Anomalous magnetic moment of an electron in a constant magnetic field in (2 + 1)-dimensional quantum electrodynamics

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The anomalous magnetic moment (AMM) for excited states of an electron in a constant magnetic field has been calculated within the framework of two-dimensional electrodynamics. The analytical results for the interaction energy of the anomalous magnetic moment with the external magnetic field are obtained in two limiting cases of nonrelativistic and relativistic energy values in a comparatively weak magnetic field. It is shown that the interaction energy of the spin with the external field does not contain infrared divergence and tends to zero as magnetic field decreases, while the electron's AMM increases logarithmically.

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I. INTRODUCTION

It is well known that the solution of the Dirac equation in external magnetic field gives a kinetic (intrinsic) magnetic moment of electron equal to one Bohr magneton

$$\mu_B = \frac{e\hbar}{2mc} \simeq 5.79 \times 10^{-19}\,\frac{\text{eV}}{G}.$$

The theoretical explanation for the origin of the electron anomalous magnetic moment (AMM) and the calculation of its magnitude were given by Schwinger [1]. Taking into account the part of the vacuum interaction energy of an electron in a magnetic field, Schwinger showed in the nonrelativistic approximation that a term linear in the field strength leads to a change in the electron g-factor. The electron behaves as if it had a static magnetic moment equal to

$$\mu = -\mu_B\left(1 + \frac{\alpha}{2\pi} + \cdots\right),$$

where α is a fine-structure constant. Therefore, the true magnetic moment of the electron has deviation from the Bohr magneton, which is called "anomalous magnetic moment." Note that this result of Schwinger actually has given an excellent quantitative explanation for contemporary experiments in hyperfine splitting of energy levels of S-states of hydrogen atom [2,3].

In previous publications [4–8], an interesting effect was predicted and analyzed, demonstrating that the AMM of electron has dynamic nature. It appears to be a complex function of the magnetic field strength and electron's energy and in sufficiently strong magnetic fields, the AMM can differ greatly from Schwinger's result, which is [6]

$$-\mu_B \frac{\alpha}{2\pi} = -\mu_B (a_e)_{\text{Schwinger}}.$$

Presently, the value $a_e = \frac{\Delta\mu}{\mu_B}$ is calculated in the standard model of Weinberg-Salam-Glashow with very high accuracy, and its experimental verification serves as an important method to verify predictions of the standard model [9–13]. The accuracy for measuring the AMM of electrons, according to [13], is evaluated as

$$\frac{\mu}{\mu_B} - 1 = \frac{g-2}{2} = 0.00115965218091 \pm 0.00000000000026,$$

$$\delta\left(\frac{\mu}{\mu_B} - 1\right) = 0.26 \times 10^{-14}.$$

The increasing accuracy of experiments for the determination of electron AMMs makes it possible to experimentally verify its dynamic nature using

$$a_e = \frac{\alpha}{2\pi} f(E, H),$$

where E is the electron energy and H is the magnetic field strength. In a constant magnetic field, under normal conditions of experiment, the numerical correction δf in the formula $f(E, H) = 1 - \delta f$ is fairly small. For instance, even at the energy of the electron of 1 GeV and the magnetic field of $10^4 G$, we have $\delta f = 5 \times 10^{-10}$.

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In the vacuum, the role of radiation effects in propagation of an electron or photon becomes especially important, when the external field strength in the rest system of the electron, or in the system where the photon has the energy of mc^2, becomes comparable to Schwinger's critical field [2,14–16]

$$H_0 = \frac{m^2c^3}{e\hbar} = 4.41 \times 10^{13} G.$$

On the other hand, in solid matter or in a magnetized plasma, the notion of a strong magnetic field at finite temperature and nonzero chemical potential differs significantly from the corresponding notion in the vacuum [9,14,17–20]. In particular, it can be noted that, at room temperature ($T = 300K$), the dynamic nature of mass shift and the AMM of electrons in QED_{3+1} becomes apparent in magnetic fields with intensity of $H \geq 10^6 G$, which is significantly less than the maximum intensity of pulsed magnetic fields of $H \approx 10^7 G$, reached in laboratory conditions.

Recently, much attention has been paid to the study of radiation and spin effects in low-dimensional models of the quantum field theory [21–24]. The results of such studies are used to explain the fractional Hall effect, high-temperature superconductivity, electronic properties of graphene [25–31], quantum optics [32], and other areas of physics. In the graphene and a number of other planar structures [33], the dynamics of electronic excitations is described by Dirac effective two-dimensional equation for both massless and massive charged fermions. The account of spin effects is usually carried out phenomenologically and reduced to addition in the Dirac effective equation of spin summands which have an analogy in QED_{3+1} (for instance, refer to [34]). The results of experimental studies of the electron spin g-factor in graphene in a constant magnetic field are given in [35,36]. Theoretical analysis of these results within the frameworks of the pseudo-QED model is provided in [37] in linear approximation upon the external field, i.e., without consideration of the dynamic nature of the AMM of an electron.

