Exact Schwarzschild-like solution in a bumblebee gravity model

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We obtain an exact vacuum solution from the gravity sector contained in the minimal standard-model extension. The theoretical model assumes a Riemann spacetime coupled to the bumblebee field which is responsible for the spontaneous Lorentz symmetry breaking. The solution achieved in a static and spherically symmetric scenario establishes a Schwarzschild-like black hole. In order to study the effects of the spontaneous Lorentz symmetry breaking we investigate some classic tests, including the advance of perihelion, the bending of light, and Shapiro's time delay. Furthermore, we compute some upper bounds, among which the most stringent associated with existing experimental data provides a sensitivity at the 10^{-15} level and that for future missions at the 10^{-19} level.

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I. INTRODUCTION

General relativity (GR) and the standard model of particle physics are examples of successful field theories describing nature. The former describes gravitation at the classical level, and the latter describes particles and the other three fundamental interactions at the quantum level. The unification of these two theories is a fundamental endeavor, and this achievement will provide us with a deeper understanding of nature.

In the pursuit of this unification some theories of quantum gravity have already been proposed, but direct tests of their properties are currently beyond the energy scale of current experiments because they would be observed at the Planck scale ($\sim 10^{19}$ GeV). However, it is possible that some signals of quantum gravity can emerge at sufficiently low energy scales, and their effects could be observed in experiments carried out at current energy scales. One of these signals would be associated with the breaking of Lorentz symmetry.

Studies involving scenarios with Lorentz symmetry breaking in nature have already been developed and are being considered as promising avenues for exploration. The violation of this symmetry principle arises as a possibility in the context of string field theory [1,2], noncommutative field theories [3], and loop quantum gravity theory [4], among other scenarios [5]. These facts suggest that looking for evidence of Lorentz violation can be an efficient way to investigate signals of the existence of an underlying theory of quantum gravity at the Planck scale. A candidate providing a general theoretical framework for testing Lorentz and *CPT* symmetries is the standard-model extension (SME) [6,7].

The SME is an effective field theory which describes the standard model coupled to GR and includes additional terms containing information about the Lorentz violation occurring at the Planck scale [7]. The electromagnetic sector of the SME has been extensively studied in the literature [8–19], as well as the electroweak sector [20], some aspects of the strong sector [21], and hadronic physics [22]. Furthermore, some effects of Lorentz violation in the gravitational sector were studied in Refs. [23–31], and the case of gravitational waves was analyzed in Ref. [32].

The aim of this manuscript is to obtain a spherically symmetric exact solution to the Einstein equations in the presence of a spontaneous breaking of Lorentz symmetry due to a nonzero vacuum expectation value of the bumblebee field. Its influence in some well-known experimental tests of general relativity is also analyzed, namely, the advance of the perihelia of the inner planets, the bending of light, and the time-delay effect owing to curvature. All tests

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analyzed allow to estimate some upper bounds for the Lorentz-violating parameter involved. The manuscript is organized as follows. In Sec. II, we present a general geometrical framework allowing for the existence of nonzero vacuum expectation values promoting the spontaneous breaking of local Lorentz invariance. Furthermore, we compute the modified Einstein equation generated by the bumblebee gravity. In Sec. III, we solve the modified Einstein equation and look for spherically symmetric solutions. In Sec. IV, we study the effects of Lorentz violation in some classical tests of general relativity and, by using the data from existing and future Solar System experiments, we establish some upper bounds for the Lorentz-violating parameter involved. Finally, in Sec. V we give our remarks and conclusions. Throughout the manuscript we consider natural units ($\hbar = c = 1$) and the metric signature $(-+++)$. The physical constants are written explicitly when appropriate.

II. THE THEORETICAL FRAMEWORK

The focus of this work is to study spherically symmetric vacuum solutions in the context of an extended gravitational model including Lorentz-violating terms. Consequently, we study the effects of Lorentz violation in some classical tests of general relativity. For this purpose, we consider the bumblebee model, which is a known example of a gravity model that extends the standard formalism of GR, where under a suitable potential the bumblebee vector field B_μ acquires a nonzero vacuum expectation value (VEV), inducing a spontaneous Lorentz symmetry breaking.

In order to investigate the possible effects of the Lorentz violation in the gravitational sector, we consider the special class of theories in which Lorentz violation arises from the dynamics of a single vector B_μ that acquires a nonzero VEV. These theories are called bumblebee models and are among the simplest examples of field theories with spontaneous Lorentz and diffeomorphism violations. It is well known in the literature that the breaking of local Lorentz symmetry is always accompanied by diffeomorphism violation [33]. In this scenario, the Lorentz violation is triggered by a potential whose functional form possesses a minimum which ensures the breaking of the $U(1)$ symmetry. In general, the action for a single bumblebee field B_μ coupled to gravity and matter can be written as

$$S_B = \int d^4x \mathcal{L}_B = \int d^4x (\mathcal{L}_g + \mathcal{L}_{gB} + \mathcal{L}_{\rm K} + \mathcal{L}_V + \mathcal{L}_{\rm M}). \quad (1)$$

In Riemann spacetime, $\mathcal{L}_g$ is the pure gravitational Einstein-Hilbert term which may also include the cosmological constant, $\mathcal{L}_{gB}$ describes the gravity-bumblebee coupling, $\mathcal{L}_{\rm K}$ contains the bumblebee kinetic and any self-interaction terms, $\mathcal{L}_V$ corresponds to the potential which includes terms that trigger the spontaneous Lorentz violation, and $\mathcal{L}_{\rm M}$ defines the matter and other field contents and their couplings to the bumblebee field. By considering the case of a spacetime with null torsion and a null cosmological constant ($\Lambda = 0$), we introduce the following Lagrangian density:

$$\mathcal{L}_B = \frac{e}{2\kappa} R + \frac{e}{2\kappa} \xi B^\mu B^\nu R_{\mu\nu} - \frac{1}{4} e B_{\mu\nu} B^{\mu\nu} - eV(B^\mu) + \mathcal{L}_{\rm M}, \quad (2)$$

where $\kappa = 8\pi G_N$ is the gravitational coupling, $e \equiv \sqrt{-g}$ is the determinant of the vierbein and ξ is the real coupling constant (with mass dimension -1) that controls the nonminimal gravity-bumblebee interaction. The corresponding bumblebee field strength is defined as

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (3)$$

where B_μ has mass dimension 1. We point out that some bumblebee models involving nonzero torsion in more general contexts were investigated in Refs. [7,31,33].

For our purposes, the particular form of the potential $V(B_\mu)$ in Eq. (2) driving its dynamics is irrelevant, but it is important to emphasize that it must be formed from scalar combinations of the bumblebee field B_μ and the metric $g_{\mu\nu}$. In any case, we choose a potential V that provides a nonvanishing VEV for B_μ, which could have the following general functional form:

$$V \equiv V(B^\mu B_\mu \pm b^2), \quad (4)$$

where b^2 is a positive real constant. Some qualitative features of the symmetry-breaking potential have been explored in Refs. [1,7,33–35]. It follows that the VEV of the bumblebee field is determined when $V(B^\mu B_\mu \pm b^2) = 0$, implying that the condition

$$B^\mu B_\mu \pm b^2 = 0 \quad (5)$$

must be satisfied. This is solved when the field B^μ acquires a nonzero VEV given by

$$\langle B^\mu \rangle = b^\mu, \quad (6)$$

where the vector b^μ is a function of the spacetime coordinates such that $b^\mu b_\mu = \mp b^2 = \text{const}$; then, the nonzero vector background b^μ spontaneously breaks the Lorentz symmetry. We note the $\pm$ signs in the potential (4) determine whether the field b^μ is timelike or spacelike.

