonequal rotational parameters and the radial coordinates r_i of isotropic subsystems, we introduce the notation

$$\{y_a\} = \{x_1, \ldots, x_p; r_1, \ldots, r_s\},$$
$$\{\tilde{m}_a\} = \{m_1, \ldots, m_p; \kappa^{(1)}, \ldots, \kappa^{(s)}\}.$$
(68)

In this notation the metric (7) can be rewritten as

$$\frac{ds^2}{r_H^2} = A(y)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{a=1}^{p+s} dy_a^2 + \sum_{b=1}^{s} y_{p+b}^2 d\Omega_{l_b-1}^{(b)}$$
$$+ \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i(y) x_j(y) D\varphi^i D\varphi^j,$$
(69)

where $\tilde{\gamma}_{ij}$, $A(y)$ are defined as in (8) and (61), respectively. Therefore, the Hamiltonian of the corresponding angular mechanics reads

$$\mathcal{I} = A\left[\sum_{a,b=1}^{p+s-1} h^{ab} \pi_a \pi_b + \sum_{a=1}^{p+s} \frac{g_a^2}{y_a^2} + g_0\right]$$
$$\{g_a^2\} = \{p_{\varphi_1}^2, \ldots, p_{\varphi_p}^2; \mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(s)}\},$$
$$\mathcal{I}^{(a)} = F_{l_a-1}^{(a)},$$
(70)

where $\mathcal{I}^{(a)}$ are the spherical subsystems resulting from the s sets of equal rotation parameters, h^{ab} is defined by (62), and

$$F_d^{(a)} = p_{\theta_d^{(a)}}^2 + \frac{(g_{d+1}^{(a)})^2}{\cos^2\theta_d^{(a)}} + \frac{F_{d-1}^{(a)}}{\sin^2\theta_d^{(a)}},$$

$$F_0^{(a)} = (g_1^{(a)})^2,$$

$$g_d^{(a)} = p_{\varphi_{p+l_1+\cdots+l_{a-1}+d}}$$

$$\{\pi_a, \lambda_b\} = \delta_{ab}, \qquad \{p_{\varphi_i}, \varphi_j\} = \delta_{ij}$$

$$\{p_{\theta_b^{(a)}}, \theta_d^{(c)}\} = \delta_{ac}\delta_{bd}. \tag{71}$$

Hence, the reduced spherical mechanics (70) has the exact form of (21) (with appropriate constants) whose integrability has already been discussed. All discussions from the previous subsection can be easily extended to this case, e.g. separation of variables may be achieved in the ellipsoidal coordinates (see [30] for a discussion on similar coordinate transformation),

$$y_a^2 = \frac{\prod_{b=1}^{p+s}(\tilde{m}_a - \lambda_b)}{\prod_{b=1;b\neq a}^{p+s}(\tilde{m}_a - \tilde{m}_b)}, \tag{72}$$

and place $\lambda_{p+s} = 0$ for resolving the constraint on latitudinal coordinates (9), which now takes the form $\sum_{a=1}^{p+s} \frac{y_a^2}{\tilde{m}_a} = 1$.

So, we separated the variables for the $(N-1)$-dimensional angular mechanics describing the geodesics in the near-horizon limit of a $(2N+1+\sigma)$-dimensional Myers-Perry black hole in arbitrary dimension with arbitrary nonzero values of rotational parameters. The number of constants of motion in this system can be easily counted: it is equal to $d + N_\sigma - p - 2s$. The generic case of nonequal m_i is recovered by $s = 0$, $p = N_\sigma$ and the fully isotropic case as $s = 1$, $p = 0$. In a similar manner one can construct associated Killing tensors.

