Extended Rindler spacetime and a new multiverse structure

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This is the first of a series of papers in which we use analyticity properties of quantum fields propagating on a spacetime to uncover a new multiverse geometry when the classical geometry has horizons and/or singularities. The nature and origin of the "multiverse" idea presented in this paper, that is shared by the fields in the standard model coupled to gravity, are different from other notions of a multiverse. Via analyticity we are able to establish definite relations among the universes. In this paper we illustrate these properties for the extended Rindler space, while black hole spacetime and the cosmological geometry of mini-superspace (see Appendix B) will appear in later papers. In classical general relativity, extended Rindler space is equivalent to flat Minkowski space; it consists of the union of the four wedges in (u, v) light-cone coordinates as in Fig. 1. In quantum mechanics, the wavefunction is an analytic function of (u, v) that is sensitive to branch points at the horizons $u = 0$ or $v = 0$, with branch cuts attached to them. The wave function is uniquely defined by analyticity on an infinite number of sheets in the cut analytic (u, v) spacetime. This structure is naturally interpreted as an infinite stack of identical Minkowski geometries, or "universes", connected to each other by analyticity across branch cuts, such that each sheet represents a different Minkowski universe when (u, v) are analytically continued to the real axis on any sheet. We show in this paper that, in the absence of interactions, information does not flow from one Rindler sheet to another. By contrast, for an eternal black hole spacetime, which may be viewed as a modification of Rindler that includes gravitational interactions, analyticity shows how information is "lost" due to a flow to other universes, enabled by an additional branch point and cut due to the black hole singularity.

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I. EXTENDED RINDLER SPACETIME

A massive particle moving in a background spacetime with metric $g_{\mu\nu}(x)$ is described by a worldline action

$$S = \int d\tau \left[\frac{1}{2e(\tau)} g_{\mu\nu}(x(\tau))\partial_\tau x^\mu(\tau)\partial_\tau x^\nu(\tau) - \frac{e(\tau)}{2}\mu^2\right]. \quad (1)$$

The einbein $e(\tau)$ is the gauge field for τ-reparametrization symmetry. Its equation of motion is a constraint that may be written in terms of the canonical conjugate momentum $p_\mu(\tau)$ as, $g^{\mu\nu}(x)p_\mu p_\nu + \mu^2 = 0$. When the system is quantized, the wavefunction in position space $\varphi(x^\mu)$ must satisfy the quantum-ordered constraint that takes the form of the Klein-Gordon equation in a curved background

$$(-\nabla^2 + \mu^2)\varphi(x) = 0, \quad \text{with}$$
$$\nabla^2\varphi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}(x)\partial_\nu\varphi(x)). \quad (2)$$

The case of $g_{\mu\nu}(x)$ for Rindler spacetime commonly refers to the coordinate frame of an observer undergoing constant proper acceleration in an otherwise flat spacetime [1]. Using lightcone coordinates (u, v) in flat spacetime,

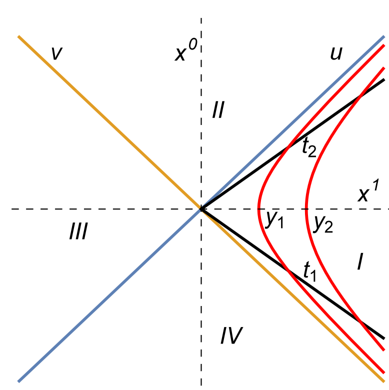

FIG. 1. Four regions of the map (u, v) to (t, y).

Rindler spacetime corresponds to just region-I in Fig. 1, namely $u > 0, v < 0$, bounded by horizons at $u = 0$ or $v = 0$. This wedge of flat spacetime can be reparametrized in terms of Rindler coordinates, $y > 0, -\infty < t < \infty$, as in Eq. (5).

By extended Rindler spacetime we mean the union of the four regions I-IV shown in Fig. 1, which seems to be equivalent to the full Minkowski space. We will motivate the study of the union of the four regions and will find new features beyond just Minkowski space that are not apparent at the classical level (such as geodesics). The new aspects emerge only at the quantum level as properties of the first quantized wave function φ, or equivalently a property of fields φ that satisfy the Klein-Gordon equation $(-\nabla^2 + \mu^2)\varphi = 0$ in extended Rindler spacetime.

We were motivated to study extended Rindler spacetime because we found that the wavefunctions for the cases of cosmology as well as black hole physics have the same features. These applications are consequences of the standard model (SM) coupled to general relativity (GR) that includes a modest modification that lifts the conventional theory to a locally scale invariant (Weyl symmetric) version of $\mathrm{GR} + \mathrm{SM}$ [2]. The conventional $\mathrm{GR} + \mathrm{SM}$ at low energies is recovered by fixing a Weyl gauge that introduces the dimensionful parameters, the Newton constant G_N, dark energy Λ and electroweak scale v_{EW}, all coming from the same source [2]. This explains that all dimensionful constants are the same everywhere in the observed spacetime because they come from the same field that, when frozen to a constant by a gauge fixing, fills the entire universe of the conventional $\mathrm{GR} + \mathrm{SM}$. The Weyl symmetry geodesically completes the universe of the conventional theory at high energies, beyond cosmological or black hole singularities, by including previously missing patches of spacetime in a way analogous to enlarging the Rindler patch in Fig. 1 to the extended Rindler spacetime. In cosmological studies, using the Friedman equation at the classical level or the Wheeler deWitt equation at the quantum level, it is found that the effective geometry of minisuperspace—as a *geometry in field space* that includes the scale factor, curvature, anisotropy, and matter in the form of radiation and the Higgs field—is closely related to the geometry of the extended Rindler spacetime discussed in this paper, including some interactions that are not part of the discussion here. In certain limits of the interactions the minisuperspace geometry reduces mathematically exactly to the extended Rindler space. Then a wedge in minisuperspace (region II) is related to the expanding spacetime after the big bang, while the other regions I,III, and IV play a role in determining a geodesically complete history of the universe. These comments are amplified in Appendix B to which the interested reader may turn anytime without having to read the rest of the paper. Full details will appear in separate papers [3,4]. Until then, we will discuss the mathematical properties of the familiar Rindler space and its extensions without any reference to minisuperspace, cosmology or black holes. The applications outlined in this paragraph are motivating factors, otherwise we emphasize that, this paper stands on its own to discuss mainly the new quantum aspects of extended Rindler spacetime.

As seen by a traditional Rindler observer in region I, during the entire time span of the Rindler universe, $-\infty < t < \infty$, geodesics of moving particles remain only within the Rindler wedge (see Sec. II). However, region I is a geodesically incomplete spacetime from the perspective of other observers, such as a Minkowski observer that uses x^0 rather than t as "time," or more generally a proper observer that uses proper time τ. So even though physical particles may escape/enter through the horizons, and physical phenomena may exist in all the four regions in Fig. 1, a Rindler observer is incapable of detecting such phenomena from his/her own perspective. Explorers that wish to understand the deeper nature of space-time beyond their own limited observational capabilities must therefore consider all possible observers, not only those observers limited by information available in some chosen coordinate system. Examples of observers with limited capabilities of observation due to geodesically incomplete coordinate systems include an observer outside of a black hole that is similar to a Rindler observer. With this thought in mind, in this paper we are interested in the "extended Rindler space" that consists of the geodesically complete union of the four regions in Fig. 1. This means that, in the absence of interactions, extended Rindler space is essentially flat Minkowski space. Indeed this is true in classical physics. However, in quantum physics, we will show that the wave functions of particles are sensitive to aspects of extended Rindler space that classical physics cannot capture even with geodesically complete spacetime. Wave functions for particles in first quantization amount to fields. Therefore, as a first exercise, we study here scalar fields in the background of extended Rindler space.

Rindler geometry has a long history of applications including the Unruh effect [5–15], therefore, it is inescapable that some of our discussion below overlaps old analyses. But for completeness, as well as for establishing notation and conceptual background, we include in this paper some familiar material along with our newer ideas to help the reader follow our views on the multiverse aspects of extended Rindler spacetime that becomes apparent only at the quantum level. The same approach will be used in future papers to make similar cases for black holes and cosmology for which the discussion and results in this paper are a prelude toward the more complicated multiverse nature of geodesically complete cosmological spacetimes [3] and eternal black hole spacetimes [4]. Therefore, in the present paper we wish to provide sufficient details to build up the ideas through the simpler case of the extended Rindler spacetime without interactions.

Minkowski spacetime in $1+1$ dimensions,[1] (x^0, x^1), may be rewritten in terms of light cone coordinates (u, v),

$$u \equiv x^0 + x^1, \qquad v \equiv x^0 - x^1, \quad \text{or} \quad x^0 = \frac{u+v}{2}, \qquad x^1 = \frac{u-v}{2}.$$

Rindler coordinates (t, y), that are convenient to describe each region separately, are given by a coordinate transformation

$$2y = -uv \quad \text{and} \quad e^{2t}\mathrm{sign}(y) = -\frac{u}{v}, \tag{3}$$

with $-\infty < t < \infty$ and $-\infty < y < \infty$. In the (t, y) coordinates, the flat Minkowski metric takes the appearance of a curved metric, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, with its corresponding Laplacian as in Eq. (2),

$$\begin{aligned} ds^2 &= -dudv = -(2y)dt^2 + (2y)^{-1}dy^2 = \pm e^{2\xi}(-dt^2 + d\xi^2), \\ \nabla^2\varphi &= -4\partial_u\partial_v\varphi = -\frac{1}{2y}\partial_t^2\varphi + \partial_y(2y\partial_y\varphi) = \pm e^{-2\xi}(-\partial_t^2\varphi + \partial_\xi^2\varphi), \end{aligned} \tag{4}$$

where $e^{2\xi} \equiv |2y|$, and the $(\pm) = \mathrm{sign}(y)$ refer to regions I&III versus II&IV. For the transformation of Eq. (3) it is useful to distinguish four regions, I,II,III,IV, as indicated in Fig. 1. In various regions (t, y) is related to (u, v) as follows

$$\begin{aligned} &I_{(u>0,v<0,y>0)}: \; u = +\sqrt{2y}e^t = e^{t+\xi}, \qquad v = -\sqrt{2y}e^{-t} = -e^{-t+\xi}, \qquad 2y = -uv, \qquad e^{2t} = -\frac{u}{v}, \\ &II_{(u>0,v>0,y<0)}: \; u = +\sqrt{-2y}e^t = e^{t+\xi}, \qquad v = +\sqrt{-2y}e^{-t} = +e^{-t+\xi}, \qquad 2y = -uv, \qquad e^{2t} = +\frac{u}{v}, \\ &III_{(u<0,v>0,y>0)}: \; u = -\sqrt{2y}e^t = -e^{t+\xi}, \qquad v = +\sqrt{2y}e^{-t} = +e^{-t+\xi}, \qquad 2y = -uv, \qquad e^{2t} = -\frac{u}{v}, \\ &IV_{(u<0,v<0,y<0)}: \; u = -\sqrt{-2y}e^t = -e^{t+\xi}, \qquad v = -\sqrt{-2y}e^{-t} = -e^{-t+\xi}, \qquad 2y = -uv, \qquad e^{2t} = +\frac{u}{v}. \end{aligned} \tag{5}$$

The sign of the square root, $\pm'\sqrt{|2y|}$ (which agree with the signs of u and v), distinguishes region I from III and II from IV. The square roots $\pm'\sqrt{|2y|}$ appear in both the classical and quantum solutions of the extended Rindler system. In particular, continuity of the solutions in the (t, y) coordinates across the horizons in Fig. 1, require the inclusion of all four Rindler regions.

An intuitive description of the extended Rindler geometry in classical physics is partially conveyed by the following comments. The horizons, that form the boundaries of the four regions, occur at either $u = 0$ or $v = 0$. The $u = 0$ horizons are indicated as the orange line in Fig. 1, where $-\infty < v < \infty$ and $t = -\infty$, $y = 0$; the $v = 0$ horizons are indicated as the blue line in Fig. 1, where $-\infty < u < \infty$ and $t = \infty$, $y = 0$. A foliation of the (u, v) plane is provided by either fixed values of y or fixed values of t within each Rindler region separately. The case of $y = -\frac{1}{2}uv = \text{fixed}$ corresponds to hyperbolas in each (u, v) region; red curves labeled by $y_{1,2}$ in Fig. 1, with $0 < y_1 < y_2 < \infty$, are examples shown only in region I. The case of $t = \frac{1}{2}\ln|u/v| = \text{fixed}$ correspond to straight rays that extend from the origin to infinity within each (u, v) region; black rays in Fig. 1, labeled by $-\infty < t_1 < t_2 < +\infty$, are examples shown only in region I. In all regions $|y|$ increases uniformly from the center or horizons ($|y| = 0$) to the outer boundaries of the region at infinity ($|y| = \infty$). On the other hand, going around in the counterclockwise direction in Fig. 1, the Rindler t that labels the rays increases from $-\infty$ to $+\infty$ in region I, followed by a decrease from $+\infty$ to $-\infty$ in region II, followed by an increase from $-\infty$ to $+\infty$ in region III, and followed by a decrease from $+\infty$ to $-\infty$ in region IV.

It is important to emphasize that in region I, the Minkowski time, $x^0 = (u + v)/2$, *increases*, while the Rindler time (the t that labels the rays) also increases counterclockwise from $-\infty$ to $+\infty$; however in region III the Minkowski time x^0 *decreases* while the Rindler time t increases counterclockwise from $-\infty$ to $+\infty$. This difference between regions I and III is important in the interpretation of particle versus antiparticle quantum waves, and it leads to an interchange of creation/annihilation symbols, $a \leftrightarrow b^\dagger$, in the construction of the field in region I versus region III, as exhibited later in Eq. (22) versus Eq. (24).

The rest of this paper is organized as follows. In Sec. II we discuss the geodesics in the classical extended Rindler

[1] We focus on $1+1$ dimensions for simplicity; this is easily generalized to any number of dimensions.

space. In Sec. III we discuss the complete and orthonormal set of modes of the Klein-Gordon equation in the extended Rindler background, construct the general first quantized wavepackets and the second quantized quantum field, insuring that these are continuous across horizons of the four Rindler quadrants in Fig. 1. In Sec. IV we determine the analyticity properties of the first quantized wavepackets and quantum field and show how, by analytic continuation, these naturally take values in an infinite stack of Minkowski sheets labeled by two integers, (n, m), that constitute the multiverse. In Sec. V we impose boundary conditions at the horizons of the four quadrants in the (0,0) universe to require that on this sheet the extended Rindler space is equivalent to Minkowski space. By analyticity, this determines the boundary conditions on all (n, m) sheets of the multiverse, and we show that the quantum oscillators at various sheets are related to each other by a specific canonical transformation determined by analyticity. In Sec. VI we display the multiverse directly in the Minkowski basis and derive a very nontrivial canonical transformation that relates the general level (n, m) Minkowski field to the level-(0,0) Minkowski field. This canonical transformation represents in the Minkowski basis the analytic continuation of the field in the Rindler basis, and it could not be obtained without going through the Rindler basis. In Sec. VII we study charge (or information) conservation and unitarity and show that, even though there is a flux of information (or charge) across the horizons of neighboring quadrants, charge is conserved within each Rindler quadrant separately at each (n, m) universe. From this we conclude that there is no leakage of information among levels of the Rindler multiverse. In Sec. VIII we summarize the essential message of this paper and then suggest that the multiverse structure discussed here in the simple context of extended Rindler space is more general and also emerges in any spacetime that has horizons, such as black holes, including the Schwarzchild black hole and others. Furthermore, we argue that in the presence of interactions, such as gravitational interactions represented by a black hole, big bang and others, the levels of the multiverse are no longer isolated from each other, and charge/information/probability do leak from one level of the modified multiverse to any other level, as discussed in other papers including the case of the Rindler-like geometry of mini-superspace with interactions [3] and the case of an eternal black hole [4]. Appendix A gives details of computations of information conservation and information fluxes across the horizons and at asymptotic regions in each Rindler wedge. Appendix B is included to clarify and amplify the physically motivating factors outlined at the beginning of this section, in particular in the case of cosmology where the new multiverse idea should be relevant to the cyclic universe scenario.

II. GEODESICS

Before discussing the first quantized wave function or equivalently the field, in this section we study the geodesics in extended Rindler space. The purpose is to first understand the motion of particles in the classical geometry. This will provide a background to better understand the flux of charge or information from the perspective of wave packets. We will see in Sec. (IV) that the wave function reveals a far richer geometry involving an infinite stack of (u, v) sheets with each sheet related to the classical geometry.

The geodesics in a curved spacetime with metric $g_{\mu\nu}(x)$ can be computed by solving the equations of motion of a massive or massless particle on a worldline $x^\mu(\tau)$ moving in the curved background. The action on the worldline in the first order formalism is given by, $S(x) = \int d\tau \{\dot{x}^\mu(\tau) p_\mu(\tau) - \frac{e(\tau)}{2}[g^{\mu\nu}(x(\tau)) p_\mu(\tau) p_\nu(\tau) + \mu^2]\}$. The equation of motion for varying the einbein $\delta e(\tau)$ gives the on-shell constraint, and the equation of motion for varying δp_μ gives the relation between the velocity and momentum. After the variations, choosing the gauge $e(\tau) = 1$ (due to τ-reparametrization), these equations take the form

$$g^{\mu\nu}(x) p_\mu p_\nu + \mu^2 = 0, \qquad \dot{x}^\mu = g^{\mu\nu}(x) p_\nu. \tag{6}$$

The equation of motion for varying δx^μ gives an expression for $\dot{p}_\mu$ which amounts to a second order differential equation for $x^\mu(\tau)$. This is the geodesic equation. A first integral of the geodesic equation is already contained in the constraint equation, therefore it can be ignored and concentrate on solving just the equations above in order to find the geodesic solution for $x^\mu(\tau)$ as a function of τ. Note that τ is invariant under target spacetime reparametrizations so, unlike observer-dependent choices of "time" in target space-time, τ is an unambiguous choice of "time" as the evolution parameter for the motion of the particle from the perspective of a proper observer in the frame of the particle itself.

In the case of the flat 2D Minkowski metric, $ds^2 = -du dv$, or $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}$, the *inverse* metric is $g^{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$, and the equations to be solved (6) are, $\dot{u} = -2p_v = p^u$, $\dot{v} = -2p_u = p^v$, where $p^u(\tau) = k^+$ and $p^v(\tau) = k^-$ are constants of motion (due to translation invariance of the action, or the $\dot{p}$ equations of motion), while the constraint is, $-4p_u p_v + \mu^2 = 0 = -k^+k^- + \mu^2$. So, the geodesic solution is

$$u(\tau) = k^+\tau + u_0,\ v(\tau) = \frac{\mu^2}{k^+}\tau + v_0, \qquad \text{and} \quad -\infty < k^+ < \infty, \tag{7}$$

where (u_0, v_0) is the initial position in the (u, v) plane. By eliminating τ between the first two equations this solution is

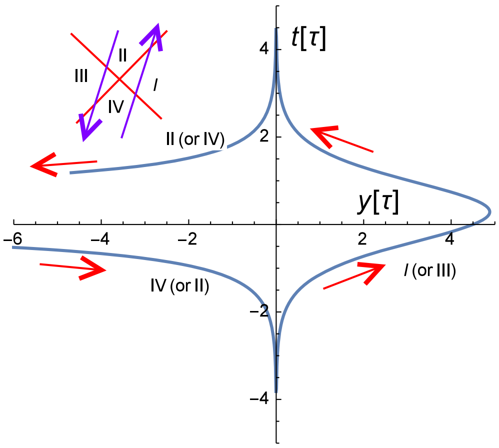

FIG. 2. Increasing $t(\tau)$ in I or III.

rewritten as a straight line in the flat (u, v) spacetime, $u(\tau) = (k^+/\mu)^2 v(\tau) + \text{constant}$. Equivalently, the solution is plotted as a parametric plot that amounts to a timelike straight line whose direction in (u, v) space is set by the timelike on-shell momentum, $k^\mu = (k^+, k^-)$, as shown in the upper left corners of Figs. 2, 3. Note that $k^+ > 0$ corresponds to a particle (both k^+ and k^- positive, so upward arrow) while $k^+ < 0$ corresponds to an antiparticle (both k^+ and k^- negative, so downward arrow).

In the case of the extended Rindler metric, $ds^2 = -(2y)dt^2 + (2y)^{-1}dy^2$, the *inverse* metric is $g^{\mu\nu} = \begin{pmatrix} -1/2y & 0 \\ 0 & 2y \end{pmatrix}$, and the equations to be solved (6) are, $\dot{t}(\tau) = -\frac{p_t(\tau)}{2y(\tau)}$, $\dot{y}(\tau) = 2y(\tau)p_y(\tau)$, where $p_t(\tau) = \omega$ is a constant of motion (due to translation invariance, $t(\tau) \to t(\tau) + c$, of the action, or the $\dot{p}_t$ equations of motion), while the constraint is, $-\frac{p_t^2}{2y} + 2yp_y^2 + \mu^2 = 0$. We rewrite this constraint by substituting the expressions for the momenta in terms of velocities,

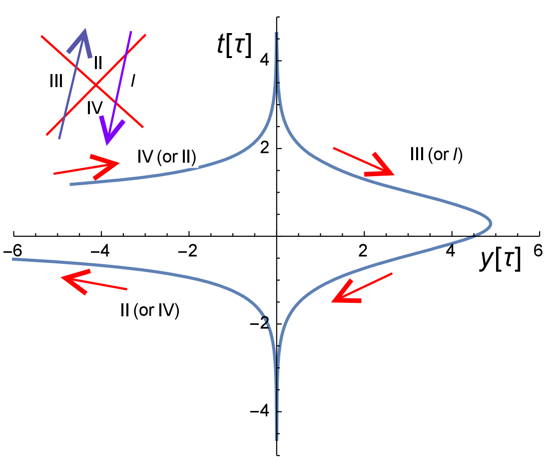

FIG. 3. Decreasing $t(\tau)$ in I or III.

$$-\frac{\omega^2}{2y(\tau)} + \frac{\dot{y}^2(\tau)}{2y(\tau)} + \mu^2 = 0. \tag{8}$$

So, $y(\tau)$ is given by the solution of a simple first order differential equation while $t(\tau)$ is an integral over $\frac{-\omega}{2y(\tau)}$,

$$\dot{y}(\tau) = \pm\sqrt{\omega^2 - 2\mu^2 y(\tau)}, \qquad t(\tau) = t_* + \int_{\tau_*}^{\tau} d\tau' \frac{-\omega}{2y(\tau')}, \tag{9}$$

where the sign change $\pm$ for the velocity $\dot{y}$ occurs at a specific time, $\tau = \tau_*$, when $\dot{y}(\tau)$ vanishes, namely at $y_* = y(\tau_*) = \frac{\omega^2}{2\mu^2}$. The solution is,

$$y(\tau) = -\frac{\mu^2}{2}(\tau - \tau_*)^2 + \frac{\omega^2}{2\mu^2},$$
$$t(\tau) = \frac{1}{2}\ln\left|\frac{\tau - \tau_* - \omega/\mu^2}{\tau - \tau_* + \omega/\mu^2}\right| + t_*. \tag{10}$$

where (ω, τ_*, t_*) are integration constants determined by initial conditions.