On the other hand, the Lorentz-violating contributions to the gravitational sector provided by the minimal SME are

$$S_{LV} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(uR + s^{\mu\nu}R_{\mu\nu} + t^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}), \quad (7)$$

where u, $s^{\mu\nu}$, and $t^{\mu\nu\alpha\beta}$ are real and dimensionless tensors carrying information about Lorentz violation. It is possible to establish a match between the bumblebee action (1) and the Lorentz-violating action (7) by considering the following parametrization of the underlying bumblebee field and the metric with the Lorentz-violating tensors u, $s^{\mu\nu}$, and $t^{\mu\nu\alpha\beta}$:

$$u = \frac{1}{4}\xi B^\mu B_\mu, \qquad s^{\mu\nu} = \xi\left(B^\mu B^\nu - \frac{1}{4}g^{\mu\nu}B^\alpha B_\alpha\right), \quad (8)$$

$$t^{\mu\nu\alpha\beta} = 0, \quad (9)$$

where $s^{\mu\nu}$ is traceless [7,30].

The next step is to establish the fields equations from the action (1) with the aim of finding vacuum solutions in the context of the extended gravitational sector.

A. The field equations

The Lagrangian density (2) yields the extended Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \quad (10)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the total energy-momentum tensor arising from the matter sector ($T^{\mathrm{M}}_{\mu\nu}$) and the contributions of the bumblebee field ($T^B_{\mu\nu}$); thus, we write

$$T_{\mu\nu} = T^{\mathrm{M}}_{\mu\nu} + T^B_{\mu\nu}, \quad (11)$$

with

$$\begin{aligned} T^B_{\mu\nu} = {} & -B_{\mu\alpha}B^\alpha{}_\nu - \frac{1}{4}B_{\alpha\beta}B^{\alpha\beta}g_{\mu\nu} - Vg_{\mu\nu} + 2V'B_\mu B_\nu \\ & + \frac{\xi}{\kappa}\left[\frac{1}{2}B^\alpha B^\beta R_{\alpha\beta}g_{\mu\nu} - B_\mu B^\alpha R_{\alpha\nu} - B_\nu B^\alpha R_{\alpha\mu}\right. \\ & + \frac{1}{2}\nabla_\alpha\nabla_\mu(B^\alpha B_\nu) + \frac{1}{2}\nabla_\alpha\nabla_\nu(B^\alpha B_\mu) \\ & \left. - \frac{1}{2}\nabla^2(B_\mu B_\nu) - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(B^\alpha B^\beta)\right]. \end{aligned} \quad (12)$$

The prime denotes differentiation with respect to the argument, as usual. Similarly, Eq. (2) provides the following equation of motion for the bumblebee field:

$$\nabla^\mu B_{\mu\nu} = J_\nu, \quad (13)$$

with $J_\nu = J^B_\nu + J^{\mathrm{M}}_\nu$, where J^{M}_ν is associated with the matter sector (acting as a source for the bumblebee field) and J^B_ν is a partial current that arises from the bumblebee self-interaction

$$J^B_\nu = 2V'B_\nu - \frac{\xi}{\kappa}B^\mu R_{\mu\nu}. \quad (14)$$

Taking the covariant divergence on the extended Einstein equations (10) and using the contracted Bianchi identities ($\nabla^\mu G_{\mu\nu} = 0$) leads to the condition

$$\nabla^\mu T_{\mu\nu} = 0, \quad (15)$$

which gives the covariant conservation law for the total energy-momentum tensor $T_{\mu\nu}$.

The trace of Eq. (10) reads

$$\begin{aligned} R = {} & -\kappa T^{\mathrm{M}} + 4\kappa V - 2\kappa V'B_\mu B^\mu \\ & + \xi\left[\frac{1}{2}\nabla^2(B_\mu B^\mu) + \nabla_\alpha\nabla_\beta(B^\alpha B^\beta)\right], \end{aligned} \quad (16)$$

where $T^{\mathrm{M}} \equiv g^{\mu\nu}T^{\mathrm{M}}_{\mu\nu}$, and by substituting this into Eq. (10) we obtain the trace-reversed version,

$$\begin{aligned} R_{\mu\nu} = {} & \kappa\left(T^{\mathrm{M}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\mathrm{M}}\right) + \kappa T^B_{\mu\nu} + 2\kappa g_{\mu\nu}V \\ & - \kappa B_\alpha B^\alpha g_{\mu\nu}V' + \frac{\xi}{4}g_{\mu\nu}\nabla^2(B_\alpha B^\alpha) \\ & + \frac{\xi}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(B^\alpha B^\beta). \end{aligned} \quad (17)$$

Note that if both the bumblebee field B_μ and the potential $V(B_\mu)$ vanish, Eq. (17) recovers the usual GR equations, as expected.

III. A SPHERICALLY SYMMETRIC SOLUTION IN THE LORENTZ-VIOLATING SCENARIO

We focus on a vacuum solution (i.e., one that describes empty space surrounding a gravitating body) by imposing $T^{\mathrm{M}}_{\mu\nu} = 0$. Furthermore, the potential in Eq. (2) is taken to trigger a nonzero VEV satisfying Eq. (6) and thereby vanishes. We will assume this scenario in the remainder of this manuscript.

Specifically, we are interested in vacuum solutions induced by a spontaneous Lorentz symmetry breaking when the bumblebee field B_μ remains frozen in its vacuum expectation value b_μ. A similar hypothesis was used in Ref. [36]. In this way, the bumblebee field is fixed to be

$$B_\mu = b_\mu, \quad (18)$$

and consequently we have $V = 0$ and $V' = 0$. Under such conditions, Eq. (17) leads to the extended Einstein equations in vacuum,

$$\bar{R}_{\mu\nu} = 0, \tag{19}$$

where we have defined

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \kappa b_{\mu\alpha} b^{\alpha}_{\nu} + \frac{\kappa}{4} b_{\alpha\beta} b^{\alpha\beta} g_{\mu\nu} + \xi b_{\mu} b^{\alpha} R_{\alpha\nu} + \xi b_{\nu} b^{\alpha} R_{\alpha\mu} - \frac{\xi}{2} b^{\alpha} b^{\beta} R_{\alpha\beta} g_{\mu\nu} - \frac{\xi}{2} \nabla_{\alpha} \nabla_{\mu} (b^{\alpha} b_{\nu}) - \frac{\xi}{2} \nabla_{\alpha} \nabla_{\nu} (b^{\alpha} b_{\mu}) + \frac{\xi}{2} \nabla^2 (b_{\mu} b_{\nu}). \tag{20}$$

The tensor $\bar{R}_{\mu\nu}$ has been introduced as a shorthand notation and for algebraic simplicity. Meanwhile, the bumblebee equation of motion (13) now becomes

$$\nabla^{\mu} b_{\mu\nu} = -\frac{\xi}{\kappa} b^{\mu} R_{\mu\nu}, \tag{21}$$

where $b_{\mu\nu} \equiv \partial_{\mu} b_{\nu} - \partial_{\nu} b_{\mu}$ is the field strength associated with the vector b_{μ}.

