# Tricritical behavior in the Chern-Simons-Ginzburg-Landau theory of self-dual Josephson junction arrays

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We examine the phase structure of a $U(1) \times U(1)$-symmetric model in three dimensional space-time with a potential of sixth order in the scalar fields coupled to dynamic gauge fields with a mixed Chern-Simons term. Using a large $N$ technique, we compute the quantum effective potential and the renormalization group functions of the various couplings to the next-to-leading order of the $1/N$ expansion in terms of the mixed Chern-Simons coefficient. The model has a phase which exhibits spontaneous breaking of scale symmetry accompanied by a massless dilaton which is a Goldstone mode. We explicitly show that the various sextic couplings beta functions vanish to leading order in the $1/N$ expansion; however, at the next order in the $1/N$ expansion, they exhibit nontrivial running that we analyze explicitly in terms of the mixed Chern–Simons coefficient. We demonstrate that the gauge fluctuations split the zeros of the beta functions and generate a variety of nontrivial fixed points. The stability of these fixed points and the requirement of the positivity of the renormalization group improved effective potential are also examined. Our study identifies a window in the parameter space of the mixed Chern-Simons coefficient where the renormalization group flow has a stable infrared fixed point where scale invariance is recovered.

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## I. INTRODUCTION

Quantum criticality describes the collective fluctuations associated with a second order phase transition at zero temperature [1–3]. It is compelled by competing interactions that promote different ground states separated by a quantum critical point (QCP) at which the system's degrees of freedom show anomalously large fluctuations on a long-wavelength scale compared with those of a normal system. Handling such large fluctuations is challenging since standard theoretical schemes that assume small long-wavelength fluctuations cease to be effective; consequently, one resorts to novel methods such as the renormalization group formalism [4] combined with an expansion around the system's critical dimension or a large N technique [5]. These methods were successfully applied to problems involving self-interacting scalar fields, which are the simplest nontrivial field theories used as phenomenological models for a variety of phenomena, notably systems exhibiting tricritical points such as He-He mixtures or magnetic materials with competing antiferromagnetic and ferromagnetic interactions [6].

This paper explores instabilities of QCPs induced by the interplay between fluctuating scalar fields and gauge fields in the context of Josephson junction arrays (JJAs). In these systems the insulator–superconductor quantum transition arises as a result of the competition between the charging energy, which localizes Cooper pairs, and the Josephson coupling, which tends to delocalize Cooper pairs [7]. JJAs have been studied intensively because of their rich phase diagram, frustration effects, and vortex dynamics; as a result, they were used as generic models that capture essential features of superconducting granular systems and inhomogeneous superconducting films. In particular, a special self-dual Josephson junction array system (SDJJA) was introduced in [8] and modeled in [9] by a topological $U(1) \times U(1)$ two-field Ginzburg–Landau theory. The appeal of such a formulation is that it describes the insulator–superconductor quantum transition as arising from the condensation of bosonic disorder fields. Furthermore, in this theory the conserved currents of Cooper pairs and vortices are represented by dynamic fictitious gauge fields whose Maxwell terms encode kinetic terms for charges and vortices, as well as a mixed Chern–Simons term that describes the Lorentz force exerted by the vortices on the charges and the Magnus force exerted by the charges on the vortices [9]. Along with these emerging gauge fields, the theory includes complex scalar fields to account for quantum disorder due to electric and magnetic excitations in the system. The resulting low energy effective model is a Ginzburg–Landau theory with two interacting disorder fields coupled to a mixed-Chern–Simons gauge theory [9]. The competition between these disordering fields leads to a rich phase diagram that exhibits phases such as the emergence of a superconducting phase which occurs when the charge disorder field condenses while the magnetic disorder field resides in a vacuum. In addition the insulating phase occurs when magnetic charges condense while the charge disorder field is in a vacuum. The model also accounts for the emergence of a Hall phase which occurs when bound objects made up of electric and magnetic charges condense [9].

The quantum gauge fluctuations add features in the critical region that are well worth studying. For instance, Coleman-Weinberg model [10] revealed that fluctuating gauge fields can qualitatively change the nature of the transition and lead to spontaneous mass generation in an initially massless theory. In other words a first order transition is induced when the mass of the scalar field passes through zero. Similarly, the study of Halperin *et al.* [11] established that the electromagnetic field fluctuations induced a first order transition from a superconducting to a metallic phase, although a mean-field model predicted a second-order phase transition. Specifically, the phase transition remained of second order only when the number of components $N$ of the order parameter was large; instead, no infrared stable charged fixed point was accessible in the physical case ($N = 1$). This fluctuation-driven change of the order of the phase transition was debated intensively; for example, the Monte Carlo simulations of [12] suggested a second order phase transition, while the investigation of [13] indicated the existence of a tricritical point in the phase diagram separating first and second-order regimes.

On the other hand, Chern-Simons gauge interactions bring about other peculiar properties. In contrast to a Maxwell term, a Chern-Simons interaction is intimately connected with odd dimensional space and cannot be universally continued to other dimensions. Chern-Simons theory proved to be an important theoretical framework to study many condensed matter phenomena in three space-time dimensions; for example, it was featured in the phase transitions between quantum Hall liquids and insulators [14], where a large $N$ analysis revealed that the Chern-Simons gauge field is a marginal perturbation to the scalar field fixed point, giving rise to critical exponents that depend on the coefficient of the Chern-Simons term. By contrast, the analysis of [15] in the physically relevant case $N = 1$ showed that the phenomenon of statistical angle-dependent critical exponents did not occur and suggested the absence of a critical point, which implied a first order transition driven by the fluctuations of the gauge field.

Our study in [9] focused on the critical behavior of a Ginzburg-Landau theory containing up to the fourth-order power of the fields $(\Phi * \Phi)^2$ coupled to gauge fields described by Maxwell terms and a Mixed-Chern-Simons term at fixed dimension $d = 3$. Two different approaches were used: a one-loop renormalization group calculation at fixed dimension as popularized by Parisi in [16], and a more systematic investigation in the framework of a $1/N$ expansion as applied to critical phenomena by Ma [17]. The latter technique provided a convenient expansion parameter $(1/N)$ that allowed the summation of an infinite class of Feynman graphs leading to non-perturbative results. The model turned out to be renormalizable in the $1/N$ expansion, and its $\beta$-functions, infrared fixed points, and its critical exponents were obtained in a systematic way [9]. Investigating these infrared fixed points was essential because they naturally controlled the long distance and low energy behavior of the theory, resulting in scale-invariant correlation functions solely characterized by universal critical exponents. However, when scale invariance at an interacting fixed point is broken spontaneously, a mass appears with a scale not determined by the fundamental parameters of the theory. This phenomenon was demonstrated by Bardeen *et al.* [18,19] in an ungauged $O(N)$ symmetric scalar field theory $(\Phi * \Phi)^3$ in the limit of infinitely many fields $N$. An interacting $UV$ fixed point was also analyzed in [20] using perturbative renormalization group equations. Other aspects of that phenomenon including $1/N$ corrections were analyzed in [21–28]. Recently, the large $N$ limit of this model was the focus of a study of the exact behavior of conformal theories in higher than two dimensions. Models with Chern-Simons gauge fields [29], and the fate of light dilaton under $1/N$ corrections [30] were also investigated.

Motivated by these issues, we propose in this paper an extension of model [9] that incorporates self-interactions terms up to sixth order in the scalar fields coupled to dynamic gauge fields with a mixed Chern-Simons term. The sextic terms are required to maintain stability of the effective Ginzburg-Landau potential when the fourth order terms change sign. The purpose is to address the tricritical behavior driven by quantum fluctuations of competing scalar and dynamic gauge fields. The critical behavior will be analyzed with a controlled large $N$ technique combined with a renormalization group procedure. We develop a formalism that allows the computation of the quantum effective potential from which we obtain information about the phase diagram and the beta functions of the various coupling at the next order in the $1/N$ expansion. This paper is organized as follows. In Sec. II, the model is introduced. In Sec. III, the effective potential is derived to leading order of the large $N$ technique. In Sec. IV, the corrections to the effective potential arising from the scalar fields and the fluctuating gauge fields are computed and used to derive the various renormalization group flow beta functions. In Sec. V, the fixed points of the model are examined analytically and numerically, and a window in the parameter space of the mixed Chern-Simons coefficient is identified where the renormalization group flow has a stable infrared fixed point. Finally Sec. VI summarizes the results and discusses their implications.