As was demonstrated in a number of previous publications, reduction in system dimensionality leads to a significant change in dependence of magnetic properties of electrons and photons on field intensity and particles' energy, as well as temperature and chemical potential. The polarization operator and elastic-scattering amplitude of the photon in QED_{2+1} in a constant magnetic field was considered in [38,39], while the polarization operator of QED_{2+1} with nonzero fermion density in the magnetic field was discussed in [40,41]. The electron self-energy in $(2+1)$ QED with the Chern-Simons term at finite temperature and density has been analyzed in the papers [42,43]. However, until now, the dynamic nature of AMM of electrons in two-dimensional electrodynamics has not been studied either with the Chern-Simons term or without it and the question of possibility for experimentally observing a dynamic nature of electron AMM in graphene remains open. The first calculations of the AMM of electrons in QED_{2+1} with the Chern-Simons term were made based on the vertex function in the field-free case, i.e., without taking into account the influence of external magnetic field [44–46].

In an external magnetic field, the radiative shift of the electron ground state energy was calculated within the framework of topologically massive two-dimensional electrodynamics in the paper [47]. A complete description of the electron stationary states in a magnetic field has been conducted using two-dimensional electrodynamics based on the spin operator proposed in our Ref. [48].

On the basis of this result, the calculation of the radiative shift of the electron ground state energy and the electron AMM in the magnetized plasma of topologically massive QED_{2+1} has been performed [48]. In particular, it was shown that the vacuum value of electron AMM in QED_{2+1} with the Chern-Simons term, obtained in [48] in a relatively weak magnetic field, coincides with the result obtained previously by other authors based on the calculation of vertex function (for instance, refer to [44–46]).

The purpose of this work is to study the dynamic nature of the AMM of electron in a constant magnetic field within the framework of the two-dimensional electrodynamics. The article has the following structure. In Sec. II, using one-loop approximation the exact expression for the anomalous magnetic moment of an electron are found as a function of magnetic field strength and the electron energy. In Secs. III–IV, the analytical results for the AMM of an electron in QED_{2+1} are obtained in limiting cases of nonrelativistic and relativistic values of the energy in a comparatively weak magnetic field. The main results of this study are formulated in the final Sec. V.

II. AMM OF AN ELECTRON IN QED_{2+1} IN A CONSTANT MAGNETIC FIELD

The Lagrangian of QED_{2+1} can be described by formula [23,49–51]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}[(\hat{p} + e\hat{A}) - m]\Psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{2.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field tensor, ξ is parameter fixing calibration, m is the electron mass, $-e < 0$ is the electron charge. We consider the four-component fermions in QED_{2+1}, connected with a four-dimensional reducible representation of the Dirac matrices [48,50–53]

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad \gamma^{1,2} = \begin{pmatrix} i\sigma_{1,2} & 0 \\ 0 & -i\sigma_{1,2} \end{pmatrix}, \tag{2.2}$$

where $\sigma_\mu (\mu = 1, 2, 3)$ are Pauli matrices. In the one-loop approximation, the electron mass operator $\Sigma(x, x')$ in a constant magnetic field is defined by the expression [9]

$$\Sigma(x, x') = -ie^2 \gamma^\mu S_c(H; x, x') \gamma^\nu D_{\mu\nu}(x - x'), \tag{2.3}$$

where

$$S_c(H; x, x') = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \exp[i\omega(t - t')] \times \sum_{s,\varepsilon=\pm 1} \frac{\Psi_s^\varepsilon(\vec{x}) \bar{\Psi}_s^\varepsilon(\vec{x'})}{\omega + \varepsilon E_s(1 - i\delta)} \tag{2.4}$$

is the causal Green function of electron in an external magnetic field [17], $D_{\mu\nu}(x - x')$ is the photon propagation function, which in QED_{2+1} in the Landau gauge are determined by the formula [42,54]:

$$D_{\mu,\nu}(p) = \frac{i g_{\mu\nu}}{p^2 - \theta^2 + i0}. \tag{2.5}$$

Here, $g_{\mu\nu} = (1, -1, -1)$, and we note that, in QED_{2+1} theory, e^2 in (2.3) is given in mass units. In order to compare our results with the contribution coming in QED_{2+1} with the Chern Simons term [46–48] only from the $g_{\mu\nu}$ term in the photon propagator, we take into account the dependence the numerator in (2.5) from the Chern-Simons parameter Θ.

The summation in (2.4) is carried out over all quantum numbers s of the positive $(\varepsilon = +1)$ and negative $(\varepsilon = -1)$ frequency states, $\Psi_s^{(\varepsilon)}(\vec{x})$—is the coordinate part of the Dirac equation solution in a static magnetic field in QED_{2+1} and E_s is the energy of electron stationary states. Choosing the vector potential for external magnetic field in the Landau gauge $(A_0 = A_1 = 0, A_2 = -xH)$, the Hamiltonian of the Dirac equation

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \tag{2.6}$$

can be represented in a magnetic field as

$$\hat{H} = \alpha_1 \hat{p}_x + \alpha_2(\hat{p}_y + exH) + m\gamma^0, \tag{2.7}$$

where the matrices $\alpha_{1,2} = \gamma^0\gamma^{1,2}$, $\hat{p}_x$ and $\hat{p}_y$ are the projections of a momentum operator, and H is the magnetic field strength. Following to [48], we require that the solution of the Dirac equation be an eigenfunction of the following operators:

(1) energy operator:

$$\hat{H}\Psi = \varepsilon E \Psi, \tag{2.8}$$

(2) operator $\hat{p}_y$

$$\hat{p}_y \Psi = p_y \Psi, \tag{2.9}$$

(3) operator $\hat{A} = i\varepsilon\gamma^0\gamma^1\gamma^2$ of spin projection on “direction” of the magnetic field:

$$\hat{A}\Psi = \xi\Psi, \tag{2.10}$$

where $\xi = \pm 1$—is the spin quantum number.