D. Contraction from fully nonisotropic to isotropic NHEMP

Having the two corner cases discussed (fully nonisotropic and isotropic), an interesting question arises. What kind of approximation would transform the first integrals of fully nonisotropic NHMEP to the first integrals of isotropic NHEMP? It is straightforward to check that simply taking all rotation parameters to be equal just transforms all the first integrals of fully nonisotropic NHMEP to the Hamiltonian of the spherical mechanics of isotropic NHEMP (with an overall constant factor and a constant term). So if $m_i = N$

$$F_a = C_a \left(\sum_{b,c=1}^{N-1} (\delta_{bc} - x_b x_c) p_b p_c + \sum_{k=1}^{N} \frac{p_{\varphi_k}^2}{x_k^2} \right) + C_a' \tag{73}$$

where C_a and C_a' are constants. To find the desired approximation, we will work with rotation parameters which have little variations from their isotropic value N ($\epsilon_i \ll N$),

$$m_i = N + \epsilon_i.$$

In such a limit, the Hamiltonian of the nonisotropic mechanics can be extended in powers of ϵ_i, keeping the first order term only

$$F_1 = N^{N-3} \left[N\tilde{\mathcal{I}}_{\text{iso}} + N^2 g_0 - \sum_{i=1}^{N} \epsilon_i x_i^2 \left[\sum_a p_a^2 + \sum_k \frac{p_{\varphi_k}^2}{x_k^2} + g_0 \right] \right.$$
$$\left. + 2\sum_{a,b}^{N-1} \epsilon_a p_a x_a p_b x_b \right] \tag{74}$$

where

$$\tilde{\mathcal{I}}_{\text{iso}} = \sum_{a,b=1}^{N-1} (N\delta_{ab} - x_a x_b) p_a p_b + N \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} \tag{75}$$

is the isotropic Hamiltonian. We should note that the linear term of F_1 still corresponds with the isotropic Hamiltonian $\tilde{\mathcal{I}}_{\text{iso}}$ but the relation $\sum x_i^2 = N$ does not hold anymore.

Now, if we find some linear combination $P(F_a)$ of first integrals of nonisotropic mechanics such that the free term of the expansion around $m_i = N$ vanishes, we can write

$$\{P(F_a), F_1\} = 0 = \left\{ \sum_{i=1}^{N} \epsilon_i P_i(p_j, x_j), \tilde{\mathcal{I}}_{\text{iso}} + \sum_{i=1}^{N} \epsilon_i(\ldots) \right\}$$
$$= \sum_{i=1}^{N} \epsilon_i \{P_i(p_j, x_j), \tilde{\mathcal{I}}_{\text{iso}}\}$$
$$\Rightarrow \{P_i(p_j, x_j), \tilde{\mathcal{I}}_{\text{iso}}\} = 0. \tag{76}$$

We see that the first order coefficients $P_i(p_j, x_j)$ of the $P(F_a)$ linear combination are first integrals for $\tilde{\mathcal{I}}_{\text{iso}}$. To construct such combination whose free term vanishes we can take any of the first integrals, let us say F_{N-1} and expand it,

$$F_{N-1} = (-1)^N \left[\tilde{\mathcal{I}}_{\text{iso}} - \frac{g_0}{N} \sum_{i=1}^{N} \epsilon_i x_i^2 + \sum_i^{N} \epsilon_i \frac{p_{\varphi_i}^2}{x_i^2} + \sum_{a=1}^{N-1} \epsilon_a p_a^2 \right] \tag{77}$$

We see from (74) and (77) that by combining F_1 and F_{N-1} the free term can be eliminated

$$N^{-(N-3)} F_1 + (-1)^{N-1} N F_{N-1} - g_0 N^2$$
$$= -\left(\sum_a^{N-1} p_a^2 + \sum_k^{N} \frac{p_{\varphi_k}^2}{x_k^2} \right) \sum_{i=1}^{N} \epsilon_i x_i^2$$
$$+ 2\sum_{a,b}^{N-1} \epsilon_a p_a x_a p_b x_b - N \left(\sum_i^{N} \epsilon_i \frac{p_{\varphi_i}^2}{x_i^2} + \sum_{a=1}^{N-1} \epsilon_a p_a^2 \right). \tag{78}$$