## II. THE CONTINUUM MODEL

The low energy effective field theory describing the dynamics of Cooper pairs and vortices in a self-dual Josephson junction array consists of two complex fields associated with disordering due to electric charges ($\Psi$) and magnetic charges ($\Phi$) interacting through fictitious gauge fields $a_\mu$ and $b_\mu$ whose dual field strengths represent the currents of Cooper pairs and vortices in the underlying microscopic model. The derivation was presented in [9], and the Euclidean Lagrangian including here sextic terms in the scalar fields is given by

$$L = |(\partial_\mu - ia_\mu)\Phi|^2 + |(\partial_\mu - ib_\mu)\Psi|^2 + i\kappa N b_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + U(|\Psi|^2, |\Phi|^2) \tag{1a}$$

$$\begin{aligned} U(|\Psi|^2,|\Phi|^2) &= r_1|\Phi|^2 + r_2|\Psi|^2 + \frac{1}{2N}(\lambda_1|\Phi|^4 + \lambda_2|\Psi|^4 + 2\lambda_3|\Psi|^2|\Phi|^2) \\ &\quad + \frac{1}{6N^2}(g_1|\Phi|^6 + g_2|\Psi|^6 + 3g_3|\Psi|^2|\Phi|^4 + 3g_4|\Psi|^4|\Phi|^2) \end{aligned} \tag{1b}$$

Because of quantum fluctuations and RG iterations we start with the most general Landau-Ginzburg-Wilson Lagrangian that is symmetric under $U(1)\times U(1)$ transformations and containing up to sextic interaction terms described by the couplings $\lambda_i$ and $g_i$. In order to facilitate the $1/N$ expansion, this theory is enlarged to an $O(N)\times O(N)$ symmetric model and all couplings are rescaled to produce a meaningful $N\to\infty$ limit. The model at hand is renormalizable in $(2+1)$ dimensions by the standard power counting procedure and possesses the usual UV divergences.

## III. EFFECTIVE POTENTIAL AT THE LEADING ORDER

To get the effective potential, we examine the fluctuations in the Euclidean functional integral

$$e^{W(J,K)} = \int D\Phi D\Phi^\dagger D\Psi D\Psi^\dagger \exp\left[-\int d^3x(L - J^\dagger\cdot\Phi - J\cdot\Phi^\dagger - K^\dagger\cdot\Psi - K\cdot\Psi^\dagger)\right], \tag{2a}$$

where two sources $J$ and $K$ are introduced in order to use the functional integral as a generating functional for the correlators of $\Phi$ and $\Psi$. For instance the two-point connected correlation functions are

$$\left.\frac{\delta^2 W}{\delta J^\dagger(x_1)\delta J(x_2)}\right|_{J=K=0} = \langle\Phi(x_1)\Phi^\dagger(x_2)\rangle \tag{2b}$$

$$\left.\frac{\delta^2 W}{\delta K^\dagger(x_1)\delta K(x_2)}\right|_{J=K=0} = \langle\Psi(x_1)\Psi^\dagger(x_2)\rangle \tag{2c}$$

The generating functional of the connected one-particle irreducible correlation functions is obtained by performing a Legendre transformation

$$\Gamma(\Phi,\Psi) = -W(J,K) + \int d^3x(J^\dagger\cdot\Phi + J\cdot\Phi^\dagger + K^\dagger\cdot\Psi + K\cdot\Psi^\dagger), \tag{3a}$$

$$\Gamma^{(n)}(x_1,x_2,\ldots,x_n) = \frac{\delta^n\Gamma}{\delta\varphi(x_1)\delta\varphi(x_2)\ldots\delta\varphi(x_n)}, \tag{3b}$$

and with $\varphi$ representing $\Phi$ or $\Psi$ and $\Phi=\delta W/\delta J^\dagger$, $\Psi=\delta W/\delta K^\dagger$. The effective potential is obtained from the action for x-independent $\Phi$ and $\Psi$,

$$\Gamma(\Phi,\Psi) = (2\pi)^3\delta^3(p=0)V_{\rm eff}(\Phi,\Psi). \tag{3c}$$

In order to facilitate a systematic $1/N$ expansion we introduce a pair of two new auxiliary fields $\sigma_i$ and $\chi_i$ with $i=1$, 2 and rewrite the scalar potential in Eq. (1b) as

$$\begin{aligned} U &= \sigma_1(|\Phi|^2 - N\chi_1) + \sigma_2(|\Psi|^2 - N\chi_2) \\ &\quad + N\sum_{i=1,2} r_i\chi_i + \frac{\lambda_i}{2}\chi_i^2 + \frac{g_i}{6}\chi_i^3 \\ &\quad + \lambda_3\chi_1\chi_2 + \frac{g_3}{2}\chi_1^2\chi_2 + \frac{g_4}{2}\chi_1\chi_2^2. \end{aligned} \tag{4}$$

The functional integrals over $\sigma_i$ from $-i\infty$ to $i\infty$ give delta functions which render the physical content of Eqs. (4) and (1) identical. The leading order (in 1/N) effective potential $V_{\rm eff}(\sigma_i,\chi_i)$ is obtained by integrating out the $\Phi$ and $\Psi$ which appear in (4) in a quadratic form

$$\begin{aligned} V_{\rm eff}(\sigma_1,\sigma_2,\chi_1,\chi_2) &= N\sum_{i=1,2}\left[-\sigma_i\chi_i + r_i\chi_i + \frac{\lambda_i}{2}\chi_i^2 + \frac{g_i}{6}\chi_i^3 + \int_p \ln(p^2+\sigma_i)\right] \\ &\quad + \lambda_3\chi_1\chi_2 + \frac{g_3}{2}\chi_1^2\chi_2 + \frac{g_4}{2}\chi_1\chi_2^2 \end{aligned} \tag{5}$$

here we adopt the convention that the Fourier integral $\int_p \equiv \int d^3p/(2\pi)^3$, which is used throughout this paper. We study the region of phase diagram where the $O(N)\times O(N)$ is unbroken, but when the system is in a phase with spontaneously broken scale invariance characterized by nonzero condensates for $\chi_i$ and $\sigma_i$. In the symmetric phase with zero expectation values of the fields $\Phi$ and $\Psi$, the vacuum structure is characterized by the gap equations:

$$\frac{\partial V}{\partial\sigma_i} = N\int_p \frac{1}{p^2+\sigma_i} - N\chi_i = 0, \qquad i=1,2 \tag{6a}$$

whose solutions are

$$\varphi_i = \frac{\sqrt{\sigma_i}}{4\pi} = \frac{m_i}{4\pi}, \qquad i = 1, 2 \tag{6b}$$

where $m_i$, $i = 1$, 2 assume the role of masses and the momentum integral has been cutoff at $p = \Lambda \gg m$. The renormalized fields are defined by $\varphi_i = -\chi_i + \Lambda/(2\pi^2)$, with $i = 1, 2$. Eliminating the unphysical fields $\sigma_i$ using the above gap equations, we obtain the leading order effective potential:

$$V^{(0)}_{\rm eff}(\varphi_1, \varphi_2)/N = \left(\frac{16\pi^2}{3} - \frac{g_1}{6}\right)\varphi_1^3 + \left(\frac{16\pi^2}{3} - \frac{g_2}{6}\right)\varphi_2^3 - \frac{g_3}{2}\varphi_1^2\varphi_2 - \frac{g_4}{2}\varphi_1\varphi_2^2 + \frac{\lambda_1}{2}\varphi_1^2 + \frac{\lambda_2}{2}\varphi_2^2 + \lambda_3\varphi_1\varphi_2 - r_1\varphi_1 - r_2\varphi_2. \tag{7}$$

In Eq. (7), the parameters $r_i$, $\lambda_i$ have been renormalized in order to make the effective potential cutoff independent. At the leading order (in $1/N$), the couplings $g_i$ remain unrenormalized. Furthermore, for the theory to make sense, we require that the effective potential to be bounded from below. In the special case corresponding to the behavior in the vicinity of the tricritical point ($r_1 = r_2 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$), nonzero masses can be generated provided the couplings satisfy the relation

$$4(g_3^2 + 2h_1g_4)(g_4^2 + 2h_2g_3) = (g_3g_4 - 4h_1h_2)^2, \tag{8a}$$

where $h_1 = 16\pi^2 - g_1/2$, $h_2 = 16\pi^2 - g_2/2$ and the masses $m_1$ and $m_2$ are related by

$$[4h_1h_2 - g_3g_4]m_1 = 2[2h_2g_3 + g_4^2]m_2 \tag{8b}$$

The physics described by Eq. (7) at the tricritical point has been studied in [31] in the limit $N = \infty$ in connection with the spontaneous breaking of scale invariance due to the nonperturbative mass generation. This was shown to occur on a compact critical surface in the coupling space defined by the couplings $g_i$, and it manifests itself in various phases, in some of which it is a consequence of internal-symmetry breaking. In the next section we examine the $1/N$ corrections to the effective potential which will introduce new Logarithmic divergences, and we show how to handle such divergences with the renormalization group equations.