In order to obtain a static, spherically symmetric vacuum solution to the extended Einstein equations, we assume a spacetime driven by a Birkhoff metric $g_{\mu\nu} = \mathrm{diag}(-e^{2\gamma}, e^{2\rho}, r^2, r^2 \sin^2\theta)$, with γ and ρ being functions of r. Hereafter, we consider a spacelike background b_{μ} assuming the form

$$b_{\mu} = (0, b_r(r), 0, 0). \tag{22}$$

Moreover, once we have assumed a background field in the form (22), it follows that all components of the corresponding field strength vanish identically, i.e., $b_{\mu\nu} = 0$.

Now by using the condition $b^{\mu} b_{\mu} = b^2 = \mathrm{const}$, we determine the explicit form of the radial background field,

$$b_r(r) = |b| e^{\rho}. \tag{23}$$

It is easy to verify that the background given by Eq. (23) is not covariantly constant, i.e., we have some nonvanishing values for $\nabla_{\mu} b_{\nu}$. It is worthwhile to point out the difference with the proposal analyzed in Ref. [36], which assumes the condition $\nabla_{\mu} b_{\nu} = 0$.

Next, we proceed to solve for the functions $\gamma(r)$ and $\rho(r)$. For this, we take the extended Einstein equations in vacuum with our metric ansatz to get the following nonvanishing components for the tensor (20):

$$\bar{R}_{tt} = \left(1 + \frac{\ell}{2}\right) R_{tt} + \frac{\ell}{r} (\partial_r \gamma + \partial_r \rho) e^{2(\gamma - \rho)}, \tag{24}$$

$$\bar{R}_{rr} = \left(1 + \frac{3\ell}{2}\right) R_{rr}, \tag{25}$$

$$\bar{R}_{\theta\theta} = (1 + \ell) R_{\theta\theta} - \ell \left(\frac{1}{2} r^2 e^{-2\rho} R_{rr} + 1\right), \tag{26}$$

$$\bar{R}_{\phi\phi} = \bar{R}_{\theta\theta} \sin^2\theta, \tag{27}$$

where we have defined the Lorentz-violating parameter $\ell = \xi b^2$. The components of the Ricci tensor $R_{\mu\nu}$ appearing above are given by

$$R_{tt} = e^{2(\gamma - \rho)} \left[\partial_r^2 \gamma + (\partial_r \gamma)^2 - \partial_r \gamma \partial_r \rho + \frac{2}{r} \partial_r \gamma\right], \tag{28}$$

$$R_{rr} = -\partial_r^2 \gamma - (\partial_r \gamma)^2 + \partial_r \gamma \partial_r \rho + \frac{2}{r} \partial_r \rho, \tag{29}$$

$$R_{\theta\theta} = e^{-2\rho} [r(\partial_r \rho - \partial_r \gamma) - 1] + 1. \tag{30}$$

We note that the $\bar{R}_{\mu\nu}$ tensor also has only three diagonal independent components, the same as in the absence of the background field.

Each of the components given by Eqs. (24)–(27) must satisfy Eq. (19), which implies that every one of them is independently null. In this way, from Eq. (25) we have $\bar{R}_{rr} = 0$, leading to the condition

$$R_{rr} = 0. \tag{31}$$

At this point we note that the equation of motion of the bumblebee vacuum background b_{μ} established in Eq. (21) is identically satisfied by considering the parametrization (22), since it provides $b_{\mu\nu} = 0$, and together with Eq. (31) gives $b^{\mu} R_{\mu\nu} = 0$.

We now focus our attention on solving the modified vacuum Einstein equations. The condition (31) allows to obtain from Eq. (29) the equation

$$\partial_r^2 \gamma + (\partial_r \gamma)^2 - \partial_r \gamma \partial_r \rho = \frac{2}{r} \partial_r \rho. \tag{32}$$

By use of Eq. (32) it follows that we can rewrite Eq. (28) as

$$R_{tt} = \frac{2}{r} (\partial_r \rho + \partial_r \gamma) e^{2(\gamma - \rho)}, \tag{33}$$

and we use it together with Eq. (31) in Eqs. (24) and (26) to get

$$\bar{R}_{tt} = \frac{2(1 + \ell)}{r} (\partial_r \rho + \partial_r \gamma) e^{2(\gamma - \rho)}, \tag{34}$$

$$\bar{R}_{\theta\theta} = (1 + \ell) R_{\theta\theta} - \ell. \tag{35}$$

Next, we take the combination

$$r^2 e^{-2\gamma} \bar{R}_{tt} + 2 \bar{R}_{\theta\theta} = 0, \tag{36}$$

which yields a differential equation for the function $\rho(r)$,

$$(1+\ell)\partial_r(re^{-2\rho}) = 1. \tag{37}$$

It is easy to show that the solution is

$$e^{2\rho} = (1+\ell)\left(1-\frac{\rho_0}{r}\right)^{-1}, \tag{38}$$

where for now ρ_0 is some arbitrary constant.

In order to find the function $\gamma(r)$ we consider the combination

$$\ell r^2 e^{-2\gamma}\bar{R}_{tt} - (2+\ell)\bar{R}_{\theta\theta} = 0, \tag{39}$$

which provides

$$0 = (2+3\ell)(1+\ell)r\partial_r\gamma + (1+\ell)(2+\ell) + (\ell+1)(\ell-2)r\partial_r\rho - (2+\ell)e^{2\rho}. \tag{40}$$

By substituting Eq. (38), we obtain an explicit differential equation for $\gamma(r)$,

$$(\rho_0 - r)\partial_r\gamma + \frac{\rho_0}{2r} = 0,$$

whose solution, written in a convenient form, is given by

$$e^{2\gamma} = e^{-2\gamma_0}\left(1-\frac{\rho_0}{r}\right), \tag{41}$$

where $e^{-2\gamma_0}$ is a constant which can be removed by means of the rescaling $t\to e^{\gamma_0}t$. In fact, it can be verified that the solutions (38) and (41) actually satisfy the set of equations (24)–(27).