Of course, the geodesics written in terms of $(t(\tau), y(\tau))$ in the extended Rindler space must be the same as those written in terms of Minkowski space $(u(\tau), v(\tau))$ given in Eq. (7). Therefore, a more elegant solution is to compute $(t(\tau), y(\tau))$ by using the map between the Rindler and Minkowski coordinates in Eqs. (3), (5) and inserting the geodesics in Eq. (7), as follows

$$y(\tau) = -\frac{u(\tau)v(\tau)}{2} = -\frac{1}{2}(\mu^2\tau^2 + (k^+ + \mu^2/k^-)\tau + u_0 v_0),$$
$$t(\tau) = \frac{1}{2}\ln\left|\frac{u(\tau)}{v(\tau)}\right| = \frac{1}{2}\ln\left|\frac{k^+\tau + u_0}{\frac{\mu^2}{k^+}\tau + v_0}\right|. \tag{11}$$

By comparing Eqs. (10), (11) one can establish the relation between the integration parameters in the two versions (ω, τ_*, t_*) versus (k, u_0, v_0) that provide different physical insights.

The parametric plots of the explicit Minkowski and Rindler solutions are given in Figs. 2, 3. The Minkowski plots appear in the upper left corner of these figures while the Rindler plots appear in the main body of these figures. These are each other's images according to the maps in Eq. (5). In both the Minkowski and Rindler plots the Rindler regions I-IV traversed by the geodesics are also shown. The bending point (y_*, t_*) where the Rindler plot turns around in region I (or III) occurs at $\tau = \tau_*$.

In these figures the arrows show the direction of motion as the proper time τ increases uniformly from $\tau = -\infty$ to $\tau = +\infty$. Proper time is the time used by an observer that travels in the frame of the particle.

Minkowski observers use $x^0(\tau)$ as "time" as measured by clocks in a static laboratory, while Rindler observers use $t(\tau)$ for that purpose noting that this is the clock that ticks in the frame of a laboratory experiencing constant proper acceleration [1]. A series of events that occur sequentially according to proper time τ, may have different interpretations when they are rearranged according to one choice of time versus another. To see this in the present case, first focus on the Minkowski plots in the upper left-hand corner of each Figs. 2, 3, where the upward (downward) trajectory indicates that the Minkowski time $x^0(\tau)$ increases (decreases) as proper time τ increases; hence the upward (downward) trajectory is for a Minkowski particle (antiparticle) that has positive (negative) energy $E = (k^+ + k^-)/2$, since both $k^\pm$ are positive (negative). In the Rindler images of these same geodesics, note that in Fig. 2, $t(\tau)$ *increases* during passage of a Minkowski particle (antiparticle) through region I (III), so these are interpreted as Rindler particles by Rindler observers in regions I and III. By contrast, in Fig. 3, $t(\tau)$ *decreases* during passage of a Minkowski particle (antiparticle) through region III (I), so these are interpreted as antiparticles by Rindler observers in regions I or III. So a Rindler observer's particle is a mixture of Minkowski particles and antiparticles, and vice versa. As is well known, this is expressed as a Bogoliubov transformation for the corresponding particle creation/annihilation operators, as rederived below in Eqs. (38), (47).

The plots clearly show that the geodesics passing through regions I or III take an infinite amount of Rindler time $t(\tau)$. Hence, only this portion of the complete geodesic is measurable by the Rindler observer in region I (or III). Meanwhile, the complete geodesic, that includes regions beyond I (or III) is measurable by the Minkowski observer as shown in the upper left corner of each figure. Of course, the proper time τ captures the full geodesics in all curved spacetimes, so τ will be our preferred choice of evolution parameter to discuss complete geodesics when analyzing the geometry of more complicated cases, such as black holes.

Using τ as the evolution parameter, there is another way to intuitively determine the complete trajectory for $y(\tau)$ without solving it explicitly. The method is exhibited here because this approach can be applied generally to any spacetime with a timelike Killing vector when an explicit solution is not available. Consider the constraint in Eq. (8) that the trajectory $y(\tau)$ must satisfy and write it in the form of a vanishing non-relativistic Hamiltonian

$$\frac{\dot{y}^2}{2\mu^2} + V(y) = 0, \quad \text{with} \quad V(y) = \left(y - \frac{\omega^2}{2\mu^2}\right). \tag{12}$$

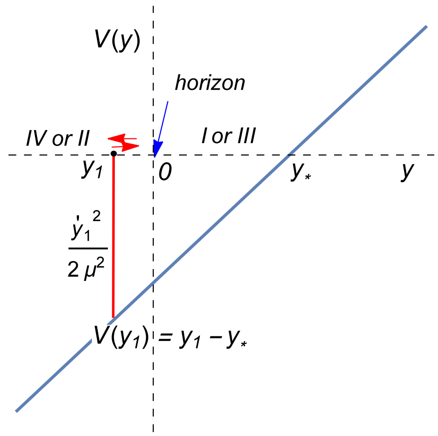

FIG. 4. $V(y)$ and kinetic energy. The particle cannot move to the region $y > y_*$.

In this form the constraint describes the dynamics of a nonrelativistic particle in 1-dimension with some potential energy $V(y)$, such that its total energy (kinetic + potential) is constrained to be zero. This is shown in Fig. 4 where the potential energy $V(y)$ in the current case is plotted as a straight blue line.

For a more general case, such as a black hole, the potential $V(y)$ is a more general curve. The 0 total energy level, which is conserved throughout the motion for all τ (because of the timelike Killing vector) is represented by the horizontal axis, and the evolving kinetic energy at any point y_1 is constrained to be, $\dot{y}_1^2/2\mu^2 = -V(y_1)$, corresponds to the length of the vertical red segment that connects the 0 energy level and the value of the potential at y_1. Without solving any equations, from this figure we see intuitively that, as $y_1(\tau)$ evolves dynamically, a classical particle/antiparticle is confined to the region, $y(\tau) \leq y_* = \omega^2/2\mu^2 = -\frac{u_* v_*}{2}$, because its kinetic energy is positive, $\frac{\dot{y}^2}{2\mu^2} = -V(y) > 0$. Its total-energy-conserving motion proceeds in the direction of the velocity $[\mathrm{sign}(\dot{y}(\tau))]$ indicated by the red arrows on the real axis. The particle (antiparticle) approaches from region $y(\tau) < 0$ which is region IV (region II); reaches the horizon at $y = 0$ and proceeds to $y(\tau) > 0$ which is region I (region III); it reaches a maximum at y_* within region I (region III); bounces back at y_* at time $\tau = \tau_*$, and then moves toward the horizon, to proceed to $y(\tau) < 0$ which is region II (region IV), in the future of region I (region III).

This physical description of the trajectory, obtained only from the physical interpretation of Fig. 4, clearly matches the behavior of the Rindler plots in Figs. 2, 3. This intuitive guide for the complete geodesics can of course be complemented by analytic methods or approximations if necessary. This approach has been applied to the case of the Schwarzchild black hole in [16] and will feature also in our upcoming work on black holes [4].

Finally we comment on the geodesics of a massless particle. The zero mass limit of the constraint in Minkowski basis, $k^+k^- = \mu^2 \to 0$, has solutions $k^\mu = (k^+, (k^- = 0))$ or $((k^+ = 0), k^-)$. Therefore the massless Minkowski geodesics are, $u(\tau) = k^+\tau + u_0$ and $v(\tau) = v_0$, or $u(\tau) = u_0$ and $v(\tau) = k^-\tau + v_0$. These correspond to lines parallel to either the u or the v axis that replace the slanted lines in the upper left hand corners of Figs. 2, 3. As for the massive case, for $k^\pm \gtrless 0$ these are particle/antiparticle trajectories. Their images in Rindler coordinates are,

$$\begin{aligned} y(\tau) &= -\frac{u(\tau)v(\tau)}{2} = \left[-\frac{1}{2}v_0(k^+\tau + u_0) \text{ or } -\frac{1}{2}u_0(k^-\tau + v_0)\right], \\ t(\tau) &= \frac{1}{2}\ln\left|\frac{u(\tau)}{v(\tau)}\right| = \left[\frac{1}{2}\ln\left|\frac{k^+\tau + u_0}{v_0}\right| \text{ or } \frac{1}{2}\ln\left|\frac{u_0}{k^-\tau + v_0}\right|\right]. \end{aligned} \tag{13}$$

By eliminating τ between these two equations one finds the geodesic relation, $t(\tau) = \pm\frac{1}{2}\ln|y(\tau)| + c_\pm$, where the constants $c_\pm$ are fixed with some initial conditions. The new parametric plots of $(y(\tau), t(\tau))$ produce a deformation of the curves in Figs. 2, 3 to two possible branches such that the remaining branch contains either the $t \to +\infty$ or the $t \to -\infty$ peak. The remaining branches are separate curves disconnected from each other and correspond to the plot of the functions, $t(\tau) = \pm\frac{1}{2}\ln|y(\tau)| + c_\pm$.

III. MINKOWSKI FREE FIELD IN EXTENDED RINDLER BASIS

Consider a complex scalar field in $1+1$ dimensional Minkowski spacetime, $-\infty < x^0 < \infty$ and $-\infty < x^1 < \infty$, that satisfies the massive or massless Klein Gordon equation

$$(\nabla^2 - \mu^2)\varphi(x^0, x^1) = 0. \tag{14}$$

The well-known general solution [17] is a superposition of relativistic plane waves, $e^{-iEx^0 + ik^1x^1}/\sqrt{4\pi E}$, and their complex conjugates, that form a properly normalized complete set of modes, where $E = \sqrt{k_1^2 + \mu^2}$ is the energy,

$$\begin{aligned} &\varphi(x^0, x^1) \\ &= \int_{-\infty}^{\infty} dk^1 \left(A(k^1)\frac{e^{-iEx^0 + ik^1x^1}}{\sqrt{4\pi E}} + B^\dagger(k^1)\frac{e^{iEx^0 - ik^1x^1}}{\sqrt{4\pi E}}\right). \end{aligned} \tag{15}$$

For this paper, it will be convenient to split the integral into two parts, $\int_{-\infty}^{\infty} dk^1 = \int_0^\infty dk + \int_{-\infty}^0 dk$, where k^1 has been renamed as k. Changing $k \to -k$ in the second integral and defining, $A_\pm(k) \equiv A(\pm k)$ with positive k, the field is rewritten as

$$\begin{aligned} \varphi(x^0, x^1) &= \int_0^\infty dk \left(A_+(k)\frac{e^{-iEx^0 + ikx^1}}{\sqrt{4\pi E}} + A_-(k)\frac{e^{-iEx^0 - ikx^1}}{\sqrt{4\pi E}} + hc_{A_\pm^\dagger \to B_\pm^\dagger}\right), \\ &= \int_0^\infty dk \left(A_+(k)\frac{e^{-i\frac{E-k}{2}u}e^{-i\frac{E+k}{2}v}}{\sqrt{4\pi E}} + A_-(k)\frac{e^{-i\frac{E+k}{2}u}e^{-i\frac{E-k}{2}v}}{\sqrt{4\pi E}} + hc.\right) \end{aligned} \tag{16}$$

where "$hc_{A^\dagger \to B^\dagger}$" stands for Hermitian conjugates of the first two terms but with $A_\pm^\dagger(k)$ replaced by $B_\pm^\dagger(k)$ for a complex field. From now on we will sometimes abbreviate this piece simply as "hc" unless some clarification is needed. For a real field, we simply replace $B_\pm^\dagger(k)$ by $A_\pm^\dagger(k)$ everywhere.

In *classical field theory,* $(A_\pm(k), B_\pm^\dagger(k))$ are complex functions of the positive momentum k. These $[A_\pm(k), B_\pm^\dagger(k)]$ could be fixed by initial/final boundary conditions that correspond to some wave packets. In *quantum field theory*, the $(A_\pm(k), A_\pm^\dagger(k))$ and $(B_\pm(k), B_\pm^\dagger(k))$ are pairs of annihilation/creation operators for particles (A) and antiparticles (B) acting in the Fock space built on the Minkowski vacuum [17].

$$\begin{aligned} [A_\pm(k), A_{\pm'}^\dagger(k')] = \delta_{\pm,\pm'}\delta(k - k') = [B_\pm(k), B_{\pm'}^\dagger(k)], \\ (A_\pm(k) \text{ or } B_\pm(k))|0_M\rangle = 0. \end{aligned} \tag{17}$$

Now we would like to setup the equivalent general superposition of the same field in terms of Rindler modes rather than the plane wave Minkowski modes. Rindler modes $\varphi_\pm(t, y)$ are the complete set of solutions to the Rindler Klein-Gordon equation given in (4). The positive frequency modes, $\varphi_\pm(t, y) = e^{-i\omega t}\varphi_{\mp\omega}(y)$, and their complex conjugate negative energy modes $\varphi_\pm^*(t, y) = e^{i\omega t}\varphi_{\mp\omega}^*(y)$, satisfy the time independent differential equation

$$\left(\partial_y^2 + \frac{1}{y}\partial_y + \frac{\omega^2}{4y^2} - \frac{\mu^2}{2y}\right)\varphi_{\mp\omega}(y) = 0. \tag{18}$$

The linearly independent solutions $\varphi_{\mp\omega}(y)$ are proportional to the Bessel functions $I_{\mp i\omega}(\sqrt{2y\mu^2})$. The *normalized*[2] positive frequency solutions in region I are conveniently written in the form

$$\varphi_{\pm}(t,y) = e^{-i\omega t}\varphi_{\mp\omega}(y) = \frac{e^{-i\omega t}}{\sqrt{4\pi\omega}}\Gamma(1\mp i\omega)\left(\frac{\mu}{2}\right)^{\pm i\omega} I_{\mp i\omega}\left(\sqrt{2y\mu^2}\right)$$
$$= \frac{e^{-i\omega t}(2y)^{\mp i\frac{\omega}{2}}}{\sqrt{4\pi\omega}} S_{\mp}(2y\mu^2) = \begin{cases} \frac{u^{-i\omega}}{\sqrt{4\pi\omega}} S_{-}(-\mu^2 uv) \\ \frac{(-v)^{i\omega}}{\sqrt{4\pi\omega}} S_{+}(-\mu^2 uv) \end{cases}. \tag{20}$$

In the last step, the region I relations, $2y = -uv$ and $e^{2t} = -u/v$, were used to re-write $\varphi_{\pm}(t,y)$ in terms of (u,v). The functions $S_{\mp}(z)$ are defined such that $\lim_{z\to 0} S_{\mp}(z) = 1$, when the argument $z = 2y\mu^2 = -\mu^2 uv$ vanishes. $S_{\mp}(z)$ are given by the hypergeometric function ${}_0F_1$

$$S_{\mp}(z) = {}_0F_1\left(1\mp i\omega, \frac{z}{4}\right) = \frac{\Gamma(1\mp i\omega) I_{\mp i\omega}(\sqrt{z})}{(\frac{1}{2}\sqrt{z})^{\mp i\omega}} = \sum_{n=0}^{\infty} \frac{(\frac{z}{4})^n \Gamma(1\mp i\omega)}{n!\Gamma(n+1\mp i\omega)}. \tag{21}$$

$S_{\mp}(z)$ are entire analytic functions of z in the finite complex z-plane and have an essential singularity at $z=\infty$ [18]. For the massless field $S_{\mp}(z)$ are both replaced by 1 since $\lim_{\mu\to 0} S_{\mp}(-\mu^2 uv) \to 1$. The analytic properties of the modes (20) will play an essential part in our discussion in Sec. IV where they will be discussed in detail.

We can now express the general solution $\varphi_1(u,v)$ in region I as the general superposition of the normalized basis in an analogous form to the Minkowski case in Eq. (16),

$$\varphi_1(u,v) = \int_0^\infty d\omega \begin{bmatrix} a_{1-}(\omega)\varphi_{-}(t,y) + b_{1-}^{\dagger}(\omega)(\varphi_{-}(t,y))^* \\ +a_{1+}(\omega)\varphi_{+}(t,y) + b_{1+}^{\dagger}(\omega)(\varphi_{+}(t,y))^* \end{bmatrix}$$
$$= \int_0^\infty d\omega \left[a_{1-}(\omega)\frac{u^{-i\omega}S_{-}(-\mu^2 uv)}{\sqrt{4\pi\omega}} + a_{1+}(\omega)\frac{(-v)^{i\omega}S_{+}(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc_{a_{1\pm}^{\dagger}\to b_{1\pm}^{\dagger}} \right]. \tag{22}$$

Note that both $a_{1\pm}$-coefficients are associated with Rindler wave packets of positive frequency ω [see Eq. (20)], so these represent Rindler particles, while the $b_{1\pm}^{\dagger}$-coefficients are associated with the complex conjugate wave packets that have negative frequency and represent Rindler antiparticles. In classical field theory (wave function in first quantization) the coefficients $(a_{1\pm}, b_{1\pm}^{\dagger})$ in region I serve to specify some Rindler wave packets that satisfy some initial/final conditions. In quantum field theory, the pairs $(a_{1\pm}, a_{1\pm}^{\dagger})$ and $(b_{1\pm}, b_{1\pm}^{\dagger})$ are creation-annihilation operators for Rindler particles/antiparticles respectively, acting on the Fock space built on the Rindler vacuum $|0_R\rangle$

[2] The so called "Klein-Gordon" dot product between two relativistic wave functions φ_1, φ_2 in curved spacetime is given by an integral over a spacelike Cauchy surface, $\langle\varphi_1|\varphi_2\rangle = -i\int d\Sigma_\mu \sqrt{-g} g^{\mu\nu}(\varphi_1^{\dagger}\partial_\nu\varphi_2 - \partial_\nu\varphi_1^{\dagger}\varphi_2)$. This relies on the Klein-Gordon current, $J_{1,2}^{\mu} = -i\sqrt{-g}g^{\mu\nu}(\varphi_1^{\dagger}\partial_\nu\varphi_2 - \partial_\nu\varphi_1^{\dagger}\varphi_2)$ which is conserved $\partial_\mu J_{1,2}^{\mu}(x) = 0$. This dot product is *independent of the choice of the Cauchy surface*. In the Minkowski case one chooses $d\Sigma_{x^0} = dx^1$ as a fixed-x^0 surface, while in the Rindler case in regions I&III one chooses $d\Sigma_t = dy$ as a fixed-t surface since $i\partial_t$ is a Killing vector. Using $\sqrt{-g} = 1$ and $g^{tt} = -(2y)^{-1}$, one finds

$$\langle\varphi_1|\varphi_2\rangle = i\int_0^\infty \frac{dy}{2y}(\varphi_1^{\dagger}\partial_t\varphi_2 - \partial_t\varphi_1^{\dagger}\varphi_2). \tag{19}$$

Hence, the basis functions $\varphi_{\mp\omega}^{\mp'} \equiv e^{\mp' i\omega t}\phi_{\mp\omega}^{\mp'}(y)$ are orthonormalized as follows

$$\langle\varphi_{\mp_1\omega_1}^{\mp_1'}|\varphi_{\mp_2\omega_2}^{\mp_2'}\rangle = \delta^{\mp_1',\mp_2'}\delta_{\mp_1,\mp_2}(\pm_1' 2\omega_1)\int_0^\infty \frac{dy}{2y}(\phi_{\mp,\omega_1}^{\mp'}(y))^*\phi_{\mp,\omega_2}^{\mp'}(y) = \pm_1'\delta(\omega_1-\omega_2)\delta_{\mp_1,\mp_2}\delta^{\mp_1',\mp_2'}.$$

As usual, in relativistic field theory, the Klein-Gordon "norm" of basis functions is proportional to the "charge" associated with the conserved current, while the sign of the frequency term in the exponent of the plane wave, i.e., $\mp'$, is minus the sign of the charge that distinguishes particle/antiparticle. In accordance with this, note the overall $\pm'$ signs in front of the delta functions in the final expression.

$$[a_{1\pm}(\omega), a^{\dagger}_{1\pm'}(\omega')] = \delta_{\pm,\pm'}\delta(\omega-\omega') = [b_{1\pm}(\omega), b^{\dagger}_{1\pm'}(\omega)],$$
$$(a_{1\pm}(\omega) \text{ or } b_{1\pm}(\omega))|0_R\rangle = 0. \tag{23}$$

This defines the quantum field $\varphi_1(u,v)$ and its Hermitian conjugate $\varphi^{\dagger}_1(u,v)$ in region I $(u>0, v<0)$.

Similarly, the quantum field is constructed region by region in every region of the extended Rindler space. For region III $(u<0, v>0)$ the modes of the Laplace equation (4) look the same as those in region I $(u>0, v<0)$, but one must introduce a new set of coefficients, $(a_{3\pm}, a^{\dagger}_{3\pm})$ and $(b_{3\pm}, b^{\dagger}_{3\pm})$, to write down the general solution (and its Hermitian conjugate $\varphi^{\dagger}_3(u,v)$)

$$\varphi_3(u,v) = \int_0^\infty d\omega \left[b^{\dagger}_{3-}(\omega)\frac{(-u)^{-i\omega}S_-(-\mu^2 uv)}{\sqrt{4\pi\omega}} + b^{\dagger}_{3+}(\omega)\frac{v^{i\omega}S_+(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc_{b_{3\pm}\to a_{3\pm}} \right]. \tag{24}$$

Note that $\varphi_3(u,v)$ is structurally very similar to $\varphi_1(u,v)$ except that the (u,v) in region I is replaced by the (u,v) in region III, including a convenient change of signs $(u,-v)_I \to (-u,v)_{III}$, where the sign change is absorbed in the definition of the corresponding coefficients. The extra signs do not affect the fact that the modes in Eq. (24) are orthonormalized solutions of Eq. (4). In addition, in comparing φ_3 to φ_1, note that the first two terms in φ_3, which are positive frequency solutions, are associated with Rindler antiparticle $b^{\dagger}_{3\mp}$-coefficients by contrast to the Rindler particle $a_{1\mp}$-coefficients in φ_1, and vice versa. The reasoning [5] for this switch of particle ↔ antiparticle interpretations of the Rindler waves in region I versus region III, is that the Minkowski time x^0 increases as the Rindler time t increases in region I, but x^0 decreases as t increases in region III; this was emphasized in the second paragraph following Eq. (4), and is also evident in the contrast of the directions of the arrows in the geodesics in Figs. 2, 3. For the quantized field $\varphi_3(u,v)$, the coefficients turn into pairs of creation-annihilation operators, $(a_{3\pm}, a^{\dagger}_{3\pm})$ and $(b_{3\pm}, b^{\dagger}_{3\pm})$, for Rindler particles/antiparticles respectively. These act on the Fock space built on the same Rindler vacuum $|0_R\rangle$ as in Eq. (23),

$$[a_{3\pm}(\omega), a^{\dagger}_{3\pm'}(\omega')] = \delta_{\pm,\pm'}\delta(\omega-\omega') = [b_{3\pm}(\omega), b^{\dagger}_{3\pm'}(\omega)],$$
$$(a_{3\pm}(\omega) \text{ or } b_{3\pm}(\omega))|0_R\rangle = 0. \tag{25}$$

Appropriate boundary conditions discussed in Sec. V will provide certain relations between the four complex functions in region I, $(a_{1\pm}, b^{\dagger}_{1\pm})$ and those in region III, $(a_{3\pm}, b^{\dagger}_{3\pm})$, consistently with the quantum commutation relations above. Before boundary conditions are applied $(a_{1\pm}, b^{\dagger}_{1\pm}), (a_{3\pm}, b^{\dagger}_{3\pm})$ are treated as if they are unrelated to each other.