Furthermore, from the expression $\sum_i^N x_i^2/m_i = 1$ we can find

$$x_N^2 = \left(\tilde{x}_N^2 + \frac{1}{N} \sum_a^{N-1} \epsilon_a x_a^2 \right) \left(1 + \frac{\epsilon_N}{N} \right),$$

$$\tilde{x}_N^2 \equiv N - \sum_a^{N-1} x_a^2$$

and replace with this relation every occurrence of x_N in (78). Doing this, we will end up with the same equation (78) with just x_N^2 replaced by $\tilde{x}_N^2$. So in further calculations we are free to consider Eq. (78) with a redefined x_N

$$\tilde{x}_N^2 \to x_N^2 = N - \sum_a^{N-1} x_a^2. \tag{79}$$

Thus, having in mind (76), we find the first integrals of isotropic mechanics to be

$$F_a^{\text{iso}} = -x_a^2 \left(\sum_b^{N-1} p_b^2 + \sum_k^N \frac{p_{\varphi_k}^2}{x_k^2} \right) + 2 p_a x_a \sum_b^{N-1} p_b x_b$$

$$- N \left(\frac{p_{\varphi_a}^2}{x_a^2} + p_a^2 \right),$$

$$F_N^{\text{iso}} = -x_N^2 \left(\sum_b^{N-1} p_b^2 + \sum_k^N \frac{p_{\varphi_k}^2}{x_k^2} \right) - N \frac{p_{\varphi_N}^2}{x_N^2}. \tag{80}$$

Now, we can see that the sum of all N first integrals results into the Casimir of isotropic mechanics

$$\sum_{i=1}^N F_i^{\text{iso}} = -2 \tilde{\mathcal{I}}_{\text{iso}}. \tag{81}$$

Thus, by definition, all F_i^{iso} commute with $\sum_{i=1}^N F_i^{\text{iso}}$, but one can check that they do not commute with each other.

V. EXTREMAL VANISHING HORIZON CASE

As seen from metric (1), the case where one of the a_i's is zero is a singular case. In fact for this case one should revisit the near-horizon limit. It has been shown that [26] for the odd dimensional extremal MP black holes the horizon area also vanishes and we are hence dealing with an extremal vanishing horizon (EVH) black hole [25]. The near-horizon EVH black holes have remarkable features which are not shared by generic extremal black holes; they constitute a different set of geometries which should be studied separately [31]. In particular, it has been proven that for EVH black holes the near-horizon geometry includes an AdS$_3$ factor (in contrast with the AdS$_2$ factor of general extremal case) [31,32], i.e. the d-dimensional NHEVHMP exhibits $SO(2,2) \times U(1)^{N-1}$ isometry. To study this case, we start by a review on black hole geometry itself. Then, by taking the near-horizon and EVH limit, we discuss the

separability of Hamilton-Jacobi equations on the NHEVH geometries.

As discussed in the special case of EVH black holes, one has to revisit the standard NH theorems for extremal black holes. Here we review EVH black holes in the family of odd-dimensional MP black holes [24]:

$$ds^2 = -d\tau^2 + \frac{\mu \rho^2}{\Pi F} \left(d\tau + \sum_{i=1}^N a_i \mu_i^2 d\phi_i \right)^2 + \frac{\Pi F}{\Pi - \mu \rho^2} d\rho^2$$

$$+ \sum_{i=1}^N (\rho^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{82}$$

where

$$F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{\rho^2 + a_i^2}, \quad \Pi = \prod_{i=1}^N (\rho^2 + a_i^2), \quad \sum_i \mu_i^2 = 1. \tag{83}$$

The extremal case happens when $\Pi - \mu \rho^2 = 0$ has double roots and the EVH case is when one of a_i parameters, which we take to be a_N, is zero. That is in the EVH case $\mu = \prod_{a=1}^{N-1} a_a^2$. We note that we could have considered a "near-EVH" metric where the black hole is at a nonzero but small temperature and the horizon area is also small, while the ratio of horizon area to the temperature is finite [25,31].

The horizon for the EVH case is at $\rho = 0$ and hence in the NH limit, the leading contributions come from

$$\Pi = \mu \rho^2 \left(1 + \frac{\rho^2}{r_0^2} \right), \quad F_0 = 1 - \sum_{a=1}^{N-1} \mu_a^2, \quad \frac{1}{r_0^2} = \sum_{b=1}^{N-1} \frac{1}{a_b^2}. \tag{84}$$

