Spectral distances on the doubled Moyal plane using Dirac eigenspinors

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We present here a novel method for computing spectral distances in the doubled Moyal plane in a noncommutative geometrical framework using Dirac eigenspinors, while solving the Lipschitz ball condition explicitly through matrices. The standard results of longitudinal, transverse, and hypotenuse distances between different pairs of pure states have been computed and the Pythagorean equality between them has been reproduced. The issue of the nonunital nature of the Moyal plane algebra is taken care of through a sequence of projection operators constructed from Dirac eigenspinors, which plays a crucial role throughout this paper. At the end, a toy model for a "Higgs field" has been constructed by fluctuating the Dirac operator and the variation on the transverse distance has been demonstrated, through an explicit computation.

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I. INTRODUCTION

Alain Connes, through his non commutative geometry, provided a new insight into the structure of the standard model of particle physics [1] (see [2] for a review). Here one essentially captures the whole gauge symmetry of the standard model (which is formulated using the "Almost commutative spacetime" model) through the group of inner-automorphism of the algebra

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \tag{1}$$

One of the most remarkable features of this formulation is that the Higgs field arises naturally here, along with the other gauge fields of Yang Mills theory, thus providing a unified conceptual perspective regarding their geometrical origin. This formulation, although quite successful in predicting or postdicting almost all the phenomenological observations made so far, including the computation of Higgs mass of 125 GeV [3], is not equipped yet to address the issues of quantum gravity, which is expected to play a key role only in the vicinity of the Plank energy scale $\sim 10^{19}$ GeV. One rather expects the differentiable manifold M, representing the (Euclidianized) spacetime, to be replaced by a truly noncommutative space as follows from some plausibility arguments due to Doplicher, fredenhagen, and Roberts, 1995 [4]. The simplest of such model noncommutative spaces are

$$\begin{aligned} &\text{Moyal plane } (\mathbb{R}^2_*)\colon\ [\hat{x}_1, \hat{x}_2] = i\theta; \\ &\text{Fuzzy sphere } (\mathbb{S}^2_*)\colon\ [\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k. \end{aligned} \tag{2}$$

The computation of distances on a generic noncommutative space has to be performed with the formalism of noncommutative geometry as pioneered by Connes [5]. In this context the computation of spectral distances for which the algebra of the spectral triple is given by (1) can be shown to be equivalent to a particular spectral triple where the algebra in (1) is replaced by just $\mathbb{C}^2$ [6,7]. This results from the fact that the incorporation of the experimental data on the Dirac operator causes the distance between any pair of states in $M_3(\mathbb{C})$ to diverge and that the algebra of quarternions $\mathbb{H}$ has only one state [6]. Besides, the algebra $\mathcal{A} = \mathbb{C}^2$ was used by Dungen & Suijlekom, 2013 [8] to discuss electrodynamics in a noncommutative geometric setup. The internal space, described by the algebra $\mathcal{A} = \mathbb{C}^2$ along with other data for spectral triple [see (A1) and for other details regarding this spectral triple see the Appendix A], when composed with that of the Moyal plane $\mathbb{R}^2_*$ [see (10)] describes the doubled Moyal plane $\mathbb{R}^2_* \cup \mathbb{R}^2_*$ [see (25)] for the corresponding spectral triple.

The question of the computability of spectral distances between pair of states associated to the same or different manifold arises naturally in this context. This is an important question in its own right as it may reveal some geometrical features of the Higgs field. This is so because the presence or absence of the Higgs field can have a nontrivial impact on these distances. These questions were addressed already in the literature [6–8]. One is particularly interested in computing the distance between states belonging to one of the copies of algebra and its "clone" (the more precise meaning of this terminology will be explained in subsequent sections) belonging to the other algebra. We refer to such a distance as "transverse." On the other hand

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distance between a pair of states belonging to the same algebra to be referred as "longitudinal." Finally, distance between any pair of pure states which are not clones of each other but belong to different algebras will be referred to as the "hypotenuse" distance. In [6] the authors have proved various theorems and laid down the conditions for the Pythagoras theorem to hold for commutative manifold M, described by a commutative C^*-algebra and also extended them to noncommutative spaces like the doubled Moyal plane [7], which is necessarily described by a noncommutative C^*-algebra. Eventually, these analyses were also extended to study distances between the nonpure states involving more general spectral triples to find that in the generic cases the Pythagoras equalities are replaced by corresponding inequalities [9] and analyzed their relationships with Wasserstein distance W of order 1, which occurs in the theory of optical transport [10].

Here we would like to revisit the problem by explicitly constructing suitable Dirac eigenspinors for the doubled Moyal plane, by making use of the corresponding eigenspinors for single Moyal plane, introduced in [11] through a Hilbert-Schimdt operator formulation to describe non relativistic noncommutative quantum mechanics [12]. This has the advantage that it involves a purely operator formulation and is quite transparent in execution as one has to deal with matrices only, albeit, big ones some times. We were therefore forced to make use of the computer algebra system *Mathematica* and with its help could reproduce all the existing results of the distances and eventually verify the Pythagoras theorem. At the end, we have explicitly shown the variation of transverse distance in the presence of a prototype "Higgs" field, which arises when the Dirac operator is fluctuated using a general one-form.

The paper is organized as follows. In Sec. II we have set up the framework of our calculation, namely the Hilbert-Schmidt operatorial formulation using which the spectral triple for the Moyal plane has been introduced. Moreover, a review of the computation of the spectral distance on the Moyal plane using Dirac eigenspinors [11] along with the aspects of translational invariance on the Moyal plane has been provided here. Then in Sec. III, the notion of doubling the spectral triple for the Moyal plane by composing it with that of the two-point space has been introduced. We then introduce the concept of the restricted spectral triple using suitable projection operator, using which the individual spectral triples of Moyal plane and that of the two-point space can be recovered. The construction of the eigenspinors corresponding to the Dirac operator of the doubled Moyal plane has been taken up in Sec. IV which has then been used extensively in Sec. V to compute distances in all the three cases viz. transverse, longitudinal, and hypotenuse ones. In Sec. VI the Dirac operator has been fluctuated using an one-form arising solely from a prototype "Higgs" field and then the variation of the transverse distance has been studied by restricting the spectral triple to that of the two-point space again using a projection operator built out of the Dirac eigenspinors. Finally we conclude in Sec. VII.

II. REVIEW OF THE HILBERT-SCHMIDT OPERATOR FORMALISM AND THE SPECTRAL DISTANCE ON THE MOYAL PLANE ($\mathbb{R}^2_*$)

The Hilbert space $\mathcal{H}_q$ furnishing a representation of the entire noncommutative Heisenberg algebra

$$[\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{ij}; \qquad [\hat{X}_i, \hat{P}_j] = i\delta_{ij}; \qquad [\hat{P}_i, \hat{P}_j] = 0 \tag{3}$$

for the Moyal plane is given by the space of Hilbert-Schmidt (HS) operators $\psi(\hat{x}_1, \hat{x}_2) \in \mathcal{H}_q$, which consists of composite algebra elements generated by the position operators $(\hat{x}_1, \hat{x}_2)$, subject to the above coordinate sub-algebra, i.e., the commutation relations $[\hat{x}_1, \hat{x}_2] = i\theta$ in (2), on which the above operators $\hat{X}_i$ and $\hat{P}_j$ act as:

$$\begin{aligned} \hat{X}_i\psi(\hat{x}_1, \hat{x}_2) &= \hat{x}_i\psi(\hat{x}_1, \hat{x}_2), \\ \hat{P}_i\psi(\hat{x}_1, \hat{x}_2) &= \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi(\hat{x}_1, \hat{x}_2)]. \end{aligned} \tag{4}$$

These operators $\hat{x}_i$ and therefore $\psi(\hat{x}_1, \hat{x}_2)$, in turn, act on an auxiliary Hilbert space $\mathcal{H}_c$, which furnishes a representation of just this coordinate sub-algebra (2) and is defined by

$$\begin{aligned} \mathcal{H}_c = \mathrm{Span}\left\{|n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle; \hat{b}|0\rangle = 0, \hat{b} = \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2\theta}}\right\}_{n=0}^{\infty}; \\ [\hat{b}, \hat{b}^\dagger] = 1. \end{aligned} \tag{5}$$

Note that here $\mathcal{H}_c$ is isomorphic to the Hilbert space of the 1-D harmonic oscillator where the role of momentum is played by another spatial coordinate operator $\hat{x}_2$ and that of $\hbar$ by θ. The inner product on $\mathcal{H}_q$ is defined through $\mathcal{H}_c$ as

$$\begin{aligned} (\psi_1(\hat{x}_1, \hat{x}_2), \psi_2(\hat{x}_1, \hat{x}_2)) &= \mathrm{Tr}_{\mathcal{H}_c}(\psi_1^\dagger(\hat{x}_1, \hat{x}_2), \psi_2(\hat{x}_1, \hat{x}_2)) \\ &= \sum_{n=0}^{\infty}\langle n|\psi_1^\dagger(\hat{x}_1, \hat{x}_2), \psi_2(\hat{x}_1, \hat{x}_2)|n\rangle. \end{aligned} \tag{6}$$

This is clearly well-defined, as $\mathcal{H}_q$ is necessarily, by definition, a Hilbert space itself and consists of elements which are necessarily compact and have finite Hilbert-Schmidt norms associated with this inner product (6). In fact, the space of trace-class operators forms a dense subspace of this Hilbert space, i.e., the completion of the space of trace-class operators with HS norm becomes identical to the Hilbert space of HS operators $\mathcal{H}_q$. Further, the product of two HS operators is a trace-class operator and this fact is used to define the inner product (6) of $\mathcal{H}_q$. Also note that we make a distinction here between $\hat{X}_i$ and

$\hat{x}_i$, depending upon their domains of action, i.e., on $\mathcal{H}_q$ and $\mathcal{H}_c$, respectively. In a sense, the former can be regarded as a representation of the latter. On the other hand, the momentum operator $\hat{P}_i$ has action on $\mathcal{H}_q$ only, as is clear from (4), where it is identified through the adjoint action of $\epsilon_{ij}\hat{x}_j$ and therefore corresponds to the difference between left and right actions on $\psi(\hat{x}_1, \hat{x}_2)$. Finally, the vectors of $\mathcal{H}_q$ and $\mathcal{H}_c$ are distinguished by using round $|.)$ and angular $|.\rangle$ kets, respectively.