As a result, the Dirac equation solution for electron may be represented in the following form [48]:

$$\Psi_{\varepsilon=+1} = \frac{(eH)^{\frac{1}{4}}}{\sqrt{2E_n}} \exp[-iE_n t + iyp_y] \left[\begin{pmatrix} \sqrt{E_n + m}u_{n-1} \\ \sqrt{E_n - m}u_n \\ 0 \\ 0 \end{pmatrix} D_1 + \begin{pmatrix} 0 \\ 0 \\ \sqrt{E_n - m}u_{n-1} \\ \sqrt{E_n + m}u_n \end{pmatrix} D_{-1} \right], \tag{2.11}$$

where with $\xi = +1$ it is necessary to set $D_1 = 1, D_{-1} = 0$ (spin is directed along the field), while at $\xi = -1$, on the contrary, $D_1 = 0, D_{-1} = 1$ (spin is directed against the field). The coefficients D_1 and D_{-1} satisfy the normalization condition

$$D_{-1}^2 + D_1^2 = 1. \tag{2.12}$$

The electron energy level in a magnetic field in $(2 + 1)$-dimensional QED determined by the main quantum number n only and defined by the formula

$$E_n = \sqrt{m^2 + 2eHn}, \qquad n = 0, 1, 2, \ldots, \tag{2.13}$$

and the argument of the Hermite functions $u_n(\eta)$ [2,48] is

$$\eta = \sqrt{eH}\left(x + \frac{p_y}{eH}\right). \tag{2.14}$$

It has been shown in [48] that the operator $\hat{A}$ in (2.10) is a $(2 + 1)$-dimensional analog of the projection of the operator three-dimensional spin vector on the direction of magnetic field in QED_{3+1}. In the absence of a longitudinal component in the momentum, it is proportional to the operator transverse polarization μ_3 [2], and the quantum number ξ does have meaning of electron spin projection on the direction of the external magnetic field. It follows from

(2.11) that in the ground state ($n = 0$) the electron spin can only be directed opposite to the direction of the magnetic field $D_1 = 0, D_{-1} = 1$.

In the one-loop approximation, the mass operator $\Sigma(x, x')$ determines the radiative correction to the electron energy in the form

$$\Delta E_q^{\xi,\xi'}(H) = -ie^2 \frac{1}{T} \iint d^3x d^3x' \bar{\Psi}_{q\xi'}(x) \times \gamma^\mu S_c(H; x, x') \times \gamma^\nu D_{\mu\nu}(x - x') \Psi_{q\xi}(x'). \tag{2.15}$$

Here, functions $S_c(H, x, x')$ and $D_{\mu\nu}(x - x')$ are defined by formulas (2.4)–(2.5), T is the interaction time, $\Psi_{q\xi}(x)$—is the Dirac equation solution for an electron in an external field in the stationary state with quantum number $(q, \xi) \equiv (n, q_y, \xi)$, which radiation energy shift is to be found, quantum numbers ξ and $\xi' = \pm 1$ characterize the dependence energy shift on the spin initial and final orientation. The value determined by this formula diverges and requires renormalization. For this reason, one should subtract from it a similar value corresponding to a zero field limit.

As is known, for the electron excited states in a magnetic field, the energy shift part proportional to bilinear combination $(D_1 D_1' - D_{-1} D_{-1}')$ is directly associated with the presence of the electron AMM [2]. In the ground state, when the electron spin can only be directed opposite to the magnetic field orientation, the part of the whole amount of the energy shift in a weak magnetic field, which is linear in the magnetic field strength, equal to the energy of interaction of the Schwinger's anomalous magnetic moment with the magnetic field [1,2].

When calculating the electron propagator in formula (2.4), we use a method proposed in [55] for calculation of two-loop contribution to thermodynamic potential of QED in a constant magnetic field. We follow this approach also to study resonant processes in the field of a plane electromagnetic wave [56].

To calculate (2.15), we expand the matrix $K^s_{\alpha\beta} = \Psi^\epsilon_s(\vec{x}) \bar{\Psi}^\epsilon_s(\vec{x'})$ in (2.4) with respect to matrices $I, \gamma^\mu, \gamma^\mu\gamma^\nu, \gamma^0\gamma^1\gamma^2$:

$$4K = I + \gamma^0 F_0 + \gamma^1 F_1 + \gamma^2 F_2 + i\gamma^1\gamma^2 F_{12} + i\gamma^0\gamma^1\gamma^2 F_{012}, \tag{2.16}$$

where

$$I = SpK, F_\mu = Sp(\gamma_\mu K),$$
$$F_{12} = Sp(i\gamma_1\gamma_2 K), \qquad F_{012} = Sp(i\gamma_0\gamma 1\gamma_2 K). \tag{2.17}$$

Next, we perform summation in (2.4) over the principle quantum number n' of the electron intermediate states using the formula [57]

$$\sum_{n'=0}^{\infty} z^{n'} L^\alpha_{n'}(x) = (1 - z)^{-(\alpha+1)} \exp\left[\frac{xz}{z - 1}\right], \tag{2.18}$$

where $L^\alpha_{n'}(x)$ is a Laguerre polynomial. As a result, in the momentum representation, the following expression is obtained for the propagator of the two-dimensional electron in a constant magnetic field (also refer to [51,52,58]):

$$S_c(k) = -i \int_0^\infty ds \exp\left[is\left(k_0^2 - m^2 + i\delta - \vec{k}^2 \frac{tg(eHs)}{eHs}\right)\right] \times \left[(\gamma^0 k^0 + m)(1 + \gamma^1\gamma^2 tg(eHs)) - \frac{(\vec{k}\vec{\gamma})}{\cos^2(eHs)}\right]. \tag{2.19}$$