## IV. EFFECTIVE POTENTIAL AT NEXT-TO-LEADING ORDER

### A. Scalar field contribution

To find the effective action to the next-to-leading order in large $N$, we expand the action to quadratic order in the shifted fields $\delta\sigma_{1,2}$, defined as $\sigma_\alpha = m_\alpha^2 + i\delta\sigma_\alpha$ with $\alpha = 1$, 2. Differentiating twice the action with respect to these fields one obtains:

$$\frac{\delta^2 V}{\delta\sigma_1\delta\sigma_1} = \frac{\delta^2 V}{\delta\sigma_2\delta\sigma_2} = 0, \tag{9a}$$

$$\frac{\delta^2 V}{\delta\sigma_1\delta\chi_1} = \frac{\delta^2 V}{\delta\sigma_2\delta\chi_2} = -i \tag{9b}$$

$$\frac{\delta^2 V}{\delta\chi_1\delta\chi_1} = \lambda_1 - g_1\varphi_1 - g_3\varphi_2 \equiv A_1, \tag{9c}$$

$$\frac{\delta^2 V}{\delta\chi_2\delta\chi_2} = \lambda_2 - g_2\varphi_2 - g_4\varphi_1 \equiv A_2, \tag{9d}$$

$$\frac{\delta^2 V}{\delta\chi_1\delta\chi_2} = \lambda_3 - g_3\varphi_1 - g_4\varphi_2 \equiv B, \tag{9e}$$

Integrating out the quadratic scalar fluctuations, we obtain the next-to-leading order contribution to the effective action

$$S/N = \int \sum_{i=1,2}\left[-\frac{m_i^3}{6\pi} + \varphi_i m_i^2\right] + V(\varphi_1, \varphi_2) + \frac{1}{2N}\int_p \ln[(1 + A_1\Pi_1)(1 + A_2\Pi_2) - B^2\Pi_1\Pi_2] \tag{10a}$$

where

$$\Pi_\alpha(p) = \int_k \frac{1}{(k^2 + m_\alpha^2)[(k + p)^2 + m_\alpha^2]} = \frac{1}{4\pi p}\tan^{-1}\left(\frac{p}{2m_\alpha}\right) \tag{10b}$$

The integral in the next-to-leading order contribution Eq. (10a) presents new divergences which require renormalization. In order to eliminate divergent terms involving powers of $m_i$ in the numerator, we express the effective action in terms of the renormalized masses $M_i$ which are defined from the self-energy $\Sigma(p, m)$ in diagram Fig. 1. The two-point function and the renormalized mass $M_i$ are given by

$$\Gamma^{(2)}(p, m_i) = p^2 + m_i^2 - \Sigma(p, m_i)$$
$$M_i^2 = m_i^2 - \Sigma_i(0, m_i) + m_i^2\Sigma_i' \tag{10c}$$

where $\Sigma_i' = [\frac{\partial\Sigma_i(p, m_i)}{\partial p^2}]_{p=0}$.

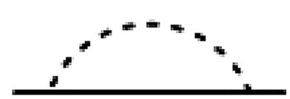

FIG. 1. Scalar field self-energy.

Evaluating diagram Fig. 1. we find

$$\Sigma_1(p,m) = -\int_q \frac{D_{\sigma\sigma}(q)}{(q+p)^2+m_1^2} = -\int_q \frac{A_1+(A_1A_2-B^2)\Pi_2}{[(q+p)^2+m_1^2][1+A_1\Pi_1+A_2\Pi_2+(A_1A_2-B^2)\Pi_1\Pi_2]}. \tag{10d}$$

At this order in the large $N$ expansion, in the pure scalar case no infinite $\Phi$-field wave-function renormalization is needed as can be seen by inspecting

$$\left[\frac{\partial\Sigma_1(p,m)}{\partial p^2}\right]_{p=0} = \frac{-1}{3}\int_q \left[\frac{1}{(q^2+m_1^2)^2} - \frac{4m^2}{(q^2+m_1^2)^3}\right]D_{\sigma\sigma}(q) \tag{10e}$$

which yields a finite contribution. The divergent terms in the self-energy for $m_1$ at zero momentum are found by expanding $D_{\sigma\sigma}(q)$ in a Taylor expansion in $A_1$, $A_2$, and $B$

$$\Sigma_1^{\rm div}(0,m_1) = -\int_q \frac{1}{q^2+m_1^2}\{A_1 - A_1^2\Pi_1(q) - B^2\Pi_2(q)+\cdots\} = -A_1\left(\frac{\Lambda}{2\pi^2}-\frac{m_1}{4\pi}\right)+\frac{A_1^2+B^2}{16\pi^2}\ln\left(\frac{\Lambda}{m_1}\right). \tag{10f}$$

Similarly the divergent terms in the self-energy for $m_2$ at zero momentum are

$$\Sigma_2^{\rm div}(0,m_2) = -A_2\left(\frac{\Lambda}{2\pi^2}-\frac{m_2}{4\pi}\right)+\frac{A_2^2+B^2}{16\pi^2}\ln\left(\frac{\Lambda}{m_2}\right). \tag{10g}$$

Now replacing $m_1$ and $m_2$ in (10a) by their renormalized quantities

$$m_1^2 = M_1^2+\Sigma_1(0,M_1)-M_1^2\Sigma_1', \qquad m_2^2 = M_2^2+\Sigma_2(0,M_2)-M_2^2\Sigma_2' \tag{10h}$$

results in the cancellation of all divergent terms proportional to $m_1$ and $m_2$ in the next-to-leading order contribution to the effective potential [these are the first, second, and third order terms in a Taylor expansion in $A_1$, $A_2$, and $B$ of the last term in (10a)]. The remaining divergent terms involving $\varphi_1$ and $\varphi_2$ can be cancelled by counterterms added to $V(\varphi_1,\varphi_2)$, which introduce a scale $\mu$, and the effective potential now reads as

$$\begin{aligned}V_{\rm eff}/N = {}& V(\varphi_1,\varphi_2)+\sum_{i=1,2}\left[-\frac{M_i^3}{6\pi}+\varphi_iM_i^2+\frac{1}{N}\left(\varphi_i-\frac{M_i}{4\pi}\right)(\Sigma_i^{\rm reg}-M_i^2\Sigma_i')\right]+\frac{1}{2N}\int_p^{\rm F.P}\ln[(1+A_1\Pi_1)(1+A_2\Pi_2)-B^2\Pi_1\Pi_2]\\ &+\frac{1}{16N\pi^2}\left[\varphi_1(A_1^2+B^2)+\varphi_2(A_2^2+B^2)+\frac{1}{128}B^2(A_1+A_2)+\frac{1}{384}(A_1^3+A_2^3)\right]\ln\left(\frac{\mu}{M}\right).\end{aligned} \tag{10i}$$

The letters F.P on the integral indicate that its divergent terms have been subtracted out to make the integral finite. The last logarithmic terms in this renormalized effective potential clearly show that scale invariance is indeed violated. Since the effective potential is a physical quantity that should not depend on the renormalization scale [32], we require its couplings to depend on that scale in such a way that the coefficients of $V_{\rm eff}$ when expanded in powers of the fields $\varphi_i$ do not depend on the scale $\mu$. This yields the following $\beta$-functions of the ungauged model:

$$\beta_0(r_1) = \frac{1}{16\pi^2N}\left(\lambda_3^2+\lambda_1^2-\frac{1}{128}g_1\lambda_3^2-\frac{1}{128}g_4\lambda_3^2-\frac{1}{128}\lambda_1^2g_1-\frac{1}{128}\lambda_2^2g_4-\frac{1}{64}\lambda_1\lambda_3g_3-\frac{1}{64}\lambda_2\lambda_3g_3\right) \tag{11a}$$

$$\beta_0(r_2) = \frac{1}{16\pi^2N}\left(\lambda_3^2+\lambda_2^2-\frac{1}{128}g_3\lambda_3^2-\frac{1}{128}\lambda_3^2g_2-\frac{1}{128}\lambda_1^2g_3-\frac{1}{128}g_2\lambda_2^2-\frac{1}{64}\lambda_1\lambda_3g_4-\frac{1}{64}\lambda_2\lambda_3g_4\right) \tag{11b}$$

$$\beta_0(\lambda_1) = \frac{-1}{16\pi^2N}\left(-4g_3\lambda_3-4g_1\lambda_1+\frac{1}{32}g_3\lambda_3(g_1+g_4)+\frac{1}{64}g_3^2(\lambda_1+\lambda_2)+\frac{1}{64}g_1^2\lambda_1+\frac{1}{64}g_4^2\lambda_2\right) \tag{11c}$$

$$\beta_0(\lambda_2) = \frac{-1}{16\pi^2 N}\left(-4g_4\lambda_3 - 4g_2\lambda_2 + \frac{1}{32}g_4\lambda_3(g_3+g_2) + \frac{1}{64}g_4^2(\lambda_1+\lambda_2) + \frac{1}{64}g_3^2\lambda_1 + \frac{1}{64}g_2^2\lambda_2\right) \tag{11d}$$

$$\beta_0(\lambda_3) = \frac{-1}{16\pi^2 N}\left(-2g_4\lambda_3 - 2g_3\lambda_3 - 2\lambda_1 g_3 - 2\lambda_2 g_4 + \frac{1}{64}g_1\lambda_3 g_4 + \frac{1}{64}g_3^2\lambda_3 + \frac{1}{64}\lambda_1 g_4 g_3 + \frac{1}{64}g_4^2\lambda_3 + \frac{1}{64}g_2\lambda_3 g_3 + \frac{1}{64}\lambda_2 g_4 g_3 + \frac{1}{64}\lambda_1 g_1 g_3 + \frac{1}{64}\lambda_2 g_4 g_2\right) \tag{11e}$$

$$\beta_0(g_1) = \frac{1}{16\pi^2 N}\left(6g_1^2 + 6g_3^2 - \frac{g_1^3}{64} - \frac{g_4^3}{64} - \frac{3g_1 g_3^2}{64} - \frac{3g_4 g_3^2}{64}\right) \tag{11f}$$