In view of the noncommutative (i.e., $\theta \neq 0$) nature of the coordinate algebra, the common eigenstates of $\hat{x}_1$ and $\hat{x}_2$ simply cannot exist. One is therefore forced to introduce a coherent state

$$|z\rangle = e^{-\bar{z}\hat{b}+z\hat{b}^\dagger}|0\rangle = e^{-|z|^2/2}e^{z\hat{b}^\dagger}|0\rangle \in \mathcal{H}_c; \qquad \hat{b}|z\rangle = z|z\rangle; \quad z = \frac{x_1 + ix_2}{\sqrt{2\theta}} \tag{7}$$

with maximal localization: $\Delta x_1 \Delta x_2 = \theta/2$. As is well known, this provides an overcomplete and nonorthonormal basis in $\mathcal{H}_c$: $\langle z'|z\rangle = e^{-|z'-z|^2/2}$. The above inner product (6) can therefore be expressed alternatively as

$$\begin{aligned}(\psi_1(\hat{x}_1, \hat{x}_2), \psi_2(\hat{x}_1, \hat{x}_2)) &= \int \frac{d^2z}{\pi} \langle z|\psi_1^\dagger(\hat{x}_1, \hat{x}_2)\psi_2(\hat{x}_1, \hat{x}_2)|z\rangle \\ &= \int \frac{d^2x}{2\pi\theta} \mathrm{Tr}_{\mathcal{H}_c}(\rho_z \psi_1^\dagger(\hat{x}_1, \hat{x}_2)\psi_2(\hat{x}_1, \hat{x}_2)); \\ \rho_z = |z\rangle\langle z| \in \mathcal{H}_q. \end{aligned} \tag{8}$$

Here we have introduced the density matrix $\rho_z \in \mathcal{H}_q$, as viewed from $\mathcal{H}_c$ and can be associated with the pure state ω_{ρ_z} corresponding to the $*$-algebra $\mathcal{H}_q = \mathcal{A}_M$ and defined as a linear functional on $\mathcal{A}_M$: $\omega_{\rho_z}(a) \in \mathbb{C}$ of the norm one. As explained in detail in [13], here too we shall be working with normal states, so that the states can be represented by density matrices: $\omega_{\rho_z}(a) = \mathrm{Tr}_{\mathcal{H}_c}(\rho_z a)$. This, in fact, follows from the fact that the representation of the algebra $\mathcal{A}_M$ on $\mathcal{H}_c$ (5) is irreducible, as $\mathcal{H}_c$ carries an irreducible representation of the oscillator $(\hat{b}, \hat{b}^\dagger)$ algebra. This implies, in turn, that the von Neumann algebra generated by $\mathcal{A}_M$ in $\mathcal{B}(\mathcal{H}_c)$ is the whole of $\mathcal{B}(\mathcal{H}_c)$ itself. For brevity, therefore, the states will be denoted just by density matrices themselves as $\rho(a) = \mathrm{Tr}(\rho a)$. Finite distance, *à la* Connes, between the pair of pure states $\rho_0 \coloneqq |0\rangle\langle 0|$ and ρ_z has already been computed in [7] to be

$$d(\rho_0, \rho_z) = \sqrt{2\theta}|z|. \tag{9}$$

This was also rederived via a somewhat different approach in [11] by employing the following spectral triple

$$\mathcal{A}_M = \mathcal{H}_q, \qquad \mathcal{D}_M = \sqrt{\frac{2}{\theta}}\begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix},$$
$$\mathcal{H}_M = \mathcal{H}_c \otimes \mathbb{C}^2 = \mathrm{Span}\left\{ \begin{pmatrix} |\psi\rangle \\ |\phi\rangle \end{pmatrix} : |\psi\rangle, |\phi\rangle \in \mathcal{H}_c \right\} \tag{10}$$

where the action of the algebra $\mathcal{A}_M$ on the Hilbert space $\mathcal{H}_M$ is given by the diagonal representation π:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{11}$$

whereas the Dirac operator $\mathcal{D}_M$ acts on $\mathcal{H}_M$—the module of spinorial sections from the left. This spectral triple is even as it admits grading or chirality operator $\gamma_M = \sigma_3$ which commutes with $\pi(a)$ for all $a \in \mathcal{A}_M$ and anticommutes with the Dirac operator $\mathcal{D}_M$. The chirality operator splits the Hilbert space into positive and negative sectors as

$$\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_- = \left(\mathcal{H}_c \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \oplus \left(\mathcal{H}_c \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Let us recall, in this context, that the spectral distance between a pair of states ρ_1 and ρ_2, which by definition are a pair of linear functionals over the algebra $\mathcal{A}_M$, is given by

$$d(\rho_1, \rho_2) = \sup_{a\in B}|\rho_1(a) - \rho_2(a)|; \quad B = \{a \in \mathcal{A}_M : \|[\mathcal{D}_M, \pi(a)]\|_{op} \leq 1\}. \tag{12}$$

In [11], in particular, the computation of the operator norm was carried out in the eigenspinor basis $|m\rangle\rangle_\pm \in \mathcal{H}_M$ (10)

$$|0\rangle\rangle_\pm = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, \qquad |m\rangle\rangle_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix} \quad \text{with} \quad m \in \{1, 2, 3\ldots\} \tag{13}$$

of the Dirac operator $\mathcal{D}_M$ with eigenvalues $\lambda_\pm^{(m)}$, given by

$$\mathcal{D}_M|m\rangle\rangle_\pm = \lambda_m^\pm|m\rangle\rangle_\pm; \qquad \lambda_\pm^{(m)} = \pm\sqrt{\frac{2m}{\theta}} \quad \text{with} \quad m \in \{0, 1, 2\ldots\}. \tag{14}$$

They satisfy the following orthonormality and completeness relations

$$_\pm\langle\langle m|n\rangle\rangle_\pm = \delta_{mn}; \qquad _+\langle\langle m|n\rangle\rangle_- = 0; \quad \sum_{m=0}^{\infty}|m\rangle\rangle_{\pm\pm}\langle\langle m| = \mathbb{1}_{\mathcal{H}_M}. \tag{15}$$

It was also necessary to introduce a projector $\mathbb{P}_N$ of rank $(2N+1)$ into the $(2N+1)$ dimensional subspace of $\mathcal{H}_M$

$$\mathbb{P}_N = \sum_{m=0}^{N} |m\rangle\!\rangle_{\pm\pm}\langle\!\langle m| = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix};$$
$$P_N = \sum_{m=0}^{N} |m\rangle\langle m| \tag{16}$$

with P_N being the projector for the $N+1$ dimensional subspace of $\mathcal{H}_c$. Following [7], the distance in (9) was first shown to be an upper bound and subsequently an optimal element

$$a_s = \sqrt{\frac{\theta}{2}}(\hat{b}e^{i\alpha} + \hat{b}^\dagger e^{-i\alpha}), \tag{17}$$

saturating the right-hand side (RHS) of (9) as the upper bound was identified (see Appendix B). Besides, it was found to saturate the ball condition as well: $[\mathcal{D}_M, \pi(a_s)] = 1$. With this one can indeed recognize the RHS of (9) as the true distance itself. We would like to mention in this context that, although $a_s \notin \mathcal{H}_q = \mathcal{A}_M$, it nevertheless belongs to the multiplier algebra and can be shown [7] to correspond to the limit point of a sequence whose elements are in $\mathcal{H}_q$. In an alternative approach, proposed in [11], the projected element

$$\mathbb{P}_N \pi(a_s)\mathbb{P}_N \in \mathcal{H}_q \otimes M_2(\mathbb{C}); \qquad N \geq 2 \tag{18}$$

was shown to satisfy the ball condition

$$\|[\mathcal{D}_M, \mathbb{P}_N\pi(a_s)\mathbb{P}_N]\|_{op} = 1 \quad \forall\ N \geq 2, \tag{19}$$

and shown to yield the correct infinitesimal distance $d(\rho_0, \rho_{dz}) = \sqrt{2\theta}|dz|$ by taking appropriate inner product in $\mathcal{H}_q \otimes M_2(\mathbb{C})$. Eventually this could be "integrated" to get (9) as the finite distance. Although the conventional notions of points and geodesics do not exist in the Moyal plane in view of the uncertainty $\Delta x_1 \Delta x_2 \geq \frac{\theta}{2}$ stemming from the noncommutative coordinate subalgebra (3), the notion of geodesics in the form of a straight line can be retrieved in some sense by explicitly constructing a one-parameter family of pure states $\rho_{zt} \coloneqq |zt\rangle\langle zt|$ with $t \in [0, 1]$ interpolating the extremal pure states ρ_0 and ρ_z, where the triangle inequality is saturated to an equality: $d(\rho_0, \rho_{zt}) + d(\rho_{zt}, \rho_z) = d(\rho_0, \rho_z)$. This is of course an exception; such a feature does not persist in generic noncommutative spaces. Indeed, for fuzzy sphere $\mathbb{S}^2_*$ the interpolating states satisfying a similar "triangle-equality" are necessarily mixed [11,14].

Note that in order to compute the spectral distance between a pair of pure states on the Moyal plane, we focus our attention on the coherent states $|z\rangle$ (7) obtained by translating $|0\rangle \in \mathcal{H}_c$ to $|z\rangle = U(z,\bar{z})|0\rangle$, where $U(z,\bar{z}) \coloneqq e^{-\bar{z}\hat{b} + z\hat{b}^\dagger}$ provides a projective unitary representation of the group of translation. Conversely, by the unitary transformation $U^{-1}(z,\bar{z})$, it is always possible to translate $|z\rangle$ back to $|0\rangle$ such that the density matrix will transform adjointly as

$$\rho_z = |z\rangle\langle z| \to U^{-1}(z,\bar{z})|z\rangle\langle z|U(z,\bar{z}) = |0\rangle\langle 0| = \rho_0; \tag{20}$$

furnishing a proper representation of the group of translation. This transformation acts on the Dirac operator (10) of the single Moyal plane as

$$\mathcal{D}_M \to U^{-1}(z,\bar{z})\mathcal{D}_M U(z,\bar{z}) = \sqrt{\frac{2}{\theta}}\begin{pmatrix} 0 & \hat{b}^\dagger - \bar{z} \\ \hat{b} - z & 0 \end{pmatrix} \tag{21}$$

where $(\hat{b} - z)|z\rangle = 0$ can be rewritten as

$$\hat{\tilde{b}}|\tilde{0}\rangle = 0; \qquad \hat{\tilde{b}} = \hat{b} - z, \qquad |\tilde{0}\rangle = |z\rangle, \tag{22}$$

so that $|z\rangle$ can be identified as the new and shifted "vacuum" $|\tilde{0}\rangle$. Indeed, we can write $(\hat{b}^\dagger - \bar{z})|z\rangle = \hat{\tilde{b}}^\dagger|\tilde{0}\rangle = |\tilde{1}\rangle$, and higher tower of states can be constructed by repeated actions of the new creation operator $\hat{\tilde{b}}^\dagger$:

$$|\tilde{n}\rangle = \frac{1}{\sqrt{n!}}(\hat{\tilde{b}}^\dagger)^n|\tilde{0}\rangle. \tag{23}$$

This shows that the new Fock space can be defined by translating the "vacuum" by a c-number, which would then be annihilated by the "translated" lowering operator. Finally note that the change in $\mathcal{D}_M$ under translation (21) is again given by an element

$$-\sqrt{\frac{2}{\theta}}\begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

which commutes with the diagonal representation of any algebra element $\pi(a)\ \forall\ a \in \mathcal{A}_M$. This implies in turn that $[U^{-1}(z,\bar{z})\mathcal{D}_M U(z,\bar{z}), \pi(a)] = [\mathcal{D}_M, \pi(a)]$ so that this unitary transformation has no impact on the ball condition B occurring in (12). The spectral distance, therefore, remains invariant under the translation [11].