Similarly, one obtains the general solutions in regions II and IV of the extended (u,v) Rindler space, and then must insure that the wave function in the full (u,v) space is continuous across all horizons. It turns out that the fields $\varphi_2(u,v)$ and $\varphi_4(u,v)$ in regions II and IV respectively are fully determined by analytic continuation of the fields $\varphi_1(u,v)$ and $\varphi_3(u,v)$ across the horizons. So, there are no new a, b coefficients beyond those already introduced above. The full continuous field throughout the extended Rindler space (u,v) is

$$\varphi(u,v) = \varphi_0 + \theta(I)\varphi_1(u,v) + \theta(II)\varphi_2(u,v) + \theta(III)\varphi_3(u,v) + \theta(IV)\varphi_4(u,v). \tag{26}$$

The theta functions, $\theta(I) \equiv \theta(u)\theta(-v)$, etc. enforce the regions I-IV as defined in Fig. 1 and Eq. (5). φ_0 is a constant zero mode that is justified in Eq. (35). The expressions for $\varphi_1(u,v)$ and $\varphi_3(u,v)$ given above, as well as $\varphi_2(u,v)$ and $\varphi_4(u,v)$ obtained by analytic continuation are

$$\begin{aligned}
\varphi_1(u,v) &= \int_0^\infty d\omega \left[a_{1-}(\omega)\frac{u^{-i\omega}S_-(-\mu^2 uv)}{\sqrt{4\pi\omega}} + a_{1+}(\omega)\frac{(-v)^{i\omega}S_+(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc_{a^{\dagger}_{1\pm}\to b^{\dagger}_{1\pm}} \right], \\
\varphi_2(u,v) &= \int_0^\infty d\omega \left[a_{1-}(\omega)\frac{u^{-i\omega}S_-(-\mu^2 uv)}{\sqrt{4\pi\omega}} + b^{\dagger}_{3+}(\omega)\frac{v^{i\omega}S_+(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc^{b_{3+}\to a_{3+}}_{a^{\dagger}_{1-}\to b^{\dagger}_{1-}} \right], \\
\varphi_3(u,v) &= \int_0^\infty d\omega \left[b^{\dagger}_{3-}(\omega)\frac{(-u)^{-i\omega}S_-(-\mu^2 uv)}{\sqrt{4\pi\omega}} + b^{\dagger}_{3+}(\omega)\frac{v^{i\omega}S_+(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc_{b_{3\pm}\to a_{3\pm}} \right], \\
\varphi_4(u,v) &= \int_0^\infty d\omega \left[b^{\dagger}_{3-}(\omega)\frac{(-u)^{-i\omega}S_-(-\mu^2 uv)}{\sqrt{4\pi\omega}} + a_{1+}(\omega)\frac{(-v)^{i\omega}S_+(-\mu^2 uv)}{\sqrt{4\pi\omega}} + hc^{a^{\dagger}_{1+}\to b^{\dagger}_{1+}}_{b_{3-}\to a_{3-}} \right].
\end{aligned} \tag{27}$$

The analytic continuation of the field across the horizons needs some explanation. Compare $\varphi_1(u,v)$ to $\varphi_2(u,v)$ at the horizon that separates regions I&II where $v=0$ and $0<u<\infty$. For continuity of the field we want to argue that, $\varphi_1(u,0) = \varphi_2(u,0)$ for $u>0$. In the first half of the field it is clear that $a_{1-}(\omega)u^{-i\omega}S_-(-\mu^2 uv)$ is analytically continued from one side of the horizon, $v<0$, to the other, $v>0$, and noting that $S_\pm(z)$, that satisfy $S_\pm(0)=1$

[see Eq. (21)], are analytic entire functions in the finite complex plane. In the second half of the field, continuity is not evident because different coefficients $a_{1+}(\omega)$ and $b_{3+}^{\dagger}(\omega)$ appear in $\varphi_1(u,v)$ versus $\varphi_2(u,v)$. However, in these apparently problematic terms, the factors $(-v)^{i\omega}$ or $(v)^{i\omega}$ vanish as *distributions* near $v\sim 0$ on both sides of the horizon. This is because $|v|^{i\omega}=e^{i\omega\ln|v|}$ oscillates wildly as $|v|\to 0$, so the integral $\int_0^\infty d\omega$ can be computed by the steepest descent method. Noting that only the neighborhood of $\omega=0$ can contribute to the leading behavior of such an integral, a typical smooth integrand $F(\omega)$ can be approximated by its value near $\omega=0$ to give

$$\lim_{v\to 0}\int_0^\infty d\omega F(\omega)e^{i\omega\ln|v|}\simeq F(0)\lim_{v\to 0}\lim_{\varepsilon\to 0^+}\int_0^\infty d\omega e^{-\varepsilon\omega}e^{i\omega\ln|v|} = \lim_{v\to 0}\frac{F(0)}{i\ln|v|}=0. \quad (28)$$

The $e^{-\varepsilon\omega}$ factor is introduced as a device to enforce the integration region to remain close to $\omega=0$, thus producing a convergent integral in regions far from $\omega=0$. This shows that the terms involving $(-v)^{i\omega}$ or $(v)^{i\omega}$ in $\varphi_{1,2}(u,v)$ vanish at the $v=0$ horizon. Hence we have shown that, despite the fact that $a_{1+}(\omega)$ and $b_{3+}^{\dagger}(\omega)$ are different, the field is continuous at the horizon and is given by

$$\varphi_1(u,0)=\varphi_2(u,0) = \int_0^\infty d\omega\left[a_{1-}(\omega)\frac{u^{-i\omega}}{\sqrt{4\pi\omega}}+b_{1-}^{\dagger}(\omega)\frac{u^{i\omega}}{\sqrt{4\pi\omega}}\right]. \quad (29)$$

Even though the field is continuous, its derivative is discontinuous at the $v=0$ horizon because $a_{1+}(\omega)$ and $b_{3+}^{\dagger}(\omega)$ are different. However continuity of the derivative is not required to have a solution of the Klein-Gordon equation in the (u,v) variables because the Laplacian (proportional to $\partial_u\partial_v$) is linear rather than quadratic in ∂_u and similarly in ∂_v. The same argument applies at every horizon, thus determining all the terms in $\varphi_{2,4}(u,v)$ by analytic continuation from regions I and III. Therefore, we can state that the full field $\varphi(u,v)$ as given in Eqs. (26), (27) has just sufficient amount of continuity throughout the extended Rindler space (u,v) to be a solution of the Klein-Gordon equation without any sources.

IV. ANALYTICITY IN THE EXTENDED RINDLER SPACE

We have already required some analyticity properties in (u,v) space in order to establish the continuity of the field. The field discussed in the previous section has branch points and corresponding branch cuts in the complex (u,v) planes. This defines an infinite number of sheets in both the complex u-plane and complex v-plane. On the real axes on each sheet, the field $\varphi(u,v)$ of Eqs. (26), (27) takes on different values that are related to each other continuously by analytic continuation from sheet to sheet. This presents itself as a geometry consisting of an infinite stack of different real Minkowski spaces (u,v) on which the first quantized wave function (i.e., the classical field) as well as the second quantized field take on values. The branch points at $u=0$ or $v=0$ correspond to the horizons within each Minkowski plane as shown in Fig. 1. So the connection between the infinite stack of real Minkowski spaces is precisely at the horizons. The field is analytically continued from one Minkowski plane to another by going slightly off the real axis at the horizons and winding around a branch cut in either the u or v complex planes and then back to the real axis on a different sheet. Hence, information could potentially flow at the horizons from one Minkowski plane to any other Minkowski plane. This shows that there are previously missed aspects of extended Rindler spacetime, namely the natural presence of a multiverse structure consisting of a spacetime with an infinite stack of Minkowski planes connected to each other at the horizons. This was not evident from the classical metric or the geodesics discussed before; it emerged only at the quantum level.

We now expand on these properties and clarify them. The analyticity properties of the basis functions, $\varphi_+(u,v)\sim u^{-i\omega}S_-(-\mu^2uv)$ and $\varphi_-(u,v)\sim(-v)^{i\omega}S_+(-\mu^2uv)$, follow from their properties in the complex u and complex v planes. Since $S_\mp(z)$ are entire functions [see remarks after Eq. (21)], the prefactors $u^{-i\omega}$ and $(-v)^{i\omega}$ determine the analyticity properties for both the massless and massive fields. Clearly, these have branch points at $u=0$ and $v=0$, respectively. We choose the branch cuts to be on the positive imaginary axes in both the complex u and v planes as shown in Figs. 5, 6. This defines an infinite number of sheets in the analytic u and v planes. The u-sheets (v-sheets) are labeled by an integer n (m), so the stack of universes is labeled by these integers (n,m).

Some typical points $(u_I,v_I),(u_{II},v_{II}),(u_{III},v_{III}),(u_{IV},v_{IV})$ in regions I-IV in universe (0,0), connected to each other with some arbitrary analyticity path, are shown in Fig. 7 (this ignores the small excursions into the complex u or v plane near the horizons). The images of the points on the real axes in the complex (u,v) planes are shown in

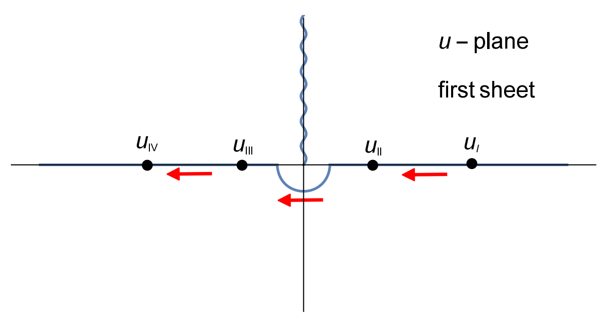

FIG. 5. A path on 0^{th} sheet in analytic u -plane.

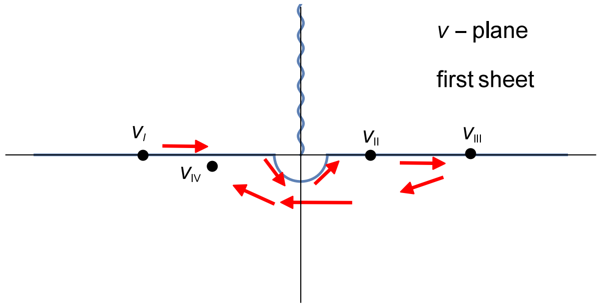

FIG. 6. A path on 0^{th} sheet in analytic v -plane.

Figs. 5, 6. The analyticity path that connects them is also shown, such that, for clarity, the path goes slightly under the branch points in the complex planes to stay within the (0,0) universe. Of course, the analyticity path within the same (0,0) level can be any other curve in the complex planes that connects the points as long as it does not cross the branch cuts in Figs. 5, 6. This is the path of analyticity on level (0,0) used to establish the continuity of the field $\varphi(u,v)$ as given in Eqs. (26), (27).

Now we can analytically continue from level (0,0) to any other level (n,m) by crossing the branch points and coming back again to the real axes of u and v. This provides the value of the field in Eqs. (26), (27) at different levels (n,m) that again look like the Minkowski plane. The basis $\varphi_\pm(u,v)$ on the real axis of universe (n,m) is related by monodromy to the basis $\varphi_\pm(u,v)$ in Eq. (20) for universe (0,0). Recall that $S_\mp(-\mu^2 uv)$ given in Eq. (21) are entire functions of their arguments, so they have no

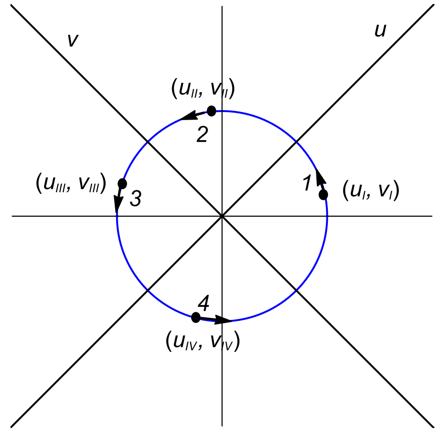

FIG. 7. Figure 7—A path on the real (u,v) plane in the (0,0) universe.

discontinuities under analytic continuation, so we only need to analytically continue $u^{-i\omega}$ and $v^{-i\omega}$.

The analytic continuation involves replacing, $u\to ue^{2\pi in}$ or $v\to ve^{2\pi im}$ with integer n or m, to indicate how many times we wind around the branch points at $u=0$ or $v=0$ in the respective u or v complex planes in Figs. 5, 6 when the horizons are crossed in an analytic path on the real (u,v) plane shown in Fig. 7. The winding numbers, that may be different at each horizon, will lead to some sheet in the multiverse. An analytic continuation of the field $\varphi(u,v)$ in Eqs. (26), (27) from level (0,0) to level (n,m), which is consistent with the boundary conditions[3] that are later explained in Sec. V, can only have the following pattern[4]

$$\varphi^{(n,m)}(u,v)=\begin{pmatrix}\varphi_0+\theta(I)\varphi_1(ue^{2\pi in},e^{-2\pi in}v)+\theta(II)\varphi_2(ue^{2\pi in},ve^{2\pi im})\\ +\theta(III)\varphi_3(ue^{-2\pi im},ve^{2\pi im})+\theta(IV)\varphi_4(ue^{-2\pi im},ve^{-2\pi in})\end{pmatrix}$$
$$\bar\varphi^{(n,m)}(u,v)=\begin{pmatrix}\varphi_0+\theta(I)\varphi_1^\dagger(ue^{2\pi in},e^{-2\pi in}v)+\theta(II)\varphi_2^\dagger(ue^{2\pi in},ve^{2\pi im})\\ +\theta(III)\varphi_3^\dagger(ue^{-2\pi im},ve^{2\pi im})+\theta(IV)\varphi_4^\dagger(ue^{-2\pi im},ve^{-2\pi in})\end{pmatrix}. \tag{31}$$

where (n,m) are integers, and the Hermitian conjugates $\varphi_{1,2,3,4}^\dagger$ are defined for real (u,v). Here $\bar\varphi^{(n,m)}(u,v)$ is the canonical conjugate to $\varphi^{(n,m)}(u,v)$. The alert reader will note that in these expressions the regional fields $\varphi_{1,2,3,4}$ and $\varphi_{1,2,3,4}^\dagger$ are

[3]A crucial consequence of boundary conditions is Eq. (43). This requires the oscillators of regions I and III to satisfy

$$\bar a_{1-}a_{1-}=\bar a_{1+}a_{1+},\qquad \bar b_{1-}b_{1-}=\bar b_{1+}b_{1+},\qquad \bar a_{3-}a_{3-}=\bar a_{3+}a_{3+},\qquad \bar b_{3-}b_{3-}=\bar b_{3+}b_{3+} \tag{30}$$

A physical consequence of these relations is that the fields $\varphi_{1,3}(u,v)$ vanish at the asymptotic regions $|u|\to\infty$ or $|v|\to\infty$ in regions I or III. Without these conditions probablity would become infinite in those asymptotic regions. This is explained in Sec. V. The same physical conditions exist also for all layers of the multiverse. To respect these boundary conditions only certain patterns of winding numbers are allowed in the definition of the layers of the multiverse as shown in Eq. (31). The consequence of these patterns leads to the oscillators for level (n,m) given Eq. (33). It can be observed that these rescaled oscillators also satisfy the physical boundary conditions at all such levels, $\bar a_{1-}^{(n)}a_{1-}^{(n)}=\bar a_{1+}^{(n)}a_{1+}^{(n)}$, etc. because the relation is true at $n=0$ as given in Eq. (43). A different pattern of windings at the horizons violates the physical boundary conditions discussed above, and this is the reason why they are not consistent.

[4]This pattern applies only to the extended Rindler space. For black hole spacetimes, $S_\mp(uv)$ have a branch cut that starts at the black hole singularity $uv=1$, therefore the monodromy in Eq. (32) as well as the pattern of analytic continuation is different for black holes (see [4]).

analytically continued in a pattern that is different in each region. In each region (u,v) are real in the respective ranges as seen in Fig. 1. The phases $e^{\pm 2\pi i n} = e^{\pm 2\pi i m} = 1$ do not change the reality of the continued (u,v) for each region, but because of the monodromy properties of the factors, $(u^{\mp i\omega}, v^{\pm i\omega})$ that appear in the expressions for $\varphi(u,v)$, such as

$$(ue^{2\pi i n})^{\mp i\omega} = u^{\mp i\omega} e^{\pm 2\pi\omega n}, \qquad (ve^{-2\pi i n})^{\pm i\omega} = v^{\pm i\omega} e^{\pm 2\pi\omega n}, \qquad S_{\mp}(-\mu^2 uve^{\pm 2\pi\omega k}) = S_{\mp}(-\mu^2 uv), \tag{32}$$

the result for $\varphi^{(n,m)}(u,v)$ in Eq. (31) is different than $\varphi(u,v)$. The analytically continued fields $\varphi_{1,2,3,4}$ in $\varphi^{(n,m)}(u,v)$ have the same form as the $\varphi_{1,2,3,4}$ in Eq. (27) except for the fact that the oscillators $(a_{1\mp}, b^{\dagger}_{1\mp}, b^{\dagger}_{3\pm}, a_{3\pm})$ in Eq. (27) are now replaced by new ones in $\varphi^{(n,m)}(u,v)$ that we label as $(a^{(n)}_{1\mp}, \bar{b}^{(n)}_{1\mp}, \bar{b}^{m}_{3\pm}, a^{m}_{3\pm})$. The relation between $(a^{(n)}_{1\mp}, \bar{b}^{(n)}_{1\mp}, \bar{b}^{m}_{3\pm}, a^{m}_{3\pm})$ and $(a_{1\mp}, b^{\dagger}_{1\mp}, b^{\dagger}_{3\pm}, a_{3\pm})$ is obtained by inserting Eq. (32) into the $\varphi_{1,2,3,4}$ in Eq. (27), and similarly for the Hermitian conjugates $\varphi^{\dagger}_{1,2,3,4}$. The result of the analytic continuation in Eq. (31) then yields the desired relations between levels (n,m) and (0,0)

$$a^{(n)}_{1\mp} = a_{1\mp} e^{2\pi\omega n}, \qquad \bar{a}^{(n)}_{1\mp} = a^{\dagger}_{1\mp} e^{-2\pi\omega n}, \qquad b^{(n)}_{1\mp} = b_{1\mp} e^{2\pi\omega n}, \qquad \bar{b}^{(n)}_{1\mp} = b^{\dagger}_{1\mp} e^{-2\pi\omega n},$$
$$a^{(m)}_{3\pm} = a_{3\pm} e^{2\pi\omega m}, \qquad \bar{a}^{(m)}_{3\pm} = a^{\dagger}_{3\pm} e^{-2\pi\omega m}, \qquad b^{(m)}_{3\pm} = a_{3\pm} e^{2\pi\omega m}, \qquad \bar{b}^{(m)}_{3\pm} = b^{\dagger}_{3\pm} e^{-2\pi\omega m}. \tag{33}$$

This analyticity-induced map is a canonical transformation since

$$[a^{(n)}_{1\mp}, \bar{a}^{(n)}_{1\mp}] = [a_{1\mp}, a^{\dagger}_{1\mp}], \text{ etc.} \tag{34}$$

We explain some notation. We used the overbar symbol in $(\bar{a}^{(n)}_{1\mp}, \bar{b}^{(n)}_{1\mp}, \bar{a}^{(m)}_{3\pm}, \bar{b}^{(m)}_{3\pm})$ to indicate the canonical conjugates of $(a^{(n)}_{1\mp}, b^{(n)}_{1\mp}, a^{(m)}_{3\pm}, b^{(m)}_{3\pm})$, respectively. For $n = m = 0$ the overbar is defined to be actually the same the Hermitian conjugate, $(\bar{a}^{(0)}_{1\mp}, \bar{b}^{(0)}_{1\mp}, \bar{a}^{(0)}_{3\pm}, \bar{b}^{(0)}_{3\pm}) \equiv (a^{\dagger}_{1\mp}, b^{\dagger}_{1\mp}, a^{\dagger}_{3\pm}, b^{\dagger}_{3\pm})$, but for general n, $\bar{a}^{(n)}_{1\mp}$ is not the Hermitian conjugate of $a^{(n)}_{1\mp}$, although it is its canonical conjugate, and similarly for the other oscillators.

This shows that the analytic continuation to universe (n,m) defined by Eqs. (31)–(34) amounts to a canonical transformation of the creation-annihilation operators. The full field $\bar{\varphi}^{(n,m)}(u,v)$ is not the naive Hermitian conjugate of $\varphi^{(n,m)}(u,v)$ but it is its canonical conjugate. The equal time quantum commutator, $[\varphi^{(n,m)}(t,y), \bar{\varphi}^{(n,m)}(t,y')]$, produces a delta function $\delta(y-y')$ at every universe (n,m) just as the (0,0) universe. It should be emphasized that the fields on different levels $(\varphi^{(n_1,m_1)}(t,y), \bar{\varphi}^{(n_2,m_2)}(t,y'))$ with (n_1,m_1) different from (n_2,m_2) are not independent of each other since the corresponding oscillators $(a^{(n_1)}_{1\mp}, \bar{b}^{(n_1)}_{1\mp}, \bar{b}^{(m_1)}_{3\pm}, a^{(m_1)}_{3\pm})$ and $(a^{(n_2)}_{1\mp}, \bar{b}^{(n_2)}_{1\mp}, \bar{b}^{(m_2)}_{3\pm}, a^{(m_2)}_{3\pm})$ are all related to the same set of basic oscillators $(a_{1\mp}, b^{\dagger}_{1\mp}, b^{\dagger}_{3\pm}, a_{3\pm})$ and their Hermitian conjugates that define the field in universe (0,0). So the quantum rules for the entire multiverse, including nontrivial commutators among fields at different levels, such as, $[\varphi^{(n_1,m_1)}(t,y), \bar{\varphi}^{(n_2,m_2)}(t,y')]$, depend only on the quantum rules established at level (0,0).

Propagators, various correlators and probabilities of processes computed with $\varphi^{(n,m)}(u,v)$ may be different for different levels because of the shift of normalizations of the Rindler coefficients as given in Eq. (33). Hence one must also specify the universe on which boundary conditions are imposed. We define level (0,0) as the reference universe at which boundary conditions are applied as shown in the next section. Then, using $\varphi^{(n,m)}(u,v)$ as given above, probabilities for various physical processes at universe (n,m), that may depend on the boundary conditions in the (0,0) universe, can be determined. In particular, we will try to answer the question: is there probability (or information) flow from one universe labeled by (n,m) to other universes labeled by (n',m')?