$$\beta_0(g_2) = \frac{1}{16\pi^2 N}\left(6g_2^2 + 6g_4^2 - \frac{g_2^3}{64} - \frac{g_3^3}{64} - \frac{3g_2 g_4^2}{64} - \frac{3g_3 g_4^2}{64}\right) \tag{11g}$$

$$\beta_0(g_3) = \frac{1}{16\pi^2 N}\left(2g_3^2 + 2g_4^2 + 4g_3 g_4 + 4g_3 g_1 - \frac{1}{32}g_1 g_4 g_3 - \frac{g_3^3}{64} - \frac{1}{32}g_4^2 g_3 - \frac{1}{64}g_2 g_3^2 - \frac{1}{64}g_3 g_1^2 - \frac{1}{64}g_2 g_4^2\right) \tag{11h}$$

$$\beta_0(g_4) = \frac{1}{16\pi^2 N}\left(2g_4^2 + 2g_3^2 + 4g_3 g_4 + 4g_2 g_4 - \frac{1}{32}g_2 g_3 g_4 - \frac{g_4^3}{64} - \frac{1}{32}g_3^2 g_4 - \frac{1}{64}g_1 g_4^2 - \frac{1}{64}g_1 g_3^2 - \frac{1}{64}g_4 g_2^2\right) \tag{11i}$$

which are indeed suppressed by $1/N$. These beta functions of the enlarged $O(N)\times O(N)$-symmetric model reproduce those in the model with only an internal $O(N)$ symmetry [23] when the system is decoupled, i.e., $\lambda_3 = g_3 = g_4 = 0$. It is worth pointing out some known facts about this special case in which the beta function for either $g_1$ or $g_2$ (denoted by $g$) reduces to $\beta_0(g) = (3/8\pi^2 N)(g^2 - g^3/384)$. For a positive potential ($g > 0$), it follows from the quadratic term in this beta function that the coupling is marginally irrelevant for small values of $g$, for which it increases toward the UV. As $g$ increases, the cubic term becomes important and a perturbative UV fixed point is reached at $g^* = 384$. In [18], a self-consistent nonperturbative UV fixed point was found, in the strict $N = \infty$ limit, at a smaller value $g = 32\pi^2 < g^*$. The effective potential in the $O(N)$ vector model at the tricritical point is $V = (N/6)(g^* - g)\varphi^3$. The system has various phases. For values of $g$ smaller than $g^*$ the system consists of N massless noninteracting $\Phi$ particles. These particles do not interact in the infinite $N$ limit and the correlation functions do not depend on $g$. For the special value $g = g^*$ a flat direction in $\varphi$ opens up. For a zero value of the expectation value of $\varphi$, the theory continues to consist of $N$ massless $\Phi$ fields. For any nonzero value of the expectation value the system has $N$ massive $\Phi$ particles, which all have the same mass due to the unbroken $O(N)$ symmetry. At the same time scale invariance is broken spontaneously and the vacuum energy still vanishes. The Goldstone boson associated with the spontaneous breaking of scale invariance, the dilaton, is massless and identified as the $O(N)$ singlet field $\varphi - \langle\varphi\rangle$. All the particles are noninteracting in the infinite $N$ limit. This theory is not conformal: in the infrared limit, it flows to another theory containing a single, massless, $O(N)$-singlet particle. For larger values of $g$ the effective potential is unbounded from below. The system is unstable. The analysis of [18] suggests that the apparent instability reflects the inability to define a renormalizable interacting theory, all masses are of the order of the cutoff and there is no mechanism to scale them down to low mass values. In other words, the theory depends strongly on its UV completion.

Going back to the $O(N)\times O(N)$-symmetric model at the tricritical point, Eqs. (11f)–(11i) have several fixed points: $g_1 = g_2 = g_3 = g_4 = g^*$, which corresponds to the noncritical fixed point of the enlarged $O(2N)$ model. The second fixed point has all couplings equal to zero. In addition we have a fixed point with $g_1 = g^*$ and $g_2 = g_3 = g_4 = 0$ and another fixed point with $g_2 = g^*$ and $g_1 = g_3 = g_4 = 0$. In Sec. IV C we show that these beta functions receive corrections from the gauge fields fluctuations which have the effect of splitting the zeros of the beta functions and generating a variety of nontrivial fixed points.

### B. Particle content of the tricritical model in the phase with broken scale invariance at next-to-leading order

Since the $O(N)\times O(N)$ symmetry is unbroken in this model, we expect two massive $O(N)$ vector fields. The broken scale invariance leads to a massless dilaton, which manifests as a zero pole in some appropriate correlator. To proceed we rewrite the effective potential in (10i) as

$$V_{\rm eff}/N = \sum_{i=1,2}\left[-\frac{M_i^3}{6\pi} + \varphi_i M_i^2\right] - \frac{g_1}{6}\varphi_1^3 - \frac{g_2}{6}\varphi_2^3 - \frac{g_3}{2}\varphi_1^2\varphi_2 - \frac{g_4}{2}\varphi_1\varphi_2^2 + \frac{1}{N}W^{(S)}(\varphi_1,\varphi_2,M_1,M_2) \tag{12a}$$

where

$$W^{(S)}(\varphi_1,\varphi_2,M_1,M_2) = \sum_{i=1,2}\left[\left(\varphi_i - \frac{M_i}{4\pi}\right)(\Sigma_i^{\rm reg} - M_i^2\Sigma_i')\right] + \frac{1}{2}\int_p^{\rm F.P} \ln[(1+A_1\Pi_1)(1+A_2\Pi_2) - B^2\Pi_1\Pi_2] \quad (12b)$$

is a homogeneous function of $\varphi_1, \varphi_2, M_1, M_2$ which can be written in the following convenient form

$$W^{(S)}(\varphi_1,\varphi_2,M_1,M_2) = \frac{f_1}{6}\varphi_1^3 + \frac{f_2}{6}\varphi_2^3 + \frac{f_3}{2}\varphi_1^2\varphi_2 + \frac{f_4}{2}\varphi_1\varphi_2^2 \quad (12c)$$

where the functions $f_i = (1/N)f_i(M_1/\varphi_1, M_2/\varphi_2)$. We seek solutions of the equations of motion,

$$\frac{\partial V_{\rm eff}}{\partial M_i} = 0, \qquad \frac{\partial V_{\rm eff}}{\partial \varphi_i} = 0 \quad (12d)$$

which read

$$\varphi_i = \frac{M_i}{4\pi} - \frac{1}{2NM_i}\frac{\partial W^{(S)}}{\partial M_i}$$
$$M_i^2 = -\frac{\partial V}{\partial \varphi_i} - \frac{1}{N}\frac{\partial W^{(S)}}{\partial \varphi_i} \quad (12e)$$

with $i = 1, 2$. This minimization of the effective potential leads to the following equations

$$(2h_1+f_1)\varphi_1^2 - (g_3-f_3)\varphi_2^2 - 2(g_3-f_3)\varphi_1\varphi_2 = 0$$
$$-(g_3-f_3)\varphi_1^2 + (2h_2+f_2)\varphi_2^2 - 2(g_4-f_4)\varphi_1\varphi_2 = 0, \quad (12f)$$

which have nontrivial solutions provided the couplings satisfy the following consistency relation:

$$[4H_1H_2 - G_3G_4]^2 = 4[G_3^2 + 2H_1G_4][G_4^2 + 2H_2G_3] \quad (12g)$$

where $H_i = 16\pi^2 - g_i/2 + f_i/2$, $G_i = g_i - f_i$ and the masses are related by

$$[4H_1H_2 - G_3G_4]M_1 = 2[G_4^2 + 2H_2G_3]M_2 \quad (12h)$$

To look for the dilaton which is a Goldstone boson for spontaneous breaking of the scale symmetry, we examine the fluctuations matrix about the solution (12e):

$$S = \frac{N}{2}\int \delta\phi_i \frac{\delta^2 S}{\delta\phi_i\delta\phi_j}\delta\phi_j \quad (13a)$$

here $\delta\phi_1 = \delta\varphi_1, \delta\phi_2 = \delta\varphi_2, \delta\phi_3 = \delta M_1, \delta\phi_4 = \delta M_2$. Inverting this matrix we find the correlator

$$\langle\delta\varphi_1(-q)\delta\varphi_1(q)\rangle = \frac{K_2}{K_1K_2 - \Delta^2} \quad (13b)$$

where $K_i = A_i + a_{ii} - 4M_i^2/D_i - 4M_ic_{ii}/D_i$, $a_{ij} = \delta^2W^{(S)}/\delta\varphi_i\delta\varphi_j$, $c_{ij} = \delta^2W^{(S)}/\delta M_i\delta\varphi_j$, $b_{ij} = \delta^2W^{(S)}/\delta M_i\delta M_j$, $D_i(q) = -M_i/\pi + 2\varphi_i + b_{ii} + q^2/(96\pi M_i)$, and $\Delta = B - 2M_1c_{12}/D_1 - 2M_2c_{22}/D_2 + 4M_1M_2b_{12}/(D_1D_2)$. Using (12g) and (12h), we find in the limit of low momenta $q$ and to leading order in $1/N$:

$$\langle\delta\varphi_1(-q)\delta\varphi_1(q)\rangle = \frac{6M_1M_2}{N\pi}\frac{M_1g_4 - 2h_2M_2}{(g_3+g_4)M_1M_2 - 2h_1M_1^2 - 2h_2M_2^2}\frac{1}{q^2} \quad (13c)$$

This correlator is singular at zero momentum reflecting the presence of the dilaton pole which is a Goldstone boson for spontaneous breaking of the scale symmetry.