III. DOUBLING THE SPECTRAL TRIPLE FOR THE MOYAL PLANE

In noncommutative geometry the notion of a usual manifold is generalized by the spectral triple $T = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ which obeys some axioms [5]. If we have two such spectral triples T_1 and T_2 associated with given two spaces, then the composite spectral triple denoted also by the tensor product notation $T_1 \otimes T_2$[1] generalizes the

[1]Although T_1 and T_2 are not vector spaces by themselves, we still use this convention of notation as in [6].

notion of the fiber bundle over a manifold [6]. If one of the spectral triples is taken to be that of a finite discrete space, say, for example the simplest known finite discrete space of the commutative two-point space for which the spectral triple is given by (A brief review of this two-point space has been provided in Appendix A, where the relevant notations are also introduced):

$$T_2 = \left(\mathcal{A}_2 = \mathbb{C}^2 \simeq M_2^d(\mathbb{C}), \mathcal{H}_2 = \mathbb{C}^2, \mathcal{D}_2 = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}\right); \quad \Lambda \in \mathbb{C}, \tag{24}$$

then, one can take the product of any even spectral triple T_1 with that of two-point space so that the resulting composite spectral triple can be written as,

$$T \coloneqq T_1 \otimes T_2 = (\mathcal{A} = \mathcal{A}_1 \otimes \mathbb{C}^2, \mathcal{H} = \mathcal{H}_1 \otimes \mathbb{C}^2, \mathcal{D} = \mathcal{D}_1 \otimes \mathbb{1}_2 + \gamma_1 \otimes \mathcal{D}_2); \quad \gamma_1 \text{ is the grading operator of} \mathrm{T}_1. \tag{25}$$

This corresponds to the space $M \cup M$ where M is the space associated with the spectral triple T_1. In particular, if the spectral triple T_1 is that of the Moyal plane (10), i.e., $M = \mathbb{R}_*^2$, with the grading operator $\gamma_1 = \gamma_M = \sigma_3$, then the product space $T \coloneqq T_1 \otimes T_2$, is known as the doubled Moyal plane and the spectral triple for which is given by

$$\mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C}), \qquad \mathcal{H}_T = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2, \quad \mathcal{D}_T = \mathcal{D}_M \otimes \mathbb{1}_2 + \sigma_3 \otimes \mathcal{D}_2. \tag{26}$$

Here the subscript T stands for total. Further, the total grading operator here takes the form

$$\gamma_T = \gamma_M \otimes \gamma_2 = \sigma_3 \otimes \sigma_3, \tag{27}$$

as the grading operator for the two-point space is also the same as that for the Moyal plane, i.e., $\gamma_2 = \gamma_M = \sigma_3$. The pure states in the doubled Moyal plane (see Fig. 1), between which we shall be computing the distances, also comes in a tensor product form

$$\Omega_i^{(z)} = \rho_z \otimes \omega_i; \qquad i \in \{1, 2\}, \tag{28}$$

where ω_1 and ω_2 are the only two pure states of the two-point space [see (A5) in Appendix]. The fact that the "composite" pure states $\Omega_i^{(z)}$ also remains pure is because of the fact that the algebra $\mathcal{A}_2 = \mathbb{C}^2$ is Abelian. In Fig. 1, the two copies of the Moyal plane ($\mathbb{R}_*^2$) are denoted by Σ_i based on the pure states ω_i. It was shown in [7] for the unital doubled Moyal plane that the transverse distance d_t between a state ρ_z of the single Moyal plane and its "clone" belonging to the other Moyal plane is the same as that of the distance between states ω_1 and ω_2 of the two-point space (see Appendix A)[2]

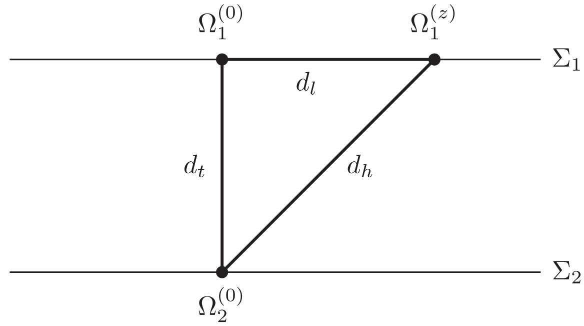

FIG. 1. $\mathbb{R}_*^2 \cup \mathbb{R}_*^2$, Space associated with the doubled Moyal plane.

$$d_t(\Omega_1^{(z)}, \Omega_2^{(z)}) = d_t(\rho_z \otimes \omega_1, \rho_z \otimes \omega_2) = d_{\mathcal{D}_2}(\omega_1, \omega_2) = \frac{1}{|\Lambda|}. \tag{29}$$

Analogously, the longitudinal distance d_l computed between the states ρ_0 and ρ_z belonging to the same copy of the Moyal plane Σ_i is equal to the distance between ρ_0 and ρ_z on the same Moyal plane

$$d_l(\Omega_i^{(0)}, \Omega_i^{(z)}) = d_{\mathcal{D}_M}(\rho_0, \rho_z) = \sqrt{2\theta}|z|. \tag{30}$$

Distance, d_h, between the pair of states like $\Omega_1^{(z)}$ and $\Omega_2^{(0)}$, as shown in Fig. 1, is known as the hypotenuse distance. In case of a commutative and unital spectral triple, these distances obey the Pythagoras equality

$$\{d_h(\Omega_1 = \rho \otimes \omega_1, \Omega_2 = \rho' \otimes \omega_2)\}^2 = \{d_t(\Omega_1 = \rho \otimes \omega_1, \Omega_2 = \rho \otimes \omega_2)\}^2 + \{d_l(\Omega_1 = \rho \otimes \omega_1, \Omega_2 = \rho' \otimes \omega_1)\}^2. \tag{31}$$

This Pythagoras theorem is shown to be valid even for certain noncommutative but unital spectral triple [6]. In fact, by unitizing the algebra associated with the Moyal plane it was shown that the Pythagoras theorem is obeyed on the doubled unitized Moyal plane [7]. Here, one of our aims is to compute the above spectral distances on the doubled Moyal plane, in the context of the Hilbert-Schmidt operatorial formulation of the noncommutative quantum mechanics [12], by making use of the Dirac eigenspinors, and verify the Pythagoras equality without unitizing the algebra $\mathcal{H}_q$.

[2]This distance was used in [7] to reconcile the nonvanishing nature of the "quantum length" between a state and itself [15] with the spectral distance between a state and its clone, thereby identifying $\frac{1}{|\Lambda|} \sim \sqrt{\theta} \sim L_p$ -the Planck length. This issue, however, is beyond the scope of this paper.

Just as the algebra elements of the Moyal plane act on its Hilbert space elements through diagonal representation (11), a generic element of

$$a_T = |\psi) \otimes \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in \mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C})$$

(25), where $M_2^d(\mathbb{C})$ is the c-valued 2×2 diagonal matrix [see (A2) in Appendix A], acts on a generic element

$$|\Phi\rangle_T = \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} \otimes \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathcal{H}_T = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2$$

(With λ_1, λ_2, μ_1, $\mu_2 \in \mathbb{C}$) as

$$\pi(a_T)|\Phi\rangle_T = \left\{ \begin{pmatrix} |\psi) & 0 \\ 0 & |\psi) \end{pmatrix} \otimes \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\} \left\{ \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} \otimes \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\} = \begin{pmatrix} |\psi)|\phi_1\rangle \\ |\psi)|\phi_2\rangle \end{pmatrix} \otimes \begin{pmatrix} \lambda_1\mu_1 \\ \lambda_2\mu_2 \end{pmatrix}. \tag{32}$$

Moreover, the total Dirac operator $\mathcal{D}_T$ (26) is obtained by substituting the forms of $\mathcal{D}_M$ from (10) and $\mathcal{D}_2$ (A1) in Appendix A to get,

$$\mathcal{D}_T = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}. \tag{33}$$

By writing $\Lambda = |\Lambda| e^{i\phi}$, we can carry out a unitary transformation of the above Dirac operator: $\mathcal{D}_T \to U(\phi)\mathcal{D}_T U^\dagger(\phi)$, where $U(\phi)$ is given by

$$U(\phi) = \begin{pmatrix} \mathbb{1}_{\mathcal{H}_q} & 0 \\ 0 & \mathbb{1}_{\mathcal{H}_q} \end{pmatrix} \otimes \begin{pmatrix} e^{\frac{-i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix}, \tag{34}$$

so that the Dirac operator becomes

$$\mathcal{D}_T = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + |\Lambda| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{35}$$

The first slot of $U(\phi)$ in (34), which is the representation of $\mathbb{1}_{\mathcal{H}_q}$ is, however, a bit troubling since the Moyal plane algebra $\mathcal{A}_M = \mathcal{H}_q$ is nonunital: $\mathbb{1}_{\mathcal{H}_q} \notin \mathcal{H}_q$. The way out is provided by the sequence of projection operators $\mathbb{P}_N$ (16), all of which lie in $\mathcal{B}(\mathcal{H}_M)$ (10) and can be considered to reproduce the identity element in the $N \to \infty$ limit:

$$\lim_{N\to\infty} \mathbb{P}_N = \pi(\mathbb{1}_{\mathcal{H}_q}). \tag{36}$$

This identification relies on the fact that the composite operators formed by multiplying any compact operators, like a Hilbert-Schmidt operator $\in \mathcal{H}_q$ by $\mathbb{P}_N$, is same as that of multiplication with the identity operator in the limit $N \to \infty$. Note that in order to arrive at a "unitary equivalent" spectral triple, i.e., $(\mathcal{A}_T, \mathcal{H}_T, U(\phi)\mathcal{D}_T U^\dagger(\phi))$, with $U(\phi)$ as in (34), one needs to make a simultaneous unitary transformation of $\pi(a_T)$ using this $U(\phi)$. This simultaneous transformation ensures that the ball condition (B) in (12) remains invariant. It is, however, quite straightforward to verify that $\pi(a_T)$ as given in (32) remains invariant under such a unitary transformation and therefore we need not worry about this issue any more. From this point onward, unless mentioned otherwise, we will therefore take $\Lambda \in \mathbb{R}$ and positive, $\Lambda > 0$, without loss of generality in order to avoid clutter in the notation. Moreover, such a unitary equivalent spectral triple is a part of several other transformations which preserves the metric properties of the triple (see [6]).

Before concluding this section, we would like to mention that it is also possible to retrieve individual spectral triples T_1 or T_2 along with their respective distances from the composite one T (25), provided certain conditions are satisfied. For the spectral triple (26) these conditions are indeed satisfied, as we shall exhibit now. To that end, we begin with a brief review of the concept of the restricted spectral triple as introduced in [6]. For a given spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, let us consider the action of a self-adjoint element $\rho \in \mathcal{A}$, satisfying the property of a projector: $\rho^2 = \rho = \rho^*$, on an arbitrary algebra element $a \in \mathcal{A}$ through the following map

$$\alpha_\rho\colon \mathcal{A} \to \mathcal{A}; \qquad a \mapsto \alpha_\rho(a) = \rho a \rho. \tag{37}$$

This transformation gives rise to the following "restricted" spectral triple

$$\mathcal{A}^{(\rho)} = \alpha_\rho(\mathcal{A}), \qquad \mathcal{H}^{(\rho)} = \pi(\rho)\mathcal{H}, \qquad \mathcal{D}^{(\rho)} = \pi(\rho)\mathcal{D}\pi(\rho), \tag{38}$$

where $\pi(\rho)$ indicates that the domain of representation π has been restricted to $\pi|_{\mathcal{H}^{(\rho)}}$. For a pair of pure states ω_1 and ω_2 of $\mathcal{A}^{(\rho)}$, the corresponding spectral distance remains unaffected by projection (see, e.g., Lemma 1 in [6]), i.e.,

$$d^{(\rho)}(\omega_1, \omega_2) = d(\omega_1 \circ \alpha_\rho, \omega_2 \circ \alpha_\rho) \quad \forall\ \omega_1, \quad \omega_2 \in \mathcal{P}(\mathcal{A}^{(\rho)}), \tag{39}$$

provided

$$[\mathcal{D}, \pi(\rho)] = 0. \tag{40}$$

This condition implies that $\pi(\rho)$ should correspond to a projection operator built out of the eigenspinors of the