V. HORIZON BOUNDARY CONDITIONS

Boundary conditions imposed in universe (0,0) will automatically fix all boundary conditions for all (n,m) as determined in the previous section. Accordingly, given the relationship between Minkowski and Rindler coordinates as given in Eq. (5), we require the (0,0) Rindler field $\varphi^{(0,0)}(u,v) = \varphi(u,v)$ given in Eqs. (26), (27), to be identical to the Minkowski field $\varphi(u,v)$ given in Eqs. (15), (16). For this, it is sufficient to impose boundary conditions at each horizon, $\varphi^{(0,0)}(u,0) = \varphi(u,0)$ and $\varphi^{(0,0)}(0,v) = \varphi(0,v)$. Other boundary conditions could be considered for the extended Rindler case, however we emphasize these horizon boundary conditions since in the case of a black hole it is also appropriate to impose the same boundary conditions as done in an upcoming paper [4]. This is because, near the horizons, the black hole metric and field behave *locally* just like the flat Minkowski metric and field. So the horizon boundary conditions used here for the Rindler case will be used also identically for black hole case. Hence, for either Rindler or a black hole metric, using $S_{\mp}(0) = 1$ at horizons in Eqs. (26), (27), we have the desired boundary conditions

$$\begin{aligned}
\varphi_0 &= \varphi(0,0) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi E}}(A(k)+B^\dagger(k)) \\
\varphi_{1\text{ or }2}(u,0) &= \int_0^\infty d\omega\left[a_{1-}(\omega)\frac{u^{-i\omega}}{\sqrt{4\pi\omega}} + b_{1-}^\dagger(\omega)\frac{u^{i\omega}}{\sqrt{4\pi\omega}}\right] = \theta(u)(\varphi(u,0)-\varphi_0) \\
\varphi_{3\text{ or }4}(u,0) &= \int_0^\infty d\omega\left[b_{3-}^\dagger(\omega)\frac{(-u)^{-i\omega}}{\sqrt{4\pi\omega}} + a_{3-}(\omega)\frac{(-u)^{i\omega}}{\sqrt{4\pi\omega}}\right] = \theta(-u)(\varphi(u,0)-\varphi_0) \\
\varphi_{2\text{ or }3}(0,v) &= \int_0^\infty d\omega\left[b_{3+}^\dagger(\omega)\frac{v^{i\omega}}{\sqrt{4\pi\omega}} + a_{3+}(\omega)\frac{v^{-i\omega}}{\sqrt{4\pi\omega}}\right] = \theta(v)(\varphi(0,v)-\varphi_0) \\
\varphi_{1\text{ or }4}(0,v) &= \int_0^\infty d\omega\left[a_{1+}(\omega)\frac{(-v)^{i\omega}}{\sqrt{4\pi\omega}} + b_{1+}^\dagger(\omega)\frac{(-v)^{-i\omega}}{\sqrt{4\pi\omega}}\right] = \theta(-v)(\varphi(0,v)-\varphi_0)
\end{aligned} \tag{35}$$

where the $\varphi(u,0)$ or $\varphi(0,v)$ on the right-hand side of Eq. (35) is the Minkowski field (16) evaluated at the horizons

$$\begin{aligned}
\varphi(u,0) &= \int_0^\infty dk\left(\frac{A_+(k)e^{-i\frac{E-k}{2}u} + B_+^\dagger(k)e^{i\frac{E-k}{2}u}}{\sqrt{4\pi E}} + \frac{A_-(k)e^{-i\frac{E+k}{2}u} + B_-^\dagger(k)e^{i\frac{E+k}{2}u}}{\sqrt{4\pi E}}\right), \\
\varphi(0,v) &= \int_0^\infty dk\left(\frac{A_+(k)e^{-i\frac{E+k}{2}v} + B_+^\dagger(k)e^{i\frac{E+k}{2}u}}{\sqrt{4\pi E}} + \frac{A_-(k)e^{-i\frac{E-k}{2}v} + B_-^\dagger(k)e^{i\frac{E-k}{2}v}}{\sqrt{4\pi E}}\right).
\end{aligned} \tag{36}$$

We see in the first line of Eq. (35) that a nontrivial zero mode φ_0 is necessary because $\varphi_{1,2,3,4}(0,0)$ all vanish according to the arguments in Eq. (28). From the second expression in Eq. (35) at the horizon I&II, a_{1-} can be extracted by using the orthonormality and completeness of the basis $u^{\mp i\omega}/\sqrt{4\pi\omega}$ on the half-line $u>0$; namely (see footnote 2) $a_{1-}(\omega) = \int_0^\infty du(\frac{u^{i\omega}}{\sqrt{4\pi\omega}} i\partial_u\varphi(u,0) - \varphi(u,0)i\partial_u\frac{u^{i\omega}}{\sqrt{4\pi\omega}})$, and similarly for $b_{1-}^\dagger$ by replacing $u^{-i\omega}$ instead of $u^{i\omega}$. After performing the integrals we obtain

$$\begin{aligned}
a_{1-}(\omega) &= \frac{\Gamma(1+i\omega)}{i\sqrt{2\pi\omega}}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{4\pi E}}\left(\frac{E-k}{2}\right)^{-i\omega}(A(k)e^{\frac{\pi\omega}{2}} + B^\dagger(k)e^{-\frac{\pi\omega}{2}}), \\
b_{1-}^\dagger(\omega) &= \frac{\Gamma(1-i\omega)}{-i\sqrt{2\pi\omega}}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{4\pi E}}\left(\frac{E-k}{2}\right)^{i\omega}(B^\dagger(k)e^{\frac{\pi\omega}{2}} + A(k)e^{-\frac{\pi\omega}{2}}).
\end{aligned} \tag{37}$$

Note that on the right-hand side the integral $\int_{-\infty}^{\infty} dk$ contains $A(k), B^\dagger(k)$ over the full momentum range. To make contact with the notation $A_\pm(k), B_\pm^\dagger(k)$, with only $k>0$, the integral can be split to the positive and negative intervals.

The Hermitian conjugate $a_{1-}^\dagger$ looks like $b_{1-}^\dagger$ above but with $A\leftrightarrow B$ interchanged on the right-hand side of (37). As a consistency check, it can then be verified that the commutation rules (23) of the Rindler modes, $[a_{1-}(\omega), a_{1-}^\dagger(\omega')] = \delta(\omega-\omega')$, etc., can be obtained by using only the commutation rules (17) of the Minkowski modes, $[A(k), A^\dagger(k')] = \delta(k-k') = [B(k), B^\dagger(k')]$, by using the relations above. Similar expressions are obtained at the 4 horizons.

This fixes the 8 Rindler complex coefficients in the level (0,0) universe, $a_{1\mp}, b_{1\mp}^\dagger, a_{3\mp}, b_{3\mp}^\dagger$, in terms of the 4 Minkowski complex coefficients, $A_\pm(k), B_\pm^\dagger(k)$ (similarly, for the black hole [4]). It is revealing to rewrite the 8 relations in level (0,0) in the form of Bogoliubov transformations as follows, where $\frac{\Gamma(1\pm i\omega)}{\sqrt{2\pi\omega}} = \frac{e^{\pm i\theta(\omega)}e^{-\pi\omega/2}}{\sqrt{1-e^{-2\pi\omega}}}$ is used,

$$\begin{aligned}
\frac{ie^{-i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{1-}(\omega)\\ b_{3+}^\dagger(\omega)\end{pmatrix} &= \int_{-\infty}^{\infty} dk\frac{(\frac{E-k}{2})^{-i\omega}}{\sqrt{4\pi E}}\begin{pmatrix}A(k)\\ B^\dagger(k)\end{pmatrix} \\
\frac{-ie^{i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{3+}(\omega)\\ b_{1-}^\dagger(\omega)\end{pmatrix} &= \int_{-\infty}^{\infty} dk\frac{(\frac{E-k}{2})^{i\omega}}{\sqrt{4\pi E}}\begin{pmatrix}A(k)\\ B^\dagger(k)\end{pmatrix} \\
\frac{-ie^{-i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{3-}(\omega)\\ b_{1+}^\dagger(\omega)\end{pmatrix} &= \int_{-\infty}^{\infty} dk\frac{(\frac{E+k}{2})^{-i\omega}}{\sqrt{4\pi E}}\begin{pmatrix}A(k)\\ B^\dagger(k)\end{pmatrix} \\
\frac{ie^{i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{1+}(\omega)\\ b_{3-}^\dagger(\omega)\end{pmatrix} &= \int_{-\infty}^{\infty} dk\frac{(\frac{E+k}{2})^{i\omega}}{\sqrt{4\pi E}}\begin{pmatrix}A(k)\\ B^\dagger(k)\end{pmatrix}
\end{aligned} \tag{38}$$

and their Hermitian conjugates. These equations can be easily inverted[5] to obtain an explicit expression for the Minkowski oscillators $(A(k), B^\dagger(k))$ in terms of the Rindler oscillators $(a_{1\mp}, b^\dagger_{1\mp}, a_{3\mp}, b^\dagger_{3\mp})$.

$$A(k) = \frac{\varphi_0/c + \sigma_0}{2\sqrt{\pi E}} + \int_0^\infty d\omega \left[\begin{array}{c} \frac{(\frac{E\mp k}{2})^{\pm i\omega}}{\sqrt{\pi E}} \frac{ie^{\mp i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}} (a_{1\mp}(\omega) - e^{-\pi\omega} b^\dagger_{3\pm}(\omega)) \\ + \frac{(\frac{E\mp k}{2})^{\mp i\omega}}{\sqrt{\pi E}} \frac{-ie^{\pm i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}} (a_{3\pm}(\omega) - e^{-\pi\omega} b^\dagger_{1\mp}(\omega)) \end{array} \right]$$

$$B^\dagger(k) = \frac{\varphi_0/c - \sigma_0}{2\sqrt{\pi E}} + \int_0^\infty d\omega \left[\begin{array}{c} \frac{(\frac{E\mp k}{2})^{\mp i\omega}}{\sqrt{\pi E}} \frac{ie^{\pm i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}} (b^\dagger_{1\mp}(\omega) - e^{-\pi\omega} a_{3\pm}(\omega)) \\ + \frac{(\frac{E\mp k}{2})^{\pm i\omega}}{\sqrt{\pi E}} \frac{-ie^{\mp i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}} (b^\dagger_{3\pm}(\omega) - e^{-\pi\omega} a_{1\mp}(\omega)) \end{array} \right] \quad (39)$$

where $-\infty < k < \infty$. Here φ_0 is the zero mode that satisfies Eq. (35) where[5] $c \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi E(k)} = \delta(0)$, while σ_0 is another Rindler zero mode. Note that the two integrands with the upper/lower signs "$\pm$" in Eq. (39) are equal to each other for the massive case on account of the relations between $a_{1\pm}(\omega)$ etc. explained below in Eqs. (42), (43).

Based on analyticity properties of the wave function, Unruh [5] gave a simple argument to derive the so called Unruh modes. An Unruh mode is the following combination of the Rindler modes that annihilates the Minkowski vacuum $|0_M\rangle$, such as

$$\frac{a_{1-}(\omega) - e^{-\pi\omega} b^\dagger_{3+}(\omega)}{\sqrt{1-e^{-2\pi\omega}}} |0_M\rangle = 0. \quad (40)$$

This is in agreement with the first line of Eq. (38) where the Unruh mode above is seen in more detail to be equal to, $-ie^{i\theta(\omega)} \int_{-\infty}^{\infty} dk \frac{A(k)}{\sqrt{4\pi E}} (\frac{E-k}{2})^{-i\omega}$, which is clearly a combination of the Minkowski annihilation operators as defined in Eq. (17). Knowing the additional detail given here, of how to write the Unruh mode in terms of the Minkowski modes as in Eq. (38), is important because this can be used to compute the action of the Unruh modes, or more generally a_{1-} or $b^\dagger_{3+}$ on their own as in Eq. (37), on any general Minkowski state, not only the vacuum state $|0_M\rangle$. Our explicit relations in Eq. (38) should be useful for various applications involving quantum effects in Rindler space. Furthermore, our expressions (38) are more general because they apply also to black holes [4].

The relations (38) reveal additional important properties. For example, the expressions for the region I coefficients $a_{1\pm}, b^\dagger_{1\pm}$ extracted from Eq. (38) are

$$a_{1\mp}(\omega) = \frac{e^{\pm i\theta(\omega)}}{\pm i\sqrt{1-e^{-2\pi\omega}}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi E}} \left(\frac{E\mp k}{2}\right)^{\mp i\omega} \times (A(k) + B^\dagger(k) e^{-\pi\omega}),$$

$$b^\dagger_{1\mp}(\omega) = \frac{e^{\mp i\theta(\omega)}}{\mp i\sqrt{1-e^{-2\pi\omega}}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi E}} \left(\frac{E\mp k}{2}\right)^{\pm i\omega} \times (B^\dagger(k) + A(k) e^{-\pi\omega}). \quad (41)$$

Now concentrate on the Rindler case because we will next use the fact that $E = \sqrt{k^2+\mu^2}$ (for a black hole E and k have a different relation [4]). Then in Eq. (41) insert

$$\frac{E-k}{2} = \frac{\mu^2}{4} \left(\frac{E+k}{2}\right)^{-1}, \quad (42)$$

and then see that $(a_{1-}, b^\dagger_{1-})$ and $(a_{1+}, b^\dagger_{1+})$ are proportional to each other with overall phases. Similar arguments hold also for $(a_{3-}, b^\dagger_{3-})$ and $(a_{3+}, b^\dagger_{3+})$, so we find

$$\begin{aligned} a_{1+}(\omega) &= -(\mu^2/4)^{i\omega} e^{-2i\theta(\omega)} a_{1-}(\omega), \\ a_{3+}(\omega) &= -(\mu^2/4)^{i\omega} e^{-2i\theta(\omega)} a_{3-}(\omega), \\ b^\dagger_{1+}(\omega) &= -(\mu^2/4)^{-i\omega} e^{2i\theta(\omega)} b^\dagger_{1-}(\omega), \\ b^\dagger_{3+}(\omega) &= -(\mu^2/4)^{-i\omega} e^{2i\theta(\omega)} b^\dagger_{3-}(\omega). \end{aligned} \quad (43)$$

A significant consequence of these relations is the vanishing of the fields $\varphi_{1,3}(u,v)$ in (27) in the asymptotic regions I and III when either $|u|$ or $|v|$ goes to infinity, namely

$$\lim_{u \text{ or } (-v)\to\infty} \varphi_1(u,v) = 0 = \lim_{(-u) \text{ or } v\to\infty} \varphi_3(u,v). \quad (44)$$

This can be verified by using the asymptotic behavior of the Bessel functions $I_{\mp i\omega}(\sqrt{-2\mu^2 uv})$ as given in Eqs. (20), (21) when $uv < 0$, and using the relations in Eq. (43).

[5] $A(k)$ and $B^\dagger(k)$ are isolated from the right-hand side of Eq. (38) by multiplying with $(E' \mp k')^{\pm' i\omega}$ as appropriate and adding two terms so that one may use $\int_0^\infty d\omega((E-k)^{-i\omega}(E'\mp k')^{i\omega} + (E-k)^{i\omega}(E'\mp k')^{-i\omega}) = \delta(k \mp k') 2\pi E$ when $E(k) = \sqrt{k^2+\mu^2}$. To check that Eq. (39) satisfies Eq. (38) use $\int_{-\infty}^{\infty} \frac{dk}{2\pi E(k)} \times (E(k) \pm k)^{i\lambda} = \delta(\lambda)$. The contributions of the zero modes to the integrals in Eq. (38) are proportional to $\delta(\omega)$, but these vanish since $\omega > 0$.

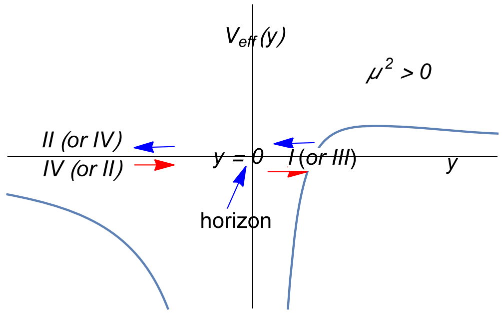

FIG. 8. Incoming/reflected waves at $y \to -\infty$ (regions II&IV); vanishing wavefunction at $y \to +\infty$ in I & III.

Quite independently, the vanishing of the field in the asymptotic regions of I&III is required on physical grounds because without it the field (or the quantum wave function) would blow up at infinity, implying infinite probability. It is gratifying that this required physical behavior emerged from the boundary conditions at the horizons automatically without having to impose it as an additional boundary condition, thus giving confidence that the horizon boundary condition is a correct physical approach.

Furthermore, note that the vanishing of the field asymptotically in regions I&III is the expected behavior for wave packets, on the basis of the classical geodesics in Figs. 2, 3, as well as on the basis of the intuitive physical approach using the effective classical mechanical potential in Fig. 4, and the effective quantum potential explained below in Eq. (45) and plotted in Fig. 8.

Turning next to regions II&IV, the asymptotic behavior of the Rindler field is oscillatory as seen from analyzing the asymptotic behavior of $I_{\mp i\omega}(\sqrt{-2\mu^2 uv})$ when $uv > 0$. Consistent with the geodesics in Figs. 2, 3, this is allowed physical behavior for incoming or outgoing particles/antiparticles, or oscillatory waves and wave packets built from them. Further boundary conditions may be imposed in regions II&IV to correspond to physical processes for either incoming or outgoing wave packets for particles or antiparticles.

There is another intuitive approach to understand the same general behavior of the wave function, without doing any calculations, which is in agreement with the results of the horizon boundary conditions given in the preceding paragraphs. Namely, by defining $\psi(y) \equiv \sqrt{2y}\varphi(y)$, Eq. (18) takes the standard form of the non-relativistic Schrödinger equation

$$[-\partial_y^2 + V_{\rm eff}(y)]\psi(y) = 0, \qquad V_{\rm eff}(y) = \frac{\mu^2}{2y} - \frac{\omega^2+1}{(2y)^2}, \tag{45}$$

where the effective quantum potential, $V_{\rm eff}(y)$, is plotted in Fig. 8.

The "Schrödinger energy level" in Eq. (45) is zero, which corresponds to the real axis in Fig. 8. The intuitive physics extracted from this figure is that of scattering of waves from the barrier presented by the "hill." Thus, oscillating waves approaching from $y \sim -\infty$ in region IV (or II) pass the horizon at $y = 0$ and move into region I (or III), they get scattered from the barrier and move within region I (or III) toward the horizon at $y = 0$, then they continue into region II (or IV) and go on to its asymptotic regions, $y \sim -\infty$, as oscillating waves. This behavior of the quantum wavefunction is fully consistent with the geodesics in Figs. 2, 3 and the intuition gained from the mechanical potential for geodesics in Fig. 4. The effective potential approaches of Figs. 4, 8 are very important especially when explicit solutions are not available (such as the case of general black holes, see, e.g., [16]). A figure of the potential $V_{\rm eff}(y)$ conveys much of the physical behavior, including boundary conditions, such as the vanishing of the wave function for $y \to +\infty$ in Fig. 8, consistent with Eq. (44).

Next, it is worth outlining the behavior of the massless field ($\mu^2 = 0$) in contrast to the massive field. In this case Eq. (42) cannot be used naively because in the massless limit either $(E-k)$ or $(E+k)$ vanishes. So the consequences of Eq. (38) for the massless case need to be analyzed separately for $(E-|k|) = 0$. For this purpose, in these equations the integral $\int_{-\infty}^{\infty} dk$ must be split to its positive and negative regions and the limit $(E-|k|) \to 0$ taken. The integrals that contain the wildly oscillating factors $(E-|k|)^{\pm i\omega}$ vanish in the limit, leaving behind the correct massless limit of Eq. (38) for region-I in the Rindler case (not black hole case),

$$\begin{aligned} a_{1\mp}(\omega) &= \frac{e^{\pm i\theta(\omega)}}{\pm i\sqrt{1-e^{-2\pi\omega}}}\int_0^\infty \frac{dk}{\sqrt{4\pi k}} k^{\mp i\omega}(A_\mp(k) + B_\mp^\dagger(k)e^{-\pi\omega}) \\ b_{1\mp}^\dagger(\omega) &= \frac{e^{\mp i\theta(\omega)}}{\mp i\sqrt{1-e^{-2\pi\omega}}}\int_0^\infty \frac{dk}{\sqrt{4\pi k}} k^{\pm i\omega}(B_\mp^\dagger(k) + A_\mp(k)e^{-\pi\omega}) \end{aligned} \tag{46}$$

noting that only half of the $A_\mp(k), B_\mp^\dagger(k)$ survive in each line. A similar set of equations hold for region III. Together, these may be written as Bogoliubov transformations that correspond to the massless limit of Eq. (38)

$$\frac{ie^{-i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{1-}(\omega)\\ b_{3+}^{\dagger}(\omega)\end{pmatrix}=\int_0^\infty dk\frac{k^{-i\omega}}{\sqrt{4\pi k}}\begin{pmatrix}A_-(k)\\ B_-^{\dagger}(k)\end{pmatrix}$$
$$\frac{-ie^{i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{3+}(\omega)\\ b_{1-}^{\dagger}(\omega)\end{pmatrix}=\int_0^\infty dk\frac{k^{i\omega}}{\sqrt{4\pi k}}\begin{pmatrix}A_-(k)\\ B_-^{\dagger}(k)\end{pmatrix}$$
$$\frac{-ie^{-i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{3-}(\omega)\\ b_{1+}^{\dagger}(\omega)\end{pmatrix}=\int_0^\infty dk\frac{k^{-i\omega}}{\sqrt{4\pi k}}\begin{pmatrix}A_+(k)\\ B_+^{\dagger}(k)\end{pmatrix}$$
$$\frac{ie^{i\theta(\omega)}}{\sqrt{1-e^{-2\pi\omega}}}\begin{pmatrix}1 & -e^{-\pi\omega}\\ -e^{-\pi\omega} & 1\end{pmatrix}\begin{pmatrix}a_{1+}(\omega)\\ b_{3-}^{\dagger}(\omega)\end{pmatrix}=\int_0^\infty dk\frac{k^{i\omega}}{\sqrt{4\pi k}}\begin{pmatrix}A_+(k)\\ B_+^{\dagger}(k)\end{pmatrix} \quad (47)$$

The contrast with the massive case in Eq. (38) is the right-hand side of these equations, noting that only half of the $A_{\mp}(k), B_{\mp}^{\dagger}(k)$ survive in each line for the massless case. Furthermore, Eqs. (46), (47) show that, unlike Eq. (43), the $a_{1\pm}$ are independent of each other for the massless case. This implies that the massless field as written in Eq. (27), but with $S(0)=1$, does not vanish at the asymptotic regions of I or III, but rather has an oscillating behavior. This is consistent with the behavior of the massless limit of the geodesics in Figs. 2, 3 as discussed in the last paragraph of Sec. II, which indicate that massless particles/antiparticles do indeed travel to such asymptotic regions. This is also evident from the intuitive effective potential approach in Figs. 4, 8 after the corresponding effective potentials are replaced by their $\mu^2=0$ counterparts. Thus, the mechanical potential for geodesics becomes a constant [see Eqs. (8), (12)] so the maximum position y_* in the modified Fig. 4 moves to infinity. Similarly the effective quantum potential in the modified Fig. 8 no longer has a barrier, so waves can move both ways from $y=\mp\infty$ to $y=\pm\infty$.

As a check of our expressions we may compute the expectation value of the Rindler number density operators $a_{1-}^{\dagger}a_{1-}$ etc. in the Minkowski vacuum, $\langle 0_M|a_{1-}^{\dagger}(\omega)\times a_{1-}(\omega')|0_M\rangle$, by using directly the Bogoliubov relation between $a_{1-}(\omega)$ and $A(k)$ & $B^{\dagger}(k)$ given in Eq. (37). Using the properties of the Minkowski vacuum, $\langle 0_M|A^{\dagger}(k)=0=B(k)|0_M\rangle$, we obtain,

$$\langle 0_M|a_{1-}^{\dagger}(\omega)a_{1-}(\omega')|0_M\rangle=\frac{e^{-\pi(\omega+\omega')}\int_{-\infty}^{\infty}\frac{dk}{4\pi E(k)}\left(\frac{1}{2}E(k)-\frac{1}{2}k\right)^{i(\omega-\omega')}}{\sqrt{(1-e^{-2\pi\omega})(1-e^{-2\pi\omega'})}}=\frac{1}{2}\frac{\delta(\omega-\omega')}{e^{2\pi\omega}-1}. \quad (48)$$

The integral is given in footnote 5. The result for other number operators, $a_{1\pm}^{\dagger}a_{1\pm}$, $b_{1\pm}^{\dagger}b_{1\pm}$, $a_{3\pm}^{\dagger}a_{3\pm}$, $b_{3\pm}^{\dagger}b_{3\pm}$, is the same. The factor $(e^{2\pi\omega}-1)^{-1}$ is the well-known thermal distribution which, as expected, is in agreement with previous results [5].