### C. Gauge fields contribution

In this section we consider the full gauge invariant action in Eq. (1). The first step is to integrate out the $\Phi$ and $\Psi$ degrees of freedom which leads to an effective action for the gauge fields that has, besides the mixed-Chern-Simons term in Eq. (1), induced Maxwell terms from the bosonic functional determinant

$$S_G(a_\mu, b_\mu) = N{\rm Tr}\ln(-(\partial_\mu - ia_\mu)^2 + m_1^2) + N{\rm Tr}\ln(-(\partial_\mu - ib_\mu)^2 + m_2^2) \quad (14a)$$

Expanding this nonlocal term about $a_\mu = 0$ and $b_\mu = 0$ and keeping only quadratic terms in the fields gives at an intermediate step

$$S_G(a_\mu, b_\mu) = \frac{N}{2}\int_q a_\mu(-q)\Gamma_a(q)\delta_{\mu\nu}^T a_\nu(q) + b_\mu(-q)\Gamma_b(q)\delta_{\mu\nu}^T b_\nu(q) - 2\kappa b_\mu(-q)\epsilon_{\mu\lambda\nu}q_\lambda a_\nu(q) \quad (14b)$$

here $\delta_{\mu\nu}^T = (\delta_{\mu\nu} - q_\mu q_\nu/q^2)$ and where the $\Gamma_\alpha(q)$ term arises from the usual bosonic one-loop polarization diagrams [9], which are expressed as

$$2\int_k \frac{\delta_{\mu\nu}}{k^2 + m_\alpha^2} - \int_k \frac{[2k_\mu + q_\mu][2k_\nu + q_\nu]}{(k^2 + m_\alpha^2)((k+q)^2 + m_\alpha^2)} = \Gamma_\alpha(q)\delta_{\mu\nu}^T. \quad (14c)$$

A full analytic evaluation of the integrals is possible using some standard steps [9], and the result is

$$\Gamma_\alpha(q) = \frac{q^2 + 4m_\alpha^2}{8\pi q}\tan^{-1}\left(\frac{q}{2m_\alpha}\right) - \frac{m_\alpha}{4\pi}. \quad (14d)$$

The resulting gauge propagators in Landau gauge are then $(\alpha, \beta = a, b)$

$$G^{\alpha\beta}_{\mu\nu}(q) = \frac{1}{N}[F_\alpha(q)\delta^T_{\mu\nu}\delta^{\alpha\beta} + G(q)\epsilon_{\mu\lambda\nu}q_\lambda(1-\delta^{\alpha\beta})] \quad (14e)$$

$$F_a(q) = \frac{\Gamma_b(q)}{\Gamma_a(q)\Gamma_b(q) + q^2\kappa^2} \quad (14f)$$

$$G(q) = \frac{\kappa}{\Gamma_a(q)\Gamma_b(q) + q^2\kappa^2}. \quad (14g)$$

Next, integrating out the gauge fluctuations yields a new next-to-leading order contribution to the effective potential

$$V^{(1)}_{\rm gauge} = \int_q \ln[\Gamma_1\Gamma_2 + \kappa^2 q^2]. \quad (15a)$$

We find it convenient to subtract out the zero mass terms from this contribution and write it as

$$V^{(1)}_{\rm gauge}(m_1,m_2) - V^{(1)}_{\rm gauge}(0,0) = \int_q \ln\left[1+\epsilon\left(\frac{16^2\Gamma_1\Gamma_2}{q^2}-1\right)\right] \quad (15b)$$

where the parameter $\epsilon = 1/[1+(16\kappa)^2]$ takes values in the interval [0, 1] with $\epsilon = 0$ corresponding to no gauge coupling (pure scalar model) and $\epsilon = 1$ corresponding to zero mixed-Chern-Simons term (gauge model with induced Maxwell terms). The gauge fields contribution to the effective potential (15b) presents divergences which are contained in the first, second, and third order terms in a Taylor expansion in $\epsilon$. To handle such divergences we first express the effective action in terms of the renormalized masses $M_i$ which involve another self-energy $\Sigma^{\rm gauge}(p,m)$ in diagram Fig. 2 and by adding counterterms to $V(\varphi_1,\varphi_2)$

$$\Sigma^{\rm gauge}_1(p,m) = \int_q \frac{(2p_\mu+q_\mu)G_{\mu\nu}(q)(2p_\nu+q_\nu)}{(p+q)^2+m_1^2} = 4\int_q \frac{[p^2-(p\cdot q)^2/q^2)]\Gamma_2(q)}{(\Gamma_1\Gamma_2+\kappa^2q^2)[(p+q)^2+m_1^2]} \quad (15c)$$

$$\Sigma^{\rm gauge}_1(0,m) = 0 \quad (15d)$$

$$\left[\frac{\partial\Sigma^{\rm gauge}_1(p,m)}{\partial p^2}\right]_{p=0} = \frac{64\epsilon}{3\pi^2}\ln\left(\frac{\Lambda}{m_1}\right). \quad (15e)$$

The divergence in (15e) introduces infinite scalar fields wave-function renormalization $\Phi = Z^{1/2}\,\Phi_R$, $\Psi = Z^{1/2}\,\Psi_R$ with $Z = 1 + \partial\Sigma^{\rm gauge}/\partial p^2$. The terms introduced by mass renormalization are

FIG. 2. Gauge fields self-energy.

$$-\sum_{i=1,2}\left(\varphi_i - \frac{M_i}{4\pi}\right)M_i^2\left[\frac{\partial\Sigma^{\rm gauge}_i(p,m)}{\partial p^2}\right]_{p=0}. \quad (15f)$$

These terms are proportional to $(\varphi_i - \frac{M_i}{4\pi})$ and vanish on shell. The remaining terms in the gauge field contribution to the effective potential including logarithmic divergent terms are

$$V^{(1)}_{\rm gauge} = \int_q^{\rm F.P.} \ln\left[1+\epsilon\left(\frac{16^2\Gamma_1\Gamma_2}{q^2}-1\right)\right] + \frac{2^{10}}{3}\epsilon\ln\left(\frac{\mu}{M}\right)\left[\left(-1+3\epsilon-\frac{16}{\pi^2}\epsilon^2\right)(\varphi_1^3+\varphi_2^3) - \epsilon(1-\epsilon)\left(1-\frac{16}{\pi^2}\epsilon\right)(\varphi_1^2\varphi_2+\varphi_1\varphi_2^2)\right], \quad (15g)$$

Collecting the leading order contribution in Eq. (7) and the two next-to-leading order contributions in Eqs. (10i) and (15g) give the complete effective potential to that order which has an explicit dependence on the renormalization scale. Since this scale is unphysical, there must be implicit dependence of $V_{\rm eff}$ on $\mu$ through the couplings and the fields so that $\mu\frac{dV}{d\mu} = 0$. This renormalization group equation leads to the new $\beta$-functions which now take account of wave-function renormalization and are expressed in terms of the ones we already found in the ungauged model as:

$$\beta(g_1) = \beta_0(g_1) - g_1\frac{64\varepsilon}{\pi^2N} - \frac{2^{11}}{N}\epsilon\left(1-3\epsilon+\frac{16}{\pi^2}\epsilon^2\right) \quad (16a)$$

$$\beta(g_2) = \beta_0(g_2) - g_2\frac{64\varepsilon}{\pi^2N} - \frac{2^{11}}{N}\epsilon\left(1-3\epsilon+\frac{16}{\pi^2}\epsilon^2\right), \quad (16b)$$

$$\beta(g_3) = \beta_0(g_3) - g_3\frac{64\varepsilon}{\pi^2N} - \frac{2^{11}}{N}\epsilon(1-\epsilon)\left(1-\frac{16}{\pi^2}\epsilon\right), \quad (16c)$$

$$\beta(g_4) = \beta_0(g_4) - g_4\frac{64\varepsilon}{\pi^2N} - \frac{2^{11}}{N}\epsilon(1-\epsilon)\left(1-\frac{16}{\pi^2}\epsilon\right). \quad (16d)$$

The correction terms with powers of $\epsilon$ arise from the gauge fields contribution. The beta functions $\beta(r_i)$, $i = 1, 2$ are as in Eqs. (11a)–(11b) but with an extra term $-r_i 64\varepsilon/(3\pi^2N)$ which accounts for the wave-function renormalization. Similarly $\beta(\lambda_i)$, $i = 1, 2, 3$ are as in Eqs. (11c)–(11e) but with an extra term $-\lambda_i 128\varepsilon/(3\pi^2N)$.