Integration over variables x^μ and $x'^\mu (\mu = 0, 1, 2)$ gives

$$\int d^3x d^3x' \exp\left[-i\left((p + k)(x - x') + \frac{eH}{2}(y - y')(x + x') - E_n(t - t') + q_y(y - y')\right)\right] u_n(\eta) u_m(\eta') = (2\pi)^2 LT \delta(p_0 + k_0 - E_n) \frac{2}{eH}(-1)^m \times \exp[i(n - m)\lambda] I_{n,m}\left(\frac{2\kappa^2}{eH}\right). \tag{2.20}$$

Here, $p_\mu = (p_0, \vec{p})$ is a 4-momentum of a virtual photon, $\vec{\kappa} = \vec{p} + \vec{k}$, $\lambda = \frac{\pi}{2} - \phi$, $tg\phi = (\frac{\kappa_2}{\kappa_1})$, $I_{n,m}(z)$ is the Laguerre function [2], T is interaction time, L is the length of periodicity in the direction of axis Oy, and the appearance of the Dirac δ-function expresses the energy conservation.

As a result, we arrive at the next representation, which is exact in one-loop approximation for the interaction energy of the spin with the external field for arbitrary excited state of a two-dimensional electron:

$$\Delta E^\xi_n(g_{\mu\nu}) = -\xi \frac{m^2 e^2}{16\pi^{\frac{3}{2}} E_n} \exp\left(i\frac{\pi}{4}\right) \int_0^1 \frac{du}{\sqrt{u}} \int_0^\infty \frac{dy}{\sqrt{y}} \times \exp\left[-im^2 uy - i\frac{y(1 - u)}{u}\Theta^2 - 2inarctg\lambda + 2ieHny(1 - u)\right] F_0. \tag{2.21}$$

Here, we assumed $\xi = \xi'$, and the notation is taken as

$$F_0 = \frac{2 - u + 2u \exp[-2ieHy]}{1 - u + u \exp[-ieHy] \frac{\sin(eHy)}{eHy}} - \frac{\exp[2iarctg\lambda](2u + (2 - u)\exp[-2ieHy])}{1 - u + u \exp[-ieHy] \frac{\sin(eHy)}{eHy}}, \tag{2.22}$$

$$\lambda = \frac{tgz}{1+\frac{u}{1-u}\frac{tgt}{t}}, \qquad t = eHy. \tag{2.23}$$

In QED_{3+1} with a constant magnetic field, the following relation takes place between the AMM of the electron and the energy shift, which explicitly dependent on the electron spin orientation [59,60]:

$$\Re(\Delta E_n^{\xi}) = -(\Delta\mu)\left[\xi B_t + \frac{m}{E_n}\xi B_l\right]. \tag{2.24}$$

Here B_t and B_l, are the transverse and longitudinal (relative to the direction of motion of the electron) vectors of the magnetic induction. Since the component of the vector of the magnetic field strength that is longitudinal with respect to the speed of electron is equal to zero, the real part of the energy (2.21) is connected with the contribution to the AMM of the electron in QED_{2+1} by the formula

$$\Re(\Delta E_n^{\xi}) = -\xi H \Delta\mu. \tag{2.25}$$

The expression for the mass shift of the electron ground state was first obtained in [47], where the two-component electron wave function is used, and then obtained independently in [48]. It can be represented in the form

$$\Delta E_0 = \frac{e^2}{8\pi^{\frac{3}{2}}}\int_0^1 \frac{du}{\sqrt{u}}\int_0^\infty \frac{dy}{\sqrt{y}}\exp[-m^2uy]A_0,$$
$$A_0 = \frac{e^{-\nu}[2-u+2ue^{-2eHy}]}{\Phi} - u - 2, \tag{2.26}$$

where the renormalization of the mass is conducted by subtracting divergent part $\Delta E(H\to 0)$ and the notations are taken as

$$\nu = \frac{y(1-u)\Theta^2}{u}, \qquad \Phi = 1 - u + u\frac{\sin(eHy)}{eHy}e^{-eHy}.$$

III. THE AMM OF AN ELECTRON IN QED_{2+1} WITH THE CHERN-SIMONS TERM: CONTRIBUTION FROM THE $g_{\mu\nu}$ TERM IN THE PHOTON PROPAGATOR

First, we consider the case of a weak magnetic field and a nonrelativistic electron, when the following conditions are satisfied:

$$\beta = \frac{H}{H_0} \ll 1, \qquad 2eHn \ll m^2. \tag{3.1}$$

The real part of the energy shift in (2.21)–(2.23) is related to the electron mass shift Δm_ξ, which in contrast to ΔE_n^{ξ} is the Lorentz invariant. The change in the electron mass Δm_ξ, which is dependent on the electron spin orientation, can be found from the formula

$$\Delta m_\xi = \frac{E_n^{\xi}}{m}\Delta E_n^{\xi}. \tag{3.2}$$

Up to a factor equal to $-i$, the exponent in formula (2.21) can be represented in the form

$$\frac{t}{\beta}\left[u + \left(\frac{1}{u} - 1\right)\rho^2\right] \equiv \frac{t}{\beta}F(u,\rho), \tag{3.3}$$

where $\rho = \frac{\Theta}{m}$, $t = eHy$, $F(u,\rho) = u + (\frac{1}{u} - 1)\rho^2$. We note that $F(u,\rho)\to\infty$ at $u\to+0$, $F(u=1,\rho) = 1$ and, without restricting the generality, it is assumed that $\rho < 2$. At the critical point $u_0 = \rho = \frac{\Theta}{m}$, which is a minimum point of the function $F(u)$ on the interval [0, 1], the following inequation is true

$$F(u_0,\rho) = 2\frac{\Theta}{m}\left(1 - \frac{\Theta}{2m}\right) > 0. \tag{3.4}$$