Dirac operator. In the case of the spectral triple (26) for example, this should involve $\mathbb{P}_N$ (16) and ω_i (A5). Indeed, it can be verified in a straightforward manner that the projection operators $P^{(\text{trans})}_{T(0)}$ and $P^{(\text{long})}_{T(i)}(N)$, defined as

$$P^{(\text{trans})}_{T(0)} \coloneqq \mathbb{P}_0 \otimes \mathbb{1}_2 = \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \in \mathcal{A}_T$$
$$P^{(\text{long})}_{T(i)}(N) \coloneqq \mathbb{P}_N \otimes \omega_i = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \omega_i; \quad i = 1, 2 \tag{41}$$

can be used to construct the following two spectral triples from (26). The first one yields

$$P^{(\text{trans})}_{T(0)} \mathcal{A}_T P^{(\text{trans})}_{T(0)} = \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \otimes M_2^d(\mathbb{C}),$$
$$P^{(\text{trans})}_{T(0)} \mathcal{H}_T = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \otimes \mathbb{C}^2,$$
$$P^{(\text{trans})}_{T(0)} \mathcal{D}_T P^{(\text{trans})}_{T(0)} = \mathbb{P}_0 \otimes \mathcal{D}_2, \tag{42}$$

which is clearly identical to the one involving two-point space (24). Note that in this case the role of representation π becomes redundant. Essentially following the same approach one can also retrieve the unitized version of the spectral triple for the single Moyal plane (10) by making use of the projector $P^{(\text{long})}_{T(i)}(N)$ in the limit $N \to \infty$. Moreover, for any finite $N (0 < N < \infty)$ this does not belong to $\mathcal{A}_T$. This stems from the fact that the projector $\mathbb{P}_N$ (16) by itself can not be identified with the diagonal representation π of $\mathcal{A}_{\mathcal{M}} = \mathcal{H}_q$ (10). It is also easily verifiable that either of the projectors (41) commute with the total Dirac operator $\mathcal{D}_T$ (26):

$$[\mathcal{D}_T, P^{(\text{trans})}_{T(0)}] = [\mathcal{D}_T, P^{(\text{long})}_{T(i)}(N)] = 0. \tag{43}$$

This is precisely the condition (40) so that the results (29), (30), regarding the transverse and longitudinal distances, follow trivially from (39). Further, the fact that the RHS of (29) is independent of "z" can be seen easily by first considering a slight variant of the projector $P^{(\text{trans})}_{T(0)}$ viz.

$$P^{(\text{trans})}_{T(z)} = \mathbb{P}_{\tilde{0}} \otimes \mathbb{1}_2 = \begin{pmatrix} |\tilde{0}\rangle\langle \tilde{0}| & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \tag{44}$$

where the states $|\tilde{0}\rangle\langle\tilde{0}|$ can be thought to be anchored to the "shifted" origin (in the spirit of Gelfand and Naimark) as discussed earlier (22) and then invoking the translational invariance of the spectral distance of the coherent states, parametrized by the complex plane [see the discussion below (22)].

IV. CONSTRUCTION OF EIGENSPINORS OF THE TOTAL DIRAC OPERATOR $\mathcal{D}_T$

Looking at the total Dirac operator (35), we realize that its eigenspinors would belong to a space spanned by the tensor product of the Moyal plane eigenspinors (13) on the left slot and eigenspinors $\frac{1}{\sqrt{2}}\binom{1}{\pm 1}$ of $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the right slot. One is immediately tempted to work with the irreducible subspaces of the eigenspinors viz. spin up ($\uparrow$) subspace $\text{Span}\{V^{(m)}_{++}, V^{(m)}_{-+}\}$, formed by tensoring with $|\uparrow\rangle \coloneqq \frac{1}{\sqrt{2}}\binom{1}{1}$ where $\sigma_1|\uparrow\rangle = +|\uparrow\rangle$ and spin down ($\downarrow$) subspace $\text{Span}\{V^{(m)}_{+-}, V^{(m)}_{--}\}$, formed by tensoring with $|\downarrow\rangle \coloneqq \frac{1}{\sqrt{2}}\binom{1}{-1}$ satisfying $\sigma_1|\downarrow\rangle = -|\downarrow\rangle$ where the (unnormalized) spinors $V^{(m)}_{\pm\pm}$ are given by

$$V^{(m)}_{\pm\pm} = \begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{45}$$

To that end, we consider an arbitrary linear combination of spinors from the spin up subspace, i.e., $C^1_m V^{(m)}_{++} + C^2_m V^{(m)}_{-+}$ and require it be an eigenspinor of the total Dirac operator (35). This yields the following eigenvalue equation

$$\begin{pmatrix} \sqrt{\frac{2m}{\theta}} & \Lambda \\ \Lambda & -\sqrt{\frac{2m}{\theta}} \end{pmatrix} \begin{pmatrix} C^1_m \\ C^2_m \end{pmatrix} = \lambda^{(m)} \begin{pmatrix} C^1_m \\ C^2_m \end{pmatrix}, \tag{46}$$

whose eigenvalues are

$$\lambda^{(m)}_{\pm} = \pm\Lambda\sqrt{\kappa m + 1}; \qquad \kappa = \frac{2}{\theta\Lambda^2}, \tag{47}$$

and eigenspinors are (up to appropriate normalization)

$$|\Phi^{(m)}_{+\uparrow}\rangle = V^{(m)}_{-+} + (\sqrt{\kappa m + 1} + \sqrt{\kappa m})V^{(m)}_{++},$$
$$|\Phi^{(m)}_{-\uparrow}\rangle = V^{(m)}_{-+} - (\sqrt{\kappa m + 1} - \sqrt{\kappa m})V^{(m)}_{++}. \tag{48}$$

Here the subscript $+$ and $-$ stands for the eigenvalue being $\lambda^{(m)}_{+}$ and $\lambda^{(m)}_{-}$, respectively. Repeating the same exercise for the spin down subspace we obtain the same set of eigenvalues (47), while the corresponding eigenspinors are as follows:

$$|\Phi^{(m)}_{+\downarrow}\rangle = V^{(m)}_{--} - (\sqrt{\kappa m + 1} + \sqrt{\kappa m})V^{(m)}_{+-},$$
$$|\Phi^{(m)}_{-\downarrow}\rangle = V^{(m)}_{--} + (\sqrt{\kappa m + 1} - \sqrt{\kappa m})V^{(m)}_{+-}. \tag{49}$$

However, these subspaces are not invariant under the action of the algebra $\mathcal{A}_T$ (26) as some of the algebra elements (e.g., in the case of transverse distance) mixes the eigenspinors of the two subspaces and one has to work with all the four of them anyway. It turns out that working with the following (symmetric) linear combination of the eigenspinors, rather than the ones obtained earlier (48)–(49), simplifies our calculations drastically:

$$
\begin{aligned}
&|\Psi_{+}^{(m)}\rangle = (\sqrt{\kappa m+1}-\sqrt{\kappa m})|\Phi_{+\uparrow}^{(m)}\rangle + |\Phi_{+\downarrow}^{(m)}\rangle, \qquad |\Psi_{-}^{(m)}\rangle = |\Phi_{-\downarrow}^{(m)}\rangle - (\sqrt{\kappa m}+\sqrt{\kappa m+1})|\Phi_{-\uparrow}^{(m)}\rangle,\\
&|\tilde{\Psi}_{+}^{(m)}\rangle = |\Phi_{+\uparrow}^{(m)}\rangle - (\sqrt{\kappa m+1}-\sqrt{\kappa m})|\Phi_{+\downarrow}^{(m)}\rangle, \qquad |\tilde{\Psi}_{-}^{(m)}\rangle = |\Phi_{-\uparrow}^{(m)}\rangle + (\sqrt{\kappa m}+\sqrt{\kappa m+1})|\Phi_{-\downarrow}^{(m)}\rangle.
\end{aligned} \tag{50}
$$

One can write a normalized version of these eigenspinors in the following compact form

$$
\begin{aligned}
&|\Psi_{\pm}^{(m)}\rangle = N_m[V_{++}^{(m)} + V_{--}^{(m)} \pm V_{-+}^{(m)}(\sqrt{\kappa m+1} \mp \sqrt{\kappa m}) \mp V_{+-}^{(m)}(\sqrt{\kappa m+1} \pm \sqrt{\kappa m})]\\
&|\tilde{\Psi}_{\pm}^{(m)}\rangle = N_m[V_{+-}^{(m)} + V_{-+}^{(m)} \pm V_{++}^{(m)}(\sqrt{\kappa m+1} \pm \sqrt{\kappa m}) \mp V_{--}^{(m)}(\sqrt{\kappa m+1} \mp \sqrt{\kappa m})], \qquad N_m = \frac{1}{4\sqrt{\kappa m+1}}
\end{aligned} \tag{51}
$$

where $m \in \{1, 2, 3\ldots\}$ and N_m is the normalization constant. Moreover the case of $m = 0$ is 2 dimensional, unlike the eigenspinors in (51), which are 4 dimensional for each m. The $m = 0$ case can be written after normalization as

$$
|\Psi_{\pm}^{(0)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{52}
$$

Here again the subscript $\pm$ represents the eigenvalues $\lambda_{\pm}^{(0)} = \pm\Lambda$. The whole set (51), (52) furnishes a complete and an orthonormal basis for the entire space:

$$
\begin{aligned}
&\langle\Psi_{\pm}^{(m)}|\Psi_{\pm}^{(n)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\tilde{\Psi}_{\pm}^{(n)}\rangle = \delta^{mn}; \qquad \langle\Psi_{\pm}^{(0)}|\Psi_{\pm}^{(0)}\rangle = 1; \qquad m, n \in \{1, 2, 3\ldots\}\\
&\langle\Psi_{\pm}^{(m)}|\Psi_{\mp}^{(n)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\tilde{\Psi}_{\mp}^{(n)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\Psi_{\pm}^{(n)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\Psi_{\mp}^{(n)}\rangle = 0\\
&\langle\Psi_{\pm}^{(m)}|\Psi_{\pm}^{(0)}\rangle = \langle\Psi_{\pm}^{(m)}|\Psi_{\mp}^{(0)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\Psi_{\pm}^{(0)}\rangle = \langle\tilde{\Psi}_{\pm}^{(m)}|\Psi_{\mp}^{(0)}\rangle = 0.
\end{aligned} \tag{53}
$$

V. COMPUTING SPECTRAL DISTANCES IN THE DOUBLED MOYAL PLANE

In this section we will compute the exact distances for all the three cases and reproduce the results obtained by Martinetti and Tomassini, 2013 [7] including the Pythagoras equality for the doubled Moyal plane. We will do so by making explicit use of the Dirac eigenspinors (51)–(52) that we have just constructed. This will involve making the right ansätze regarding the structure of the optimal algebra elements belonging to $\mathcal{A}_T$ (26). Necessary hints for the same can be obtained from the corresponding structure of (17) for the single Moyal plane. In an alternative approach, we show how the projection operators built out of these same Dirac eigenspinors can be used to compute the transverse distance, instead of unitizing the algebra $\mathcal{A}$ as in [7].