Finally, we write down the Lorentz-violating spherically symmetric solution

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + (1+\ell)\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{42}$$

where we have conveniently identified the arbitrary constant $\rho_0 \equiv 2M$ ($M = G_N m$ is the usual geometrical mass) such that in the limit $\ell\to 0$ we recover the usual Schwarzschild metric. The metric (42) represents a purely radial Lorentz-violating solution outside a spherical body characterizing a modified black hole solution. Furthermore, we compute the Kretschmann scalar

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{4(12M^2 + 4\ell M r + \ell^2 r^2)}{r^6(\ell+1)^2}, \tag{43}$$

which clearly differs from the Schwarzschild Kretschmann invariant for non-null ℓ. This ensures that the metric (42) is a true solution containing Lorentz-violating corrections, i.e., there exists no coordinate transformation connecting the metric (42) to the Schwarzschild one; otherwise, the scalar invariant (43) would be the same for both metrics. We observe, for $r = 2M$, that the Kretschmann invariant is finite so such a singularity can be removed (by an adequate coordinate transformation). However, $r = 0$ is a physical (or not removable) singularity due to the fact that the Kretschmann invariant is divergent. Therefore, we point out that the nature of the singularities $r = 0$ and $r = 2M$ (event horizon) remains unchanged.

Once the solution (42) has been obtained, we can look for observational signatures or constraints on the Lorentz-violating parameter ℓ by analyzing some fundamental tests of GR. For this purpose, we can pursue these Lorentz-violating corrections in a gravitational environment such as the Solar System. An alternative treatment could be done using the parametrized post-Newtonian (PPN) formalism [37] which allows for the comparison and testing of metric theories of gravity by using the experimental data when the weak-field limit is considered. Thus, we can investigate the possibility that this formalism allows for some sort of match between the weak-field limit of our solution and an equivalent post-Newtonian metric; the latter could withstand a partial overlap with the isotropic limit of the SME, as discussed in Ref. [30].

In the GR scenario an analysis of the post-Newtonian corrections to the gravitational field of a static spherical body can be established by taking the weak-field limit and expanding to post-Newtonian accuracy from the already known Schwarzschild solution. Following this idea, a common choice for a weak-field analysis is the isotropic coordinate system. Thus, the metric solution (42) expressed in isotropic coordinates assumes the form

$$ds^2 = \bar{g}_{00}(\bar{r})dt^2 + [f(\bar{r})]^2(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2), \tag{44}$$

where $\bar{r}$ is the isotropic radial coordinate defined by

$$\bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2}, \tag{45}$$

and

$$d\bar{r}^2 + \bar{r}^2 d\Omega^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2. \tag{46}$$

The metric component $\bar{g}_{00}(\bar{r})$ is given by

$$\bar{g}_{00}(\bar{r}) = -1 + \frac{2M}{r(\bar{r})}. \tag{47}$$

The radial coordinate transformation $r\to\bar{r}$ is ruled by the equation

$$r = \bar{r}\left(\frac{M}{2\bar{r}}\right)^{1-1/\sqrt{1+\ell}}\left[1+\left(\frac{M}{2\bar{r}}\right)^{1/\sqrt{1+\ell}}\right]^2, \tag{48}$$

where the integration constant has been considered in such a way that by setting $\ell = 0$ one recovers the result in the absence of Lorentz violation. The spatial metric function $f(\bar{r})$ is

$$f(\bar{r}) = \left(\frac{M}{2\bar{r}}\right)^{1-1/\sqrt{1+\ell}}\left[1+\left(\frac{M}{2\bar{r}}\right)^{1/\sqrt{1+\ell}}\right]^2, \quad (49)$$

and the metric components written in the isotropic form read

$$\bar{g}_{00}(\bar{r}) = -\frac{[1-(\frac{M}{2\bar{r}})^{1/\sqrt{1+\ell}}]^2}{[1+(\frac{M}{2\bar{r}})^{1/\sqrt{1+\ell}}]^2}, \quad (50)$$

$$\bar{g}_{ij}(\bar{r}) = [f(\bar{r})]^2\delta_{ij}, \qquad \bar{g}_{0i}(\bar{r}) = 0. \quad (51)$$

Immediately, the tensor field $s^{\mu\nu}$ defined in Eq. (8) can be expressed in isotropic coordinates. The nonzero components of the tensor $\bar{s}^{\mu\nu}$ are

$$\bar{s}^{00} = \frac{\ell}{4}\frac{[1+(\frac{M}{2\bar{r}})^{1/\sqrt{1+\ell}}]^2}{[1-(\frac{M}{2\bar{r}})^{1/\sqrt{1+\ell}}]^2}, \quad (52)$$

$$\bar{s}^{ij} = \frac{\ell}{[f(\bar{r})]^2}\left(\frac{\bar{x}^i\bar{x}^j}{\bar{r}^2} - \frac{1}{4}\delta^{ij}\right). \quad (53)$$

We now analyze the match between the Lorentz-violating parameter ℓ and the pure-gravity sector of the minimal SME. For this purpose, we consider the weak-field limit, $M/\bar{r} \ll 1$ and $\ell \ll 1$, and obtain

$$\langle\bar{s}^{00}\rangle_\Omega = \bar{s}^{00} = \frac{\ell}{4} + \cdots, \quad (54)$$

$$\langle\bar{s}^{11}\rangle_\Omega = \langle\bar{s}^{22}\rangle_\Omega = \langle\bar{s}^{33}\rangle_\Omega = \frac{\ell}{12} + \cdots, \quad (55)$$

$$\langle\bar{s}^{12}\rangle_\Omega = \langle\bar{s}^{13}\rangle_\Omega = \langle\bar{s}^{23}\rangle_\Omega = 0, \quad (56)$$

where $\langle\cdot\rangle_\Omega$ means the average over the sphere. It is important to emphasize that the tensor $\langle\bar{s}^{\mu\nu}\rangle_\Omega$ satisfies the traceless property and, besides that, these components restrict the SME coefficients to the isotropic limiting form, which is consistent with what was discussed in Ref. [30]. Therefore, it is reasonable to conclude that the estimates of the attainable experimental sensitivities for Lorentz violation associated with the nonzero components of $\langle\bar{s}^{\mu\nu}\rangle_\Omega$ will be of approximately the same order of magnitude as the parameter ℓ.

We attempt to find some matching with the PPN parameters by expanding each metric component in a series of $U = M/\bar{r} \ll 1$ and $\ell \ll 1$. So the metric expansion becomes

$$\bar{g}_{00}(\bar{r}) \simeq -1 + 2U - 2U^2 + \cdots - \ell[U - 2U^2 + \cdots]\ln\left(\frac{U}{2}\right) + \cdots, \quad (57)$$

$$\bar{g}_{ij}(\bar{r}) \simeq \delta_{ij}\left[1 + 2U + \frac{3}{2}U^2 + \cdots + \ell(1 + U + \cdots)\ln\left(\frac{U}{2}\right) + \cdots\right]. \quad (58)$$

This result reveals that the exact solution (42) does not admit a PPN expansion, i.e., it is not possible to expand only in powers of $U(\bar{r})$ due to the emergence of logarithmic contributions in the metric series. Hence, we conclude that the adoption of the parametrization (22) does not allow a post-Newtonian version of the Lorentz-violating metric (42). Therefore, this is not a promising avenue for searching for upper bounds on the Lorentz-violating parameter ℓ.

In the next section, we will use our exact solution in order to obtain estimates of the sensitivities to the Lorentz-violating coefficient ℓ from key GR tests by adopting a standard framework where the treatment of metric fluctuations about a Minkowski background is not required.