VI. MULTIVERSE LEVELS IN MINKOWSKI BASIS

In this section we display the multiverse directly in the Minkowski basis by obtaining the relation between the general level (n,m) field and the level-(0,0) field of Eq. (16), both expressed in terms of Minkowski plane waves. The level (n,m) field $\varphi^{(n,m)}(u,v)$ can be written in terms of level (n,m) Minkowski oscillators $A^{(n,m)}(k), \bar{B}^{(n,m)}(k)$ as follows, just like Eq. (16),

$$\varphi^{(n,m)}(u,v)=\int_{-\infty}^{\infty}dk\left(A^{(n,m)}(k)\frac{e^{-i\frac{E-k}{2}u}e^{-i\frac{E+k}{2}v}}{\sqrt{4\pi E}}+\bar{B}^{(n,m)}(k)\frac{e^{i\frac{E-k}{2}u}e^{i\frac{E+k}{2}v}}{\sqrt{4\pi E}}\right). \quad (49)$$

This is equivalent to the same field $\varphi^{(n,m)}(u,v)$ given in Eq. (31) in terms of Rindler oscillators. We will derive a very non-trivial canonical transformation between the oscillators $(A^{(n,m)}(k), \bar{B}^{(n,m)}(k))$ and the level (0,0) oscillators $(A(k), B^{\dagger}(k))$ that appear in Eqs. (15), (16). This relation represents, in the Minkowski basis, the consistent analytic continuation of the field in the Rindler basis, as given in Eq. (31), and it could not be obtained without going through the Rindler basis.

In Sec. V we related the level (0,0) Rindler oscillators $((a_{1\mp}, b_{3\pm}^{\dagger}), (a_{3\pm}, b_{1\mp}^{\dagger}))$ to the Minkowski oscillators $(A, \bar{B})$ and vice versa via Bogoliubov transformations in Eqs. (38), (39). The same arguments can be given for level (n,m) to claim the analogous forward and inverse

Bogoliubov transformations that relate $(A^{(n,m)}(k), \bar{B}^{(n,m)}(k)) \leftrightarrow ((a^{(n)}_{1\mp}, \bar{b}^{(m)}_{3\pm}), (a^{(m)}_{3\pm}, \bar{b}^{(n)}_{1\mp}))$. These have the same formal appearance as Eqs. (38), (39) except for inserting the level (n,m) oscillators instead of the level (0,0) ones. Now, consider the pair $(A^{(n,m)}(k), \bar{B}^{(n,m)}(k))$ in the Bogoliubov relation analogous to (39), and on the right hand side insert the level (n,m) doublets in the following form [using Eq. (33)]

$$\begin{pmatrix} a^{(n)}_{1\mp}(\omega) \\ \bar{b}^{(m)}_{3\pm}(\omega) \end{pmatrix} = \begin{pmatrix} e^{2\pi\omega n} & 0 \\ 0 & e^{-2\pi\omega m} \end{pmatrix} \begin{pmatrix} a_{1\mp}(\omega) \\ b^{\dagger}_{3\pm}(\omega) \end{pmatrix},$$
$$\begin{pmatrix} a^{(m)}_{3\pm}(\omega) \\ \bar{b}^{(n)}_{1\mp}(\omega) \end{pmatrix} = \begin{pmatrix} e^{2\pi\omega m} & 0 \\ 0 & e^{-2\pi\omega n} \end{pmatrix} \begin{pmatrix} a_{3\pm}(\omega) \\ b^{\dagger}_{1\mp}(\omega) \end{pmatrix}. \quad (50)$$

Moreover, replace the (0,0) doublets $((a_{1\mp}, b^{\dagger}_{3\pm}), (a_{3\pm}, b^{\dagger}_{1\mp}))$ that appear in (50) in terms of the (0,0) level Minkowski doublets $(A(k), \bar{B}(k))$ by using Eq. (38). This gives the relation between $(A^{(n,m)}(k), \bar{B}^{(n,m)}(k))$ and $(A(k), \bar{B}(k))$. The result takes the following form

$$\begin{pmatrix} A^{(n,m)}(k) \\ \bar{B}^{(n,m)}(k) \end{pmatrix} = \int_{-\infty}^{\infty} dk' M^{(n,m)}(k,k') \begin{pmatrix} A(k') \\ B^{\dagger}(k') \end{pmatrix}, \quad (51)$$

where the 2×2 matrix $M^{(n,m)}(k,k')$ in infinite momentum space, $-\infty < k, k' < \infty$, is given by,

$$M^{(n,m)}(k,k') = \frac{1}{2\pi\sqrt{E(k)E(k')}} \int_{-\infty}^{\infty} d\omega \left(\frac{E(k)+k}{E(k')+k'}\right)^{-i\omega} \times M^{(n,m)}(\omega), \quad (52)$$

with $E(k) = \sqrt{k^2+\mu^2}$, while the 2×2 matrices $M^{(n,m)}(\omega)$ in Rindler frequency space $-\infty < \omega < \infty$ are given by

$$M^{(n,m)}(\omega) \equiv \frac{1}{1-e^{-2\pi\omega}} \begin{pmatrix} 1 & -e^{-\omega} \\ -e^{-\omega} & 1 \end{pmatrix} \times \begin{pmatrix} e^{2\pi\omega n} & 0 \\ 0 & e^{-2\pi\omega m} \end{pmatrix} \begin{pmatrix} 1 & e^{-\omega} \\ e^{-\omega} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{e^{\pi\omega(2n+1)} - e^{-\pi\omega(2m+1)}}{e^{\pi\omega} - e^{-\pi\omega}} & \frac{e^{\pi\omega(2n)} - e^{-\pi\omega(2m)}}{e^{\pi\omega} - e^{-\pi\omega}} \\ -\frac{e^{\pi\omega(2n)} - e^{-\pi\omega(2m)}}{e^{\pi\omega} - e^{-\pi\omega}} & -\frac{e^{\pi\omega(2n-1)} - e^{-\pi\omega(2m-1)}}{e^{\pi\omega} - e^{-\pi\omega}} \end{pmatrix}. \quad (53)$$

For example, for $(n,m) = (0,1)$ or (1,0), they are

$$M^{(0,1)}(\omega) = \begin{pmatrix} 1+e^{-2\pi\omega} & e^{-\pi\omega} \\ -e^{-\pi\omega} & 0 \end{pmatrix},$$
$$M^{(1,0)}(\omega) = \begin{pmatrix} 1+e^{2\pi\omega} & e^{\pi\omega} \\ -e^{\pi\omega} & 0 \end{pmatrix}. \quad (54)$$

Note the property of $M^{(n,m)}(\omega)$, that when the integers are interchanged $(n,m) \to (m,n)$ and $\omega \to -\omega$, we obtain the same matrix $M^{(n,m)}(\omega)$

$$M^{(m,n)}(-\omega) = M^{(n,m)}(\omega). \quad (55)$$

The diagonal matrices in Eq. (50) are interchanged under the same transformation. This explains how we end up with an integral $\int_{-\infty}^{\infty} d\omega$ over positive and negative Rindler frequency in Eq. (52) even though the integrals in the Bogoliubov transformation (38) are only over positive Rindler frequency. Thus, when the integral $\int_{-\infty}^{\infty} d\omega$ is split into its positive and negative pieces, and in the negative piece we replace $\omega \to -\omega$ and interchange $(n,m) \to (m,n)$, we see that the positive (negative) piece comes from the contribution of the first (second) doublet in Eq. (50).

The canonical conjugates that appear in the field $\bar{\varphi}^{(n,m)}(u,v)$, written in a row matrix form, are

$$(\bar{A}^{(n,m)}(k) B^{(n,m)}(k)) = \int_{-\infty}^{\infty} dk' (A^{\dagger}(k') B(k')) \bar{M}^{(n,m)}(k',k). \quad (56)$$

This is obtained by taking the Hermitian conjugate of Eq. (51) and replacing (n,m) by $(-n,-m)$ on the right side. This gives the matrix $\bar{M}^{(n,m)}(k',k)$ as follows

$$\bar{M}^{(n,m)}(k',k) = \frac{1}{2\pi\sqrt{E(k)E(k')}} \int_{-\infty}^{\infty} d\omega \left(\frac{E(k)+k}{E(k')+k'}\right)^{i\omega} \times (M^{(n,m)}(\omega))^{-1T}, \quad (57)$$

where the exponent $(-1T)$ in $(M^{(n,m)})^{-1T}$ means inverse and transpose of the 2×2 matrix $M^{(n,m)}$, noting that the matrices in Eq. (53) satisfy, $M^{(-n,-m)}(\omega) = (M^{(n,m)}(\omega))^{-1}$ and $M^{\dagger(-n,-m)}(\omega) = (M^{(n,m)}(\omega))^{-1T}$.

These matrices satisfy the following remarkable properties in Rindler frequency space,

$$M^{(n,m)}(\omega) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (M^{(n,m)}(\omega))^{-1T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (58)$$

and in Minkowski momentum space[6]

$$\int_{-\infty}^{\infty} dk M^{(n,m)}(k_1,k) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{M}^{(n,m)}(k,k_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta(k_1 - k_2). \quad (59)$$

[6]To prove this property we use the following integrals, $\int_{-\infty}^{\infty} \frac{dk}{2\pi E(k)} (E(k)+k)^{i(\omega_1-\omega_2)} = \delta(\omega_1-\omega_2)$, and $\int_{-\infty}^{\infty} d\omega \times (\frac{E(k_1)+k_1}{E(k_2)+k_2})^{-i\omega} = 2\pi\sqrt{E_1 E_2}\delta(k_1-k_2)$.

These matrix properties indicate that the transformations in Eqs. (51), (56) are canonical transformations since it can be verified that they satisfy the standard oscillator commutation rules in momentum space for all (n,m)

$$\left[\begin{pmatrix} A^{(n,m)}(k_1) \\ \bar{B}^{(n,m)}(k_1) \end{pmatrix}, (\bar{A}^{(n,m)}(k_2) B^{(n,m)}(k_2))\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(k_1 - k_2). \tag{60}$$

This includes the original Minkowski commutators given in Eq. (17) that are reproduced in the case of $n = m = 0$. Clearly, by the construction of Eq. (51), the level (n,m) commutator follows directly from the level (0,0) commutator and the remarkable properties of the matrix $M^{(n,m)}(k,k')$ that relates the (0,0) and (n,m) levels to each other as a canonical transformation.

This establishes the quantum properties of the multiverse in the Minkowski basis for all levels (n,m). It is evident that the Minkowski-basis field $\varphi^{(n,m)}(u,v)$ at all levels given in Eqs. (49), (51), (52), (53), including $n = m = 0$, inherits its properties only from the analyticity properties of the (0,0) level field $\varphi(u,v)$ in the extended Rindler basis.

VII. CHARGE CONSERVATION AND INFORMATION FLOW

In this section we address the question on whether information flows from one level of the Rindler multiverse to other levels. To do this we consider the probability associated with a wave packet. As expected, a wave packet will on the average follow the path of a geodesic as it develops as a function of time. Earlier in the paper we discussed the geodesics at the classical level, and of course at the classical level, since there is no multiverse, the geodesics cannot give information on our question. However, a wave packet may leak to other levels of the multiverse when it crosses the horizons. The question is whether it does or not.

For the Klein-Gordon field in curved spacetime, that is normalized according to the Klein-Gordon dot product in footnote 2, probability is directly related to the conserved charge current up to the sign of the charge. While the probability density is always positive, the charge density is positive/negative for particles/antiparticles respectively (i.e., a versus b symbols in the wave packet). Therefore, to understand probability (or information) flow we study the flow of the charge current with all a and b coefficients included in order to undestand the flow based on the most general wave packet including particles and antiparticles.

The conserved current for the Klein-Gordon equation, $(\nabla^2 - \mu^2)\varphi = 0$, in curved spacetime is

$$J^\mu(x) = -i\sqrt{-g}g^{\mu\nu}(\varphi^\dagger\partial_\nu\varphi - \partial_\nu\varphi^\dagger\varphi), \qquad \partial_\mu J^\mu(x) = 0. \tag{61}$$

The conservation $\partial_\mu J^\mu(x) = 0$ is verified by using the Klein-Gordon equation $(\sqrt{-g})^{-1}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = \mu^2\varphi$. The conserved charge associated with this current is computed as an integral over a spacelike Cauchy surface Σ

$$Q = \int_\Sigma J^\mu d\Sigma_\mu \tag{62}$$

The conserved charge is independent of the surface Σ.

The Cauchy surface can be specified differently in the Minkowski versus Rindler bases. In the Minkowski version the surface is defined by taking a slice of constant x^0 and evaluating the integral as given in textbooks

$$Q_M(x^0) = \int_{-\infty}^{\infty} dx^1 J^0(x^0,x^1) = \int_{-\infty}^{\infty} dk^1(A^\dagger(k^1)A(k^1) - B^\dagger(k^1)B(k^1)), \tag{63}$$

where the computation is performed by using the Minkowski version of the field in Eq. (16) at constant finite values x^0. The time derivative $\partial_{x^0}Q_M(x^0)$ is

$$\partial_0 Q_M(x^0) = \int_{-\infty}^{\infty} dx^1\partial_0 J^0 = \int_{-\infty}^{\infty} dx^1(\partial_\mu J^\mu - \partial_1 J^1) = -J^1(x^0,\infty) + J^1(x^0,-\infty). \tag{64}$$

where the Klein-Gordon equation is used to set $\partial_\mu J^\mu = 0$, and then Stoke's theorem is applied to write the result in terms of the current $J^1(x^0,x^1)$ evaluated at the asymptotic boundaries. In general the current at the boundaries,

$$J^1(x^0,\pm\infty) = \lim_{x^1\to\pm\infty}(-i(\varphi^\dagger(x^0,x^1)\partial_1\varphi(x^0,x^1) - \partial_1\varphi^\dagger(x^0,x^1)\varphi(x^0,x^1))), \tag{65}$$

does not vanish as this represents the charge flux of incoming/outgoing particles, so in such physical processes $\partial_0 Q_M(x^0)$ cannot vanish at asymptotic boundaries. On the other hand, Eq. (63) shows that $Q_M(x^0)$ is time independent at finite x^0. These observations are reconciled by noting that the support of $J^1(x^0,\pm\infty)$ is not only at space infinity $x^1 = \pm\infty$, but also at time infinity $x^0 = \pm\infty$, such as $J^1(x^0,\pm\infty) = \pm\delta(x^0 \pm \infty)J$, where J is a constant determined in terms of $(A(k), B^\dagger(k))$ as shown in Eq. (81) below. Then the charge conservation equation takes the form

$$\partial_{x^0}Q_M(x^0) = (\delta(x^0 + \infty) - \delta(x^0 - \infty))J. \tag{66}$$

This result implies that $Q_M(x^0)$ is not in general a constant at the asymptotic past and future boundaries of Minkowski space. Furthermore, the conservation of charge Q_M in Minkowski space at finite x^0 is explained by the fact that the flux of charge J into Minkowski space at $x^0 = -\infty$ is exactly equal to the flow of charge J out of the space at $x^0 = \infty$. This is the statement of conservation of charge and it implies conservation of information within the Minkowski spacetime. It also leads to unitarity in the complete Hilbert space in the quantum field theory.

In the Rindler case the *spacelike* Cauchy surface needs to be specified differently in each region because the roles of (t, y) alternate between time and space in regions I&III versus regions II&IV. For example, in region I, a spacelike surface correspond to a fixed value of the Rindler time t (any ray in Fig. 1) so that the charge of a field configuration is given by integrating over $d\Sigma_t = dy$ at fixed t,

$$q_1(t) = \int_\Sigma J^\mu(x) d\Sigma_\mu = \int_{y_1}^{y_2} dy J^t(t, y), \quad \text{with}$$
$$J^t = \frac{i}{2y}(\varphi_1^\dagger \partial_t \varphi_1 - \partial_t \varphi_1^\dagger \varphi_1), \tag{67}$$

where, in J^t we used $\sqrt{-g} = 1$, $g^{tt} = -(2y)^{-1}$, and $\varphi_1(t, y)$ given in Eq. (27). Here $J^t(t, y)$ is the charge density, so $\int_{y_1}^{y_2} dy J^t$ is the total charge contained in the interval $y_1 < y < y_2$ at time t. Changing the value of t in the range, $-\infty < t < \infty$, covers the spacetime bounded by the hyperbolas shown in Fig. 1 within region I. Sending $y_1 \to 0$ and $y_2 \to \infty$ covers the entire region I. Then $q_1(t)$, with $y_1 = 0$ and $y_2 = \infty$, is the total regional charge within region I at an arbitrary time t.

By contrast to Eq. (67), in region II a *spacelike* Cauchy surface[7] corresponds to a fixed value of the time y (a fixed hyperbola in region II in Fig. 1, not shown) so that the charge of a field configuration is given by integrating over $d\Sigma_y = dt$ at fixed y,

$$q_2(y) = \int_\Sigma J^\mu(x) d\Sigma_\mu = -\int_{-\infty}^{\infty} dt J^y(t, y), \quad \text{with}$$
$$J^y = (-2yi)(\varphi_2^\dagger \partial_y \varphi_2 - \partial_y \varphi_2^\dagger \varphi_2), \tag{68}$$

where, in J^y we used $\sqrt{-g} = 1$, $g^{yy} = 2y$, and $\varphi_2(t, y)$ given in Eq. (27). The reason for the extra overall sign in the integral $-\int_{-\infty}^{\infty} dt$ will be explained below after Eq. (71).

[7]One may be tempted to ignore the spacelike requirement of the Cauchy surface, and based on the fact that ∂_t is the conserved Killing vector in all regions, including region II, one may take the surface of integration in region II to be again $d\Sigma_t = dy$ just like region I. Applying this reasoning uniformly to every region, one may wish to define $q_{1,2,3,4}$ as integrals over y at fixed t, just as in Eq. (67). This turns out to give the wrong set of sign patterns for $q_{1,2,3,4}$ contrary to the correct patterns displayed in our results in Eqs. (72)–(71): i.e. $+a^\dagger a$ for the charges associated to particles and the opposite signs $-b^\dagger b$ for antiparticles. The wrong set of signs that occur differently in different regions fail the self consistency check involving the Bogoliubov transformations as given in Eqs. (80), (81).

The Rindler version of the total charge Q_R for the full extended Rindler space in the (0,0) universe [the equivalent of Q_M for the Minkowski space in Eq. (63)] is given by integrals at *spacelike* Cauchy surfaces of constant t in regions I & III and constant y in regions II & IV. This is because in regions I & III t is the timelike coordinate because $\mathrm{sign}(y) = +1$ for the spacetime geometry described by the line element in Eq. (4), while in regions II & IV y is the timelike coordinate because $\mathrm{sign}(y) = -1$. The computation of the total charge Q_R is to be performed by using the regional Rindler fields $\varphi_{1,2,3,4}$ given in Eq. (27). We define the total charge, $Q_R = \int_\Sigma J^\mu(x) d\Sigma_\mu$, as an integral over the union of spacelike Cauchy surfaces that were used in the definition of $q_{1,2,3,4}$. Then we find

$$Q_R = q_1 + q_2 + q_3 + q_4, \tag{69}$$

where

$$\begin{aligned}
q_1(t) &= \int_0^\infty dy J_1^t = \int_0^\infty dy \frac{i}{2y}(\varphi_1^\dagger \partial_t \varphi_1 - \partial_t \varphi_1^\dagger \varphi_1) \\
q_2(y) &= -\int_{-\infty}^\infty dt J_2^y = -\int_{-\infty}^\infty dt (-2yi)(\varphi_2^\dagger \partial_y \varphi_2 - \partial_y \varphi_2^\dagger \varphi_2) \\
q_3(t) &= -\int_0^\infty dy J_3^t = -\int_0^\infty dy \frac{i}{2y}(\varphi_3^\dagger \partial_t \varphi_3 - \partial_t \varphi_3^\dagger \varphi_3) \\
q_4(y) &= \int_{-\infty}^\infty dt J_4^y = \int_{-\infty}^\infty dt (-2yi)(\varphi_4^\dagger \partial_y \varphi_4 - \partial_y \varphi_4^\dagger \varphi_4)
\end{aligned} \tag{70}$$

Furthermore, the rate of change of these charges with respect to time is given by the time derivatives in the respective regions

$$\partial_t q_1(t), \quad -\partial_y q_2(y), \quad -\partial_t q_3(t), \quad \partial_y q_4(y). \tag{71}$$

Note the extra overall minus signs in the definitions of the charges $q_2(t)$ and $q_3(y)$ as well as the extra signs in taking their time derivatives. The justification for such extra signs is the comparison of the time increments for the Minkowski $\mathrm{sign}(dx^0)$ to the Rindler $\mathrm{sign}(dT)$ where T is the monotonically *increasing* time in the corresponding regions. Thus, in region III we have $T \equiv t$ and $\mathrm{sign}(dx^0) = -\mathrm{sign}(dt)$ because in region III as t decreases as x^0 increases. This explains why $q_3(t)$ and $-\partial_t q_3(t)$ have an extra sign: it is because the extra sign is absorbed into $\partial/\partial(-t)$ both in the definition of the current J^t in Eq. (67) and in the rate of change, so that $\partial/\partial(-t)$ implies an increment of time with the same sign as $\partial/\partial x^0$. The same explanation works for region II, where $T \equiv -y$ since y is negative, and noting that in region II $\mathrm{sign}(dx^0) = -\mathrm{sign}(dy) = \mathrm{sign}(d|y|)$; consequently the overall sign is absorbed into $\partial/\partial(-y) = \partial/\partial|y|$. By contrast, in region IV where y is negative, we have $T = -y$ but $\mathrm{sign}(dx^0) = \mathrm{sign}(dy) = -\mathrm{sign}(d|y|)$, therefore no extra signs are needed in region IV. When these extra signs are combined with the signs produced when the currents are integrated on *spacelike* Cauchy surfaces (see footnote 7) one obtains the correct sign patterns for particle/antiparticle charges and charge fluxes in our results given below.

The explicit computation of $q_{1,2,3,4}$ shows that they are constants within each region, but $(\partial_t q_1(t), \partial_{-y} q_2(y), \partial_{-t} q_3(t), \partial_y q_4(y))$ receive nontrivial contributions at the horizons and the asymptotic boundaries of each region [analogous to constant Q_M but nontrivial ∂Q_M at boundaries as in Eqs. (63), (66)]. The results are as follows.