In the limiting case,

$$\frac{F(u_0,\rho)}{\beta} \gg 1, \tag{3.5}$$

when the field parameter β is small as compared to the Chern-Simons parameter ρ, the main contribution to the radiation energy shift (2.21) provides the domain $t = eHy \ll 1$. Expanding the expression in formula (2.22) in a series in the variable t, further integration is carried out taking into account the infinitesimal imaginary part of the electron mass in the causal propagator:

$$\lim_{\delta\to 0}\int_0^\infty t^{\frac{1}{2}}\exp[-\delta t]\left[\cos\frac{tF(u,\rho)}{\beta} - i\sin\frac{tF(u,\rho)}{\beta}\right]dt$$
$$= \exp\left[-i\frac{3\pi}{4}\right]\frac{\sqrt{\pi}\beta^{\frac{3}{2}}}{2}\frac{u^{\frac{3}{2}}}{[u^2 + (1-u)(\frac{\Theta}{m})^2]^{\frac{3}{2}}}. \tag{3.6}$$

As a result, in the leading order on the small parameter $\frac{eH}{m\Theta}$, the contribution to the mass shift of an electron (3.2), is defined by the formula:

$$\Delta m_\xi(g_{\mu\nu}) \simeq \xi e^2\frac{\beta}{16\pi}\int_0^1 \frac{u^2(2-u)du}{[u^2 + (1-u)^2\rho^2]^{\frac{3}{2}}}$$
$$= \xi e^2\frac{\beta}{16\pi}\left[3 - 3\rho - \left(2 - \frac{3\rho^2}{2}\right)\ln\frac{\rho+2}{\rho}\right],$$
$$\beta \ll F(u_0,\rho). \tag{3.7}$$

As it follows from formula (3.7), when the spin is directed opposite to the field, the result (3.7) for a nonrelativistic electron in a weak magnetic field agrees with result (38) obtained in [48] for the ground state of an electron, in which the electron spin can be only directed opposite to the magnetic field.

In the limiting case of small values of the parameter ρ, from formula (3.7) we find

$$\Delta m_\xi(g_{\mu\nu}) = \frac{\xi e^2\beta}{8\pi}\left(\frac{3}{2}+\ln\frac{\rho}{2}\right), \qquad \beta \ll \rho \ll 1. \quad (3.8)$$

Further, let us refer to the case of a weak magnetic field and ultrarelativistic values of energy of the electron, when the following condition

$$\beta \ll 1, \qquad p_\perp = \sqrt{2eHn} \gg m \quad (3.9)$$

is justified. In this area, the movement of electrons obeys the quasiclassical laws. Maintaining two first expansion terms in formula (2.23) in the significant region ($t \ll 1$),

$$arctg\lambda \simeq t(1-u) + \frac{t^3 u(1-u)^2}{3},$$

we find the following representation for the part of the electron mass shift, defined by the interaction energy of the spin with the external magnetic field:

$$\Delta m_\xi(g_{\mu\nu}) = \xi\frac{e^2}{8\pi^{\frac{3}{2}}}\int_o^\infty \frac{u-2}{1-u}G(z)du. \quad (3.10)$$

Here, we introduced the function (also refer to [39])

$$G(z) = \int_0^\infty \sqrt{t}\exp\left[-i\left(\frac{\pi}{4}+zt+\frac{t^3}{3}\right)\right]dt, \quad (3.11)$$

which depends on the argument

$$z = \left[\frac{1}{\sqrt{u}(1-u)\kappa}\right]^{\frac{2}{3}} F(u,\rho), \quad (3.12)$$

where the function $F(u,\rho)$ is defined by the formula (3.3). The expression on the right-hand side of (3.10) depends on the external field and electron energy only as a function of a characteristic dynamic parameter of synchrotron radiation,

$$\kappa = \frac{1}{m^3}\sqrt{-(eF_{\mu\nu}q^\nu)^2} = \frac{H}{H_0}\frac{p_\perp}{m}. \quad (3.13)$$

Note that in the region $u \in [0,1]$, the following inequality holds:

$$z \geq \frac{3F(u_0,\rho)}{(2\kappa)^{\frac{2}{3}}}.$$

Therefore, in the limiting case, when

$$\frac{3F(u_0,\rho)}{(2\kappa)^{\frac{2}{3}}} \gg 1, \quad (3.14)$$

the main contribution to the integral in (3.11) comes from the region $t \ll 1$, and in the first approximation, the term $\frac{t^3}{3}$ in the exponent can be neglected. According to the assumption (3.14), the dynamic parameter κ is small compared with the Chern-Simons parameter.

