Topology and strong four fermion interactions in four dimensions

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We study massless fermions interacting through a particular four-fermion term in four dimensions. Exact symmetries prevent the generation of bilinear fermion mass terms. We determine the structure of the low-energy effective action for the auxiliary field needed to generate the four-fermion term and find it has an novel structure that admits topologically nontrivial defects with nonzero Hopf invariant. We show that fermions propagating in such a background pick up a mass without breaking symmetries. Furthermore, pairs of such defects experience a logarithmic interaction. We argue that a phase transition separates a phase where these defects proliferate from a broken phase where they are bound tightly. We conjecture that, by tuning one additional operator, the broken phase can be eliminated with a single BKT-like phase transition separating the massless from massive phases.

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I. INTRODUCTION

In this paper, we construct a continuum theory of strongly interacting fermions in four dimensions in which exact symmetries prohibit the appearance of mass terms. We argue that the fermions nevertheless acquire masses at strong coupling by virtue of their interactions with a nontrivial vacuum corresponding to a symmetric four-fermion condensate. Our work points out the existence of new classes of theories of strongly interacting fermions which may be important in the search for candidate theories of BSM physics.

Furthermore, we show that the theory when discretized yields a staggered fermion lattice theory which has been the focus of several recent studies both in the particle physics and condensed matter communities [1–7] in both three and four dimensions. The numerical work in three dimensions is consistent with the absence of symmetry breaking bilinear condensates for all values of the four-fermion coupling. The model nevertheless has a two phase structure with a continuous phase transition with non-Heisenberg exponents separating a massless phase from a phase with a symmetric four-fermion condensate and massive fermions. Progress in understanding the nature of this phase diagram was given recently in [8]. In four dimensions, it appears that a very narrow symmetry broken phase emerges between the massless and massive phases.

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The ingredients of the theory are somewhat unusual; the fermions appear as components of a (reduced) Kähler-Dirac field and as a consequence the theory is invariant only under a diagonal subgroup of the Lorentz and flavor symmetries together with an additional $SO(4)$ symmetry. It is this reduced symmetry, which is enforced by the structure of the four-fermion term, that plays a key role in prohibiting conventional Dirac mass terms.

Our paper offers a way to understand the structure of the four-dimensional models from a continuum perspective where we will see that topological features of the continuum theory can play an important role.

II. FOUR-FERMION THEORY

To start consider a theory comprising four flavors of free massless Dirac fermion with (Euclidean) action:

$$S = \int d^4x \bar{\psi}^a \gamma_\mu \partial_\mu \psi^a(x). \tag{1}$$

This is invariant under the global symmetry $SO_{\text{Lorentz}}(4) \times SU_{\text{flavor}}(4)$. To build the model of interest, let us focus on the diagonal subgroup of the Lorentz symmetry and an $SO(4)$ subgroup of the original $SU(4)$ flavor symmetry which we call $\mathcal{T}$:

$$\mathcal{T} = SO'(4) = \text{diag}[SO_{\text{Lorentz}}(4) \times SO_{\text{flavor}}(4)]. \tag{2}$$

Under this symmetry, we may rewrite the action as

$$S = \int d^4x \text{Tr}(\bar{\Psi}\gamma_\mu \partial_\mu \Psi), \tag{3}$$

where we now treat the fermions as 4×4 matrices and the trace operation Tr occurring here and throughout the paper acts only on the matrix indices associated with the $\mathcal{T}$

symmetry. Actually, since the theory is massless we can decompose these matrices into two independent components using the twisted chiral projectors:

$$\Psi_{\pm} = \frac{1}{2}(\Psi \pm \gamma_5 \Psi \gamma_5), \tag{4}$$

and the fermion action can be reduced to two Dirac flavors with action

$$S = \int d^4x \mathrm{Tr}(\bar{\Psi}_+ \gamma_\mu \partial_\mu \Psi_-). \tag{5}$$