IV. SOME CLASSICAL TESTS AND UPPER-BOUND ESTIMATES

In this section, our focus is to identify dominant signatures of Lorentz violation from the motion of particles in a spacetime described by the Lorentz-violating spherically symmetric solution (42). Thus, with the aim of imposing upper bounds on the Lorentz-violating coefficient ℓ, we consider some classical key tests: the precession of the perihelia of inner planets, the bending of light, and the Shapiro time-delay effect.

The motion of test particles along the geodesics described by $x^\mu(\lambda)$ obeys the equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \quad (59)$$

where λ is an affine parameter. However, due to the metric compatibility, it is always possible to consider a constant of motion χ defined by

$$\chi = -g_{\mu\nu}\mathcal{U}^\mu\mathcal{U}^\nu, \quad (60)$$

where the vector $\mathcal{U}^\mu$ is defined as

$$\mathcal{U}^\mu = \frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu, \quad (61)$$

where a dot denotes differentiation with respect to the affine parameter. For massive particles, the affine parameter is typically chosen to be the proper time τ and $\chi = +1$ (timelike geodesics). On the other hand, for massless particles we have $\chi = 0$ and the parameter λ is not fixed (null geodesics).

A. Advance of perihelion

From the geodesic equation (59), we obtain the equations describing the trajectory of a massive test particle moving in the spacetime (42):

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0, \tag{62}$$

$$\ddot{r} + \frac{M(r-2M)}{r^3(\ell+1)}\dot{t}^2 - \frac{M}{r(r-2M)}\dot{r}^2 - \frac{r-2M}{\ell+1}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0, \tag{63}$$

$$\frac{d}{d\tau}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{64}$$

$$\frac{d}{d\tau}(r^2\sin^2\theta\dot{\phi}) = 0. \tag{65}$$

By considering the initial conditions $\theta(\tau_0) = \pi/2$ and $\dot{\theta}(\tau_0) = 0$, from Eq. (64), it follows that $\ddot{\theta}(\tau)$ and any other higher-order derivatives are equal to zero, so the particle motion is confined to the plane $\theta = \pi/2$. Therefore, we have a spherically symmetric spacetime with two Killing vectors corresponding to the conserved energy (E) and the conserved angular momentum (L). The timelike Killing vector $K^\mu = (\partial_t)^\mu$ is related to the conserved particle energy, given by

$$E = -g_{\mu\nu}K^\mu\mathcal{U}^\nu = \left(1-\frac{2M}{r}\right)\dot{t}. \tag{66}$$

The rotational Killing vector $\psi^\mu = (\partial_\phi)^\mu$ provides the conserved angular momentum of the particle,

$$L = g_{\mu\nu}\psi^\mu\mathcal{U}^\nu = r^2\dot{\phi}. \tag{67}$$

Clearly, Eqs. (66) and (67) are consistent with Eqs. (62) and (65), respectively.

Then, from the conserved quantities in Eq. (60) for timelike geodesics, this yields a single differential equation for the coordinate r in terms of the proper time τ,

$$(1+\ell)\dot{r}^2 + \left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+1\right) = E^2. \tag{68}$$

We now introduce the variable $u = r^{-1}$, such that

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = -L\frac{du}{d\phi}. \tag{69}$$

By substituting it into Eq. (68), we obtain

$$(1+\ell)\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2-1}{L^2} + \frac{2M}{L^2}u + 2Mu^3. \tag{70}$$

As is usually done in this treatment, it is preferable to solve the second-order equation which is obtained by differentiating the above equation with respect to ϕ, providing

$$(1+\ell)\frac{d^2u}{d\phi^2} + u - \frac{M}{L^2} - 3Mu^2 = 0. \tag{71}$$

It only presents Lorentz-violating contributions into the coefficient of the first term, maintaining the total structure of that obtained in the context of GR. In order to perturbatively solve Eq. (71) and due to the fact that we are assuming $\ell \ll 1$, it is still valid to consider the last term as a relativistic correction when compared with the Newtonian case. The perturbative solution is defined in terms of a small parameter $\epsilon = 3M^2/L^2$:

$$u \simeq u^{(0)} + \epsilon u^{(1)}. \tag{72}$$

The differential equation at zeroth order in ϵ yields

$$(1+\ell)\frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} - \frac{M}{L^2} = 0, \tag{73}$$

whose solution is given by

$$u^{(0)} = \frac{M}{L^2}\left[1 + e\cos\left(\frac{\phi}{\sqrt{1+\ell}}\right)\right]. \tag{74}$$

It is analogous to the Newtonian result. Here, the integration constants we have considered are the orbital eccentricity e (considered to be small as that of GR) and the initial value $\phi_0 = 0$.

The differential equation at first order in ϵ is

$$(1+\ell)\frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} - \frac{L^2}{M}(u^{(0)})^2 = 0, \tag{75}$$

which admits an approximate solution of the form

$$u^{(1)} \simeq \frac{M}{L^2}e\frac{\phi}{\sqrt{1+\ell}}\sin\left(\frac{\phi}{\sqrt{1+\ell}}\right) + \frac{M}{L^2}\left[\left(1+\frac{e^2}{2}\right) - \frac{e^2}{6}\cos\left(\frac{2\phi}{\sqrt{1+\ell}}\right)\right]. \tag{76}$$

For our purposes, the second term can be neglected once it consists of a constant displacement and a quantity that oscillates around zero.

TABLE I. Theoretical and observed data of perihelion shifts given in arcseconds per century ($''C^{-1}$).

Planet	GR prediction[a]	Observed	Uncertainty[b]
Mercury (☿)	42.981	42.979 ± 0.0030	-0.0020 ± 0.0030
Venus (♀)	8.6247	8.6273 ± 0.0016	-0.00026 ± 0.00016
Earth (⊕)	3.83877	3.83896 ± 0.00019	-0.000019 ± 0.000019
Mars (♂)	1.350938	1.350918 ± 0.000037	-0.000020 ± 0.000037
Jupiter (⊛)	0.0623	0.1210 ± 0.0283	0.0587 ± 0.0283
Saturn (⊘)	0.01370	0.01338 ± 0.00047	-0.00032 ± 0.00047
Icarus	10.1	9.8 ± 0.8	-0.3 ± 0.8

[a]Computed from the database of Refs. [40,43].
[b]From Refs. [39,42].