Plugging the above into the metric (82) and taking

$$\rho = r_0 r \epsilon, \quad \tau = r_0 t / \epsilon, \quad \psi = \varphi_N / \epsilon,$$

$$\varphi_a = \phi_a + \tau / a_a, \quad a = 1, \ldots, N-1, \quad \epsilon \to 0,$$

we obtain the NHEVHMP metric [26]:

$$ds^2 = F_0 r_0^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\psi^2 \right] + \sum_{b=1}^{N-1} a_b^2 d\mu_b^2$$

$$+ \sum_{a,b=1}^{N-1} \gamma_{ab} d\varphi_a d\varphi_a,$$

$$\gamma_{ab} \equiv a_a^2 \mu_a^2 \delta_{ab} + a_a a_b \frac{\mu_a^2 \mu_b^2}{F_0}, \tag{85}$$

where in the above a, b run from 1 to $N-1$. Had we started from the near-EVH geometry, the AdS$_3$ factor (the r, t, ψ part) of (85) would have turned into a generic Bañados,

Teitelboim and Zanelli black hole geometry [25,31]. The NH geometry (85) has $SO(2,2) \times U(1)^{N-1} \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times U(1)^{N-1}$ isometry. This is to be compared with $SL(2,\mathbb{R}) \times U(1)^N$ of the non-EVH NHEMP discussed in previous sections.

To discuss separability of the particle dynamics on (85), as in the previous sections, we introduce coordinates,

$$x_a \equiv \frac{a_a \mu_a}{r_0} \qquad m_a \equiv \frac{a_a^2}{r_0^2} \qquad \sum_{a=1}^{N-1} \frac{1}{m_a} = 1, \quad (86)$$

in which (85) takes the form

$$\frac{ds^2}{r_0^2} = F_0 ds_{\text{AdS}_3}^2 + \sum_a^{N-1} dx_a^2 + \sum_{a,b}^{N-1} \tilde{\gamma}_{ab} x_a x_b d\varphi_a d\varphi_b, \quad (87)$$

with

$$ds_{\text{AdS}_3}^2 = r^2(-dt^2 + d\psi^2) + \frac{dr^2}{r^2}, \qquad F_0 = 1 - \sum_a^{N-1} \frac{x_a^2}{m_a},$$

$$\tilde{\gamma}_{ab} x_a x_b = \frac{1}{r_0^2} \gamma_{ab}, \qquad \tilde{\gamma}_{ab} = \delta_{ab} + \frac{1}{F_0} \frac{x_a}{\sqrt{m_a}} \frac{x_b}{\sqrt{m_b}}. \quad (88)$$

The generators of the two $SL(2,\mathbb{R})$ Killing vectors may be written as

$$H_+ = \partial_v, \quad D_+ = v\partial_v - r\partial_r \quad K_+ = v^2\partial_v + \frac{1}{r^2}\partial_u - 2rv\partial_r,$$

$$H_- = \partial_u, \quad D_- = u\partial_u - r\partial_r \quad K_- = u^2\partial_u + \frac{1}{r^2}\partial_v - 2ru\partial_r, \quad (89)$$

where $v = t + \psi$ and $u = t - \psi$. The Casimir of $SL(2,\mathbb{R})$'s are

$$\mathcal{I}_{\pm} = H_{\pm} K_{\pm} - D_{\pm}^2 \quad (90)$$

and one can readily check that both Casimirs are equal to $\mathcal{I} = \frac{1}{r^2}(\partial_t^2 - \partial_\psi^2) - r^2\partial_r^2$.

The mass-shell equation of the probe particle (11) then reads

$$\frac{(p_0)^2 - (p_\psi)^2}{r^2} = (rp_r)^2 + \mathcal{I}(p_a, x_a, p_{\varphi_a}) \quad (91)$$

where

$$\{p_a, x_b\} = \delta_{ab}, \quad \{p_{\varphi_a}, \varphi^b\} = \delta_{ab},$$
$$\{p_\psi, \psi\} = 1, \quad \{p_r, r\} = 1, \quad (92)$$

and

$$\mathcal{I}(p_a, x_a, p_{\varphi_a}) = \left(1 - \sum_{c=1}^{N-1} \frac{x_c^2}{m_c}\right) \left[\sum_{a=1}^{N-1} p_a^2 + \sum_{a=1}^{N-1} \frac{p_{\varphi_a}^2}{x_a^2} + g_0\right],$$

$$g_0 = -\left(\sum_a^{N-1} \frac{p_{\varphi_a}}{\sqrt{m_a}}\right)^2 + m_0^2 r_0^2, \quad (93)$$

where $\mathcal{I}$ in (93) is the Casimir. Note that while the background has $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times U(1)^{N-1}$ isometry the Casimirs of the two $SL(2,\mathbb{R})$ factors happen to be identical and hence we are dealing with a single $\mathcal{I}$; appearance of an extra $SL(2,\mathbb{R})$ does not add to the number of constants of motion compared to the non-EVH case.