Notice that this projection only commutes with the $SO(4)$ subgroup of the original $SU(4)$ flavor symmetry. In the Appendix, we show that this reduction is equivalent to imposing the reality condition $\bar{\Psi} = \Psi$ with action

$$S = \int d^4x \mathrm{Tr}(\Psi \gamma_\mu \partial_\mu \Psi). \tag{6}$$

The equation of motion that follows from this action can be interpreted as the (reduced) Kähler-Dirac equation if one expands the fermion matrices on products of Dirac gamma matrices [9]. For the model we want to discuss, we will consider four copies of this system by taking these matrix fermions to additionally transform in the fundamental representation of an independent $SO(4)$ symmetry $\mathcal{S}$ i.e $\Psi^\alpha \to R^{\alpha\beta}\Psi^\beta$ with R an element of $SO(4)$.

Up to this point, everything we have done merely corresponds to a change of variables that serves to highlight a particular subgroup of the global symmetries—the diagonal subgroup of the Lorentz and flavor symmetries. The field content of the model still corresponds to eight flavors of massless Dirac fermions. However, this situation changes when I add four-fermion interactions of the following form:

$$\delta S = \frac{G^2}{4} \int d^4x \epsilon_{\alpha\beta\gamma\delta} \mathrm{Tr}(\Psi^\alpha \Psi^\beta) \mathrm{Tr}(\Psi^\gamma \Psi^\delta). \tag{7}$$

This interaction locks the Lorentz and flavor symmetries together and ensures that the global symmetries $\mathcal{G}$ of the theory are

$$\mathcal{G} = \mathcal{T} \times \mathcal{S} = SO'(4) \times SO(4). \tag{8}$$

It is of crucial importance to notice that the resultant theory does *not* admit any bilinear mass terms since $\mathrm{Tr}\Psi^\alpha\Psi^\alpha = 0$ and any terms of the form $\mathrm{Tr}\Psi^\alpha\Psi^\beta$ break the symmetry $\mathcal{S}$.

III. ASIDE: CONNECTION TO (REDUCED) STAGGERED FERMIONS

The motivation for this work derives in part from recent numerical investigations of lattice models involving four reduced staggered fermions interacting through the corresponding unique four-fermion interaction. In this section, we will show that the continuum model described earlier when discretized naturally leads to those lattice models. One way to discretize the continuum theory is to expand the fermion matrices on position dependent products of Dirac gamma matrices [10]. Consider the original Ψ,

$$\Psi(x) = \sum_b \gamma^{x+b} \chi(x+b), \tag{9}$$

where the components of the vector $b^i = 0, 1$ label points in the unit hypercube attached to site x in a four-dimensional hypercubic lattice and

$$\gamma^b = \prod_i (\gamma_i)^{b_i}. \tag{10}$$

Plugging this expansion into Eq. (6) and doing the trace over the gamma matrices yields the free reduced staggered fermion action comprising one single component lattice fermion at each lattice site:

$$\sum_{x,\mu} \chi(x) \eta_\mu(x) \Delta_\mu \chi(x) \tag{11}$$

with Δ_μ the symmetric difference operator and $\eta_\mu(x) = (-1)^{\sum_{i=0}^{\mu-1} x_i}$ the usual staggered fermion phase [11,12]. Equipping each of these fields with an index under the S symmetry and adding the four-fermion terms one arrives at

$$S_{\text{stag}} = \sum_{x,\mu} \chi^a(x) \eta_\mu(x) \Delta_\mu \chi^a(x) + \frac{G^2}{4} \sum_x \epsilon_{abcd} \chi^a \chi^b \chi^c \chi^d \tag{12}$$

which is precisely the action studied in [4,5]. Thus we expect that the continuum arguments described in this paper can be applied to understand the numerical results reported for this staggered fermion system.