Before analyzing the fixed points of this model, we comment here on how the gauge interactions modify the calculation of the previous section. The equations of motion (12d) now include additional terms induced by the gauge interactions. The homogeneous function $W$ consists of two contributions $W = W^{(S)} + W^{(G)}$ where

$$W^{(G)} = \int_q^{\mathrm{F.P.}} \ln\left[1+\epsilon\left(\frac{16^2\Gamma_1\Gamma_2}{q^2}-1\right)\right] \quad (17a)$$

The masses $M_1$ and $M_2$ are related as in Eq. (12h), but with the parameters $f_i$ receiving contribution from $W^{(S)}$ and $W^{(G)}$. Since these parameters are of order $1/N$, the expansion of the mass ratio to that same order gives

$$R = \frac{M_1}{M_2} = \alpha\left\{1 - f_1\frac{g_3(2g_3h_2+g_4^2)}{(g_3g_4-4h_1h_2)(g_3^2+2g_4h_1)} - 2f_2\frac{h_1}{(g_3g_4-4h_1h_2)} + f_3\frac{(g_3^2g_4-8g_3h_1h_2-2g_4^2h_1)}{(g_3g_4-4h_1h_2)(g_3^2+2g_4h_1)} - f_4\frac{(g_3^3+8h_1^2h_2)}{(g_3g_4-4h_1h_2)(g_3^2+2g_4h_1)}\right\} \quad (17b)$$

where $\alpha = M_1^{(0)}/M_2^{(0)}$ is mass ratio at the leading order Eq. (8b) and $f_2 = \alpha^3 f_1$, $f_3 = \alpha f_1/3$, $f_4 = \alpha^2 f_1/3$ with $f_1$ given by

$$f_1 = \frac{3\pi}{N}\cdot 2^6\int_0^\infty dx x^2 \ln\left\{\left(1-\frac{(g_1+g_3/\alpha)\arctan(x)}{32\pi^2 x}\right)\times\left(1-\frac{(g_4+g_2/\alpha)\arctan(\alpha x)}{32\pi^2 x}\right) - \frac{(g_3+g_4/\alpha)^2\arctan(x)\arctan(\alpha x)}{(32\pi^2)^2x^2}\right\} + \frac{3\pi}{N}\cdot 2^7\int_0^\infty dx x^2\ln\left[1-\epsilon\left(1-\frac{4}{\pi^2}F(x)F(\alpha x)\right)\right] \quad (17c)$$

where $F(x) = [1+1/x^2]\arctan(x) - 1/x$. As before, the divergent parts in these integrals have been subtracted, which means that their Taylor expansion in terms of $g_i$ and $\epsilon$ starts at the fourth order terms. It is not easy to compute these integrals analytically, but one can compute their Taylor expansion term by term. For example the $\epsilon^4$ term from the gauge field contribution involves the following integral

$$I(\alpha) = \int_0^\infty dx x^2\left[1-\frac{4}{\pi^2}F(x)F(\alpha x)\right]^4 \quad (17d)$$

whose computation for some values of $\alpha$ is given in the following table. Note that $I(1/\alpha) = \alpha^3 I(\alpha)$.

| $\alpha$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $I(\alpha)$ | 8.33684 | 3.47153 | 2.3985 | 1.95064 | 1.7085 |

To examine how the gauge interactions modify the two-point correlator $\langle\delta\varphi_1(-q)\delta\varphi_1(q)\rangle$, we start from a gap equation similar to Eq. (12d) but including additional terms induced by the gauge interactions. Making use of a consistency relation between the couplings similar to Eq. (12g) where the parameters $f_i$ now receive contributions from the scalar fields as well as from the gauge fields, the scalar two-point correlator $\langle\delta\varphi_1(-q)\delta\varphi_1(q)\rangle$ is found to be equal to

$$\langle\delta\varphi_1(-q)\delta\varphi_1(q)\rangle = \frac{6M_1}{N\pi}\frac{1}{1+R^2\frac{g_3/R-2h_1}{Rg_4-2h_2}}\frac{1}{q^2} \quad (17e)$$

where the mass ratio $R$ is given by Eq. (17b). As before, the scalar current correlator is singular at zero momentum reflecting the presence of the dilaton pole in the massive phase of the theory. The residue of the pole is determined by the masses $M_1$, $M_2$, and the physical couplings. Eq. (17e) shows that the dilaton retains its feature at the next to leading order in 1/N.

## V. FIXED POINTS ANALYSIS

The analysis of the fixed point of the beta functions Eqs. (16a)–(16d) and the evolution of all the different couplings with the energy scale is a formidable task. To make some progress, we specialize to the more manageable case $g_1 = g_2$ and $g_3 = g_4$ redefined in what follows respectively as $384x$ and $384y$, and we examine the theory at its tricritical point $r_1 = r_2 = \lambda_1 = \lambda_2 = \lambda_3 = 0$. The reduced beta functions are

$$\dot{x} = x^2 - \frac{9\epsilon x}{4} - x^3 + y^2 - 4y^3 - 3xy^2 - \frac{\pi^2\epsilon}{27}\left(1-3\epsilon+\frac{16}{\pi^2}\epsilon^2\right) \quad (18a)$$

$$\dot{y} = \frac{2xy}{3} + \frac{4y^2}{3} - \frac{9\epsilon y}{4} - x^2y - 4xy^2 - 3y^3 - \frac{\pi^2\epsilon}{27}(1-\epsilon)\left(1-\frac{16\epsilon}{\pi^2}\right) \quad (18b)$$

where $\dot{x}$ stands for $\frac{\pi^2 N}{144}dx/d\ln(\mu)$ and the condition expressing the positivity of the effective potential is

$$x < \frac{\pi^2}{12}, \qquad \left(x-y-\frac{\pi^2}{12}\right)\left(x+3y-\frac{\pi^2}{12}\right) > 0 \quad (18c)$$

### A. Pure scalar case

The limit $\epsilon$ going to zero corresponds to the decoupling of the gauge fields. In this case the theory is characterized only by the scalar interactions and the solutions of the beta functions are easy to find

$$x = y = 0; \qquad x = 1, \qquad y = 0;$$
$$x = y = 1/4; \qquad x = \frac{35}{36}, \qquad y = -\frac{5}{36}. \quad (19)$$

The only stable fixed point, as shown in Fig. 3, is the Gaussian one $x = y = 0$. The other solutions lie in the instability region.

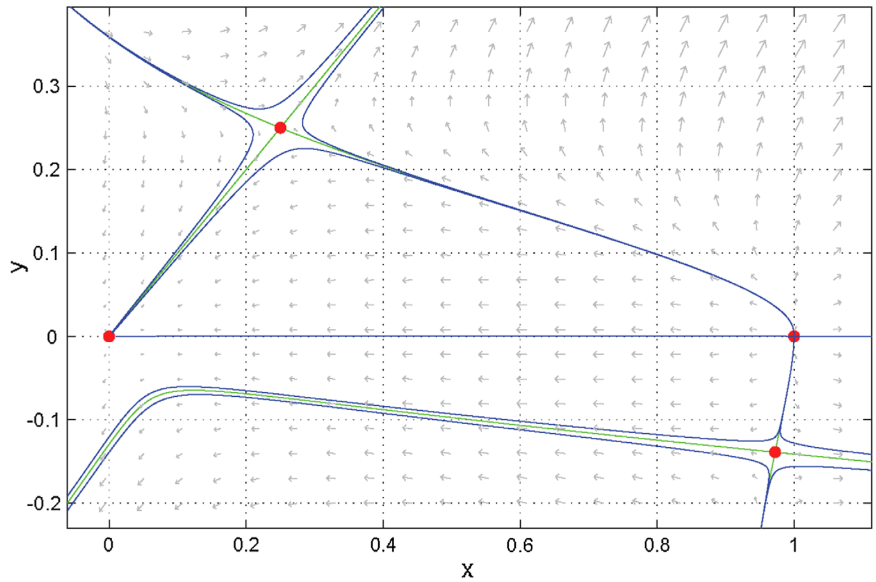


FIG. 3. Fixed points and flow diagram of the pure scalar case ($\epsilon = 0$).

### B. Fixed points in the gauged model

The inclusion of the gauge fields corrections in the nonlinear system Eqs. (18a) and (18b) makes the analytical task of finding the flow trajectories challenging. Even with the aid of a Computer Algebra System the formulas obtained are too complicated to provide much insight; instead, we will first examine analytically the behavior of the solutions around the Gaussian fixed point found in the previous section in order to show the phenomenon of splitting of the zeros of the beta functions induced by the gauge fluctuations. After that we provide results of a numerical computation of the complete system Eqs. (18a) and (18b) showing the most salient features of the phase portraits.