For region I, according to the computations shown in Appendix A we have

$$q_1 = \int_0^\infty d\omega((a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) + (a^\dagger_{1+}a_{1+} - b^\dagger_{1+}b_{1+})),$$

$$\partial_t q_1(t)|_{\mu^2=0} = \int_0^\infty d\omega \begin{bmatrix} (\lim_{v\to-\infty} - \lim_{v\to 0})\delta_\varepsilon(\ln u)(a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) \\ +(\lim_{u\to 0} - \lim_{u\to\infty})\delta_\varepsilon(\ln|v|)(a^\dagger_{1+}a_{1+} - b^\dagger_{1+}b_{1+}) \end{bmatrix}$$

$$\partial_t q_1(t)|_{\mu^2\neq 0} = \int_0^\infty d\omega \begin{bmatrix} -\lim_{v\to 0}\delta_\varepsilon(\ln u)(a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) \\ +\lim_{u\to 0}\delta_\varepsilon(\ln|v|)(a^\dagger_{1+}a_{1+} - b^\dagger_{1+}b_{1+}) \end{bmatrix} \quad (72)$$

Here the symbol $\delta_\varepsilon(z)$ is a smeared delta function defined in Eqs. (A7)–(A11). We discuss briefly the meaning of these equations. First note that the charge q_1 is conserved *within* region I by itself, $\partial_t q_1(t) = 0$ at finite t, since q_1 is explicitly time independent. However, the expression for massless particles $\partial_t q_1(t)|_{\mu^2=0}$ shows that charge is not conserved locally at both horizons $u = 0$ or $v = 0$ and at both asymptotic boundaries $u = \infty$ or $v = \infty$; similarly for massive particles $\partial_t q_1(t)|_{\mu^2\neq 0}$ shows that charge is not conserved locally at both horizons (the wave function and current in regions I&III vanish asymptotically for massive particles, so $\partial_t q_1(t)|_{\mu^2\neq 0}$ at $y \to \pm\infty$).

That we should expect nontrivial charge flow at the horizons for the massive particle, $\partial_t q_1(t)|_{\mu^2\neq 0} \neq 0$ at $t \to \pm\infty$ and $y = 0$, was evident in Figs. 2, 3 that depict the classical geodesics for massive particles that show geodesics crossing the horizons ($y = 0$) at $t = \pm\infty$. In the quantum computation, using general wave packets with particles and antiparticles, we see in Eq. (72) and Fig. 9 that at each frequency ω there is a charge flux $(-|a_{1-}(\omega)|^2 + |b^\dagger_{1-}(\omega)|^2)$ due to outgoing particles (overall − sign) and incoming antiparticles (+sign) at the future horizon in region I ($v = 0$), and another charge flux $(+|a_{1+}(\omega)|^2 - |b^\dagger_{1+}(\omega)|^2)$ due to incoming particles (+sign) and outgoing antiparticles (−sign) at the past horizon in region I ($u = 0$). These incoming and outgoing fluxes sum up to zero because, for the massive particle, the boundary conditions are $|a_{1-}(\omega)| = |a_{1+}(\omega)|$ and $|b^\dagger_{1-}(\omega)| = |b^\dagger_{1+}(\omega)|$ as seen in Eq. (43). For massless particles $|a_{1\pm}(\omega)|$ and similarly $|b^\dagger_{1\pm}(\omega)|$ are unrelated to each other, but the result for $\partial_t q_1(t)|_{\mu^2=0}$ given above indicates that again the total incoming flux of charge into region I is equal to the total outgoing flux of charge.

We emphasize the fact that the incoming and outgoing particle/antiparticle fluxes sum up to zero *separately* for every species of particle or antiparticle, for either massless or massive particles of every frequency ω. This indicates that what comes into region I goes out fully in the same form (species of particle or antiparticle) at each frequency ω. This is why the total charge *within* region I remains a constant $\partial_t q_1 = 0$ at finite t, for every species *separately*, and not by cancellation among the different species (i.e. $a_{1\pm}, a_{3\pm}, b_{1\pm}, b_{3\pm}$).

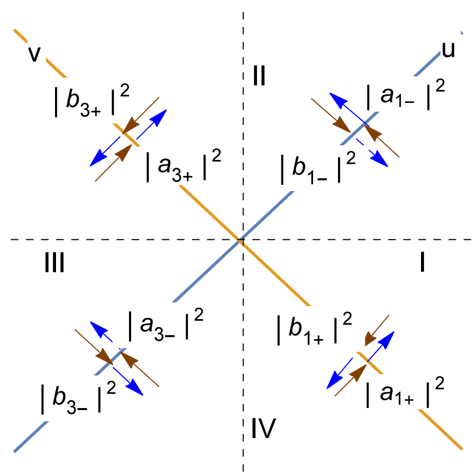

FIG. 9. Magnitudes of incoming and outgoing fluxes at the horizons of each region. Blue arrows = +sign, and red arrows = −sign. For example, for region I, at the future horizon ($v = 0$), the outgoing particle current is proportional to $|a^2_{1-}(\omega)|$ (red) and the incoming antiparticle current is proportional to $|b_{1-}(\omega)|^2$ (blue). Similarly at the past horizon of region I ($u = 0$), the incoming particle current is proportional to $|a_{1+}(\omega)|^2$ (blue) and the outgoing antiparticle current is $|b^\dagger_{1+}(\omega)|^2$ (red). Similar in and out currents are indicated for each region.

For region II, the result of the computations shown in Appendix A is

$$q_2 = \int_0^\infty d\omega((a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) + (-b^\dagger_{3+}b_{3+} + a^\dagger_{3+}a_{3+})),$$
$$\partial_{-y}q_2(y)|_{\mu^2=0} = \int_0^\infty d\omega \begin{bmatrix} (\lim_{v\to 0} - \lim_{v\to\infty})\delta_\varepsilon(\ln u)(a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) \\ +(\lim_{u\to 0} - \lim_{u\to\infty})\delta_\varepsilon(\ln v)(-b^\dagger_{3+}b_{3+} + a^\dagger_{3+}a_{3+}) \end{bmatrix}$$
$$\partial_{-y}q_2(y)|_{\mu^2\neq 0} = \int_0^\infty d\omega \begin{bmatrix} +\lim_{v\to 0}\delta_\varepsilon(\ln u)(a^\dagger_{1-}a_{1-} - b^\dagger_{1-}b_{1-}) \\ +\lim_{u\to 0}\delta_\varepsilon(\ln v)(-b^\dagger_{3+}b_{3+} + a^\dagger_{3+}a_{3+}) \\ -\lim_{v\to\infty}\delta_\varepsilon(\ln u)\frac{|a^\dagger_{1-}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}b^\dagger_{3+}|^2 - |b_{1-}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}a_{3+}|^2}{1-e^{-2\pi\omega}} \\ -\lim_{u\to\infty}\delta_\varepsilon(\ln v)\frac{-|b^\dagger_{3+}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}a_{1-}|^2 + |a_{3+}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}b^\dagger_{1-}|^2}{1-e^{-2\pi\omega}} \end{bmatrix} \quad (73)$$

The interpretation of these expressions for region II is similar to the one above for region I. The total charge q_2 is explicitly time independent *within* region II, but its rate of change locally at each horizon $u = 0$ or $v = 0$, or asymptotic boundaries $u \to \infty$ or $v \to \infty$, is generally nonzero. However, again what comes into region II goes out of region II in the same total form (either particle or antiparticle) at each frequency ω. The vanishing of the sum of incoming and outgoing fluxes for all boundaries of region II is evident without any computation for the massless particle. For the massive particle, simple algebra such as

$$\frac{-|a^\dagger_{1-} + e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}b^\dagger_{3+}|^2 + |b^\dagger_{3+} + e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}a_{1-}|^2}{1-e^{-2\pi\omega}} = -|a_{1-}|^2 + |b^\dagger_{3+}|^2, \quad (74)$$

shows that the *sum* of asymptotic fluxes at $u \to \infty$ and $v \to \infty$, namely $[(-|a_{1-}|^2 + |b^\dagger_{3+}|^2) + (-|a_{3+}|^2 + |b^\dagger_{1-}|^2)]$, matches the *sum* of the fluxes at the horizons $u = 0$ and $v = 0$ except for an overall sign. Hence, the total sum over all boundaries vanishes. We conclude (similar to region I) that charge, information or probability, are conserved within region II by itself.

Observe that the incoming (outgoing) particle (antiparticle) charge at the $v = 0$ horizon of region II, is identical to the particle (antiparticle) flux that leaves (enters) region I. This shows that the flux of particles and antiparticles is continuous across the horizon at the boundary of regions I&II as indicated in Fig. 9.

For region III, the computations are parallel to those for region I. The result is obtained from Eq. (72) simply by replacing $(u, -v) \to (-u, v)$ (see Fig. 1) and $(a_{1-}, a_{1+}) \to (b^\dagger_{3-}, b^\dagger_{3+})$ [see Eq. (27)], and multiplying by an overall minus sign for q but not for ∂q [see Eqs. (70), (71)]. The result is

$$q_3 = \int_0^\infty d\omega((-b^\dagger_{3-}b_{3-} + a^\dagger_{3-}a_{3-}) + (-b^\dagger_{3+}b_{3+} + a^\dagger_{3+}a_{3+}))$$
$$\partial_{-t}q_3(t)|_{\mu^2=0} = \int_0^\infty d\omega \begin{bmatrix} (\lim_{v\to\infty} - \lim_{v\to 0})\delta_\varepsilon(\ln|u|)(b^\dagger_{3-}b_{3-} - a^\dagger_{3-}a_{3-}) \\ +(\lim_{u\to 0} - \lim_{u\to-\infty})\delta_\varepsilon(\ln v)(b^\dagger_{3+}b_{3+} - a^\dagger_{3+}a_{3+}) \end{bmatrix}$$
$$\partial_{-t}q_3(t)|_{\mu^2\neq 0} = \int_0^\infty d\omega \begin{bmatrix} -\lim_{v\to 0}\delta_\varepsilon(\ln|u|)(b^\dagger_{3-}b_{3-} - a^\dagger_{3-}a_{3-}) \\ +\lim_{u\to 0}\delta_\varepsilon(\ln v)(b^\dagger_{3+}b_{3+} - a^\dagger_{3+}a_{3+}) \end{bmatrix} \quad (75)$$

Recall that for the massive particle $|a_{3+}| = |a_{3-}|$ and $|b_{3+}| = |b_{3-}|$ according to the boundary conditions obtained in Eq. (43). For the massless particle there are no such relations. The interpretation is parallel to the discussion above for region I. Furthermore, at the horizon at the common boundary for regions II&III we see that what leaves (enters) region II fully enters (leaves) region III. From this we conclude (similar to region I or II) that charge, information or probability, are conserved *within* region III by itself, independent of what goes on in other regions of the extended

Rindler space. This leads also to unitarity of the scattering matrix in the quantum Hilbert space of region III by itself.

For region IV, the computations are parallel to those for region II. The result is obtained from Eq. (73) simply by replacing $(u,v)\to(-u,-v)$ (see Fig. 1) and $(a_{1-},b_{3+}^{\dagger})\to(b_{3-}^{\dagger},a_{1+})$ [see Eq. (27)], and multiplying by an overall minus sign for q but not for ∂q [see Eqs. (70), (71)]. The result is

$$
\begin{aligned}
q_4 &= \int_0^\infty d\omega((-b_{3-}^{\dagger}b_{3-}+a_{3-}^{\dagger}a_{3-})+(a_{1+}^{\dagger}a_{1+}-b_{1+}^{\dagger}b_{1+}))\\
\partial_y q_4(y)|_{\mu^2=0} &= \int_0^\infty d\omega\begin{bmatrix}(\lim_{v\to0}-\lim_{v\to-\infty})\delta_\varepsilon(\ln|u|)(b_{3-}^{\dagger}b_{3-}-a_{3-}^{\dagger}a_{3-})\\ +(\lim_{u\to-\infty}-\lim_{u\to0})\delta_\varepsilon(\ln|v|)(a_{1+}^{\dagger}a_{1+}-b_{1+}^{\dagger}b_{1+})\end{bmatrix}\\
\partial_y q_4(y)|_{\mu^2\neq0} &= \int_0^\infty d\omega\begin{bmatrix}+\lim_{v\to0}\delta_\varepsilon(\ln|u|)(b_{3-}^{\dagger}b_{3-}-a_{3-}^{\dagger}a_{3-})\\ -\lim_{u\to0}\delta_\varepsilon(\ln|v|)(a_{1+}^{\dagger}a_{1+}-b_{1+}^{\dagger}b_{1+})\\ -\lim_{v\to-\infty}\delta_\varepsilon(\ln|u|)\frac{|b_{3-}^{\dagger}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}a_{1+}|^2-|a_{3-}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}b_{1+}^{\dagger}|^2}{1-e^{-2\pi\omega}}\\ +\lim_{u\to-\infty}\delta_\varepsilon(\ln|v|)\frac{|a_{1+}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}b_{3-}^{\dagger}|^2-|b_{1+}^{\dagger}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}a_{3-}|^2}{1-e^{-2\pi\omega}}\end{bmatrix}
\end{aligned}
\tag{76}
$$

The interpretation is similar to those of regions I or II or III, and again we conclude that charge, information or probability, are conserved within region IV by itself, independent of what goes on in other regions of the extended Rindler space.

We may now compute the sum of the charges Q_R, Eq. (69), in all the regions I-IV for universe (0,0), and find

$$
\begin{aligned}
Q_R &= 2\sum_{\pm}\int_0^\infty d\omega((a_{1\pm}^{\dagger}a_{1\pm}-b_{1\pm}^{\dagger}b_{1\pm})-(b_{3\pm}^{\dagger}b_{3\pm}-a_{3\pm}^{\dagger}a_{3\pm})),\\
\partial Q_R|_{\mu^2=0} &= \int_0^\infty d\omega\begin{pmatrix}+(\lim_{v\to-\infty}-\lim_{v\to\infty})\delta_\varepsilon(\ln|u|)\begin{bmatrix}(a_{1-}^{\dagger}a_{1-}-b_{1-}^{\dagger}b_{1-})\\ -(b_{3-}^{\dagger}b_{3-}-a_{3-}^{\dagger}a_{3-})\end{bmatrix}\\ +(\lim_{u\to-\infty}-\lim_{u\to\infty})\delta_\varepsilon(\ln|v|)\begin{bmatrix}(a_{1+}^{\dagger}a_{1+}-b_{1+}^{\dagger}b_{1+})\\ -(b_{3+}^{\dagger}b_{3+}-a_{3+}^{\dagger}a_{3+})\end{bmatrix}\end{pmatrix}\\
\partial Q_R|_{\mu^2\neq0} &= \int_0^\infty\frac{d\omega}{1-e^{-2\pi\omega}}\begin{bmatrix}\lim_{u\to-\infty}\delta_\varepsilon(\ln|v|)\begin{pmatrix}|a_{1+}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}b_{3-}^{\dagger}|^2\\ -|b_{1+}^{\dagger}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}a_{3-}|^2\end{pmatrix}\\ +\lim_{v\to-\infty}\delta_\varepsilon(\ln|u|)\begin{pmatrix}|a_{3-}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}b_{1+}^{\dagger}|^2\\ -|b_{3-}^{\dagger}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}a_{1+}|^2\end{pmatrix}\\ +\lim_{u\to\infty}\delta_\varepsilon(\ln v)\begin{pmatrix}|b_{3+}^{\dagger}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}a_{1-}|^2\\ -|a_{3+}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}b_{1-}^{\dagger}|^2\end{pmatrix}\\ +\lim_{v\to\infty}\delta_\varepsilon(\ln u)\begin{pmatrix}|b_{1-}+e^{-\pi\omega}(\mu^2)^{i\omega}e^{-i2\theta}a_{3+}|^2\\ -|a_{1-}^{\dagger}+e^{-\pi\omega}(\mu^2)^{-i\omega}e^{i2\theta}b_{3+}^{\dagger}|^2\end{pmatrix}\end{bmatrix}
\end{aligned}
\tag{77}
$$

The last two equations for ∂Q_R are the sums of all the fluxes $\partial q_{1,2,3,4}$ given in Eqs. (72)–(76); in this sum the fluxes at each horizon cancel out and only the asymptotic fluxes in each region remain as shown in Eq. (77). Note that the sum of all incoming terms in $\sum(\partial Q_R)_{\rm in}$ is exactly equal to the sum of all outgoing terms in $\sum(\partial Q_R)_{\rm out}$. This is easy to see for $\partial Q_R|_{\mu^2=0}$. Simple algebra, like Eq. (74), shows that it is also true for $\partial Q_R|_{\mu^2\neq0}$ when we take into account the results of the boundary conditions given in Eq. (43), namely

$$
\mu^2\neq0\colon\ |a_{1-}|=|a_{1+}|,\qquad |a_{3-}|=|a_{3+}|,\qquad |b_{1-}|=|b_{1+}|,\qquad |b_{3-}|=|b_{3+}|. \tag{78}
$$

Equation (77) is the statement of charge conservation for the entire (0,0) universe: Q_R is conserved within the (0,0) universe by itself because the charges that flow in and out its asymptotic regions balance each other exactly such that the sum of all influxes is equal to the sum of all outflows. Moreover, each *type* of charge $(a_{1\pm}, a_{3\pm}, b_{1\pm}, b_{3\pm})$ and corresponding total flux is *separately conserved*. This amounts to conservation of probability and information for the overall (0,0) level.

This result for the Rindler total charge, Q_R in the (0,0) universe, may be compared to the total charge Q_M defined in Minkowski space as given above in Eq. (63). We expect the total charge and total boundary in or out fluxes to be the same in either computation,

$$Q_R = Q_M \quad \text{and} \quad \sum (\partial Q_R)_{\text{in/out}} = \sum (\partial Q_M)_{\text{in/out}}. \tag{79}$$

To relate the Rindler/Minkowski results to each other we use the Bogoliubov transformations in Eqs. (38), (47) and find

$$Q = \begin{cases} = 2\sum_{\pm} \int_0^\infty d\omega[(a^\dagger_{1\pm}a_{1\pm} - b^\dagger_{1\pm}b_{1\pm}) - (b^\dagger_{3\pm}b_{3\pm} - a^\dagger_{3\pm}a_{3\pm})] \\ = \int_{-\infty}^\infty dk^1(A^\dagger(k^1)A(k^1) - B^\dagger(k^1)B(k^1)), \end{cases} \tag{80}$$

showing that indeed $Q_R = Q_M$ according to Eqs. (63), (77). Similarly, the identity for the sum of the in or out fluxes can also be proven by using the Bogoliubov transformations to find

$$\sum (\partial Q)_{\text{in/out}} = \begin{cases} = \int_0^\infty d\omega[(a^\dagger_{1\pm}a_{1\pm} - b^\dagger_{1\pm}b_{1\pm}) - (b^\dagger_{3\pm}b_{3\pm} - a^\dagger_{3\pm}a_{3\pm})] \\ = \frac{1}{2}\int_0^\infty dk(A^\dagger_\pm(k)A_\pm(k) - B^\dagger_\pm(k)B_\pm(k)) \end{cases}. \tag{81}$$

These checks verify that our approach is self consistent according to Eqs. (66), (77), (79).

This result implies that charge, information or probability, is conserved in the (0,0) universe by itself and furthermore that the (0,0) universe formulated in the extended Rindler space is equivalent to a full Minkowski universe on one sheet. From this we may also conclude that in the absence of interactions or perturbations, Rindler information does not leak from the (0,0) universe to any other (n, m) universe.

The same arguments can now be applied at each level by using the Rindler or Minkowski forms of the same field $\varphi^{(n,m)}(u, v)$ that we have discussed in the previous sections. A little thought is sufficient to go over the same computations by simply changing the symbols for the oscillators, and be convinced that charge or information is again conserved separately within every level (n, m).

Thus, it seems the first quantized level-(0,0) wave function or the quantum field $\varphi(u, v)$, analytically continued to all levels in the extended Rindler spacetime, describes parallel Minkowski universes. Since all levels are predictably related to each other by analiticity, one should not think of phenomena in these parallel universes as being independent from each other, at least not in the present context of free fields. This is because there is only one set of oscillators to construct wave packets, namely those of level-(0,0), and as we have shown, all oscillators at other levels are dependent on the level-(0,0) oscillators.

VIII. DISCUSSION

In summary, we have shown that, although information does flow between neighboring regions of the (0,0) universe, regional information remains constant for each species of particles/antiparticles $(a_{1\pm}, a_{3\pm}, b_{1\pm}, b_{3\pm})$ due to the balance of in/out fluxes for each region *separately*. The conserved regional charges, q_1, q_2, q_3, q_4, are generally different in each region and they are determined by the wave packet coefficients of the fields in Eq. (27) for each region in universe (0,0). Note that the constant $q_{1,2,3,4}$ as well as the fluxes at boundaries depend on the wave packet coefficients only in the combinations, $a^\dagger_{1\pm}(\omega)a_{1\pm}(\omega), b^\dagger_{1\pm}(\omega)b_{1\pm}(\omega), a^\dagger_{3\pm}(\omega)a_{3\pm}(\omega), b^\dagger_{3\pm}(\omega)b_{3\pm}(\omega)$, which turn into number operators in the second quantized field theory.

This argument is repeated for each (n, m) universe for which the corresponding fields are fully determined by analyticity. Recall that the field in the (n, m) universe differs from the field in the (0,0) universe by the canonical transformations in Eqs. (33), (34) or Eqs. (51), (56). We find that the regional constant charges q_1, q_2, q_3, q_4, and the fluxes at the boundaries, of the Rindler regions in the (n, m) universe, are identical to those of the (0,0) universe, because, according to Eqs. (33), (34), the number operators, $a^\dagger_{1\pm}a_{1\pm}$ etc., in any (n, m) universe are the same as in the (0,0) universe since these number operators are invariant under the canonical transformations. This is true despite the fact that the wavepacket coefficients $a^{(n,m)}_{1\pm}$ etc. in the (n, m) universe are different than the (n', m') universe by real factors (not just phases). Therefore, as far as information flow and conservation is concerned, the Rindler multiverse seems to consist of parallel universes that may not communicate with each other.

This conclusion emerged because of information conservation *separately* in each Rindler quadrant of Minkowski space, at all levels of the multiverse, which holds as long as the Rindler multiverse system is not disturbed by interactions that may alter the current J^μ or induce interuniverse transitions.

Note however that there are nontrivial interuniverse propagators or more general multipoint correlators with one leg in the (n, m) universe and the other(s) in a different (n', m') universe(s), such as

$$G_{(n,m)}^{(n',m')}(u_i, v_i; u_j, v_j) \equiv \langle 0_M | \varphi_i^{(n,m)}(u, v) \varphi_j^{\dagger(n',m')}(u', v') | 0_M \rangle, \tag{82}$$

where $i,\ j = 1,\ 2,\ 3,\ 4$, indicate the regions I-IV. The creation/annihilation operators in the analytically continued fields $\varphi_i^{(n,m)}, \varphi_j^{\dagger(n',m')}$ are related to each other but have different real factors that depend on (n, m) or (n', m') as given in Eqs. (33), (34). When $(n, m) = (n', m') = (0, 0)$ these propagators or more general n-point functions are guaranteed to be identical to the well-known propagators or n-point functions of a Klein-Gordon complex scalar field in Minkowski space. However, in general they will differ because of the n, m, n', m' dependent factors that modify computations of the (0,0) universe, such as the modification of the example in Eq. (48) by the additional factor as seen below

$$\langle 0_M | a_{1-}^{\dagger(n,m)}(\omega) a_{1-}^{(n',m')}(\omega') | 0_M \rangle = \frac{1}{2} \frac{\delta(\omega - \omega')}{e^{2\pi\omega} - 1} e^{-2\pi\omega(n-n')}. \tag{83}$$

The propagator $G_{(n,m)}^{(n',m')}(u, v; u', v')$ is easily computed by using such relations that include the extra factor $e^{-2\pi\omega(n-n')}$. The physical meaning of $G_{(n,m)}^{(n',m')}(u, v; u', v)$ is unclear at the moment when there are no interactions. In any case, these propagators will surely play a role if there are interactions that cause inter-universe transitions.