IV. AUXILIARY FIELD ACTION

As usual our subsequent analysis requires replacing the four-fermion term given in Eq. (7) by a Yukawa coupling to an auxiliary scalar field

$$S_0 = \int d^4x \left[iG\phi_+^{\alpha\beta}(x) \mathrm{Tr}(\Psi^\alpha \Psi^\beta) + \frac{1}{4}(\phi_+^{\alpha\beta})^2 \right] \tag{13}$$

The auxiliary field is a antisymmetric matrix and satisfies a self-dual condition $\phi_+ = \mathcal{P}^+ \phi$ where the projector $\mathcal{P}_+$ is defined as

$$\mathcal{P}^+_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\right) \tag{14}$$

Notice that the original four-fermion interaction can be written as

$$[\mathrm{Tr}\Psi^\alpha\Psi^\beta]_+^2 = \frac{1}{4G^2}(\phi_+^{\alpha\beta})^2 \tag{15}$$

This structure ensures that ϕ_+ transforms in the adjoint representation under a $SU_+(2)$ subgroup of the $\mathcal{S}$ symmetry $SO(4) = SU_+(2) \times SU_-(2)$. It is a singlet under both $SU_-(2)$ and the internal $\mathcal{T}$ symmetries (see the Appendix for more details). Furthermore, it is easy to see that the eigenvalues of the resultant fermion operator come in complex conjugate pairs. In addition, each eigenvalue is doubly degenerate since the fermion operator also commutes with $SU_-(2)$. These facts ensure that the Pfaffian that results from integration over the fermions is in fact real, positive definite.

V. EFFECTIVE ACTION

Returning to Eq. (13) we now integrate out the fermions using positivity of the Pfaffian and consider the form of the one loop effective action.

$$S_{\mathrm{eff}} = -\frac{1}{4}\mathrm{Tr}\ln\left(-\Box + G^2\mu^2 + G\gamma_\mu\partial_\mu\phi_+\right) \tag{16}$$

where $\phi_+^2 = \mu^2 I$ and we have absorbed the explicit factor of i into the auxiliary field to render ϕ_+ hermitan. Let us first consider the Coleman-Weinberg effective potential obtained by assuming a constant auxiliary field

$$V_{\mathrm{eff}}(\mu) = -\frac{1}{4}\mathrm{Tr}\ln\left(\frac{-\Box + G^2\mu^2}{-\Box}\right) + \mu^2 \tag{17}$$

where we have subtracted off the value of V_{eff} at $G = 0$ and added in the classical action for ϕ_+. If we expand the remainder in powers of G it should be clear that V_{eff} develops a minimum away from the origin for sufficiently large $G > G_c$. Thus naively one expects the system to enter a symmetry broken state for some value of the four-fermion coupling. This is the usual NJL scenario and in this case will correspond to a breaking pattern $SU_+(2) \to U(1)$ corresponding to a vacuum manifold with the topology of S^2.

Of course to understand the dynamics of the theory in more detail we need to compute the leading terms in the effective action for ϕ_+ for *nonconstant* fields. Expanding the latter on a suitable 4×4 basis T (see the Appendix for more details) we find

$$\phi_+(x) = \sum_{a=1}^{3}\phi_+^a(x)T_a = \sum_{a=1}^{3} n^a(x)\sigma^a \otimes I \tag{18}$$

In this basis, the fermion operator has a trivial dependence on $SU_-(2)$ and we will suppress it in our subsequent analysis. For $G > G_c$ the field $n^a(x)$ obeys the $O(3)$ constraint $n^a n^a = 1$. The effective action governing the fluctuations in $n^a(x)$ is now given by a derivative expansion of

$$-\frac{1}{4}\mathrm{Tr}\ln\left(I + m\frac{\gamma_\mu\partial_\mu n^a\sigma^a}{-\Box + m^2}\right) \tag{19}$$

where $m = G\mu$. At leading order, one encounters an $O(3)$ symmetric term quadratic in the derivatives of $n^a(x)$ (see the Appendix):

$$a(G)\int d^4x(\partial_\mu n^a)^2. \tag{20}$$

However, at higher orders in $1/m$, one also encounters an additional quartic term which can play an important role in understanding the possible phases of the theory:

$$b(G)\int d^4x(\epsilon^{abc}\partial_\mu n^a\partial_\nu n^b)^2. \tag{21}$$

The combination of these two terms defines the Fadeev-Skyrme model which is known to possess topologically stable field configurations which we will argue can play a role in the current theory.