To proceed with the analytical approach, we note that for those fixed points near the origin (0,0), it is safe to omit cubic terms in Eqs. (18a) and (18b) and keep only the quadratic ones. The reduced system has the form:

$$\dot{x} = F(x, y); \qquad \dot{y} = G(x, y) \tag{20}$$

$F(x,y) = x^2 + y^2 - 9\epsilon x/4 - a$, $G(x, y) = 4y^2/3 + 2xy/3 - 9\epsilon y/4 - b$, with $a = \pi^2\epsilon(1-3\epsilon+16\epsilon^2/\pi^2)/27$, $b = \pi^2\epsilon(1-\epsilon)(1-16\epsilon/\pi^2)/27$. To find the fixed points for this system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four real fixed points are found for small $\epsilon$, provided that $a^2 + 12ab - 9b^2 \geq 0$: these are $(x_-, y_-)$, $(-x_-, -y_-)$, $(x_+, y_+)$,

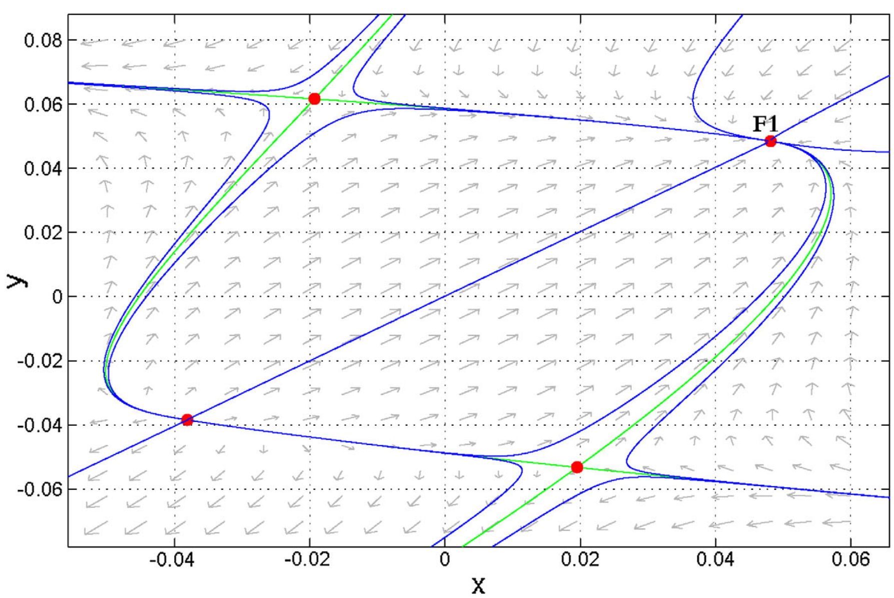


FIG. 4. Gauge field fluctuation effects on the RG flow of the couplings $x$ and $y$ showing the splitting of the zeros of the beta functions around the Gaussian fixed point of the pure scalar model.

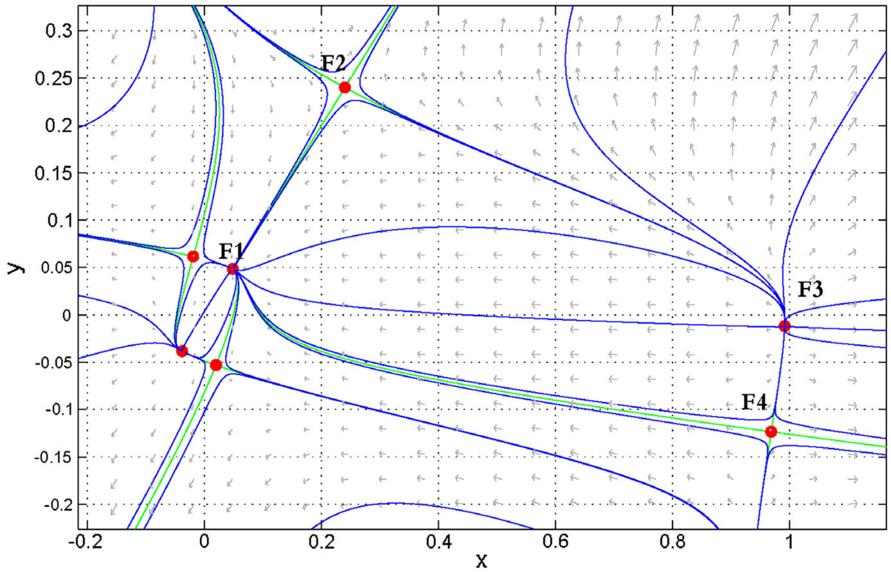


FIG. 5. All seven fixed points and flow diagram for small $\epsilon$ here $\epsilon = 0.01$.

$(-x_+, -y_+)$ with $y_\pm^2 = (a + 6b \pm \sqrt{a^2 + 12ab - 9b^2})/10$ and $x_\pm = 3b/2y_\pm - 2y_\pm$. To examine the stability, we approximate the phase portrait near these fixed points by that of a corresponding linear system. Denoting by $(x^*, y^*)$ one of the four fixed points and $u = x - x^*$, $v = y - y^*$ the components of a small disturbance from the fixed point, the linearized system which describes the growth or decay of these disturbances is given in terms of the Jacobian matrix

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 6x^* & 6y^* \\ 2y^* & 2x^*y^* + 8y^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{21}$$

The infrared stability of the fixed points is done as usual through the positivity analysis of the eigenvalues of the Jacobian matrix at each fixed point. The eigenvalues of the Jacobian are $\lambda_{1,2} = (\tau \pm \sqrt{\tau^2 - 4\Delta})/2$, its determinant is $\Delta = \lambda_1 \lambda_2 = \mp \sqrt{a^2 + 12ab - 9b^2}$ and its trace $\tau = \lambda_1 + \lambda_2 = 8(x^* + y^*)$. IR stable fixed points correspond to positive real eigenvalues ($\Delta > 0$, $\tau > 0$ and $\tau^2 - 4\Delta > 0$). In this case $(x_-, y_-)$ (denoted by $F_1$ in Fig. 4) turns out to be the only stable fixed point with two attractive directions, while $(-x_-, -y_-)$ is infrared unstable since its has two repulsive eigenvalues. The other two fixed points $(x_+, y_+)$ and $(-x_+, -y_+)$ are also infrared unstable since they each have one repulsive eigenvalue.

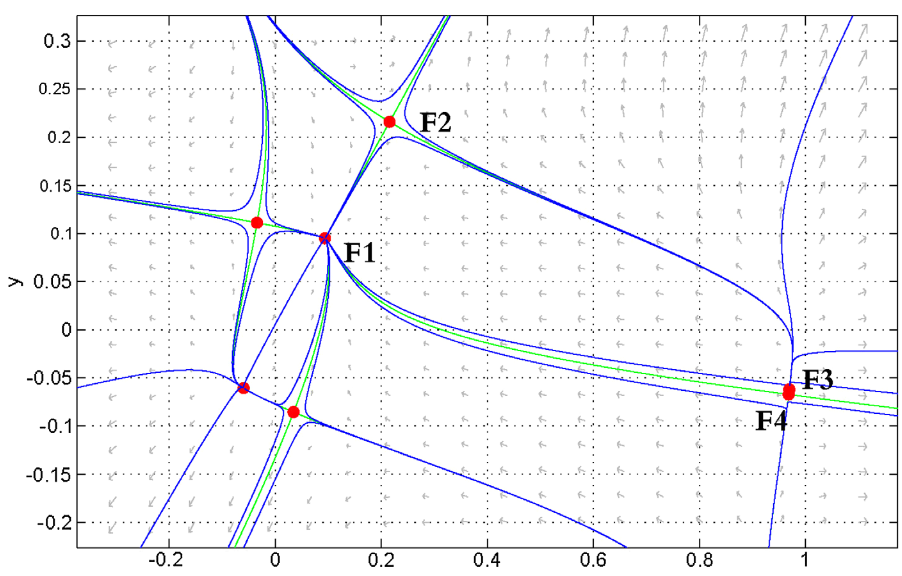


FIG. 6. Collision between fixed points F3 and F4.

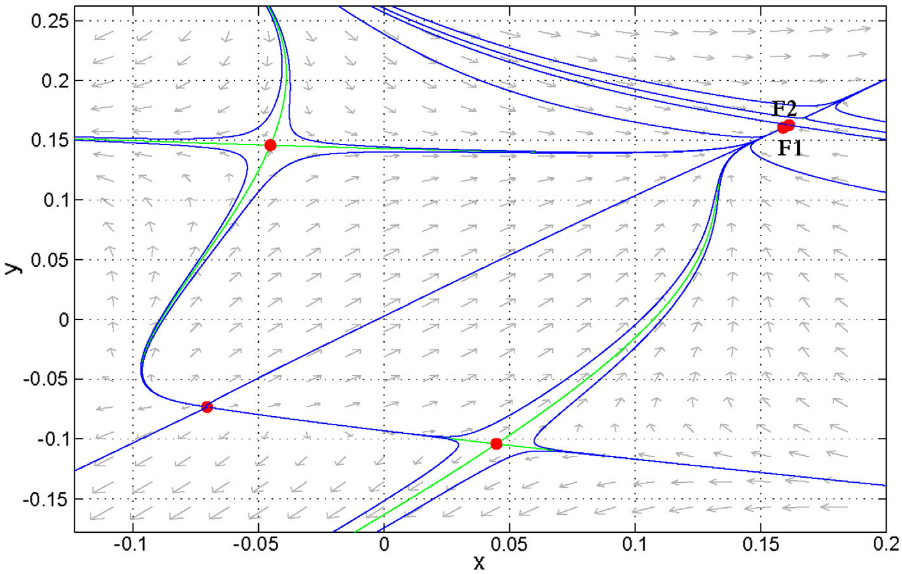


FIG. 7. Collision between F1 and F2.