As examples of disturbances of the Rindler parallel universes, we may consider the geometry of an eternal black hole or the cosmological geometry of the minisuperspace described in Appendix B. Either spacetime may be considered as introducing some gravitational interaction that deforms the extended Rindler spacetime nonperturbatively. The approach of this paper may be applied similarly to cosmology as in [3] or black holes, as in [4]. We find that, although information conservation as discussed above holds for the noninteracting Rindler multiverse, it can fail for cases like these. In particular, for black holes it is found that there is leakage of information precisely at the black hole singularity through which the current flows between different levels of the multiverse. The information loss for black holes [19,20] may be redefined as a loss of information for the (0,0) universe, but still conserved in the full eternal black hole multiverse. The flow of information away from the (0,0) universe can be tracked quantitatively by computing the amount of information that leaks to specific regions in other universes in the extended black hole multiverse [4]. The question remains as to what happens to information if the black hole can fully evaporate.

We have shown that even something as simple as the extended Rindler space is far richer at the quantum level than the Minkowski geometry specified by the metric or the geodesics at the classical level. New phenomena of physical interest may occur due to the natural multiverse predicted by the quantum field. Even for the Rindler multiverse, it would be interesting to explore which types of perturbative or nonperturbative interactions (such as black holes, big bang, and others) may induce communication among the otherwise apparently noninteracting parallel Rindler universes.

In this paper we discussed a new multiverse concept in an idealized setting and established certain technical properties of the first quantized wave function or classical field and its second quantization, in the extended Rindler spacetime. Although this spacetime is related to flat Minkowski spacetime by a simple coordinate transformation at the classical level, we showed that the presence of horizons in the Rindler coordinate system led to subtleties at the quantum level due to cuts in analytic (u, v) spacetime, and that this naturally implied the presence of a multiverse in the first and second quantized treatment of the field in such a spacetime. Analyticity of the field in the (u, v) coordinates guarantees that unavoidably $\varphi(u, v)$ takes unique values throughout the multiverse. We claim that similar multiverse properties are also shared by any spacetime that has horizons and/or singularities, such as the full spacetime of an eternal black hole [4] as well as the cosmological minisuperspace geometry (in field space) described in Appendix B and in more detail in [3]. The presence of the multiverse structure does not seem to be directly detectable by an observer in Rindler region I, or the analogous region I observer outside of a black hole, because, as we have already emphasized such an observer is incapable of directly detecting anything beyond the horizons of region I. Possible observable physical effects, that even observers in region I may notice as indirect consequences of a multiverse, could arise in cosmological or black hole phenomena. The possibility of transitions through gravitational singularities (see, e.g., [16,21–24]) may also include transitions in the multiverse. How such new mathematical properties of the field are relevant for some new physical phenomena is under investigation.

It may be worthwhile to emphasize how our multiverse for extended Rindler spacetime differs from ordinary Minkowski spacetime. Clearly they are quite different. A field in ordinary Minkowski spacetime has only the level-(0,0) field of our multiverse. Analytic continuation of the ordinary Minkowski plane-wave basis as in Eq. (16), $e^{-i\frac{E-k}{2}u}e^{-i\frac{E+k}{2}v}$, by $u \to ue^{\pm i2\pi}$ or $v \to ve^{\pm i2\pi}$, does not lead to any new analyticity results. This is because the Minkowski coordinate basis is adequate to describe the multiverse one level at a time and lacks the analyticity information that is available in the extended Rindler coordinate basis. An analogy to this is the Schwarzchild coordinate basis for a black hole, that describes only the region outside of the horizon, versus the Kruskal-Szekeres coordinate basis that provides the extension to the full eternal black hole spacetime. In a similar way, the extended Rindler coordinate basis captures the entire multiverse through its analyticity behavior. What could not be captured directly in the Minkowski basis is clarified in Sec. VI. Namely, the level-(n, m) field in the Minkowski basis in Eq. (49) is related by a very non-trivial canonical transformation to the level-(0,0) field. This canonical transformation is just the result of the nontrivial analytic continuation in the extended Rindler basis, resulting from $u \to ue^{i2\pi n}$ or $v \to ve^{i2\pi m}$ with integers n, m with the patterns given in detail in Eqs. (31), (33). Furthermore, as seen via the interlevel correlators that appear in Eqs. (82), (83), there is a wealth of information in our multiverse that is absent in ordinary Minkowski spacetime.

The notion and description of a multiverse that emerged in this paper is new and different than other multiverse notions that originated in the past from other considerations, such as the multiverse of the many worlds of quantum mechanics, the multiverse that arises from eternal inflation, or the multiverse that arises in the landscape of string theory. In particular, our multiverse contains many levels that are *predictably* connected to each other by the analyticity properties of the wave function. This predictable aspect is unlike other concepts of a multiverse in the literature. However, in a complete theory perhaps the different concepts of a multiverse could be connected to each other; see, e.g., [25,26] for some possible relations, which however does not address our new brand of multiverse. Note that in our case, analyticity connects the different universes and makes predictions of relations among them. In future investigations we will consider the physical significance of the ideas expressed in this paper in a complete realistic theory of fundamental physics (possibly in cosmology and/or black holes), including models that address the effects of quantum gravity, such as string theory. The analog of the quantum wave function of a particle is the string field. So, in a deeper investigation of the multiverse in the sense of the current paper may be possible in string field theory in which nontrivial backgrounds [27] and string-string interactions are included. This may be a context in which various notions of a multiverse, including our new one, may be connected to each other.

We have shown that the multiverse, in the quantum version of certain spacetimes, is an immutable structure of the wave function—there is no choice here because it directly follows from quantum mechanics. Our result, that was not known before, cannot be captured by any amount of analysis of *classical* general relativity. It is conceivable that indirect observational consequences of our findings could be analyzed through gravitational waves, since the fluctuations in such gravitational backgrounds, that are emitted as waves, may encapsulate the predicted multiverse structure already embedded in the quantum field.

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APPENDIX A: COMPUTATION OF CHARGE AND BOUNDARY FLUXES

In this Appendix we show the computation of q_1, $\partial_t q_1$ and q_2, $\partial_{-y} q_2$ whose results appear in Eqs. (72), (73) respectively. The remaining q_3, $\partial_{-t} q_3$ and q_4, $\partial_y q_4$ are obtained by simple substitution of variables as given just before Eqs. (75) and (76) respectively.

For region I the definitions of q_1, $\partial_t q_1$ are given in Eq. (70),

$$\begin{aligned} q_1(t) &= \int_0^\infty dy J_1^t = \int_0^\infty dy \frac{i}{2y}(\varphi_1^\dagger \partial_t \varphi_1 - \partial_t \varphi_1^\dagger \varphi_1) \\ \partial_t q_1 &= \int_0^\infty dy \partial_t J_1^t \\ &= \int_0^\infty dy (\partial_\mu J_1^\mu - \partial_y J_1^y) = -J_2^y(t,\infty) + J_2^y(t,0) \end{aligned} \tag{A1}$$

where the Klein-Gordon equation is used to set $\partial_\mu J_1^\mu = 0$, and then Stoke's theorem is applied to write the result in terms of the current $J^y(t, y)$ evaluated at the asymptotic boundaries. Here φ_1 that is given in Eq. (27) is written in terms of (t, y), and the y-component of the current at the boundaries is given by the following limits

$$\varphi_1(t,y) = \int_0^\infty d\omega\left[e^{-i\omega t}\left(a_{1-}(\omega)\frac{(2y)^{-i\frac{\omega}{2}}S_-(2\mu^2 y)}{\sqrt{4\pi\omega}} + a_{1+}(\omega)\frac{(2y)^{+i\frac{\omega}{2}}S_+(2\mu^2 y)}{\sqrt{4\pi\omega}}\right) + \mathrm{H.c.}\right]$$
$$J_1^y(t,\infty \quad \text{or} \quad 0) = \lim_{y\to\infty \text{ or } 0}(-i2y(\varphi_1^\dagger\partial_y\varphi_1 - \partial_y\varphi_1^\dagger\varphi_1)). \tag{A2}$$

where the expression for $J^y(t,y)$ follows from J^μ in Eq. (61) after using $\sqrt{-g} = 1$ and $g^{yy} = 2y$ for the Rindler spacetime.

To compute $q_1(t)$ one uses the orthonormality of the positive and negative frequency modes described in footnote 2. Then the integral in Eq. (A1) yields

$$q_1 = \int_0^\infty d\omega((a_{1-}^\dagger a_{1-} - b_{1-}^\dagger b_{1-}) + (a_{1+}^\dagger a_{1+} - b_{1+}^\dagger b_{1+})), \tag{A3}$$

as given in Eq. (72). This shows that $q_1(t)$ is time independent, so the charge is conserved $\partial_t q_1 = 0$ within region I at finite t. We will see that in general it is not conserved, $\partial_t q_1(t) \neq 0$, at the $t \to \pm\infty$ boundaries.

Next we compute the nontrivial fluxes $J_1^y(t,\infty/0)$ at the y-boundaries of region I. Consider at first $J_1^y(t,\infty)$ for the massive field $\mu^2 > 0$. We had argued in Eqs. (43), (44) that the horizon boundary conditions in Sec. V relate $a_{1\pm}(\omega)$ to each other by a definite phase, and that this implies also the correct physical asymptotic behavior, $\varphi_1(t, y\sim\infty) \to 0$. In this case the boundary current $J_I^y(t,\infty)$ vanishes asymptotically, and therefore the charge flow at the asymptotic boundary of region I vanishes for the massive field, i.e.,

$$\mu^2 > 0\colon\ J_1^y(t,\infty) = 0, \quad \text{at all } t, \quad \text{including } t = \pm\infty. \tag{A4}$$

This result is different for the massless field, $\mu^2 = 0$, since the asymptotic $\varphi_1(t,\infty)$ does not vanish in that case. However, due to masslessness, we have $\lim_{\mu\to 0} S_\mp(2\mu^2 y) = 1$, so the field $\varphi_1(t,y)$ in Eq. (A2) simplifies. The $2iy\partial_y$ derivatives that occur in $J_1^y(t,y)$ in Eq. (A2) are then easily computed by using $-2iy\partial_y(2y)^{\mp i\frac{\omega}{2}} = \mp\,\omega(2y)^{\mp i\frac{\omega}{2}}$, and we obtain the following double integral for $J_1^y(t,y)$

$$\int_0^\infty \frac{d\omega' d\omega e^{i(\omega'-\omega)t}}{\sqrt{4\pi\omega'}\sqrt{4\pi\omega}}\left[\begin{array}{l}(\omega+\omega')(-a_{1-}^\dagger(\omega')a_{1-}(\omega)(2y)^{i\frac{\omega'-\omega}{2}} + a_{1+}^\dagger(\omega')a_{1+}(\omega)(2y)^{-i\frac{\omega'-\omega}{2}}) + \cdots \\ +(\omega-\omega')(-a_{1+}^\dagger(\omega')a_{1-}(\omega)(2y)^{-i\frac{\omega'+\omega}{2}} + a_{1-}^\dagger(\omega')a_{1+}(\omega)(2y)^{i\frac{\omega'+\omega}{2}}) + \cdots\end{array}\right] \tag{A5}$$

where "$\cdots$" represent the Hermitian conjugate and mixed terms that are not shown. As $2y \to \infty$ these integrals are evaluated by using the steepest descent method because of the fast oscillating exponentials $(2y)^{\mp' i(\omega\pm\omega')/2}$. The leading contribution comes only from the neighborhood $\omega' \simeq \omega$ in the first line of (A5); then the double integral is approximated by

$$\lim_{y\to\infty}\int_0^\infty \frac{d\omega 2\omega}{4\pi\omega}\left[\begin{array}{l}-|a_{1-}(\omega)|^2\int_{-\infty}^\infty d\zeta e^{-\varepsilon|\zeta|}(e^t\sqrt{2y})^{i\xi} + \cdots \\ +|a_{1+}(\omega)|^2\int_{-\infty}^\infty d\zeta e^{-\varepsilon|\zeta|}(e^{-t}\sqrt{2y})^{-i\xi} + \cdots\end{array}\right] \tag{A6}$$

where the factor $e^{-\varepsilon|\zeta|}$ is inserted to insure the $\int_{-\infty}^\infty d\xi$ integrations are limited to the neighborhood of $\zeta = \omega' - \omega \simeq 0$. The ζ integrals produce smeared delta functions $\delta_\varepsilon(\ln z)$,

$$\int_{-\infty}^\infty d\zeta e^{-\varepsilon|\zeta|}z^{\pm i\xi} = 2\pi\frac{\varepsilon/\pi}{(\ln z)^2 + \varepsilon^2} \equiv 2\pi\delta_\varepsilon(\ln z). \tag{A7}$$

The result is

$$\mu^2 = 0\colon\ -J_1^y(t, y\sim\infty) = \left(\begin{array}{l}+(\lim\limits_{v\to-\infty}\delta_\varepsilon(\ln u))\int_0^\infty d\omega|a_{1-}(\omega)|^2 + \cdots \\ -(\lim\limits_{u\to\infty}\delta_\varepsilon(\ln|v|))\int_0^\infty d\omega|a_{1+}(\omega)|^2 + \cdots\end{array}\right). \tag{A8}$$

where we have used,

$$\lim_{y\to\infty}\delta_\varepsilon(\ln(\sqrt{2y}e^t)) = \lim_{y\to\infty}\delta_\varepsilon(t+\infty) = \lim_{y\to\infty}\delta_\varepsilon(\ln(u)) = \lim_{v\to-\infty}\delta_\varepsilon(\ln u), \tag{A9}$$

and similarly for the second term. This shows that there are nonvanishing asymptotic contributions proportional to $|a_{1-}(\omega)|^2$ when $v \to -\infty$ and u is finite, as well as $|a_{1+}(\omega)|^2$ when $u \to \infty$ and v is finite. These contributions are at the $\mathcal{I}^{\mp}$ boundaries in a Penrose diagram for region I.

To compute $J_1^y(t,0)$ for the massive or massless field, only the $y=0$ neighborhood of the field $\varphi_1(t, y\sim 0)$ is sufficient, which means $S_{\mp}(2\mu^2 y)$ in (A2) may be approximated by $S_{\mp}(0)=1$. Then $J_1^y(t, y\sim 0)$ takes the same form as Eq. (A5) except for setting $2y\sim 0$. The fast oscillations argument is valid again, and the integral is evaluated as

$$\mu^2 \geq 0\colon\ J_1^y(t, y\sim 0) = \begin{bmatrix} -(\lim_{v\to 0}\delta_\varepsilon(\ln u))\int_0^\infty d\omega |a_{1-}(\omega)|^2 + \cdots \\ +(\lim_{u\to 0}\delta_\varepsilon(\ln|v|))\int_0^\infty d\omega |a_{1+}(\omega)|^2 + \cdots \end{bmatrix}. \tag{A10}$$

where we have used,

$$\lim_{y\to 0}\delta_\varepsilon(\ln(\sqrt{2y}e^t)) = \lim_{y\to 0}\delta_\varepsilon(t-\infty) = \lim_{v\to 0}\delta_\varepsilon(\ln u), \quad \text{etc.} \tag{A11}$$

This shows that there are non-vanishing contributions when $v\to 0$ and u is finite as well as when $u\to 0$ and v is finite. These are the future and past horizons in region I.

Altogether, from Eqs. (71), (A4), (A8), (A10) we have

$$\begin{aligned} \partial_t q_1(t)|_{\mu^2=0} &= \int_0^\infty d\omega[-\lim_{v\to 0}\delta_\varepsilon(\ln u)|a_{1-}(\omega)|^2 + \lim_{u\to 0}\delta_\varepsilon(\ln|v|)|a_{1+}(\omega)|^2 + \cdots] \\ \partial_t q_1(t)|_{\mu^2\neq 0} &= \int_0^\infty d\omega \begin{bmatrix} -\lim_{v\to 0}\delta_\varepsilon(\ln u)|a_{1-}(\omega)|^2 + \lim_{u\to 0}\delta_\varepsilon(\ln|v|)|a_{1+}(\omega)|^2 + \cdots \\ +\lim_{v\to -\infty}\delta_\varepsilon(\ln|u|)|a_{1-}(\omega)|^2 - \lim_{u\to\infty}\delta_\varepsilon(\ln v)|a_{1+}(\omega)|^2 + \cdots \end{bmatrix} \end{aligned} \tag{A12}$$

After including the contributions "$\cdots$" from the Hermitian conjugate terms in φ_1, the results are given in Eq. (72).

We now turn to regions II and IV. Since space/time are interchaged in regions II and IV, we define the conserved charge and its derivative as an integral over t at fixed y as explained after Eqs. (70), (71)

$$\begin{aligned} q_2(y) &= -\int_{-\infty}^\infty dt J_2^y = -\int_{-\infty}^\infty dt(-2yi)(\varphi_2^\dagger\partial_y\varphi_2 - \partial_y\varphi_2^\dagger\varphi_2) \\ q_2(y) &= \int_{-\infty}^\infty dt\partial_{-y}(-J_2^y) = \int_{-\infty}^\infty dt(\partial_\mu J_2^\mu - \partial_t J_2^t) = -J_2^t(\infty, y) + J_2^t(-\infty, y) \end{aligned} \tag{A13}$$

The field $\varphi_2(u,v)$ in Eq. (27) is now rewritten in the (t,y) coordinates

$$\varphi_2(t,y) = \int_0^\infty d\omega\left[e^{-i\omega t}\left(a_{1-}(\omega)\frac{(2y)^{-i\frac{\omega}{2}}S_-(2\mu^2 y)}{\sqrt{4\pi\omega}} + b_{3+}^\dagger(\omega)\frac{(2y)^{+i\frac{\omega}{2}}S_+(2\mu^2 y)}{\sqrt{4\pi\omega}}\right) + \text{H.c.}\right]. \tag{A14}$$

Apply this first to the massless case to compute $q_2(y)$ when $S_\pm(0)=1$. Then, using $i2y\partial_y(2y)^{-i\frac{\omega}{2}} = \pm\omega(2y)^{-i\frac{\omega}{2}}$, gives

$$q_2(y) = \int_0^\infty d\omega((a_{1-}^\dagger(\omega)a_{1-}(\omega) - b_{3+}^\dagger(\omega)b_{3+}(\omega)) + \cdots) \tag{A15}$$

where "$\cdots$" represents the contribution from the H.c. part of the field φ_2 above. Note that the signs of the charges are consistent with the definition of particle/antiparticle as represented by a/b symbols respectively.

Now compute the nontrivial fluxes $J_2^t(\pm\infty, y)$ at the $t\to\pm\infty$ boundaries of region II. For the massless case we have

$$\begin{aligned} J_2^t(\infty, y) &= \lim_{t\to\infty}\left(\frac{i}{2y}(\varphi_2^\dagger\partial_t\varphi_2 - \partial_t\varphi_2^\dagger\varphi_2)\right) \\ &= \lim_{t\to\infty}\int_0^\infty\int_0^\infty \frac{d\omega_1 d\omega_2(\omega_1+\omega_2)}{2y\sqrt{4\pi\omega_1}\sqrt{4\pi\omega_2}}\begin{pmatrix} e^{i(\omega_1-\omega_2)}[a_{1-}^\dagger(\omega_1)(-2y)^{i\frac{\omega_1}{2}} + b_{3+}(\omega_1)(-2y)^{-i\frac{\omega_1}{2}}] \\ \times[a_{1-}(\omega_2)(-2y)^{-i\frac{\omega_2}{2}} + b_{3+}^\dagger(\omega_2)(-2y)^{+i\frac{\omega_2}{2}}] \end{pmatrix} + \cdots \end{aligned} \tag{A16}$$

The "$+\cdots$" represents the Hermitian conjugate and mixed terms that are not shown. Due to wild oscillations, at large t only the neighborhood of $\omega_1 \sim \omega_2$ can contribute to this integral. Furthermore, because the support of $\varphi_2(t,y)$ at large t is either at large $|2y| \to \infty$, or small $|2y| \to 0$, terms with $(2y)^{\pm i(\omega_1+\omega_2)/2}$ in this integral are also negligible since they too vanish at either limit $|2y| \to (0 \text{ or } \infty)$ due to wild oscillations. Therefore the expression above is simplified by keeping the leading terms and using the same arguments that followed Eq. (A5)

$$J_2^t(\infty,y) = \int_0^\infty \frac{d\omega 2\omega}{4\pi\omega \times (-|2y|)} \begin{bmatrix} |a_{1-}(\omega)|^2 \lim_{t\to\infty} \int_{-\infty}^{\infty} d\zeta e^{-\varepsilon|\zeta|} (e^t \sqrt{|2y|})^{i\xi} + \cdots \\ +|b_{3+}(\omega)|^2 \lim_{t\to\infty} \int_{-\infty}^{\infty} d\zeta e^{-\varepsilon|\zeta|} (e^{-t} \sqrt{|2y|})^{-i\xi} + \cdots \end{bmatrix}, \tag{A17}$$

where we recall that y is negative in region II to rewrite everything in terms of $|2y|$. Using the definition of the smeared delta function in Eqs. (A7)–(A11) we evaluate the result as follows

$$\begin{aligned} \lim_{t\to\infty} \int_{-\infty}^{\infty} d\zeta e^{-\varepsilon|\zeta|} \frac{(e^t\sqrt{|2y|})^{i\xi}}{-|2y|} &= \lim_{t\to\infty} \frac{\delta_\varepsilon \ln(e^t\sqrt{|2y|})}{-|2y|} \simeq \lim_{t\to\infty} \frac{\delta_\varepsilon(|y|)}{-1} = -\lim_{v\to 0} \delta_\varepsilon(\ln u), \\ \lim_{t\to\infty} \int_{-\infty}^{\infty} d\zeta e^{-\varepsilon|\zeta|} \frac{(e^{-t}\sqrt{|2y|})^{i\xi}}{-|2y|} &= \lim_{t\to\infty} \frac{\delta_\varepsilon \ln(e^{-t}\sqrt{|2y|})}{-|2y|} \simeq \lim_{t\to\infty} \frac{\delta_\varepsilon(|y|-\infty)}{-1} = -\lim_{u\to\infty} \delta_\varepsilon(\ln v). \end{aligned} \tag{A18}$$

Hence we obtain

$$J_2^t(\infty,y) = -\lim_{v\to 0} \delta_\varepsilon(\ln u) \int_0^\infty d\omega |a_{1-}(\omega)|^2 - \lim_{u\to\infty} \delta_\varepsilon(\ln v) \int_0^\infty d\omega |b_{3+}(\omega)|^2 + \cdots \tag{A19}$$

The evaluation of $J_2^t(-\infty,y)$ at $t \to -\infty$ proceeds in the same way, leading to

$$J_2^t(-\infty,y) = -\lim_{v\to\infty} \delta_\varepsilon(\ln u) \int_0^\infty d\omega |a_{1-}(\omega)|^2 - \lim_{u\to 0} \delta_\varepsilon(\ln v) \int_0^\infty d\omega |b_{3+}(\omega)|^2 + \cdots \tag{A20}$$

The combined result gives the rate of change of the charge at the boundaries of region II for the massless particle

$$\begin{aligned} \partial_{-y} q_2(y)|_{\mu^2=0} &= -J_2^t(\infty,y) + J_2^t(-\infty,y) \\ &= \int_0^\infty d\omega \begin{bmatrix} (\lim_{v\to 0} \delta_\varepsilon(\ln u) - \lim_{v\to\infty} \delta_\varepsilon(\ln u))(|a_{1-}(\omega)|^2 + \cdots) \\ (\lim_{u\to 0} \delta_\varepsilon(\ln v) - \lim_{u\to\infty} \delta_\varepsilon(\ln v))(-|b_{3+}(\omega)|^2 + \cdots) \end{bmatrix} \end{aligned} \tag{A21}$$

For the massive particle, the presence of $S_\pm(2\mu^2 y)$ in $\varphi_2(t,y)$ in Eq. (A14) complicates the calculation somewhat. The integral for $q_2(y)$ in Eq. (A13) is performed by using the properties of Bessel functions and the result is just like the massless case given in Eq. (A15). The computation of $J_2^t(\pm\infty,y)$ is more complicated because as $|2y| \to \infty$ the nontrivial asymptotic behavior of $S_\pm(2\mu^2 y)$ must be taken into account, although for $|2y| \to 0$ one still has $S_\pm(0) = 1$, as in the massless case. Hence compared to the massless case only the terms involving the $u \to \infty$ or $v \to \infty$ boundaries are altered while the terms at the horizons are the same. The result is

$$\begin{aligned} \partial_{-y} q_2(y)|_{\mu^2\neq 0} &= -J_2^t(\infty,y) + J_2^t(-\infty,y) \\ &= \int_0^\infty d\omega \begin{bmatrix} \lim_{v\to 0} \delta_\varepsilon(\ln u)(|a_{1-}(\omega)|^2 + \cdots) \\ +\lim_{u\to 0} \delta_\varepsilon(\ln v)(-|b_{3+}(\omega)|^2 + \cdots) \\ -\lim_{v\to\infty} \delta_\varepsilon(\ln u) \frac{|a_{1-}^\dagger + e^{-\pi\omega}(\mu^2)^{-i\omega} e^{i2\theta} b_{3+}^\dagger|^2 + \cdots}{1-e^{-2\pi\omega}} \\ -\lim_{u\to\infty} \delta_\varepsilon(\ln v) \frac{-|b_{3+}^\dagger + e^{-\pi\omega}(\mu^2)^{i\omega} e^{-i2\theta} a_{1-}|^2 + \cdots}{1-e^{-2\pi\omega}} \end{bmatrix} \end{aligned} \tag{A22}$$

After including the contributions "$\cdots$" from the Hermitian conjugate terms in φ_2, the results are given in Eq. (73).