The analysis of the dynamics is facilitated by a further change of variables in which the $O(3)$ vector n^a is replaced by a $SU(2)$ matrix field which rotates $n^a\sigma^a$ to a fixed matrix, say σ_3:

$$n^a(x)\sigma^a = U^\dagger(x)\sigma_3 U(x). \tag{22}$$

This has the immediate advantage that the nonlinear constraint $n^a n^a = 1$ is simply replaced by the unitarity property of $U = e^{i\theta^a\sigma^a}$ with the angular variables θ's unconstrained. Of course, this mapping cannot be the whole story since the manifold of $SU(2)$ is S^3 not S^2 and indeed it is easy to see that n^a is invariant under *local* left multiplication of $U(x)$ by an element of $U(1)$:

$$U(x) \to e^{i\sigma_3\beta(x)}U(x). \tag{23}$$

The action is also manifestly invariant under right multiplication by a *global* $SU(2)$ rotation $U \to UG$. Thus the final effective action for U should respect both this global $SU(2)$ symmetry and the local $U(1)$ gauge symmetry. We can make the local invariance explicit if we replace ordinary derivatives by covariant derivatives with the leading term now being

$$S_{\mathrm{eff}} = a(G)\int d^4x\mathrm{tr}[(D_\mu U)^\dagger(D_\mu U)] + \cdots, \tag{24}$$

where $D_\mu = \partial_\mu + iA_\mu\sigma_3$ and A_μ is an Abelian gauge field needed to enforce the $U(1)$ symmetry given in Eq. (23).

This action is classically equivalent to the original one. However, in this case one would also expect to find a Maxwell term corresponding to this exact local $U(1)$ invariance:

$$\delta S_{\rm eff} = b(G) \int d^4x F_{\mu\nu} F_{\mu\nu}. \tag{25}$$

Indeed, classically, the field strength can be expressed in terms of $O(3)$ vector n [13] as

$$F_{\mu\nu} = n.(\partial_\mu n \times \partial_\nu n), \tag{26}$$

and we see that the Maxwell term just represents the higher order term in Eq. (21).

In this picture, a conventional broken phase for the sigma model eg $n^a = \delta^{a3}$ leads to $U = I$ up to gauge transformations and corresponds to a Higgs phase with photon mass $\sqrt{a(G)}$. Close to G_c the photon mass is large and the gauge field decouples from long-distance physics so that this regime is governed by the usual $O(3)$ sigma model action.

VI. TOPOLOGICAL DEFECTS

While the uniform phase is always a possible vacuum solution additional possibilities arise at strong coupling where the quartic term plays a role. Let us search for nontrivial field configurations. To try to keep the action finite forces us to look for solutions where $D_\mu U \to 0$ as $r \to \infty$ and corresponding to vanishing photon mass. This implies

$$\partial_\mu U = -i A_\mu \sigma_3 U \tag{27}$$

or

$$A_\mu = \frac{i}{2} \mathrm{tr}(\partial_\mu U U^\dagger \sigma_3). \tag{28}$$

The long-distance contribution to the action of such a configuration is then determined by the Maxwell term

$$b(G) \int d^4x \frac{1}{4} (\mathrm{tr} \partial_\mu U \partial_\nu U^\dagger \sigma_3)^2. \tag{29}$$