The integral over variable t is again calculated using formula (3.6), and we obtain the following result in the quasiclassical approximation:

$$\Delta m_\xi = \xi\frac{\kappa e^2}{16\pi}\int_0^1 \frac{u^2(2-u)du}{[u^2+(1-u)\rho^2]^{\frac{3}{2}}}, \qquad \kappa \ll \left(\frac{\Theta}{m}\right)^{\frac{3}{2}} < 1. \quad (3.15)$$

Thus, the contribution to the AMM, which is caused by the term in the photon propagator, which is proportional to $g_{\mu\nu}$, is equal to

$$\frac{\Delta\mu(g_{\mu\nu})}{\mu_B} \approx -\frac{e^2}{8\pi m}\left[-(3-3\rho)+\left(2-\frac{3\rho^2}{2}\right)\ln\frac{\rho+2}{\rho}\right]. \quad (3.16)$$

IV. THE INTERACTION ENERGY OF THE SPIN WITH A MAGNETIC FIELD IN THE QED_{2+1} WITHOUT THE CHERN-SIMONS TERM

The results of Sec. III were obtained in the limiting case, when the parameters β and κ depending on the strength of the magnetic field were small compared with the value $\rho = \frac{\Theta}{m}$, where Θ is the Chern-Simons parameter. Otherwise, when the parameter ρ is small in comparison with the parameters β or κ, in order to obtain the main term of the expansion in the small parameter $\frac{m\Theta}{eH}$, it suffices to pass to the limit $\Theta \to 0$ in the results of Sec. II, which corresponds usually to QED_{2+1} without the Chern-Simons term. The entire interaction energy of the AMM with external magnetic field in this case is determined by the formula (2.21), where it should be assumed that $\Theta = 0$.

Here, we consider several limiting cases of most physical interest. Let us assume that conditions (3.1) are met; i.e., the nonrelativistic electron moves in a weak magnetic field. In order to obtain the asymptotics for the value of $\Delta E_n^\xi(\Theta = 0)$, determined by formula (2.21), we divide the integration region for variable u into two parts. In the first region $u \in [0, u_0]$ and in the second region $u \in [u_0, 1]$, where the value of the magnitude u_0 satisfies the condition

$$2\beta \ll u_0 \ll 1. \quad (4.1)$$

Then, in the first region, we expand the integrand in formula (2.21), except for corresponding exponent, in a series in the variable u, since $u \leq u_0 \ll 1$, and in the second region, where $u \geq u_0 \gg 2\beta$, the main contribution to the integral is given by the region of integration where the variable $t \ll 1$, and we expand the integrand in a series in the variable t. In both cases, integrals over variable t are taken using the following formulas [57]:

$$\int_0^\infty t^{\mu-1}\sin at \sin bt dt = \frac{\Gamma(\mu)}{2}\cos\frac{\pi\mu}{2} \times (|b-a|^{-\mu} - (b+a)^{-\mu}), \quad a > b, \quad b > 0, \quad a \neq b, \quad -2 < \Re\mu < 1, \tag{4.2}$$

$$\int_0^\infty t^{\mu-1}\sin at \cos bt dt = \frac{\Gamma(\mu)}{2}\sin\frac{\pi\mu}{2}((a+b)^{-\mu} + |a-b|^{-\mu} sign(a-b)), \quad a > 0, \quad b > 0, \quad |\Re\mu| < 1. \tag{4.3}$$

After the integration over variable t, we obtain

$$\Re(\Delta E_n^\xi) = -\xi\frac{e^2}{16\pi^{\frac{3}{2}}\sqrt{\beta}}[I_1 + I_2]. \tag{4.4}$$

Here,

$$I_1 = 2\sqrt{\pi\beta}\int_0^{u_0}\left[\left(\frac{u}{u+2\beta}\right)^{\frac{1}{2}} - \left(\frac{u}{u-2\beta}\right)^{\frac{1}{2}}\Theta(u-2\beta)\right]du + 2\sqrt{\frac{\pi}{\beta}}\int_0^{u_0}\sqrt{u}[u^{\frac{1}{2}} - (u+2\beta)^{\frac{1}{2}}]du + 2\sqrt{\frac{\pi}{\beta}}\int_0^{u_0}\sqrt{u}[u^{\frac{1}{2}} - (u-2\beta)^{\frac{1}{2}}\Theta(u-2\beta)]du + \cdots, \tag{4.5}$$

$$I_2 = \sqrt{\pi}\beta^{\frac{3}{2}}[1 - u_0 + 2\ln u_0] + \cdots.$$

The dots in (4.5) correspond to terms of a higher order of smallness in the parameter β, and the Heaviside step function $\Theta(x)$ is introduced.

As a result, after integration over variables u and t, the terms that depend on the quantity u_0 cancel each other in the same order of perturbation theory, and in the first approximation, we obtain the following result:

$$\Re(\Delta m_\xi) \simeq \Re(\Delta E_n^\xi) \simeq -\xi\frac{e^2}{8\pi}\left[\frac{3\beta}{2} + \beta\ln\left(\frac{\beta}{2}\right)\right], \quad \beta \ll 1, \qquad n \ll \beta^{-1}. \tag{4.6}$$