Therefore, the perturbative solution (72) reads

$$u \simeq \frac{M}{L^2}\left[1 + e\cos\left(\frac{\phi}{\sqrt{1+\ell}}\right) + \epsilon e \frac{\phi}{\sqrt{1+\ell}}\sin\left(\frac{\phi}{\sqrt{1+\ell}}\right)\right]. \tag{77}$$

Because $\epsilon \ll 1$, the perturbative solution (77) can be rewritten in the form of an ellipse equation,

$$u \simeq \frac{M}{L^2}\left[1 + e\cos\left(\frac{\phi(1-\epsilon)}{\sqrt{1+\ell}}\right)\right]. \tag{78}$$

Despite the presence of the Lorentz violation, the orbit remains periodic with period Φ,

$$\Phi = \frac{2\pi\sqrt{1+\ell}}{1-\epsilon} \approx 2\pi + \Delta\Phi. \tag{79}$$

The advance of perihelion ($\Delta\Phi$) is obtained by taking the lowest order in the ϵ and ℓ expansion, which reads

$$\Delta\Phi = 2\pi\epsilon + \pi\ell = \Delta\Phi_{\mathrm{GR}} + \delta\Phi_{\mathrm{LV}}. \tag{80}$$

The term $\Delta\Phi_{\mathrm{GR}}$ is the prediction of GR, given by

$$\Delta\Phi_{\mathrm{GR}} = 2\pi\epsilon = \frac{6\pi G_N m}{c^2(1-e^2)a}, \tag{81}$$

where c is the speed of light, m is the mass of the gravitational source, e is the orbital eccentricity, and a is the semimajor axis of the orbital ellipse. The term $\delta\Phi_{\mathrm{LV}}$ is the contribution per period due to spontaneous Lorentz violation,

$$\delta\Phi_{\mathrm{LV}} = \pi\ell. \tag{82}$$

Consequently, Eq. (80) shows the Lorentz-violating effects as an additional correction to the standard result of GR.

From perihelion-shift data of some planetary motions [37–39], we can establish estimates of the attainable sensitivities for the Lorentz-violating parameter ℓ by taking the uncertainty of experimental data obtained from the orbits of the inner planets. Table I provides some values of the perihelion advance—both observed and predicted by GR [40]—for some bodies in the Solar System, according to current measurements found in Ref. [39]. Note that, in addition to the planets, we also present the observational data for the asteroid Icarus, which agree with GR predictions to within 20% of the estimated uncertainty [41,42].

Considering, e.g., the motion of Mercury around the Sun we have an observational error yielding $0.003''C^{-1}$ (or 72.3×10^{-7} arcseconds per orbit); see Table I. Thus, one assumes that the contribution of the Lorentz-violating term $\delta\Phi_{\mathrm{LV}}$ is less than the observational error. Such a procedure allows us to estimate an upper bound at the level of $\ell_☿ < 1.1 \times 10^{-11}$. By applying the same procedure for the observational data of the other planets in Table I, we have achieved the set of estimates of the attainable sensitivities (upper bounds) for the Lorentz-violating parameter ℓ displayed in Table II. We observe that the most stringent upper bound attained from the advance of perihelion is at the level of 10^{-12}.

B. Bending of light

Unlike the previous case, we now have massless test particles whose trajectories correspond to null geodesics. Thus, $\chi = 0$ in Eq. (60), which after substituting the conserved quantities becomes

TABLE II. Estimates of upper bounds obtained from some observational uncertainties of perihelion shifts.

Lorentz-violating parameter	Upper bound	References
$\ell_♂$	1.1×10^{-12}	[39]
$\ell_⊕$	2.9×10^{-12}	[39]
$\ell_☿$	1.1×10^{-11}	[39]
$\ell_♀$	1.5×10^{-11}	[39]
$\ell_⊘$	2.1×10^{-10}	[39]
$\ell_⊛$	5.2×10^{-9}	[39]
ℓ_{Icarus}	1.3×10^{-8}	[42]

$$(1+\ell)\dot{r}^2 + \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2} = E^2, \tag{83}$$

where a dot now denotes differentiation with respect to some affine parameter.

Again, we consider $u = r^{-1}$ with $r \equiv r(\phi)$ and the differentiation with respect to ϕ in Eq. (83), which gives

$$(1+\ell)\frac{d^2u}{d\phi^2} + u - 3Mu^2 = 0. \tag{84}$$

We observe that in the limit $\ell \to 0$, Eq. (84) recovers the corresponding GR result providing the deflection of light rays, as expected. In analogy with the previous subsection, we use a perturbative method to achieve a solution by considering the quantity Mu as sufficiently small. Thus, we write the approximate solution in the form

$$u \simeq u^{(0)} + 3Mu^{(1)}. \tag{85}$$

Inserting this into Eq. (84) gives the following differential equation for $u^{(0)}$:

$$(1+\ell)\frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} = 0, \tag{86}$$

whose solution is

$$u^{(0)} = \frac{1}{d}\sin\left(\frac{\phi}{\sqrt{1+\ell}}\right), \tag{87}$$

where d is a constant of integration and we have considered the initial angle $\phi_0 = 0$, for convenience. This result corresponds to the equation of a straight line, which is analogous to the Newtonian prediction.

The differential equation for $u^{(1)}$, in turn, becomes

$$(1+\ell)\frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} - \frac{1}{d^2}\sin^2\left(\frac{\phi}{\sqrt{1+\ell}}\right) = 0, \tag{88}$$

and its solution is written as

$$u^{(1)} = \frac{1}{3d^2}\left[1 + A\cos\left(\frac{\phi}{\sqrt{1+\ell}}\right) + \cos^2\left(\frac{\phi}{\sqrt{1+\ell}}\right)\right]. \tag{89}$$

Hence, a general solution for $u(\phi)$ has the form

$$u \simeq \frac{1}{d}\sin\left(\frac{\phi}{\sqrt{1+\ell}}\right) + \frac{M}{d^2}\left[1 + A\cos\left(\frac{\phi}{\sqrt{1+\ell}}\right) + \cos^2\left(\frac{\phi}{\sqrt{1+\ell}}\right)\right], \tag{90}$$

where A is an arbitrary constant.

Since we are interested in determining the angle of deflection of a light ray, the boundary conditions are determined by assuming that (i) the source is located at $r \to \infty$ such that $u(r \to \infty) \to 0$ and $\phi = -\delta_1$, and (ii) the observer is localized at $r \to \infty$ such that $u(r \to \infty) \to 0$ and $\phi = +\delta_2$, so the total angle of deflection is given by $\delta = \delta_1 + \delta_2$. By using these boundary conditions in Eq. (90) and taking into consideration $\ell \ll 1$ and δ_1, $\delta_2 \ll 1$, the first-order equation gives

$$\delta_1 = \frac{M}{d}(2 + A), \tag{91}$$

$$\delta_2 = \frac{M}{d}(2 - A) + \frac{\pi\ell}{2}. \tag{92}$$

Hence, the light-ray deflection angle in the metric (42) is

$$\delta = \delta_{\rm GR} + \delta_{\rm LV} = \frac{4G_N m}{c^2 d} + \frac{\pi\ell}{2}, \tag{93}$$

where m is the mass of the deflecting body and d is the so-called impact parameter (defined as the distance of closest approach of the light ray to the center of mass of the deflecting body). The first term $\delta_{\rm GR}$ gives the usual deviation of light predicted by GR,

$$\delta_{\rm GR} = \frac{4G_N m}{c^2 d}. \tag{94}$$

The second term $\delta_{\rm LV}$,

$$\delta_{\rm LV} = \frac{\pi\ell}{2}, \tag{95}$$

is the correction coming from the Lorentz-violating effects. Of course, by taking the limit $\ell \to 0$ in Eq. (93) we recover the usual result established by GR for the bending of light.