A topological defect must then correspond to a $U(x)$ configuration that maps nontrivially at infinity into the S^2 target space. Such a mapping exists, is termed the Hopf map, and corresponds to $\Pi_3(S^2) = Z$. If we parametrize a general U matrix as

$$U = \begin{pmatrix} \alpha_1 + i\alpha_2 & -\alpha_3 + i\alpha_4 \\ \alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix} \tag{30}$$

with $\sum_i \alpha_i^2 = 1$, then the simplest topological defect corresponds to setting $\alpha_i = \frac{x_i}{r}$ where x_i are the four-dimensional coordinates. This parametrization yields a $S^3 \to S^3$ map but this is reduced to the Hopf map when U fields which are gauge equivalent are identified. A similar topological defect solution was constructed in a four-dimensional Yang-Mills-Higgs system in [14]. The α_i correspond to trigonometric functions of angles in four-dimensional polar coordinates and it can easily seen that the action given in Eq. (29) corresponding to such a defect diverges logarithmically with system size.[1] Furthermore the topological charge of this object can be obtained from the theta term corresponding to the $U(1)$ field.

$$\frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\lambda} \mathrm{tr}(\partial_\mu U \partial_\nu U^\dagger \sigma_3) \mathrm{tr}(\partial_\rho U \partial_\lambda U^\dagger \sigma_3) \tag{31}$$

Unlike the action this term does not diverge logarithmically since it may be recast as a Chern-Simons term which can be computed on the boundary sphere at infinity.

While such a background corresponds asymptotically to a point on the vacuum manifold it clearly does not break the $\mathcal{S}$ symmetry since $\langle \sum_x \phi_+(x) \rangle = 0$. Of course the key question is whether such defects can play a role in determining the phase structure of the model. At first glance, they should not—the logarithmically divergent action corresponding to such defects will ensure that a single defect is completely suppressed in the infinite volume limit. This situation is analogous to the behavior of vortices in the two-dimensional XY model which also possess a log divergent action. In the latter case, a configuration of finite action can be constructed consisting of a vortex and anti-vortex. The action for such a configuration depends logarithmically on the separation of the two vortices which hence bind tightly together at low temperatures. However, since the entropy associated with a vortex also increases logarithmically with system size, a BKT phase transition develops as the temperature is raised and vortices unbind and populate the ground state.

We propose that a similar phenomena may occur in this four-dimensional model—that is, the ground state for $G \sim G_c$ consists of tightly bound Hopf–anti-Hopf defects. In such a scenario, the disordering effects of the defects are suppressed and one expects a conventional symmetry broken (Higgs) phase to appear as has been observed in the numerical simulations [4,7]. However, as the coupling is increased still further, the defects may unbind via another transition to populate and disorder the ground state. This condensate of Hopf defects with $\langle \phi_+^2 \rangle \neq 0$ would then correspond to the four-fermion condensate in the original four-fermion model consistent with Eq. (15). An estimate for the critical coupling can be arrived at by comparing the entropy associated to the location of a single defect $S \sim \ln V$ with its action $E \sim b(G) \ln V$ yielding $b(G)^{\rm crit} \sim 1$.

It is interesting to compute the fermion propagator in the background of such a defect. Consider the $\mathcal{S}$-symmetric correlator

[1]For a Hopf defect the gauge field corresponds to a large gauge transformation.

$$G_F(x,y) = \mathrm{tr}\langle\Psi(x)\Psi(y)\rangle = \mathrm{tr}\left[\frac{-\gamma_\mu\partial_\mu + mn^a\sigma^a}{(-\partial_\mu^2 + m^2 + mP)}\right] \quad (32)$$

where

$$P = \gamma_\mu(\partial_\mu U^\dagger(x)\sigma_3 U(x) + U^\dagger(x)\sigma_3\partial_\mu U(x)) \quad (33)$$

and the trace is to be carried out over the $\mathcal{S}$-indices. Using the fact that the covariant derivative vanishes far from the core of the defect allows us to show that $P = 0$ and the propagator in that region simplifies to

$$G_F(x,y) = \frac{-2\gamma_\mu\partial_\mu}{-\Box + m^2} \quad (34)$$

Thus the fermion acquires a mass $m = \mu G$ in the background of such a defect. This gives a concrete realization of the mechanism discussed in [15] and is consistent with strong coupling expansions for staggered fermions [5].