Hence, as in the regular case, we have to consider separately three cases:

(i) *Generic, non-isotropic case, all m_a are nonequal.*

To separate the variables in (93), in the special case when none of the rotational parameter is equal, we introduce the ellipsoidal coordinates

$$x_a^2 = \frac{\prod_{b=1}^{N-1}(m_a - \lambda_b)}{\prod_{b \neq a}^{N-1}(m_a - m_b)}. \quad (94)$$

In this terms the angular Hamiltonian reads

$$\mathcal{I} = \left(\prod_a^{N-1} \frac{\lambda_a}{m_a}\right) \left[\sum_{a=1}^{N-1} \frac{4 \prod_b^{N-1}(m_b - \lambda_a)}{\prod_{b \neq a}^{N-1}(\lambda_b - \lambda_a)} \pi_a^2 \right.$$
$$\left. + \sum_a^{N-1} \frac{p_{\varphi_a}^2}{x_a^2} + g_0\right], \quad (95)$$

where $\{\pi_a, \lambda_b\} = \delta_{ab}$. One can see that (95) has a very similar form to (21), and using the identities (26) and (27), it can be rewritten as follows (after fixing the Hamiltonian $\mathcal{I} = \mathcal{E}$):

$$\sum_{a=1}^{N-1} \frac{R_a - \tilde{\mathcal{E}}}{\lambda_a \prod_{b=1, a \neq b}^{N-1}(\lambda_b - \lambda_a)} = 0, \quad (96)$$

where

$$R_a = 4\lambda_a \pi_a^2 \prod_b^{N-1}(m_b - \lambda_a) + (-1)^{N-1} \sum_b^{N-1} \frac{\lambda_a g_b^2}{\lambda_a - m_b}$$
$$- g_0(-\lambda_a)^{N-1},$$

$$g_a^2 = p_{\varphi_a}^2 \prod_{b=1}^{N-1}(m_a - m_b), \qquad \tilde{\mathcal{E}} = \mathcal{E} \prod_a^{N-1} m_a. \quad (97)$$

Separation of variables and the constants of motion is similar to Sec. III, where (96) corresponds to (24).

(ii) *Isotropic case, all m_a are equal.*

In this case ($m_a = N - 1$), we separate the variables in (93) by introducing spherical coordinates $\{u, , y_\alpha, \theta_{N-2}\}$

$$x_{N-1} = u \cos \theta_{N-2},$$

$$x_{N-1-\alpha} = u y_\alpha \sin \theta_{N-2}, \qquad \sum_{\alpha=1}^{N-2} y_\alpha^2 = 1 \qquad (98)$$

where $\alpha = 1 \ldots N - 2$. In these coordinates (93) will take the following form:

$$\mathcal{I} = \left(1 - \frac{1}{N-1} u^2\right)\left[p_u^2 + \frac{F_{N-2}}{u^2} + g_0\right] \qquad (99)$$

with F_a defined in (53), where the separation of variables and the derivation of integrals of motion was carried out according to Sec. IV A.

(iii) *Partially isotropic case.*

The last case is the most general one which involves sets of equal and a set of nonequal rotational parameters. With the discussions of the two previous cases (fully isotropic and fully nonisotropic) in view and recalling the analysis of the partially isotropic NHEMP case of the previous section, it is straightforward to separate the variables in a partially isotropic NHEVHMP. Following the steps in Sec. IV C, one should first introduce different spherical coordinates for each set of equal rotational parameters and ellipsoidal coordinates for the joint set of nonequal rotational parameters and the radial parts of spherical coordinates. This will result into spherical mechanics similar to (70) where the Hamiltonians of spherical subsystems will be included as parameters.