For a ray grazing the Sun we have $m = M_\odot$ and $d \approx R_\odot$. Using, e.g., the values from Ref. [43], one can verify that GR predicts an angle given by $\delta_{\rm GR} = 4G_N M_\odot / c^2 R_\odot \approx 1.75166872''$. Therefore, if there is indeed Lorentz violation in nature, the effects arising from the term $\delta_{\rm LV}$ must be smaller than the observational errors for the bending of light. The error bars obtained in recent measurements of existing and future light-bending tests [44–52] allow us to provide an interesting sensitivity for Lorentz violation in the gravity sector.

The analysis of available data from recent observations of very-long-baseline radio interferometry (VLBI) [49] for light deflection yielded an agreement with GR to 0.01%. Taking this accuracy, the upper bound will satisfy the inequality $\delta{\rm LV} < 0.0001051''$, which allows to establish the constraint $\ell < 3.2 \times 10^{-10}$. In addition, detailed simulations for future missions, e.g., the proposed Laser Astrometric Test of Relativity (LATOR) [47], can also lead to interesting estimates of the sensitivities for Lorentz

TABLE III. Estimates of sensitivities in current and future light-bending tests for the Lorentz-violating parameter ℓ.

Experiment	Upper bound	References
LATOR	7.3×10^{-15}	[47]
GAIA	8.1×10^{-13}	[48]
VLBI	3.2×10^{-10}	[49]
Hipparcos	8.1×10^{-9}	[50]
Optical	5.9×10^{-7}	[51]

violation; namely, LATOR may achieve sensitivities to coefficients for Lorentz violation at the level of 10^{-15}.

In Table III we provide upper-bound estimates for available data from past and future missions in addition to those described above. We have included the GAIA astrometric mission [44,48] which provides an accuracy to GR of $10^{-5}\%$ of unity, the optical astrometry satellite mission Hipparcos [50] with measurements reaching 0.1% of uncertainty, and past ground-based optical observations [51] with an error bar of 11%. All values computed correspond to the Lorentz-violating parameter ℓ and are based on the accuracy of each experiment referenced. Note that, if future experiments can be used for these measurements, the peak constraints of ℓ could lead us to reach the 10^{-15} level, providing even more sensitive Lorentz-violating parameters than previous tests.

C. Time delay of light

A further measurable relativistic phenomenon involving light rays is the Shapiro time-delay effect [53]. The Solar System tests involving this effect can yield interesting sensitivities to Lorentz violation. For this purpose we will derive an expression involving the Lorentz-violating corrections for the time-delay effect from the result already obtained in Sec. IV B. Namely, we are interested in an equation providing the change in the round-trip travel time of light to an object due to the influence of a massive body such as the Sun.

By considering the motion of light in the equatorial plane ($\theta = \pi/2$) and because it travels along a null geodesic in the spacetime (42), i.e., the condition $ds^2 = 0$ is satisfied, we can write

$$-\left(1-\frac{2M}{r}\right)dt^2 + (1+\ell)\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\phi^2 = 0. \tag{96}$$

Next, we consider the zeroth-order solution (87) characterizing the straight-line approximation,

$$r\sin\left(\frac{\phi}{\sqrt{1+\ell}}\right) = d, \tag{97}$$

and we use it to establish the following relation:

$$r^2 d\phi^2 = (1+\ell)\left(\frac{d^2}{r^2-d^2}\right)dr^2. \tag{98}$$

Thus, Eq. (96) can be rewritten as

$$dt^2 = \frac{1+\ell}{1-2M/r}\left(\frac{1}{1-2M/r} + \frac{d^2}{r^2-d^2}\right)dr^2. \tag{99}$$

Expanding it in terms of M/r and considering the contributions at first order, we get

$$dt \simeq \pm\frac{\sqrt{1+\ell}}{\sqrt{r^2-d^2}}\left(1+\frac{2M}{r}-\frac{Md^2}{r^3}\right)rdr. \tag{100}$$

The setup for the Shapiro delay effect involves two stations at large distances from the massive source (or curvature source). By assuming a light ray (or radar signal) from an emitter located at r_{E} traveling to a receiver at r_{R}, the travel time is given by

$$t = \sqrt{1+\ell}\Bigg\{t_0 + 2M\ln\left(\frac{r_{\mathrm{E}} + (r_{\mathrm{E}}^2-d^2)^{1/2}}{d}\right) + 2M\ln\left(\frac{r_{\mathrm{R}} + (r_{\mathrm{R}}^2-d^2)^{1/2}}{d}\right) - M\left(\frac{(r_{\mathrm{R}}^2-d^2)^{1/2}}{r_{\mathrm{R}}} + \frac{(r_{\mathrm{E}}^2-d^2)^{1/2}}{r_{\mathrm{E}}}\right)\Bigg\}, \tag{101}$$

where t_0 represents the travel time in flat spacetime,

$$t_0 = (r_{\mathrm{R}}^2-d^2)^{1/2} + (r_{\mathrm{E}}^2-d^2)^{1/2}. \tag{102}$$

It should be noted that in the absence of the Lorentz violation, $\ell = 0$, Eq. (101) recovers the time-delay expression predicted by GR, as expected. It is evident that the first term $t_0\sqrt{1+\ell}$ stands for the travel time of a radar signal along a straight line including Lorentz-violating corrections in a flat spacetime (special relativity). The other terms represent time-delay contributions due to both the curved spacetime and Lorentz violation. Such a delay effect may be interpreted as an effective increase in the distance between the emitter and receiver of the radar signal.

With the aim of using the result (101), we explore the Solar System by considering the round-trip of a light signal, under the influence of a gravitational field (e.g., that of the Sun), emitted from a source (e.g., the Earth) towards a reflective body (e.g., a planet or a spacecraft) and returning to the source. So, we take the spacetime near the Sun ($M = M_{\odot}$) and a radar signal emitted from the Earth located at r_{E} traveling towards a receiver (a planet or a spacecraft) at r_{R}; both distances are measured respectively from the Sun's center. The time-delay effect is maximum when the spacecraft is at superior conjunction and the radar signal just grazes the Sun's surface such that the radius of closest approach is $d \propto R_{\odot}$, satisfying the condition $d \ll r_{\rm E}, r_{\rm R}$. Therefore, from Eq. (101) the total round-trip time for a radar signal traveling from the emission source to another planet (or spacecraft) and returning is approximately given by

$$T \approx \sqrt{1+\ell}\left\{T_0 + \frac{4G_N M_{\odot}}{c^3}\left[\ln\left(\frac{4r_{\rm E}r_{\rm R}}{d^2}\right) - 1\right]\right\}, \quad (103)$$

where d stands for the impact parameter and T_0 is the total travel time in flat spacetime,

$$T_0 = \frac{2}{c}(r_{\rm R}^2 - d^2)^{1/2} + \frac{2}{c}(r_{\rm E}^2 - d^2)^{1/2}. \quad (104)$$