VII. BKT TRANSITION

We have argued that the model possesses a conventional broken phase (or Higgs phase) which gives way to a symmetric phase at stronger coupling due to unbinding of topological defects. Since mechanisms for giving fermions a mass are quite different in the two regimes one might expect a discontinuous phase transition separates the broken phase and the defect phase. To obtain a true BKT-like transition requires one to pass directly between the massless and massive symmetric phases. To effect such a scenario one can generalize the original four-fermion model to a true Higgs-Yukawa model by the addition of a kinetic term for the auxiliary field ϕ_+. One can then imagine tuning the coupling of this kinetic operator so as to cancel out the effects of the leading gradient term Eq. (24). This sets the photon mass to zero and eliminates the Higgs phase of the model. We conjecture that in this limit a true single BKT transition separates the massless and massive phases.

VIII. SUMMARY

We have argued that a particular four-dimensional continuum theory possesses an interesting phase structure as a function of the coupling to a particular four-fermion interaction. For sufficiently weak four-fermion coupling, we expect the theory to describe massless noninteracting fermions. As the coupling is increased, the system should undergo a NJL-like phase transition to a phase in which the $SO(4)$ symmetry is spontaneously broken via a bilinear fermion condensate. In the auxiliary field picture, this phase is characterized by tightly bound pairs of Hopf defects and a nonzero expectation value for the scalar field. As the coupling is increased further we argue that these defects may unbind at a transition to populate and disorder the vacuum restoring the symmetry. In the background of such defects, the fermions acquire a mass without breaking symmetries. This phase is interpreted as a four-fermion condensate in the original fields. We also argue that by an additional tuning of the kinetic energy the broken phase can be eliminated and a single BKT transition would separate the massless from massive phases.

The continuum theory we describe possesses an unusual Lorentz symmetry which is locked via the four-fermion interaction with an internal flavor symmetry. At weak coupling, we expect the four-fermion term to be irrelevant and the IR description of the theory will correspond to sixteen flavors of free Majorana fermion with the symmetry enhancing to the usual Lorentz and flavor symmetries. Correspondingly, the beta function for the four-fermion coupling has an IR attractive fixed point at $G = 0$. The transition to a phase of broken symmetry is likely of the NJL type and hence the corresponding (IR unstable) fixed point would lie in the universality class of the usual Higgs-Yukawa theory. However, if an additional continuous transition were to separate this phase from the four-fermion condensate phase, this would correspond to a new strongly coupled IR fixed point. This would be a fascinating prospect. The BKT limit would correspond to a situation where the two fixed points bounding the broken phase merge into a single continuous transition.

We have also argued that this continuum theory naturally discretizes to yield a theory of strongly interacting reduced staggered fermions. This lattice model has received some recent attention and the numerical phase diagram that has been uncovered matches quite closely with the gross features described in this paper. Indeed, in the condensed matter literature, there has recently been a great deal of interest in models which are able to gap fermions without breaking symmetries using carefully chosen quartic interactions [16]. This work has even been used to revive an old approach to lattice chiral gauge theories due to Eichten and Preskill [17] in which mirror states of a definite chirality can be gapped out of an underlying vector like lattice theory using four-fermion interactions [18]. It will be interesting to see whether the current model can be generalized to implement such constructions. Independent of this potential connection, the possibility of new phases and critical points in strongly interacting fermion systems in four dimensions is very interesting in its own right, and we hope the current work stimulates further work in this area.