A. Longitudinal distance

To start with, we consider a generic algebra element $a_T \in \mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C})$, which can be written in a separable form as $a_T = a \otimes a_2$ where $a \in \mathcal{H}_q$ and

$$
a_2 \coloneqq \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in \mathbb{C}^2.
$$

Since the search for the optimal algebra element can be restricted to the ones which are self-adjoint: $a_T^{\dagger} = a_T$ [7], we can set $a^{\dagger} = a$ and restrict c_1 and c_2 to the real numbers: $c_1, c_2 \in \mathbb{R}$. To compute the longitudinal distance $d_{PQ} = d_{P'Q'}$ as shown in Fig. 2, we substitute the states $\Omega_i^{(z)}$ and $\Omega_i^{(0)}$ from (28) in the Connes distance formula (12), along with the algebra element a_T to obtain

$$
d_l(\Omega_i^{(z)}, \Omega_i^{(0)}) = \sup_{a_T \in B_T} |\Omega_i^{(z)}(a_T) - \Omega_i^{(0)}(a_T)| \tag{54}
$$

$$
= \sup_{a_T \in B_T} |\mathrm{Tr}_{\mathcal{H}_M}(d\Omega_i a_T)|; \qquad d\Omega_i = \Omega_i^{(z)} - \Omega_i^{(0)} \tag{55}
$$

$$
= \sup_{a_T \in B_T} |c_i| \cdot |\mathrm{Tr}_{\mathcal{H}_c}((\rho_z - \rho_0)a)| \tag{56}
$$

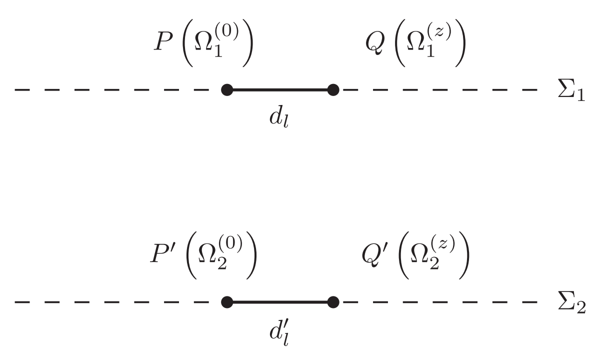

FIG. 2. Different states belonging to the same Moyal plane.

$$= \sup_{a_T \in B_T} |c_i| \cdot |\rho_z(a) - \rho_0(a)|. \quad (57)$$

Moreover, the ball condition B_T is built from the total Dirac operator D_T (35), i.e., $\|[\mathcal{D}_T, \pi(a_T)]\|_{op} \leq 1$ of the doubled Moyal plane and upon simplification looks like

$$[\mathcal{D}_T, \pi(a_T)] = [\mathcal{D}_M, \pi(a)] \otimes a_2 + a(c_1 - c_2)\sigma_3 \otimes \mathcal{D}_2. \quad (58)$$

Now, the symmetry of the doubled Moyal plane requires the longitudinal distance be same on both the sheets. By imposing this requirement on (57), we get $|c_1| = |c_2|$ or $c_1 = \pm c_2$. It is clear from (58) that the condition $c_1 = c_2 = X$ (say) makes the total ball condition B_T identical to the ball condition B_M of the single Moyal plane, thus yielding the Moyal plane distance (9). The other case: $c_1 = -c_2$ gives a distance which is lower than $\sqrt{2\theta}|z|$ (9), as we will demonstrate shortly using an algebra element a_l, and therefore will be rejected.

Let us now take as an ansatz the optimal element

$$a_s^{(l)} \coloneqq (b + b^\dagger) \otimes a_2 \in \mathcal{A}_T \quad (59)$$

(superscript "l" stands for longitudinal) for the computation of the longitudinal distance. This is motivated from (17) where we have set the phase α of $z = |z|e^{i\alpha}$ to $\alpha = 0$ by invoking the rotational symmetry on the Moyal plane (see Appendix B). This means that the longitudinal distance will be computed along the real axis only. A simple calculation then yields the following trace

$$\mathrm{Tr}_{\mathcal{H}_M}(d\Omega_i(a_s^{(l)})) = 2c_i z. \quad (60)$$

The matrix representation of $[\mathcal{D}_T, \pi(a_s^{(l)})]$ is, in general, infinite dimensional and one needs to project it onto some finite dimensional subspace (see, e.g., the case of single Moyal plane in [11]) in order to get hold of the ball condition B_T. There is a natural way to achieve this by using the projection operators

$$\mathcal{P}_N = |\Psi_+^{(0)}\rangle\langle\Psi_+^{(0)}| + |\Psi_-^{(0)}\rangle\langle\Psi_-^{(0)}| + \sum_{m=1}^{N}[|\Psi_+^{(m)}\rangle\langle\Psi_+^{(m)}| + |\Psi_-^{(m)}\rangle\langle\Psi_-^{(m)}|$$
$$|\tilde{\Psi}_+^{(m)}\rangle\langle\tilde{\Psi}_+^{(m)}| + |\tilde{\Psi}_-^{(m)}\rangle\langle\tilde{\Psi}_-^{(m)}|] \in (\mathcal{H}_q \otimes M_2(\mathbb{C})) \otimes M_2(\mathbb{C}), \quad (61)$$

constructed from the Dirac eigenspinors (51), (52). These projection operators $\mathcal{P}_N$ belong to the same space where $\pi(a_T)\ \forall\ a_T \in \mathcal{A}_T$ belong, which is $(\mathcal{H}_q \otimes M_2^d(\mathbb{C})) \otimes M_2^d(\mathbb{C})$. After explicit calculation the projection operator, quite remarkably, split as

$$\mathcal{P}_N = \mathbb{P}_N \otimes \mathbb{1}_2 \quad (62)$$

where the projection operator $\mathbb{P}_N$ is given by (16). We can make use of (61) to obtain various finite dimensional matrix representations of $[\mathcal{D}_T, \pi(a_s^{(l)})]$ in the form of $[\mathcal{D}_T, \mathcal{P}_N\pi(a_s^{(l)})\mathcal{P}_N]$, by varying N. For $\mathcal{P}_2$ and $c_1 = c_2 = X$ (one of the two cases of $c_1 = \pm c_2$ as discussed earlier), the matrix $[\mathcal{D}_T, \mathcal{P}_2\pi(a_s^{(l)})\mathcal{P}_2] \eqqcolon M_l$ (say) takes the following form

$$M_l = \begin{pmatrix}
0 & 0 & \gamma_-\delta_3 & \gamma_+\delta_4 & \gamma_-\delta_1 & \gamma_+\delta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma_+\delta_4 & -\gamma_-\delta_3 & -\gamma_+\delta_2 & -\gamma_-\delta_1 & 0 & 0 & 0 & 0 \\
-\gamma_-\delta_3 & \gamma_+\delta_4 & 0 & 0 & 0 & 0 & \eta(\beta_- - \epsilon_+) & -\eta(\beta_+ + \epsilon_-) & -\eta(\beta_- + \epsilon_-) & \eta(\beta_+ - \epsilon_+) \\
-\gamma_+\delta_4 & \gamma_-\delta_3 & 0 & 0 & 0 & 0 & \eta(\beta_+ + \epsilon_-) & -\eta(\beta_- - \epsilon_+) & -\eta(\beta_+ - \epsilon_+) & \eta(\beta_- + \epsilon_-) \\
-\gamma_-\delta_1 & \gamma_+\delta_2 & 0 & 0 & 0 & 0 & \eta(\beta_- - \epsilon_-) & -\eta(\beta_+ + \epsilon_+) & -\eta(\beta_- + \epsilon_+) & \eta(\beta_+ - \epsilon_-) \\
-\gamma_+\delta_2 & \gamma_-\delta_1 & 0 & 0 & 0 & 0 & \eta(\beta_+ + \epsilon_+) & -\eta(\beta_- - \epsilon_-) & -\eta(\beta_+ - \epsilon_-) & \eta(\beta_- + \epsilon_+) \\
0 & 0 & -\eta(\beta_- - \epsilon_+) & -\eta(\beta_+ + \epsilon_-) & -\eta(\beta_- - \epsilon_-) & -\eta(\beta_+ + \epsilon_+) & 0 & 0 & 0 & 0 \\
0 & 0 & \eta(\beta_+ + \epsilon_-) & \eta(\beta_- - \epsilon_+) & \eta(\beta_+ + \epsilon_+) & \eta(\beta_- - \epsilon_-) & 0 & 0 & 0 & 0 \\
0 & 0 & \eta(\beta_- + \epsilon_-) & \eta(\beta_+ - \epsilon_+) & \eta(\beta_- + \epsilon_+) & \eta(\beta_+ - \epsilon_-) & 0 & 0 & 0 & 0 \\
0 & 0 & -\eta(\beta_+ - \epsilon_+) & -\eta(\beta_- + \epsilon_-) & -\eta(\beta_+ - \epsilon_-) & -\eta(\beta_- + \epsilon_+) & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (63)$$

where the rows and columns are labeled by $|\Psi_+^{(0)}\rangle$, $|\Psi_-^{(0)}\rangle$, $|\Psi_+^{(1)}\rangle$, $|\Psi_-^{(1)}\rangle$, $|\tilde{\Psi}_+^{(1)}\rangle$, $|\tilde{\Psi}_-^{(1)}\rangle$, $|\Psi_+^{(2)}\rangle$, $|\Psi_-^{(2)}\rangle$, $|\tilde{\Psi}_+^{(2)}\rangle$, and $|\tilde{\Psi}_-^{(2)}\rangle$ of (51)–(52), respectively. The coefficients $\beta_\pm$, $\gamma_\pm$, η, $\epsilon_\pm$ and δ_i for $i \in \{1,2,3,4\}$ are given as follows:

$$\beta_\pm = \frac{\Lambda}{4}(\sqrt{2\kappa+1} \pm \sqrt{\kappa+1}), \qquad \gamma_\pm = (1 \pm \sqrt{\kappa+1}), \qquad \epsilon_\pm = \frac{\Lambda}{4}(\sqrt{2\kappa}\sqrt{\kappa+1} \pm \sqrt{\kappa}\sqrt{2\kappa+1}), \tag{64}$$

$$\eta = \frac{X\sqrt{\kappa}}{\sqrt{(\kappa+1)(2\kappa+1)}}, \qquad \delta_1 = \frac{X\Lambda}{2\sqrt{2(\kappa+1)}}(1+\sqrt{\kappa}+\sqrt{\kappa+1}), \qquad \delta_2 = \frac{X\Lambda}{2\sqrt{2(\kappa+1)}}(1+\sqrt{\kappa}-\sqrt{\kappa+1}),$$
$$\delta_3 = \frac{X\Lambda}{2\sqrt{2(\kappa+1)}}(1-\sqrt{\kappa}+\sqrt{\kappa+1}), \qquad \delta_4 = \frac{X\Lambda}{2\sqrt{2(\kappa+1)}}(1-\sqrt{\kappa}-\sqrt{\kappa+1}). \tag{65}$$

Now the largest eigenvalue of the matrix corresponding to $M_l^\dagger M_l$ (Using *Mathematica*) comes out to be $X^2\Lambda^2\kappa$. Therefore, by using the C^* algebra property $\|M_l^\dagger M_l\|_{op} = \|M_l\|_{op}^2$, the ball condition (B_T) becomes

$$\|[\mathcal{D}_T, \mathcal{P}_2\pi(a_s^{(l)})\mathcal{P}_2]\|_{op} = X\Lambda\sqrt{\kappa} \le 1. \tag{66}$$

Now substituting this and (60) in (55), while noting that $\kappa = \frac{2}{\theta\Lambda^2}$ we obtain the Moyal plane distance (9) as expected. Moreover, as we increase the rank of the projection operators $\mathcal{P}_N$ from $N=2$ to $N=3$ and so on, we find that the ball condition (66) remains unaffected and so does the distance (9) for all orders in N.