In the quasiclassical approximation (3.9), the radiative mass shift of the electron, which is associated with the anomalous magnetic moment of an electron in QED without the Chern-Simons term, is described by the formula

$$\Delta m_\xi = \xi\frac{e^2}{8\pi^{\frac{3}{2}}}\int_0^1\frac{u-2}{1-u}du\int_0^\infty\sqrt{t}\exp\left[-i\left(\frac{\pi}{4} + \lambda t + \frac{t^3}{3}\right)\right]dt, \tag{4.7}$$

where

$$\lambda = \left[\frac{u}{(1-u)\kappa}\right]^{\frac{2}{3}}. \tag{4.8}$$

For calculation of integrals in formula (4.7), we use the Mellin transformation with respect to the parameter $a = \frac{1}{\kappa}$:

$$F(s) = \int_0^\infty a^{s-1}F(a)da, \tag{4.9}$$

$$F(a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)a^{-s}ds. \tag{4.10}$$

As a result, we can then recast Eq. (4.7) into the form

$$\Delta m_\xi = -\xi\frac{e^2}{16\pi^{\frac{1}{2}}}\frac{1}{2\pi i}\exp\left[-i\frac{\pi}{4}\right] \times \int_{\gamma-i\infty}^{\gamma+i\infty}a^{-s}\exp\left[-i\frac{\pi}{4}(2s+1)\right]\Gamma\left(-\frac{s}{2}+\frac{1}{2}\right)3^{-\frac{s}{2}+\frac{1}{2}} \times \frac{(s+1)\Gamma(\frac{3s}{2})}{\sin\pi s}ds, \qquad 0 < \gamma < \frac{2}{3}, \tag{4.11}$$

where $\Gamma(z)$—is the Euler gamma-function, which is a meromorphic function with poles of the first order at the points $z = 0, -1, -2, \ldots$ and with a residue equal to $\frac{(-1)^n}{n!}$ at $z = -n$. Further, if $\kappa \ll 1$, we close the integration contour in (4.11) in the right half-plane and obtain an asymptotic series in powers κ. For $\kappa \gg 1$, we must close the integration contour in the left half-plane of the complex variable s and obtain a convergent series in inverse powers κ. Thus, Eq. (4.11) allows the following asymptotic expansions to be derived:

$$\Re(\Delta m_\xi) = -\xi\frac{e^2}{8\pi}\left[\kappa\ln\kappa + \frac{\kappa}{2}\left(1 - \ln 3 - \psi(1) + 3\psi\left(\frac{3}{2}\right)\right)\right] + \cdots, \kappa \ll 1, \tag{4.12}$$

where $\psi(x)$ is the Euler ψ-function. Calculation of energy shift of the ground electron state, which is similar to calculation shown above for nonrelativistic electron, gives the result matching the right-hand part of formula (4.6) at $\xi = -1$:

$$\Delta m(n=0) = \frac{e^2}{8\pi}\left[\frac{3\beta}{2} + \beta\ln\frac{\beta}{2}\right]. \tag{4.13}$$

In superstrong magnetic fields ($H \gg H_0$) electron even in the first excited state ($n = 1$) is relativistic. Assuming that, similarly to QED_{3+1}, the main contribution to the integral is given by the region of integration where

$$t\frac{1-u}{u} \gg 1,$$

we find that the mass shift of the ground state electron $(n=0, \xi=-1)$ and the AMM in the excited state with $(n=1)$ are described by the following asymptotic formulas:

$$\Delta m(n=0) \simeq \frac{e^2}{8\pi} \ln 2\beta, \qquad \beta \gg 1, \qquad (4.14)$$

$$\Re(\Delta E_1^{\xi}) \simeq \xi \frac{e^2}{16\pi\sqrt{\beta}} [C + 4 \ln 2], \qquad \beta \gg 1, \qquad (4.15)$$

where $C = 0.577\ldots$—is the Euler constant. Asymptotics similar to (4.13)–(4.14) were first obtained in [47], where only the case of the ground electronic state was considered.

V. CONCLUSION

The one-loop vertex calculation of the AMM in QED_{2+1} was performed earlier in the approximation of the small transferred momentum of an electron [46]. It was shown that there is an infrared divergence in the magnetic moment, which can be regularized by including the Chern-Simons term [46,48]. This is in contrast to the AMM in QED_{3+1} that is infrared finite. In [61], in order to avoid the logarithmic divergences in the computation of the AMM, the spectral representation for the dressed photon propagator was used. In this paper, it was shown that at weak field $(H \ll H_0)$ in the one-loop approximation, the interaction energy of the spin with the external field in QED_{2+1} without the Chern-Simons term is proportional to the product $\kappa \ln \kappa$, where κ is the dynamic parameter of synchrotron radiation (3.13). It means that the external magnetic field plays the role of the infrared regulator in the calculation of the anomalous magnetic moment in QED_{2+1}.

It should be noted that the contribution to the AMM coming from the term in the photon propagator, which is proportional to the Chern-Simons parameter Θ, is convergent at $\rho = 0$ and does not contain infrared divergence [46,48].

Formulas (3.8) and (3.16) for the contribution to the AMM of an electron from the $g_{\mu\nu}$ term in the photon propagator, as was noted above, were obtained in [46] as a result of the vertex function calculation. On their basis, in [46], it was concluded that the AMM of an electron in QED_{2+1} without a Chern-Simons term contains an infrared divergence $(F_2 \propto \ln \rho)$ which is eliminated in topologically massive QED_{2+1}.

However, this statement, if understood literally, does not correspond to the results (4.6) and (4.12) for the interaction energy of the electron AMM with the relatively weak external magnetic field in QED_{2+1} without the Chern-Simons term. As it was demonstrated in Sec. IV, the interaction energy of the spin with a magnetic field in QED_{2+1} without the Chern-Simons term is proportional to $\beta \ln \beta$ and tends to zero at $\beta \to 0$ [formula (4.6)], or at $\kappa \to 0$ [formula (4.12)], and the AMM of an electron in these limiting cases increases logarithmically.