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APPENDIX A: OBTAINING THE TWISTED MAJORANA FORM

Setting $\bar{\Psi}_+ = C^{-1}\Psi_+^T C$ where C is the charge conjugation operator the action can be rewritten

$$S = \int d^4x \mathrm{Tr}(C^{-1}\Psi_+^T C\gamma_\mu \partial_\mu \Psi_-) \tag{A1}$$

Taking the transpose of this equation yields

$$S = \int d^4x \mathrm{Tr}(C^{-1}\Psi_-^T C\gamma_\mu \partial_\mu \Psi_+) \tag{A2}$$

Adding these two expressions the action can be expressed entirely in terms of the field $\Psi = \Psi_+ + \Psi_-$.

$$S = \int d^4x \mathrm{Tr}(C^{-1}\Psi^T C\gamma_\mu \partial_\mu \Psi) \tag{A3}$$

But $C^{-1}\Psi^T C = \Psi$ if one expresses the matrix Ψ as a sum over the Clifford algebra formed from the product of Dirac gamma matrices so that the action in (twisted) Majorana form is simply

$$S = \int d^4x \mathrm{Tr}(\Psi\gamma_\mu \partial_\mu \Psi) \tag{A4}$$

APPENDIX B: CHANGING BASIS TO $SU(2)\times SU(2)$

We can verify the mapping into the $O(3)$ nonlinear sigma model by starting from an explicit 4×4 basis for the hermitian self-dual field $\phi_+ = \sum_{a=1}^3 \phi_+^a T_a$

$$T_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

These matrices clearly obey an $SU(2)$ algebra which is part of the original $SO(4)$ $\mathcal{S}$ algebra and the self-dual condition is clearly equivalent to the statement that ϕ_+ transforms in the adjoint representation of that $SU(2)$. The other independent $SU(2)$ contained in $\mathcal{S}$ is given the generators

$$U_1 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix} \quad U_3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$$

Using the similarity transformation P given by

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{pmatrix} \tag{B1}$$

one can verify that the generators T and U take the form

$$T^a = \sigma^a \otimes I \quad \text{and} \quad U^a = I \otimes \sigma^a \tag{B2}$$

This makes it clear that T^a (and hence ϕ_+) are singlets under $SU_-(2)$.

APPENDIX C: LARGE MASS EXPANSION

Starting from the expression

$$S_{\mathrm{eff}} = -\frac{1}{4}\mathrm{Tr}\ln(-\Box + m^2 + m\gamma_\mu\partial_\mu n^a\sigma^a), \tag{C1}$$

we first subtract the contribution at $m = 0$ and write

$$S_{\mathrm{eff}} = -\frac{1}{4}\mathrm{Tr}\ln\left[\left(\frac{-\Box + m^2}{-\Box}\right)\left(I + \frac{m\gamma_\mu\partial_\mu n^a\sigma^a}{-\Box + m^2}\right)\right]. \tag{C2}$$

The first factor inside the logarithm yields the effective potential previously described. So we focus on the second factor. Clearly one can imagine expanding this term in powers of $1/m$. To yield a nonzero result one must arrange for a nonzero trace over products of Dirac gamma matrices and Pauli matrices. The leading term clearly arises at second order in $1/m$ and is

$$\int d^4x \frac{\Lambda^4}{m^2}(\partial_\mu n^a)^2, \tag{C3}$$

where Λ is a UV cutoff. This term is quite generic and would arise independent of the structure of the Yukawa term. The structure of the quartic term depends crucially on the interplay of the $SU(2)$ and Dirac structures:

$$\int d^4x \frac{\Lambda^4}{m^4}(\partial_\mu n^a \partial_\nu n^b)^2. \tag{C4}$$

Since these operators contain the cutoff the coefficients must be renormalized to yield a finite effective action. We will not attempt that process here but merely note that the coefficients of the effective action will have an explicit dependence on the mass m and hence coupling G and we write them as $a(G)$ and $b(G)$.

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