As for the case of $c_1 = -c_2 = X$ we find, after following the same procedure, that the ball condition gets modified to

$$\|[\mathcal{D}_T, \mathcal{P}_2\pi(a_S^{(l)})\mathcal{P}_2]\|_{op} = X\Lambda\sqrt{8+\kappa+4\sqrt{1+\kappa}} \le 1. \tag{67}$$

This, together with (60) and (55), gives the following estimate of the distance

$$d(\Omega_i^{(z)}, \Omega_i^{(0)})_{est} = \frac{2|z|}{\Lambda\sqrt{8+\kappa+4\sqrt{1+\kappa}}}, \tag{68}$$

which is clearly less than (9) and is therefore rejected.

B. Transverse distance

For the transverse case, we take $a_s^{(t)} \coloneqq \mathbb{1}_{\mathcal{H}_q} \otimes a_2$ as an ansatz for the optimal algebra element (superscript "t" stands for transverse) such that its representation looks like

$$\pi(a_s^{(t)}) = \begin{pmatrix} \mathbb{1}_{\mathcal{H}_q} & 0 \\ 0 & \mathbb{1}_{\mathcal{H}_q} \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{69}$$

The transverse distance $d_{PP'}$ (see Fig. 3) between the states $\Omega_1^{(z)}$ and $\Omega_2^{(z)}$, when computed using this a_t, i.e.,

$$d_t(\Omega_1^{(z)}, \Omega_2^{(z)}) = \sup_{a_t \in B_T} |\mathrm{Tr}_{\mathcal{H}_M}(d\Omega^{(z)} a_t)|;$$
$$d\Omega^{(z)} = \rho_z \otimes (\omega_1 - \omega_2) \tag{70}$$

comes out to be same as the spectral distance on the two-point space (29). However, the algebra of the Moyal plane is nonunital and thus $\mathbb{1}_{\mathcal{H}_q} \notin \mathcal{A}_M = \mathcal{H}_q$. In order to get around this problem, we again make use of the projection operator $\mathbb{P}_N$ (16) as in (36) and instead of $\pi(a_s^{(t)})$ (69) we work with

$$\pi_N(a_s^{(t)}) = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{71}$$

With this choice of representation we get the trace in (70), as

$$\mathrm{Tr}_{\mathcal{H}_M}(d\Omega(z)a_s^{(t)}) = \frac{1}{2}\mathrm{Tr}_{\mathcal{H}_T}(\pi(d\Omega(z))\pi_N(a_s^{(t)})) = c_1 - c_2 \tag{72}$$

where the extra factor of $\frac{1}{2}$ takes care of the double counting stemming from the diagonal representation (11).

Now the matrix representation of $[\mathcal{D}_T, \pi_N(a_s^{(t)})] \eqqcolon M_t$ (say) comes out to be a block-diagonal matrix

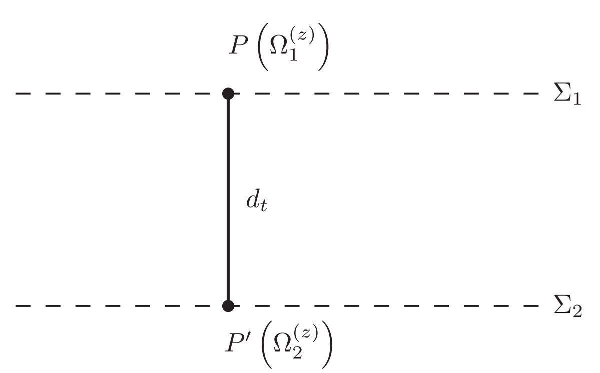

FIG. 3. Identical (cloned) states belonging to different Moyal planes.

$$M_t = \begin{pmatrix} Q & 0_{2\times4} & \cdots & 0_{2\times4} \\ 0_{4\times2} & R^{(1)} & \cdots & 0_{4\times4} \\ \vdots & \vdots & \ddots & \\ 0_{4\times2} & 0_{4\times4} & & R^{(N)} \end{pmatrix}, \tag{73}$$

where $0_{m\times n}$ are the $m\times n$ null rectangular matrices whereas the square matrices Q and $R^{(m)}$ are given by

$$Q = \begin{pmatrix} 0 & \Lambda Y \\ -\Lambda Y & 0 \end{pmatrix};$$
$$R^{(m)} = \frac{\Lambda Y}{\sqrt{m\kappa+1}} \begin{pmatrix} 0 & -\sqrt{m\kappa} & 0 & 1 \\ \sqrt{m\kappa} & 0 & -1 & 0 \\ 0 & 1 & 0 & \sqrt{m\kappa} \\ -1 & 0 & -\sqrt{m\kappa} & 0 \end{pmatrix};$$
$$Y = c_1 - c_2, \tag{74}$$

where m is a positive integer in the range $1 \le m \le N$. With this the matrix $M_t^\dagger M_t$ becomes proportional to the identity matrix

$$M_t^\dagger M_t = \Lambda^2 Y^2 \begin{pmatrix} \mathbb{1}_{2\times2} & 0_{2\times4} & \cdots & 0_{2\times4} \\ 0_{4\times2} & \mathbb{1}_{4\times4} & \cdots & 0_{4\times4} \\ \vdots & \vdots & \ddots & \\ 0_{4\times2} & 0_{4\times4} & & \mathbb{1}_{4\times4} \end{pmatrix} \tag{75}$$

enabling us to just read off the operator norm as $\|M_t^\dagger M_t\|_{op} = |Y|^2\Lambda^2$, which is independent of both m and N. We, therefore, get the ball condition (B_T) for arbitrarily large N, as

$$\|[\mathcal{D}_T, \pi_N(a_s^{(t)})]\|_{op} = |Y|\Lambda \le 1. \tag{76}$$

By substituting this and (72) in (70), we obtain the transverse distance (29). Finally, in the limit $N\to\infty$ the representation (71) becomes diagonal: $\pi_N(a_s^{(t)}) \to \pi(a_s^{(t)})$, enabling us to recover the same result as in [7]. Note, however, that we have achieved this result without explicitly unitizing the Moyal plane algebra.

C. Hypotenuse distance

To find the hypotenuse distance $d_{P'Q}$, we consider the states $\Omega_1^{(z)}$ and $\Omega_2^{(0)}$ as shown in Fig. 4. We then proceed by making an ansatz about the optimal element $a_s^{(h)}$ of the algebra $\mathcal{A}_T$ (superscript "h" stands for hypotenuse), as a linear combination of the optimal elements of the longitudinal (59) and the transverse (71) cases, so that its representation takes the form:

$$\pi_N(a_s^{(h)}) = \begin{pmatrix} b+b^\dagger & 0 \\ 0 & b+b^\dagger \end{pmatrix} \otimes \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} + \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{77}$$

Note that we have only considered the case $c_1 = c_2 = X$, as discussed in Sec. V A. With this $\pi_N(a_s^{(h)})$ we obtain, after a bit of calculation, the following trace

$$\mathrm{Tr}_{\mathcal{H}_T}(\pi_N(a_s^{(h)})\pi(\Omega_1^{(z)} - \Omega_2^{(0)})) = 4zX + 2c_1 e^{-|z|^2}\left(1 + |z|^2 + \cdots + \frac{|z|^{2N}}{N!}\right) - 2c_2, \tag{78}$$

which in the limit $N\to\infty$, takes the form

$$\mathrm{Tr}_{\mathcal{H}_T}(\pi(a_s^{(h)})\pi(\Omega_1^{(z)} - \Omega_2^{(0)})) = 2(2zX + Y) \tag{79}$$

where Y is same as in (74). Therefore, the spectral distance (12), in this case, takes the following form

$$d_h(\Omega_1^{(z)}, \Omega_2^{(0)}) = \sup_{a_s^{(h)}\in B_T} |\mathrm{Tr}_{\mathcal{H}_M}((\Omega_1^{(z)} - \Omega_2^{(0)})a_s^{(h)})| \tag{80}$$

$$= \sup_{a_s^{(h)}\in B_T} \frac{1}{2}|\mathrm{Tr}_{\mathcal{H}_T}(\pi(\Omega_1^{(z)} - \Omega_2^{(0)})\pi(a_s^{(h)}))| \tag{81}$$

$$= \sup_{a_s^{(h)}\in B_T} |2zX + Y|. \tag{82}$$

Now the matrix representation of $[\mathcal{D}_T, \mathcal{P}_N\pi_N(a_s^{(h)}) \times \mathcal{P}_N] =: M_h$ is just the sum of the matrices appearing in the longitudinal (63) and transverse (73) cases. The largest eigenvalue of the matrix $M_h^\dagger M_h$ comes out to be $\Lambda^2(\kappa X^2 + Y^2)$ thus, yielding the following ball condition (B_T):

$$\|[\mathcal{D}_T, \mathcal{P}_N\pi(a_s^{(h)})\mathcal{P}_N]\|_{op} = \Lambda\sqrt{\kappa X^2 + Y^2} \le 1. \tag{83}$$

In order to solve the expression (82), subject to the ball condition (83), let us consider the following lemma which can easily be proven (see, e.g., Lemma 7 of [9]).

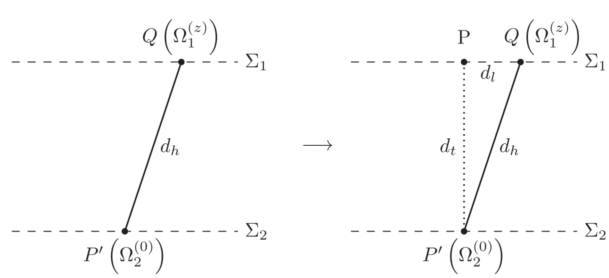

FIG. 4. Different states belonging to different Moyal planes.

Lemma: For any α, $\beta \geq 0$,

$$\sup_{x^2+y^2\leq 1} (\alpha x + \beta y) = \sqrt{\alpha^2 + \beta^2} \tag{84}$$

Now, by appropriately identifying the symbols in the above lemma as $\alpha \coloneqq \frac{2|z|}{\sqrt{\kappa}\Lambda} = \sqrt{2\theta}|z|$, $\beta \coloneqq \frac{1}{\Lambda}$, $x \coloneqq \Lambda\sqrt{\kappa}X$ and $y \coloneqq \Lambda Y$, we obtain the following hypotenuse distance:

$$d_h(\Omega_1^{(z)}, \Omega_2^{(0)}) = \sqrt{2\theta|z|^2 + \frac{1}{|\Lambda|^2}}. \tag{85}$$

Now, by making use of (30), (29), this can be recasted as

$$d_h(\Omega_1^{(z)}, \Omega_2^{(0)}) = \sqrt{(d_l(\Omega_i^{(z)}, \Omega_i^{(0)}))^2 + (d_t(\Omega_1^{(0)}, \Omega_2^{(0)}))^2}, \tag{86}$$

which is exactly the Pythagoras equality (see Fig. 4) that we discussed earlier (31).