Similarly to what is done in GR, from Eq. (103) we define, in this Lorentz-violating framework, the total excess delay as

$$\delta T = T - T_0 = \delta T_{\rm GR} + \delta T_{\rm LV}, \quad (105)$$

where we have considered only first-order terms in $\ell \ll 1$. The quantity $\delta T_{\rm GR}$ representing the excess delay due to pure GR is given by

$$\delta T_{\rm GR} = \frac{4G_N M_{\odot}}{c^3}\left[\ln\left(\frac{4r_{\rm E}r_{\rm R}}{d^2}\right) - 1\right]. \quad (106)$$

The term $\delta T_{\rm LV}$ representing the Lorentz-violating contribution to the excess delay reads

$$\delta T_{\rm LV} = \frac{\ell}{2}(\delta T_{\rm GR} + T_0) \approx \frac{\ell}{2} T_0. \quad (107)$$

This will be used to obtain estimates of the sensitivities to Lorentz violation which could be achieved from the passive radar measurements of the inner planets or active ranging experiments of interplanetary spacecrafts. For example, the measurements of radar signals reflected by Venus have provided an excess-delay predicted by GR within the experimental uncertainty of 20% [54] and, subsequently, within 2% [55]. We take this latter as an upper bound for Lorentz-violating effects which would correspond to a sensitivity of $\ell_{♀} < 5.0 \times 10^{-9}$.

Time-delay effects have also been measured using artificial satellites as active retransmitters of the radar signals, such as the Mariner 6 and 7 spacecrafts in orbit around the Sun. An analysis of the Mariner 6 (M6) and 7 (M7) data suggest that a realistic estimate of the total uncertainty, for both cases, is perhaps less than 3% [56], so that the estimates of the sensitivities for the Lorentz violation parameters $\ell_{\rm M6}$ and $\ell_{\rm M7}$ are 2.2×10^{-9} and 1.6×10^{-9}, respectively.

Another major advance was made using an active transmitter on a spacecraft stationed on a planet. An example can be given by experiments conducted during the mission of the Viking spacecraft to Mars. This consisted of space probes that orbited Mars, equipped with a lander to study the planet at its surface. The measurement from the Viking Mars (VM) landers resulted in an estimated accuracy of 0.1% [57], which allows us to establish a sensitivity of $\ell_{\rm VM} < 1.8 \times 10^{-10}$.

TABLE IV. Estimates of attainable sensitivities in some gravitational time-delay tests for the Lorentz-violating parameter ℓ.

Experiment	Upper bound	References
BEACON	3.0×10^{-19}	[60]
ASTROD	3.8×10^{-15}	[59]
Cassini	5.9×10^{-13}	[58]
Viking Mars	3.2×10^{-10}	[57]
Mariner 6	2.2×10^{-9}	[56]
Mariner 7	1.6×10^{-9}	[56]
Venus	5.0×10^{-9}	[55]

The most precise measurement of the Shapiro time delay from spacecraft measurements so far was made by the Cassini mission during its trip to Saturn [58]. Performing a detailed analysis of the data obtained in the 2002 superior conjunction of Cassini, it is verified that the resulting measurement error must be within at most 0.0012% of unity. From this value, we obtain an attainable sensitivity of $\ell_{\rm Cassini} < 5.9 \times 10^{-13}$.

We now consider some future key experiments which can also provide reasonable estimates of the upper bounds for the parameter ℓ of this current treatment. Some proposed methods of measuring the time delay of light at high accuracies are the missions Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) and Beyond Einstein Advanced Coherent Optical Network (BEACON), which will allow for the search for new physics beyond general relativity by measuring the curvature of relativistic spacetime. The ASTROD mission expects an accuracy at the level of 10^{-8} [59], while the BEACON mission expects an accuracy at the 10^{-9} level [60], in agreement with GR. We should point out that the BEACON mission is an attempt to measure the spacetime warping due to Earth's gravitational field.

The values of the upper bounds for the Lorentz-violating parameter ℓ, concerning all experiments related to the time delay of light discussed here, are listed in Table IV. There are additional experiments whose data could be of interest to perform a similar analysis [60–64].

V. CONCLUSIONS AND REMARKS

We have achieved a new static and spherically symmetric vacuum solution by investigating a Lorentz-violating gravity contained in the framework of a Riemannian bumblebee gravity model. Such a new vacuum solution looks like the Schwarzschild one; however, its Kretschmann invariant (43) guarantees that they are very different.

TABLE V. Summary of the estimates of constraints in some key classical tests for Lorentz-violating parameter ℓ.

Classical test	Constraint
Advance of perihelion	10^{-8}–10^{-12}
Bending of light	10^{-7}–10^{-15}
Time delay of light	10^{-9}–10^{-19}

We have studied its implications by exploring three classical gravitational tests: the advance of perihelion, the bending of light, and Shapiro's time delay. We have noted that the tests have corrections coming from Lorentz violation even in the absence of a massive gravitational source, which is compatible with a Kretschmann invariant (43) that is nonvanishing in the limit $M \to 0$. This result indicates that the Lorentz-violating background also deforms spacetime; nevertheless, it remains asymptotically flat due to the fact that the deformation is very small.

We have analyzed the match between the Lorentz-violating parameter ℓ and the pure-gravity sector of the minimal SME. For this purpose, we have considered the weak-field limit in the isotropic coordinates. It has been verified that the nonzero components of the tensor $\langle \bar{s}^{\mu\nu} \rangle_\Omega$ represent an isotropic form of the gravity sector of the SME similar to the one discussed in Ref. [30]. In this way, we conclude that the estimates of the attainable experimental sensitivities for Lorentz violation associated with the nonzero components of $\langle \bar{s}^{\mu\nu} \rangle_\Omega$ have approximately the same order of magnitude as the parameter ℓ. On the other hand, we have verified that the weak-field limit of our solution (42) does not admit a PPN version due to the emergence of logarithmic contributions in the metric series.

In all of the classical tests considered, the effect induced by Lorentz violation can be interpreted as a correction to GR's result. The additional terms carrying Lorentz-violating signals are given explicitly by Eqs. (82), (95), and (107), and they clearly vanish in the limit $\ell \to 0$. The Lorentz-violating contribution obtained in each classical test together with the accuracy of the corresponding experimental data allow to estimate attainable sensitivities for the parameter ℓ (see Table V).

Among the perihelion-shift data of the inner planets of the Solar System that we analyzed, the high experimental accuracy for Mars and Earth have provided the most stringent upper bound, reaching the 10^{-12} level. Furthermore, for light-bending tests we have used data from key existing and future experiments, which have constrained the Lorentz-violating parameter ℓ at the 10^{-7}–10^{-15} level. The relativistic effect involving the Shapiro time delay also yields estimates of the sensitivities that might be attainable for the parameter ℓ. The Cassini spacecraft has provided the most accurate measurement to date, yielding an upper bound of $\ell < 10^{-13}$, while the future experiment with BEACON mission predicts an upper bound with an order of magnitude of about 10^{-19}. The upper bounds for the Lorentz-violating parameter ℓ achieved by means of the three tests are summarized in Table V.

We are currently exploring other possible vacuum configurations of the bumblebee field that produce new Lorentz-violating solutions. One of them is exploring the effect of the bumblebee field on some black hole solutions, such as the charged and rotating ones. The results of these studies will be reported elsewhere.

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