Next we performed a numerical solution of the complete Eqs. (18a) and (18b) for various values of $\epsilon$. A sudden qualitative change in the solutions is observed as the parameter $\epsilon$ is moved passed a threshold value. Figure 5 shows the phase plane pictures and all possible seven fixed points for very small $\epsilon$. Among these seven fixed points, $F_1$ is the only infrared stable one and the other six fixed points have different degree of instability. As $\epsilon$ rises, the numerical analysis shows neighboring fixed points (F1, F2) and (F3, F4) migrating toward each other as is shown in Figs. 6 and 7. Of particle interest is the fate of the infrared stable fixed point $F_1$ which approaches the unstable $F_2$ as $\epsilon$ increases. These two fixed points are observed to collide when $\epsilon = \epsilon_C = 0.047127065$ and form a semistable fixed point. For $\epsilon > \epsilon_C$ those two fixed points annihilate each other and disappear as shown in Fig 8. As $\epsilon$ approaches 1, only one infrared unstable fixed point survives as shown in Fig. 9. This limit corresponds to zero Chern-Simons term in Eq. (1), but the theory has an induced Maxwell terms.

The numerical study confirms the splitting of the zeros of the beta function around the Gaussian fixed point of the pure scalar case and the existence of other nontrivial fixed points. It also indicates that there is a window in the parameter $\epsilon < \epsilon_C$ where the RG flows has one infrared stable fixed point among other fixed points. We should note that the numerical analysis is restricted to the tricritical region of the coupling space, which corresponds to the region where a second order critical line is ending. The flow

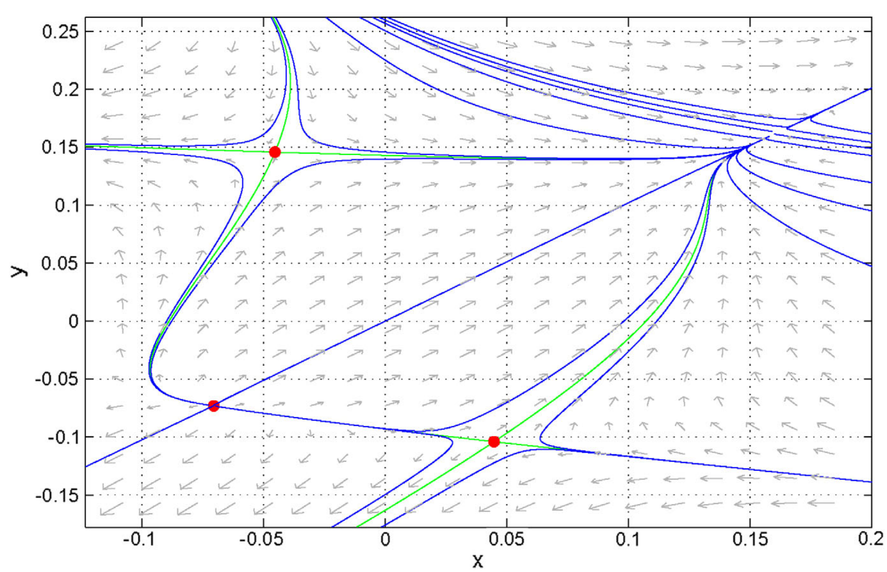


FIG. 8. Disappearance of fixed points F1 and F2.

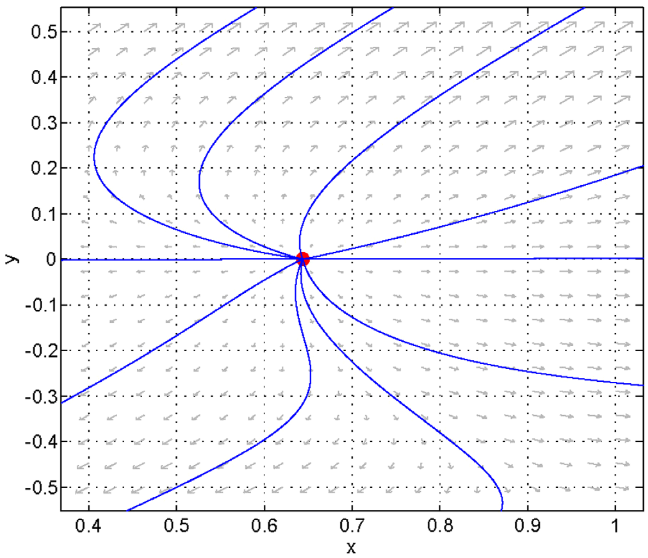


FIG. 9. Fixed point when $\epsilon = 1$.

diagrams above correspond to fluctuation induced first order quantum phase transition between the normal and the scale symmetry breaking phases. It is the Chern-Simons gauge coupling which is driving the remaining couplings toward the fixed point. At the infrared stable fixed point $F_1$, the dimensionless couplings remain unchanged under a change of renormalization scale. This implies that scale invariance is recovered at the stable fixed point $F_1$. The picture that emerges is that for $\epsilon < \epsilon_C$ the model exhibits a quantum transition between broken and unbroken scale symmetry. One phase has RG trajectories terminating at the IR-fixed point and scale symmetry is recovered and hence no dilaton. Away from the basin of attraction of the infrared fixed point, the RG-trajectories run away from it and this defines a second phase where the couplings $g_i$ are scale dependent running couplings and can be arranged so that spontaneous breakdown of scale invariance occurs and a dilaton appears in the spectrum. For $\epsilon > \epsilon_C$, the model has no stable-infrared fixed point and the only phase that exists is the one with broken scale symmetry.

## VI. SUMMARY AND DISCUSSION

In summary, the tricritical behavior of a $U(1) \times U(1)$ Ginzburg-Landau theory with a potential of sixth order in the scalar fields coupled to a mixed Chern-Simons term was investigated in the framework of a $1/N$ expansion at fixed dimension $d = 3$. The sextic coupling terms play crucial roles for stability when the quartic terms are negative. We computed the quantum effective potential which included contributions from the self-interactions of the scalar fields and from the gauge fields fluctuations, and used it to derive the renormalization group functions of the various couplings to the next-to-leading order of the $1/N$ expansion in terms of the mixed Chern-Simons coefficient. We showed that the beta functions for the various sextic couplings at the next order in the $1/N$ expansion exhibit nontrivial running which we analyzed explicitly in terms of the mixed Chern–Simons coefficient. Near the tricritical point, we demonstrated that dynamic gauge field fluctuations split the zeros of the beta functions and generate a variety of non-trivial fixed points. The stability conditions for these fixed points and the requirement of the positivity of the renormalization group improved effective potential were also examined. Both an analytical investigation for small couplings and a numerical calculation in the whole coupling space confirmed the splitting of the zeros of the beta functions around the Gaussian fixed point of the pure scalar model and the existence of an infrared stable fixed point and other fixed points of various instability degree. Our numerical study identified a window in the parameter $\epsilon \leq 0.047$ space of the mixed Chern-Simons coefficient ($\kappa \geq 0.28$) where the RG flow had an infrared stable fixed point at which scale invariance is recovered. Outside that window there was no infrared stable fixed point accessible under renormalization flows. The emerging picture is that the model exhibits a quantum transition between broken and unbroken scale symmetry when $\epsilon \leq 0.047$. In one phase RG-trajectories terminate at the IR-fixed point at which the scale symmetry is explicitly realized and it becomes exact and hence no dilaton. In the second phase RG-trajectories do not reach the IR-fixed point and scale symmetry is spontaneously broken and a dilaton appears in the spectrum. For $\epsilon > 0.047$, the stable-infrared fixed point is annihilated by an unstable fixed point and the only phase that exists is the one with broken scale symmetry and a dilaton in the spectrum.

These results are relevant for Josephson junction arrays which remain an inspiring problem and offer a generic model that captures essential features of the superconducting-insulating transition in a wide class of systems ranging from artificially manufactured Josephson junction arrays to superconducting granular systems and inhomogeneous superconducting films. Indeed, based on the expectation that the transition in these systems is continuous, one usually attempts a description in terms of an effective Ginzburg-Landau functional which is obtained from a coarse grained approach path integral decoupling procedure for the Josephson energies [7]. However, such a procedure generates fourth order terms in the Ginzburg-Landau potential which change sign when the screening length exceeds a few lattice constants; consequently, the sixth order terms in the scalar fields become decisive for stability of the effective Ginzburg-Landau potential and the associated tricritical point where the second order transition becomes first order is of paramount importance. On the other hand, the gauge fields fluctuations describe physically the fluctuations expected to appear in vortex currents and Cooper pairs currents in these systems. The resulting competition of the effects of the gauge fields fluctuations against the scalar fluctuations is of great importance for the critical behavior. The possibility that these systems exhibit fluctuations driven first order transitions has important experimental implications such as the appearance of coexisting phases that give rise to the phenomenon of hysteresis for example when changing magnetic fields.

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*Note added.*—Recently, we became aware that the new added Ref. [33] also deals with a mutual Chern-Simons theory in which are coupled bosonic fields denoting $Z_2$ charge and $Z_2$ vortex quasiparticles. But the approach of [33] uses K-matrices representation as developed in the fractional quantum Hall hierarchy theory in order to examine possible phases that such quasiparticles can have, and utilizes that as a means to access the phase transition involving topological states. It suggests that the effective theory at the critical point is a $2+1$ dimensional quartic model. Our approach uses sextic terms in the complex fields in order to obtain the correct tricritical behavior. Our paper showed that these couplings are ultimately responsible for driving the system to the tricritical point, and paid special attention to their behavior under the renormalization group iterations.


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