VI. DISCUSSION

In this work, continuing analysis of [22,23], we studied separability of geodesic motion on the near-horizon geometries of a Myers-Perry black hole in d, even or odd, dimensions and established the integrability by explicit construction of d constants of motion. In the general case, $\left[\frac{d-1}{2}\right] + 1$ of these constants of motion are related to the Killing vectors of the background [note that the background in general has $\left[\frac{d-1}{2}\right] + 3$ Killing vectors, but three of them form an $sl(2, \mathbb{R})$ algebra and hence there is only one independent conserved charge from this sector]. Our analysis reconfirms the earlier observations that although the near-horizon limit in the extremal black holes enhances the number of Killing vectors by 2 [17], the number of independent conserved charges from the Killing vectors does not change. Our system, in the general case, has $\left[\frac{d}{2}\right]$ constants of motion associated with second rank Killing tensors that the system possesses. We also constructed the explicit relation between these Killing tensors and the conserved charges and one may check that our Killing tensors and those in [14] match. We note that the Killing tensors of [14] were obtained using the near-horizon limit

on the Killing tensors of Myers-Perry black hole in a coordinate system which makes the geodesics of the black hole separable itself, whereas we directly worked with ellipsoidal coordinates for the NHEMP, introduced in [22]. Comparing the two systems before and after the NH limit, it was argued in [14] that a combination of Killing tensors is reducible to the Killing vectors; however, we obtain other second rank Killing tensors, through which the system remains integrable. Moreover, by explicitly showing the separability, one concludes that there is no inconsistency with the theorems in [10]. There is an extra conserved charge related to the Casimir of the $SL(2, \mathbb{R})$ symmetry group which intrinsically exists in the NHEG's. We have shown that the charge of the Casimir is independent of the other conserved charges. In this sense, one of the "hidden symmetries," symmetries which are associated with equations of motion and are not isometries of the background, becomes explicit in the NH limit [14].

Following the discussions in [8], we showed that for special cases where some of the rotation parameters of the background are equal, the geodesic problem on NHEMP is superintegrable. We established superintegrability by establishing existence of other constants of motion. Our methods here, combined with those in [8], allows one to read the extra second rank Killing tensors obtained in these cases. The rough picture is as follows: We started with a system with $2N + 1 + \sigma$ variables with N isometries. Fixing the momenta associated with the isometries, we obtained and focused on the $N - 1 + \sigma$ dimensional "angular mechanics" part. In this sector, whenever N number of rotation parameters m_i of the background metric are equal the $U(1)^N$ isometry is enhanced to $U(N)$ and this latter brings about other second rank Killing tensors. All in all, for the fully isotropic case in odd dimensions with $U(\frac{d-1}{2})$ isometry, the $d - 2$ dimensional spherical mechanics part is maximally superintegrable, it has $N + (N - 2) = 2N - 2$ extra constants of motion. The fully isotropic case in even dimensions, however, is not maximally superintegrable; it has still $2N - 1$ extra Killing tensors (one less than the N constants of motion to make the system fully superintegrable). We discussed the "special cases" in two different ways. First, we reanalyzed the system from scratch (in Sec. IV A) and also took the equal rotation parameter limit of the generic case (in Sec. IV D). As expected, these two cases matched. Our preliminary analysis, which we did not show here, indicates that the above statements are also true for the NH limit of extremal MP black holes in (A)dS backgrounds.

We also discussed the EVH case, which happens for odd-dimensional extremal MP when one of the rotation parameters a_i vanishes. In the general NHEVHMP case, where the background isometry is $SO(2, 2) \times U(1)^{\frac{d-3}{2}}$, the number of independent charges associated with Killing vectors is $\frac{d+1}{2}$. Despite enhancement of the isometry group compared to the generic NHEMP case, we found that this symmetry

enhancement does not add to the number of independent constants of motion; the system in general does not pose extra constants of motion and remains just integrable.

Here we explored second rank Killing tensors; one may suspect the system has independent higher rank Killing tensors too. Although it is unlikely, if it happens, the system for the generic rotation parameters becomes superintegrable. It is interesting to explore this question. Finally, as already pointed out in the Introduction, one can consider other probes including scalar, Dirac field, or gauge or tensor perturbations on the NHEMP backgrounds and study their integrablitiy. To this end, the study of Killing Yano tensor and principal tensor [33,34] should be completed. We hope to address this in our future publications.

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