VI. HIGGS FIELD FROM THE INTERNAL FLUCTUATION OF THE TOTAL DIRAC OPERATOR

In this penultimate section, we first provide a brief review of the fluctuated Dirac operator. We then discuss the emergence of a "Higgs field" in the framework of noncommutative geometry and its impact on the metric. For this, we follow [2,6] in particular. The notion of the Higgs field emerges automatically, along with the other gauge fields, when the tensor product of the spectral triples like the one which describes the doubled Moyal plane (25) are considered. The concept of the charge conjugation operator in the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, is brought about by the so-called real structure J which is an antiunitary operator on $\mathcal{H}$ satisfying $J^2 = \epsilon$, $J\mathcal{D} = \epsilon' \mathcal{D}J$ and $J\gamma = \epsilon''\gamma J$, where ϵ, ϵ' and ϵ'' are restricted to the values ± 1 only. The KO-dimension[3] $n \in \mathbb{Z}_8$ is then determined by the values of ϵ, ϵ' and ϵ'' according to a standard table (see [2]). To preserve the operator J under a unitary transformation $u \in \mathcal{A}$, satisfying $u^*u = uu^* = 1$ of $\pi(\mathcal{A})$, as in Sec. III, one has to use $U \coloneqq uJuJ^{-1}$, which fluctuates the Dirac operator as

$$\begin{aligned} \mathcal{D}_A &= U\mathcal{D}U^\dagger = \mathcal{D} + A + \epsilon' JAJ^{-1}; \\ A &= \pi(a_i)[\mathcal{D}, \pi(b_i)] \quad \text{with} \quad a_i, b_i \in \mathcal{A}. \end{aligned} \tag{87}$$

Here $A's$ are nothing but the Clifford algebra valued one-forms of the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_A)$. We suppress the summation index "i" from now on for brevity. The last term JAJ^{-1} in (87) plays no role in the distance calculation, as it commutes with $\pi(a)$ $\forall$ $a \in \mathcal{A}$ as can be shown easily by employing the so-called first order axiom and the axiom of reality (see, for instance, Lemma 5 in [6]). This leaves the ball condition B (12) unchanged and we therefore drop this term. Actually, the charge conjugation operator J maps an algebra element $a \in \mathcal{A}$ to the opposite algebra element a^o, satisfying $(ab)^o = b^o a^o$, so that $\pi(a^o) \coloneqq J\pi(a)^\dagger J^{-1}$ $\forall$ $a \in \mathcal{A}$ can act from the right on the elements of the Hilbert space $\mathcal{H}$.

This, however, is not allowed in the present construction of the spectral triple (26) as here we have only a left module, whereas we need a bi-module where both the left and the right actions are defined. To this end, consider the following spectral triple

$$\begin{aligned} \tilde{\mathcal{A}}_T &\coloneqq \mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C}); \\ \tilde{\mathcal{H}}_T &= (\mathcal{H}_q \otimes M_2(\mathbb{C})) \otimes M_2^d(\mathbb{C}) \ni \tilde{\Psi}; \\ \tilde{\mathcal{D}}_T\tilde{\Psi} &= \mathcal{D}_T\tilde{\Psi} + \tilde{\Psi}\mathcal{D}_T, \end{aligned} \tag{88}$$

instead of (26). Note that we have just replaced $\mathcal{H}_T$ in (26) by $\tilde{\mathcal{H}}_T$ here, by replacing the two factors of $\mathbb{C}^2$ in $\mathcal{H}_T$ by $M_2(\mathbb{C})$ and $M_2^d(\mathbb{C})$ respectively. Here the representation π is as before (32) and the grading operator $\tilde{\gamma}_T\tilde{\Psi} = \gamma_T\tilde{\Psi} + \tilde{\Psi}\gamma_T$, with the γ_T as in (27). Note that there is no difference in the structures of $\mathcal{D}_T$ and $\tilde{\mathcal{D}}_T$; the difference arises from their action on $\tilde{\Psi} \in \tilde{\mathcal{H}}_T$. More precisely, the action on the latter is given by the sum of the left and the right actions of the former. Likewise for γ_T and $\tilde{\gamma}_T$. On this spectral triple we can define the charge conjugation operator J_T, as $J_T\tilde{\Psi} = \tilde{\Psi}^\dagger$ (i.e., Hermitian conjugate of $\tilde{\Psi}$), which implies that $J_T^{-1} = J_T$ as $J_T^\dagger = J_T^{-1}$. With this real structure J_T the right action $\mathcal{D}_T$ can be represented by $J_T\mathcal{D}_T J_T^{-1}\tilde{\Psi} = J_T(\mathcal{D}_T\tilde{\Psi}^\dagger) = \tilde{\Psi}\mathcal{D}_T$ by using the fact that the Dirac operator is Hermitian. One can now check easily that $\epsilon = \epsilon' = \epsilon'' = 1$ and the KO-dimension of this triple comes out to be 0 modulo 8. We therefore have $\tilde{\mathcal{D}}_T = \mathcal{D}_T + J_T\mathcal{D}_T J_T^{-1}$, which again enables us to easily verify that $[\tilde{\mathcal{D}}_T, \pi(a_T)] = [\mathcal{D}_T, \pi(a_T)]$ for all $a_T \in \mathcal{A}_T$. In light of this we see that the ball condition B (12) remains unaffected with the new spectral triple (88) thus producing the same distance as in previous triple $(\mathcal{A}_T, \mathcal{H}_T, \mathcal{D}_T)$ making the two triples equivalent. We can therefore revert back to the previous definition of the spectral triple of the doubled Moyal plane (26) except that the Dirac operator is now augmented by a "Higgs" term:

$$\mathcal{D}_A = \mathcal{D}_T + A. \tag{89}$$

To understand the splitting of the one-form A (87) in case of the doubled Moyal plane, we consider the algebra

[3]KO dimension is the shift in the grading on K-theory, which is involved in a Poincaré duality for spectral triple [16]. It perhaps owes this terminology from the fact that the Dirac operator defines a class in K-homology and not in the ordinary homology.

elements to be of separable form like $a_T \coloneqq a \otimes a_2$ and $b_T \coloneqq b \otimes b_2 \in \mathcal{A}_T$ as in Sec. V A so that we get

$$\pi(a_T)[\mathcal{D}_T, \pi(b_T)] = \pi(a)[\mathcal{D}_M, \pi(b)] \otimes a_2 b_2 + \sigma_3 \pi(ab) \otimes a_2[\mathcal{D}_2, b_2]. \tag{90}$$

Note that we have used the fact that the chirality operator commutes with the representation of the algebra elements in the above calculation. The first term on the right-hand side of (90) contains a generic one-form of the Moyal plane, i.e., $\pi(a)[\mathcal{D}_M, \pi(b)]$, which gives rise to gauge fields in the noncommutative geometry. On the other hand, the additional second term on the RHS of (90) has the one-form of the two-point space, i.e., $a_2[\mathcal{D}_2, b_2]$, which gives rise to a prototype of the scalar Higgs field. For our discussion here we only focus on the implications of this "Higgs field" on the metric aspects particularly on the transverse distance. We thus retain only this term so that the one-form A becomes

$$A = c\sigma_3 \otimes a_2[\mathcal{D}_2, b_2]; \qquad c \coloneqq ab \in \mathcal{H}_q. \tag{91}$$

It is noteworthy that we can write this as $A = \sigma_3 \otimes H$, where $H = ca_2[\mathcal{D}_2, b_2]$ is referred to as the "Higgs field" in the literature. This is, however, in our case just a prototype scalar Higgs field as we are dealing only with the 2 dimensional Moyal plane. We now want to compute the transverse distance $d_t(\rho_z \otimes \omega_1, \rho_z \otimes \omega_2)$ between a generic state $\rho_z = |z\rangle\langle z|$ in one of the Moyal planes and its clone on the other one, i.e., the counterpart of (29) in the presence of the "Higgs" field. To do so we construct the restricted spectral triple using the prescription (38), with the projection operator $\mathcal{P}_{N=\tilde{0}}$ (62), which is same as $P^{(\mathrm{trans})}_{T(z)}$ (44). But to ensure that this transverse distance can again be computed through this restricted triplet itself, we have to ensure that the condition (40) is satisfied here too, i.e.,

$$[\mathcal{D}_A, P^{(\mathrm{trans})}_{T(z)}] = 0. \tag{92}$$

By making use of (44), (91) this essentially boils down to

$$[c, |\tilde{0}\rangle\langle\tilde{0}|] = [c, |z\rangle\langle z|] = 0, \tag{93}$$

which can easily be shown to satisfy, for example, if c belongs to the subspace $\mathcal{H}_q^{\tilde{0}} \coloneqq \mathrm{Span}\{|\tilde{m}\rangle\langle\tilde{n}|\colon m, n \in \{1,2,3\ldots\}\}$. We shall, for simplicity, assume in the following that this condition holds. Otherwise, one has to clearly make use of complete spectral triple. Clearly, even if (93) holds for a particular state $\rho_z = |z\rangle\langle z|$, it does not imply that it holds for the other states $\rho_{z'} = |z'\rangle\langle z'|$ as well. Now, any arbitrary algebra element $a_T \in \mathcal{A}_T$ (26) gets projected (for $a \in \mathcal{A}_M$ and c_1, $c_2 \in \mathbb{C}$) with this projector as

$$\begin{aligned} \pi(a_T) &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \mapsto \mathcal{P}_{N=\tilde{0}} \pi(a_T) \mathcal{P}_{N=\tilde{0}} \\ &= \begin{pmatrix} |\tilde{0}\rangle\langle\tilde{0}| & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} fc_1 & 0 \\ 0 & fc_2 \end{pmatrix}; \\ f &= \langle\tilde{0}|a|\tilde{0}\rangle = \langle z|a|z\rangle. \end{aligned} \tag{94}$$

Likewise, using the algebra element

$$a_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \in M_2^d(\mathbb{C})$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ in (91) for the two-point space, the fluctuated Dirac operator (89), gets projected as

$$\mathcal{D}_A \mapsto \mathcal{P}_{N=\tilde{0}} \mathcal{D}_A \mathcal{P}_{N=\tilde{0}} = \begin{pmatrix} |\tilde{0}\rangle\langle\tilde{0}| & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Lambda(1 + g\alpha_1(\beta_2 - \beta_1)) \\ \bar{\Lambda}(1 + g\alpha_2(\beta_1 - \beta_2)) & 0 \end{pmatrix}. \tag{95}$$

Here $g = g(x_1, x_2) = \langle\tilde{0}|c(\hat{x}_1, \hat{x}_2)|\tilde{0}\rangle = \langle z|c(\hat{x}_1, \hat{x}_2)|z\rangle$ is some function of the dimensionful coordinates x_1 and x_2 (7) and $\Lambda \in \mathbb{C}$ (Note that the unitary transformation using $U(\phi)$ (34) to render Λ real, as in Sec. III, has not been performed here). Further, the demand that the projected Dirac operator be Hermitian yields $\overline{g\alpha_1(\beta_2 - \beta_1)} = g\alpha_2(\beta_1 - \beta_2)$. Moreover, any arbitrary element of the Hilbert space

$$\Psi \coloneqq \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathcal{H}_T$$

(26), gets projected as

$$\Psi \mapsto \mathcal{P}_{N=\tilde{0}} \Psi = \begin{pmatrix} |\tilde{0}\rangle \\ 0 \end{pmatrix} \otimes \begin{pmatrix} hc_1 & 0 \\ 0 & hc_2 \end{pmatrix}; \qquad h = \langle\tilde{0}|\psi_1\rangle. \tag{96}$$

Finally, the projected spectral triple as a whole takes the following form:

$$\begin{aligned} \mathcal{A}_T^{(\mathcal{P}_{N=\tilde{0}})} &= \begin{pmatrix} |\tilde{0}\rangle\langle\tilde{0}| & 0 \\ 0 & 0 \end{pmatrix} \otimes M_2^d(\mathbb{C}), \\ \mathcal{H}_T^{(\mathcal{P}_{N=\tilde{0}})} &= \begin{pmatrix} |\tilde{0}\rangle \\ 0 \end{pmatrix} \otimes \mathbb{C}^2, \\ \mathcal{D}_T^{(\mathcal{P}_{N=\tilde{0}})} &= \begin{pmatrix} |\tilde{0}\rangle\langle\tilde{0}| & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Lambda(x_1, x_2) \\ \bar{\Lambda}(x_1, x_2) & 0 \end{pmatrix} \end{aligned} \tag{97}$$

where $\Lambda(x_1, x_2) = \Lambda(1 + \alpha_1(\beta_2 - \beta_1)g(x_1, x_2))$ is just another complex number. This has the same form as in (42); therefore, it essentially has the same structure as that of the two-point space. The transverse spectral distance, i.e., the distance between the pure states ρ_z and its clone is identical to the spectral distance between the pair of states ω_1 and ω_2 of the restricted spectral triple and is given, in this case, by

$$d_t(\rho_z \otimes \omega_1, \rho_z \otimes \omega_2) \coloneqq d_t^{(\mathrm{rest})}(\omega_1, \omega_2) = \frac{1}{|\Lambda(x_1, x_2)|}. \tag{98}$$

This clearly fluctuates along the Moyal plane and reproduces (29) in the absence of the "Higgs" field. Thus this variation of the transverse distance in the presence/absence of the all pervading "Higgs" field provides an alternative geometrical perspective about the "Higgs" field itself.