Therefore, we think, the statement that the Chern-Simons parameter plays the role of a regulator for infrared divergence appears unsatisfactorily proven in [46], primarily because the authors of [46] did not verify the validity of their result (18) for a weak magnetic field.

The AMM of electrons subjected to an external magnetic field is derived from the spin-dependent part of the radiative shift in the total interaction energy [1,2,6–8,50,51].

In the weak magnetic field, the energy shift of the nonrelativistic electron in topologically massive QED_{2+1} depends, as was demonstrated in Sec. III, on the Chern-Simons parameter $\rho = \frac{\Theta}{m}$ and the field parameter $\beta = \frac{H}{H_0}$. For the ground electron state, this important result was obtained in [47], where the mass shift asymptotics of the electron mass were found for the case when simultaneously $\beta \ll 1$, $\rho \ll 1$, with arbitrary ratio between these parameters. Note that just in this particular case, when the following condition is met,

$$\beta \ll \rho \ll 1, \qquad (5.1)$$

Equation (11) in [47] for the mass shift of electrons within the logarithmic approximation for the parameter ρ coincides with the result (3.8) in [48] and agrees with formula (3.8) in this work. Therefore, as it was demonstrated in Sec. III, limiting transition $\rho \to 0$ in formulas (3.8) and (3.15)–(3.16) in a constant magnetic field is impossible in principle.

In addition, for the QED_{2+1} without the Chern-Simons term in the one-loop approximation, at $\beta \to 0$ the electron AMM increases logarithmically, which differs entirely from the case of QED_{3+1} [2,6–9]. It should also be noted that the result (4.9) for the radiation-induced electron mass shift in the theory with the Chern-Simons term leads to the result (4.13) obtained in the framework of the theory without the Chern-Simons term using a mere substitution $\rho \to \beta$.

It is interesting to compare expressions (4.14)–(4.15) with the corresponding results obtained in QED_{3+1}. In the region $H \gg H_0$ in QED_{3+1}, the mass shift of the ground state electron $(\xi = -1, n = 0)$ [6,62] and the AMM in the weakly excited states are described by the formulas [8,63]:

$$\Delta m(n=0) = \frac{\alpha m}{4\pi} \ln^2 \frac{2H}{H_0},$$

$$\Delta \mu(n=1) = -\frac{\alpha e}{4\pi m} \frac{H_0}{H} \ln \frac{2H}{H_0}. \qquad (5.2)$$

It follows from (4.14)–(4.15) and (5.2) that reducing the dimension from four to three drastically changes the behavior of these values as functions of the magnetic field:

$$\frac{\Delta m^{(2+1)}(n=0)}{\Delta m^{(3+1)}(n=0)} \sim \frac{1}{\ln\beta},$$
$$\frac{\Delta \mu^{(2+1)}(n=0)}{\Delta \mu^{(3+1)}(n=0)} \sim \frac{\ln\beta}{\beta}.$$

The general conclusion can be made that the space dimension reduction leads to significant changes in dependence of the electron AMM on intensity of the magnetic field and energy of electron. In order to establish the limits of applicability for the one-loop approximation and to clarify whether logarithmic contributions to the electron AMM remain in higher orders of perturbation theory, as in QED_{3+1} [7], it would be necessary to study the contribution of fourth-order diagrams to the elastic scattering amplitude of an electron.

The possibility for experimental observation of the dynamic nature of electron AMM in graphene [35,36] or other planar structures [33] is especially interesting. Experiments demonstrate that spin g-factor in graphene takes values in the range of $g_s \simeq 2.2$–2.7 [35,36]. In this case the energy $(\Delta E_s = g_s \mu_B B)$ of electron AMM interaction with external magnetic field $B \propto (11\text{–}14)T$, has the value of order $(1.5-2)$ meV and depends weakly on magnetic field.

For theoretical explanation of these experiments, a model of pseudo-QED is used in [37]. Mass operator is calculated in linear approximation in the magnetic field strength [37], i.e. dynamic nature of AMM of electron is not taken into consideration. For the effective mass of electron, a value $M \approx (0.6-10)$ meV is used. Similar M values range was found in estimate [30,35,36] resulting from dynamic generation due to electron-electron and other interactions in graphene without a magnetic field (see also [64–66]). For the quoted values of fermion mass, condition $H \ll H_0^{(2)} = (\frac{M}{m})^2 \cdot 4.41 \times 10^{13} G$ in experiments [35,36] is invariably true. For instance, at $M = 0.5 \times 10^{-3}$ eV, we have $H_0^{(2)} = 4.41 \times 10^{21} G$ and the induction of external magnetic field (in Gaussian system) is $\propto 10^5 G$. Then, according to formula (4.6), for nonrelativistic electron states with the main quantum number of $n \ll 2 \times 10^6$, the energy of electron AMM interaction with external magnetic field appears proportional to magnitude of $\beta \ln \beta$ in a sufficiently broad range of values for the induction of magnetic field. Therefore, a reliable experimental detection of nonlinear magnetic field strength dependence for the electron AMM interaction energy would be a direct evidence for the dynamic nature of the electron AMM in graphene. This can also become an important stimulus for the further developments in theoretical studies for the impact of dynamic generation on electron masses in two-dimensional systems.

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