VII. CONCLUSION

Finite matrix spaces play a crucial role in the formulation of the "standard model" in the framework of the noncommutative geometry [1,2]. On the other hand, the Moyal plane is one of the promising candidate for modeling the geometry near the Planck scale. The merger of these two spaces with the matrix space given, specifically by (A1), yields the doubled Moyal plane which presents itself as a very interesting toy model. We have explored its metric structure using the Connes prescription of spectral distances in a Hilbert-Schmidt operatorial framework, which facilitates the construction of the Dirac eigenspinors and provides a natural basis to work with. Working with this basis also helps us in economizing our calculations and reproducing several results existing in the literature [6,7].

The actual calculations of various distances are done using specific "optimal" algebra elements, which are partially motivated from our previous work [11]. The distances comes out very neatly and satisfy the Pythagorean equality exactly as expected. We did encounter the need to introduce the identity element of the Moyal algebra to render it unital, but, as an alternative approach to [7], we worked with a sequence of projection operators constructed from the Dirac eigenspinors, which in the limiting case ($N \to \infty$) represents the identity element.

Finally, we fluctuate the Dirac operator and focus on the "Higgs" field part and analyze its impact on the variation of the transverse distance in the doubled Moyal plane. In order to fluctuate the Dirac operator we had to modify the spectral triple at an intermediate stage to arrive at an equivalent triple (in the sense that the ball condition and hence the metric is not altered), so as to be able to incorporate the real structure. At the end, however, we reverted back to our earlier triple, except that the Dirac operator is now augmented with a "Higgs" term. As a consequence, we found how the transverse distance, obtained by constructing a suitable projection operator to restrict the spectral triple to the two-point space, depends on the coordinates of the Moyal plane. This projection operator, in turn, had to be necessarily constructed with these eigenspinors, thus illustrating the important role played by the eigenspinors, which, we feel, were not emphasized adequately in the literature.

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APPENDIX A: THE SPECTRAL DISTANCE ON TWO-POINT SPACE

Two-point space is an abstract mathematical space of two complex numbers, for which the spectral triple is given by (see [2])

$$\left(\mathcal{A}_2 = \mathbb{C}^2, \mathcal{H}_2 = \mathbb{C}^2, \mathcal{D}_2 = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}\right) \tag{A1}$$

where Λ is a constant complex parameter of length-inverse dimension. Let us consider

$$a = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{A}_2 = \mathbb{C}^2.$$

One can replace the canonical bases $\binom{1}{0}$ and $\binom{0}{1}$ of $\mathbb{C}^2 = \mathrm{Span}\{\binom{1}{0}, \binom{0}{1}\}$ by the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ respectively, such that $\mathbb{C}^2 = \mathrm{Span}\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. In this new basis, an arbitrary element $a \in \mathbb{C}^2$ can be written as

$$a = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in M_2^d(\mathbb{C}), \tag{A2}$$

where $M_2^d(\mathbb{C})$ is the c-valued 2×2 diagonal matrix. In this construction, we see that $\mathcal{A}_2$ has a manifest structure of an algebra through usual matrix multiplication.

The action of $a \in \mathcal{A}_2$ on

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathcal{H}_2 = \mathbb{C}^2$$

is now given in a straightforward manner as follows:

$$\pi(a)\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = a\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}\lambda_1\mu_1\\ \lambda_2\mu_2\end{pmatrix}. \tag{A3}$$

Note that the role of representation π becomes redundant here as before (A3). Moreover, the pure states $\omega_i(a) \in \mathbb{C}$; $i = \{1, 2\}$ of $\mathcal{A}_2 \ni a$ are given by

$$\omega_i(a) = \mathrm{Tr}(\omega_i \pi(a)) = \mathrm{Tr}(\omega_i a), \tag{A4}$$

so that they correspond to the evaluation maps $\omega_1(a) = \lambda_1$ and $\omega_2(a) = \lambda_2$ of the two points of this space. Clearly, the choice

$$\omega_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad \omega_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{A5}$$

does the job, as can be seen easily by using (A4). Any other generic state are given by a convex sum of these two pure states and therefore are necessarily mixed in nature. These are clearly the density matrices associated to $\binom{1}{0}$ and $\binom{0}{1}$ and corresponds precisely to the 2×2 matrix basis, chosen for $\mathcal{A}_2 = \mathbb{C}^2$. Now, in order to compute the distance between these two pure states ω_i, let us consider a generic algebra element

$$a = \begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} \in \mathbb{C}^2 \cong M_2^d(\mathbb{C})$$

and insert it in the distance formula (12) to obtain

$$d(\omega_1, \omega_2) = \sup_{a\in B}|\omega_1(a) - \omega_2(a)| = \sup_{a\in B}|\alpha_1 - \alpha_2|. \tag{A6}$$

Moreover, the commutator $[\mathcal{D}_2, \pi(a)]$ takes the form

$$[\mathcal{D}_2, \pi(a)] = (\alpha_1 - \alpha_2)\begin{pmatrix}0 & -\Lambda\\ \bar{\Lambda} & 0\end{pmatrix} \tag{A7}$$

where we have used $\pi(a) = a$ in ω_1, ω_2 basis as before. Now invoking the C^* property of the algebra $\mathcal{A}_2$ here, i.e., $\|[\mathcal{D}_2, \pi(a)]\|_{op}^2 = \|[\mathcal{D}_2, \pi(a)]^\dagger[\mathcal{D}_2, \pi(a)]\|_{op}$ and noting that

$$[\mathcal{D}_2, \pi(a)]^\dagger[\mathcal{D}_2, \pi(a)] = |\alpha_1 - \alpha_2|^2|\Lambda|^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{A8}$$

we obtain $\|[\mathcal{D}_2, \pi(a)]\|_{op} = |\alpha_1 - \alpha_2||\Lambda|$. With this the Lipschitz ball condition yields

$$|\alpha_1 - \alpha_2| \leq \frac{1}{|\Lambda|}. \tag{A9}$$

Substituting this in (A6), we obtain the known result of spectral distance between the pure states ω_1, ω_2 of the two-point space [2], which is

$$d(\omega_1, \omega_2) = \frac{1}{|\Lambda|}. \tag{A10}$$

APPENDIX B: ON THE OPTIMAL ELEMENT FOR THE MOYAL PLANE

We provide here a brief review about how an upper bound of the spectral distance is obtained in the case of Moyal plane and the subsequent role of the optimal algebra element $a_s \in \mathcal{A}_M$ (10) (multiplier algebra), which, by definition, saturates this bound as well as the one occurring in the ball condition: $[\mathcal{D}_M, \pi(a_s)] = 1$. This was explained in detail in our previous work [11]. As explained already in Sec. II, the action of the pure state ρ_z, on a generic algebra element $a \in \mathcal{H}_q$ is given by

$$\begin{aligned}\rho_z(a) &= \mathrm{Tr}_{\mathcal{H}_c}(\rho_z a) = \mathrm{Tr}_{\mathcal{H}_c}(U(z,\bar{z})|0\rangle\langle 0|U^\dagger(z,\bar{z})a)\\ &= \langle 0|(U^\dagger(z,\bar{z})aU(z,\bar{z}))|0\rangle\end{aligned} \tag{B1}$$

where $U(z,\bar{z}) := e^{-\bar{z}\hat{b}+z\hat{b}^\dagger}$ and $|z\rangle$ is the coherent state (7). We, therefore, get the following form of the spectral distance (12)

$$\begin{aligned}d(\rho_z, \rho_0) &= d(\omega_z, \omega_0)\\ &= \sup_{a\in B}|\langle 0|(U^\dagger(z,\bar{z})aU(z,\bar{z}))|0\rangle - \langle 0|a|0\rangle|.\end{aligned} \tag{B2}$$

Now let us consider a one-parameter family of density matrices $\rho_{zt} = |zt\rangle\langle zt|$, with a real affine parameter $t \in [0, 1]$, interpolating ρ_z and ρ_0. We can then introduce the following function:

$$W(t) = \rho_{zt}(a) = \mathrm{Tr}_{\mathcal{H}_c}(\rho_{zt}a). \tag{B3}$$

Consequently, we have the inequality,

$$|\omega_z(a) - \omega_0(a)| = \left|\int_0^1 \frac{\mathrm{d}W(t)}{\mathrm{d}t}dt\right| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|dt. \tag{B4}$$

As in Sec. V A, here also we work with a Hermitian element ($a = a^\dagger \in \mathcal{A}_M$) and obtain an upper bound for $|\frac{\mathrm{d}W(t)}{\mathrm{d}t}|$ by making use of the Cauchy-Schwarz inequality as

$$\begin{aligned}\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| &= |\bar{z}\rho_{zt}([b,a]) + z\rho_{zt}([b,a]^\dagger)|\\ &\leq \sqrt{2}|z|\sqrt{|\rho_{zt}([b,a])|^2 + |\rho_{zt}([b,a]^\dagger)|^2}\\ &\leq \sqrt{2}|z|\sqrt{\|[b,a]\|_{op}^2 + \|[b,a]^\dagger\|_{op}^2}.\end{aligned} \tag{B5}$$

The "ball" condition $[\mathcal{D}_M, \pi(a)] \leq 1$ [with the Dirac operator $\mathcal{D}_M$ (10)] reduces to a simpler form:

$$\|[b,a]\|_{op} = \|[b^\dagger, a]\|_{op} \leq \sqrt{\frac{\theta}{2}} \quad \text{for } a \in B. \tag{B6}$$

From (B5) and (B6), one can therefore write

$$\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| \leq \sqrt{2\theta}|z|. \tag{B7}$$

Hence from Eqs. (B2), (B4), and (B7) we have the following upper bound for the Connes' distance:

$$d(\omega_z, \omega_0) \leq \sqrt{2\theta}|z|. \tag{B8}$$

It is now quite easy to verify the identity that $\omega_z(a_s) = \mathrm{Tr}(\rho_z a_s) = |z|\cos\gamma$, for some angle γ, with the optimal element being a_s (17), so that $\omega_0(a_s) = 0$ holds trivially. It then clearly follows from (B4), (B8) that the supremum is indeed reached by an optimal element a_s of the form (17), so that the inequality (B8) is saturated. Besides, it saturates the ball condition also, as mentioned earlier. This optimal element, however, fails to be a Hilbert-Schmidt operator. In fact it is not even a compact operator. Consequently $a_s \notin \mathcal{H}_q = \mathcal{A}$, but can be thought of as belonging to the multiplier algebra (see [7] for a resolution of this issue and also [11] for a variant).

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