APPENDIX B: MINISUPERSPACE AND COSMOLOGICAL MULTIVERSE

The Lagrangian for the geodesically complete version of the standard model (SM) coupled to general relativity (GR) is given in [2],

$$\mathcal{L}(x) = \sqrt{-g}\begin{pmatrix} L_{\mathrm{SM}}(A_\mu^{\gamma,W,Z,g},\psi_{q,l},\nu_R,\chi) \\ +g^{\mu\nu}(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - D_\mu H^\dagger D_\nu H) \\ -(\frac{\lambda}{4}(H^\dagger H-\omega^2\phi^2)^2+\frac{\lambda'}{4}\phi^4) \\ +\frac{1}{12}(\phi^2-2H^\dagger H)R(g) \end{pmatrix}. \quad \text{(B1)}$$

This action is invariant under local scale transformations (Weyl symmetry) and has a noteworthy unique coupling of conformal scalars to gravity of the form that appears in the last line above. The relative minus sign in $(\phi^2-2H^\dagger H)R(g)$ is mandatory so that a positive gravitational constant G_N can be generated by Weyl gauge fixing, $\frac{1}{12}(\phi^2-2H^\dagger H)(x^\mu)\to(16\pi G_N)^{-1}$ at least in some patch of spacetime x^μ, but the relative sign is also essential for geodesic completeness as outlined below. An attractive feature of the Weyl invariant formulation is that the universe-filling dimensionful constants G_N, dark energy Λ and the electroweak scale v_{EW}, are also generated simultaneously with G_N from the same source [2]. The uniqueness and completeness of this form for a Weyl invariant and geodesically complete approach to the SM + GR was discussed in [2], where its emergence from a deeper gauge symmetry perspective of 2T-physics [28] is also outlined (for a summary see [29]). See also [2,30,31] for the occurrence of the same structure in a supergravity setting.

In this Appendix, and with more detail in [3], we reexamine the minisuperspace derived from this theory for cosmological applications. This was discussed in a series of papers during 2009-2014 in collaborations between one of the authors of the current paper and C. H. Chen, Paul Steinhardt, and Neil Turok, as summarized in [29]. The mini-superspace consists of the cosmologically most relevant homogeneous (only time dependent) degrees of freedom, including scalar fields $(\phi(x^0),h(x^0))$, where h represents[8] the Higgs doublet in a unitary gauge, $H=(0,h/\sqrt{2})$, and the cosmological metric, $ds^2=a^2(x^0)(-(dx^0)^2e^2(x^0)+\gamma_{ij}(x^0,\vec{x})dx^i dx^j)$, where a is the cosmological scale factor, e is the lapse function (redefined up the factor a, i.e., $N=ae$) and $\gamma_{ij}(x^0,\vec{x})$ may include spacial curvature and anisotropies. Moreover, the matter energy-momentum tensor T_{00} includes the radiation density, $\rho_r(x^0)/a^4(x^0)$, to represent an average "fluid" behavior of all conformally invariant relativistic matter (photons, gluons, quarks, leptons, neutrinos, etc.).

The Weyl invariant form of the minisuperspace action was given in [29,32]. Here we are interested in its Weyl-fixed form in the so-called γ-gauge

$$S_{\mathrm{mini}} = \int d\tau\begin{Bmatrix} \frac{1}{2e}[-(\partial_\tau\phi_\gamma)^2+(\partial_\tau h_\gamma)^2+(\phi_\gamma^2-h_\gamma^2)((\partial_\tau\alpha_1)^2+(\partial_\tau\alpha_2)^2)] \\ -e\left[\phi_\gamma^4 f\left(\frac{h_\gamma}{\phi_\gamma}\right)-\frac{1}{2}(\phi_\gamma^2-h_\gamma^2)v(\alpha_1,\alpha_2)+\rho_r\right] \end{Bmatrix}, \quad \text{(B2)}$$

where $\alpha_{1,2}(\tau)$ are anisotropy degrees of freedom with the anisotropy potential $v(\alpha_1,\alpha_2)$ given in [29,32]; $\tau\equiv x^0$ is called the "conformal time" when the $e(\tau)=1$ gauge is chosen by fixing the τ-reparametrization symmetry of S_{mini}. The Weyl-symmetric version of S_{mini} starts out with three Weyl-dependent degrees of freedom, namely (a,ϕ,h), that transform according to $\phi\to\Omega\phi$, $h\to\Omega h$, $a\to\Omega^{-1}a$. The $e,\alpha_{1,2}$ and ρ_r degrees of freedom are Weyl invariant. Furthermore, $a\phi$, ah, h/ϕ and arbitrary functions of these, are also Weyl invariant.

Physics depends only on Weyl invariants, but Weyl gauges that simplify computations or clarify the physics are welcome. There is an interesting interplay of four Weyl gauge choices: E-gauge, c-gauge, γ-gauge [29,32] and string gauge or s-gauge [27]. The action above is in the γ-gauge which is defined by freezing the scale factor for all conformal times τ, and labelling the gauge dependent quantities with γ when they are in the γ-gauge, namely $a_\gamma(\tau)=1$, and dynamical $\phi_\gamma(\tau),h_\gamma(\tau)$. So, (ϕ_γ,h_γ) are gauge invariant since they can be written as, $\phi_\gamma=a_\gamma\phi_\gamma=a\phi$ and $h_\gamma=a_\gamma h_\gamma=ah$, where $(a\phi,ah)$ may be evaluated in any other gauge (see below for the case of the E-gauge). The γ-gauge is most useful to grasp the geodesic completeness and transitions through singularities (see, e.g., [21–23,29]). Note the light-cone-type structure in (ϕ_γ,h_γ) field space in Fig. 10 where, in accordance with the signatures in Eq. (B2), the fields $\phi_\gamma(\tau)$ ($h_\gamma(\tau)$) play the role of timelike (spacelike) coordinates [just like $x^\mu(\tau)$ in Eq. (1)]. We may define $u\equiv\phi_\gamma+h_\gamma$ and $v\equiv\phi_\gamma-h_\gamma$ analogous to lightcone coordinates. The quantity $z(\tau)=(\phi_\gamma^2(\tau)-h_\gamma^2(\tau))=u(\tau)v(\tau)$ is positive in regions II&IV and negative in regions I&III, while the blue and orange solid lines, where either u or v vanish, indicate where $z(\tau)$ vanishes. The hyperbolas in region II labeled by

[8] In addition to the Higgs boson there may be more scalar fields [2]. In that case the h in minisuperspace represents a combination of all the scalars. The most economical cosmological scenario is to have just the Higgs, as this seems to be not impossible [22].

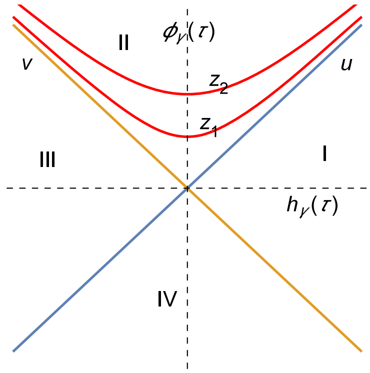

FIG. 10. The $(\phi_\gamma(\tau), h_\gamma(\tau))$ field space.

$0 < z_1 < z_2 < \infty$ correspond to the curves $(\phi_\gamma, h_\gamma)|_{z\text{fixed}}$ for two fixed values of the field $z(\tau)$; imagine similar hyperbolas in all regions I-IV. The analogy to the extended Rindler space in Fig. 1 is already apparent. We will soon explain more precisely the physical relation of the (ϕ_γ, h_γ) *field-space* to the mathematical structure of the extended Rindler spacetime discussed in the main body of the paper.

The E-gauge, which puts the full action (B1) directly in the Einstein frame, is useful for interpreting the physics because traditionally physics is discussed in the E-frame. It is defined by freezing the Weyl invariant, $\int \frac{1}{12}\sqrt{-g}(\phi^2 - 2H^\dagger H)R(g)$, to the Einstein-Hilbert form, $\int (\pm 16\pi G_N)^{-1}\sqrt{-g_E}R(g_E)$, where the Weyl-fixed fields are labeled with an extra letter "E", such as $g^E_{\mu\nu}, \phi_E, H_E$ to indicate that they are in the E-gauge. The overall sign, $\pm 1 = \mathrm{sign}(\phi^2(x) - 2H^\dagger(x)H(x))$, implies that there are patches of field space $(\phi, h)^\pm$, and corresponding regions of spacetimes x^μ, where the E-gauge condition is satisfied [29]. The $\pm$ signs, which imply a passage through zero or infinity, are Weyl-invariant because the sign of $(\phi^2(x) - 2H^\dagger(x)H(x))$ cannot be changed by Weyl transformations. One may ask if a universe can be complete in a patch with only the $+$ sign. The answer is no, because the $\mathrm{sign}(\phi^2 - 2H^\dagger H)$ does flip dynamically multiple times very generically as a function of x^μ, as was established with an extensive study of analytic solutions in [29,32]. The dynamics show that the field solutions, and similarly the geodesics, are stopped artificially if only one sign of $(\phi^2 - 2H^\dagger H)$ is imposed by hand. Hence, quite clearly the traditional SM + GR, that artificially keeps only the positive sign, is a geodesically incomplete theory. When both signs are kept to complete the E-gauge field solutions and geodesics, the suddenness of the sign flip, is just an artifact of the E-gauge. By contrast, the sign change occurs smoothly in other gauges, such as the γ-gauge or the c-gauge. We see that, as compared to the traditional GR + SM, the Weyl invariant GR + SM in (B1) describes a larger field space for the same degrees of freedom, as well as a corresponding larger spacetime. This is how geodesic completeness is achieved.

Accordingly, in the geodesically complete E-gauge, that freezes $\frac{1}{12}(\phi_E^2(\tau) - h_E^2(\tau)) = (\pm 16\pi G_N)^{-1}$, the minisuperspace degrees of freedom include the two fields $(a_E(\tau), \sigma_E(\tau))$ instead of the three fields (a, ϕ, h). Here $a_E(\tau)$ is the scale factor and the scalar $\sigma_E(\tau)$ is basically a rewriting of the Higgs in the E-gauge. Naturally, (a_E, σ_E) are related to the γ-gauge dynamical degrees of freedom (ϕ_γ, h_γ) by Weyl transformations as given in [29,32]. Consider the Weyl invariants $a^2(\phi^2 - h^2)$ and $\ln(\frac{\phi-h}{\phi+h})$; by evaluating them in the E-gauge and γ-gauge and equating them to each other we find

$$\frac{12a_E^2(\tau)}{16\pi G_N} = |\phi_\gamma^2(\tau) - h_\gamma^2(\tau)| = |z(\tau)|,$$
$$\sqrt{\frac{12}{16\pi G_N}}\sigma_E(\tau) = \frac{1}{2}\ln\left|\frac{\phi_\gamma(\tau) + h_\gamma(\tau)}{\phi_\gamma(\tau) - h_\gamma(\tau)}\right|. \quad \text{(B3)}$$

This relation is the exact analog of the Rindler-Minkowski relation in Eq. (3); it shows that $(\phi_\gamma \pm h_\gamma)$ or (u, v) are Minkowski-like global coordinates in Fig. 10, while (σ_E, a_E^2) are non-global Rindler-like coordinates similar to (t, y) that reparametrize the four different patches I-IV. Indeed there is a precise correspondence to the Minkowski and Rindler coordinates used in the rest of this paper; the translation dictionary is

$$\frac{12a_E^2\mathrm{sign}(\phi^2 - h^2)}{16\pi G_N} = z \leftrightarrow -2y, \qquad \sqrt{\frac{12}{16\pi G_N}}\sigma_E \leftrightarrow t,$$
$$(\phi_\gamma + h_\gamma) = u \leftrightarrow (x^0 + x^1),$$
$$(\phi_\gamma - h_\gamma) = v \leftrightarrow (x^0 - x^1). \quad \text{(B4)}$$

Then we can insert this information in Eq. (5) to establish the E-gauge to γ-gauge relations for every region I-IV in Fig. 10 in exact correspondence to Fig. 1. With this, we now have a precise Rindler $\leftrightarrow$ Minkowski type map for our cosmological degrees of freedom (u, v) versus (σ, z). This shows that the cosmological geometry in field space has the same properties as ordinary extended Rindler spacetime discussed in this paper, but now there are also interactions that make it much more interesting.

The E-gauge to/from γ-gauge map described above is helpful to transform the smooth γ-gauge solutions [21,29,32] to the geodesically complete but singular E-gauge solutions and vice-versa. It is then understood that at the instant $z(\tau) = (\phi_\gamma^2(\tau) - h_\gamma^2(\tau)) = u(\tau)v(\tau)$ vanishes in the γ-gauge, there is a scalar-curvature singularity in the E-gauge where $a_E^2(\tau) = 0$ at the same τ (although not so in γ-gauge where $a_\gamma(\tau) = 1$ for all τ). Hence in Fig. 10 the "horizons" at $u = 0$ or $v = 0$ correspond to big-crunch or big-bang instants as interpreted in the E-frame. Also during the periods of τ when the quantity $z(\tau) = (\phi_\gamma^2(\tau) - h_\gamma^2(\tau)) = u(\tau)v(\tau)$ is positive (negative) in the γ-gauge, the $\mathrm{sign}(\phi^2(\tau) - h^2(\tau))$ in any Weyl gauge, including in the E-gauge $(\phi_E^2(\tau)-h_E^2(\tau))=(\pm 16\pi G_N)^{-1}$, must be the same sign as $\mathrm{sign}(\phi_\gamma^2(\tau)-h_\gamma^2(\tau))$, since Weyl transformations cannot change it. Therefore, in regions II&IV (versus I&III) in Fig. 10, gravity is an attractive (repulsive) force as interpreted in the E-frame ($+G_N$ versus $-G_N$). The constants $z_{1,2}$ that label the hyperbolas in region II correspond to two fixed values of the scale factor at two instances $z_{1,2}\sim a_E^2(\tau_{1,2})$. So the successive hyperbolas in region II describe the expanding universe as τ changes, while similar hyperbolas in region IV describe a contracting universe in a region of ordinary gravity (+sign in E-gauge). By contrast, regions I&III are antigravity regions that are unavoidably probed by geodesically complete generic cosmological solutions as shown in [21,29,32], as well as by the quantum wave function of minisuperspace. Therefore, all four regions are required in a *geodesically complete theory* of SM + GR.

We are now ready for the connection of the minisuperspace in S_{mini} with the multiverse ideas discussed in the current paper. The dynamics of the cosmological fields in S_{mini} in Eq. (B2) may be compared to the dynamics of a "particle" on the worldline parametrized by τ [like Eq. (1)]. The target spacetime is four dimensional, $X^\mu\sim(\phi_\gamma,h_\gamma,\alpha_1,\alpha_2)$; the "particle" (i.e., the universe) moves in a background gravitational field with metric

$$\begin{aligned} ds^2 &= -d\phi_\gamma^2+dh_\gamma^2+(\phi_\gamma^2-h_\gamma^2)(d\alpha_1^2+d\alpha_2^2)\\ &= -du\,dv+uv(d\alpha_1^2+d\alpha_2^2)\\ &= -\frac{1}{4z}dz^2+z(d\sigma^2+d\alpha_1^2+d\alpha_2^2). \end{aligned} \tag{B5}$$

Note this is a conformally flat metric in field space. The scalar curvature is $R=6(\phi_\gamma^2-h_\gamma^2)^{-1}$. There is also a potential energy,

$$\begin{aligned} \tilde V &= \left[\phi_\gamma^4 f(h_\gamma/\phi_\gamma)-\frac{1}{2}(\phi_\gamma^2-h_\gamma^2)v(\alpha_1,\alpha_2)+\rho_r\right]\\ &= \left[z^2v(\sigma)-\frac{1}{2}zv(\alpha_1,\alpha_2)+\rho_r\right], \end{aligned} \tag{B6}$$

where a constant $\rho_r>0$ plays the role of "mass^2", thus generalizing Eq. (1) with additional interactions. Note that z (equivalently the scale factor a_E^2) plays the role of Rindler time in the gravity regions II&IV[9] where $z>0$. In the antigravity regions I&III, where $z<0$, the overall sign of the metric seems to be wrong, but this is simply equivalent to replacing G_N by $-G_N$ in the Einstein-Hilbert Lagrangian, so the meaning of the overall sign is physically interpreted as being in the gravity versus antigravity patches of the E-gauge. See [16,24] for further applications and interpretations of this overall sign switch of the metric in the E-gauge.

The quantum wave function satisfies the Wheeler-deWitt equation (WdWe) that is derived from S_{mini} in Eq. (B2) just like Eq. (2).[10] In either the Minkowski-like $(\phi_\gamma,h_\gamma)\leftrightarrow(u,v)$ or the Rindler-like (σ,z) coordinate systems, the WdWe was constructed and analyzed in [24], where the physical meaning of an antigravity region behind cosmological singularities, as interpreted by observers in the gravity regions, and the related issues of unitarity (no problem), were discussed. Explicitly, the WdWe written in both coordinate systems is given by

$$\begin{gathered} \begin{pmatrix} \frac{1}{2}(\partial_{\phi_\gamma}^2-\partial_{h_\gamma}^2)-\frac{1}{2(\phi_\gamma^2-h_\gamma^2)}(\partial_{\alpha_1}^2+\partial_{\alpha_2}^2)+\rho_r\\ +\frac{1}{2}\phi_\gamma^4 f\left(\frac{h_\gamma}{\phi_\gamma}\right)-\frac{1}{2}(\phi_\gamma^2-h_\gamma^2)v(\alpha_1,\alpha_2)\end{pmatrix}\Psi(\phi_\gamma,h_\gamma,\alpha_1,\alpha_2)=0,\\ \begin{pmatrix}\partial_z^2+\frac{1}{4z^2}(-\partial_{\alpha_1}^2-\partial_{\alpha_2}^2-\partial_\sigma^2+1)\\ +\frac{z}{2}v(\sigma)-\frac{1}{4}v(\alpha_1,\alpha_2)+\frac{\rho_r}{2z}\end{pmatrix}(z^{1/2}\Psi(z,\sigma,\alpha_1,\alpha_2))=0. \end{gathered} \tag{B7}$$

Close to the singularity in the E-frame we have, $a_E^2\sim z\sim(\phi_\gamma^2-h_\gamma^2)\sim 0$, which is equivalent to being close to the horizons in Fig. 10. In that neighborhood, assuming that the terms $(\frac{z}{2}v(\sigma)-\frac{1}{4}v(\alpha_1,\alpha_2))$ can be neglected compared to the dominant and subdominant z^{-2},z^{-1} terms, the wave function may be determined from the approximate equation

$$\begin{gathered} \left(\partial_z^2+\frac{1}{4z^2}(-\partial_{\alpha_1}^2-\partial_{\alpha_2}^2-\partial_\sigma^2+1)+\frac{\rho_r}{2z}\right)(z^{1/2}\Psi(z,\sigma,\alpha_1,\alpha_2))=0,\quad \text{or}\\ \left(\partial_z^2+\frac{1}{4z^2}(p_1^2+p_2^2+p_3^2+1)+\frac{\rho_r}{2z}\right)(z^{1/2}\psi_p(z))=0, \end{gathered} \tag{B8}$$

[9]Compare a similar timelike role of $2y<0$ in Rindler regions II&IV that was explained following Eq. (70). This played a crucial role in the treatment and interpretation of Eqs. (70), (71).

[10]The ordering ambiguity of canonical variables allows an additional term in the Laplacian, i.e., instead of ∇^2 consider $(\nabla^2-\xi R)$ where R is the curvature of the metric in field space. In the following equations taken from [24] the conformally exact choice $\xi=1/6$ was made, and then the equation was simplified by rescaling the wave function Φ with a factor, $\Psi=(\phi_\gamma^2-h_\gamma^2)^{1/2}\Phi$ to simplify it to the form of Eq. (B7).

where the second equation applies to solutions of separable form, $\Psi(z,\sigma,\alpha_1,\alpha_2) \sim e^{-i(p_1\alpha_1+p_2\alpha_2+p_3\sigma)}\psi_p(z)$. The general wave packet has a form analogous to the $\varphi_{1,2,3,4}$ of Eq. (27) in various regions I-IV, and continuity across horizons is required. For example, for region I, the general solution is

$$\Psi_1(z,\sigma,\alpha_1,\alpha_2) = \sum_{\pm}\int d^3p[a(\vec{p})e^{-i(p_1\alpha_1+p_2\alpha_2+p_3\sigma)}\psi_p^{\pm}(z) + hc]. \tag{B9}$$

where $\psi_p^{\pm}(z)$ are the two independent solutions of the simplified equation in the single variable z. The exact solutions are known in this case (see below), but it is useful to first intuitively understand their physical behavior in the union of the four regions by comparing the $\partial_z^2 + \cdots$ equation to a nonrelativistic Schrödinger equation, $(-\partial_z^2 + V(z))\psi_0(z) = 0$, with a potential energy, $V(z) = -\frac{1}{4z^2}(p_1^2+p_2^2+p_3^2+1) - \frac{\rho_r}{2z}$, and a wave function $\psi_0(z) \equiv z^{1/2}\psi_p(z)$ for the 0 eigenvalue. The plot of the potential $V(z)$ is given in Fig. 11 The physical solution for $\psi_0(z)$, with correct boundary conditions, can be described intuitively as a wave packet approaching from the region $z > 0$ (a contracting universe in gravity region IV in Fig. 10), passing through $z = 0$ (a cosmological crunch) and entering the antigravity region where $z < 0$, then necessarily reflecting from the barrier (that forms due to radiation $\rho_r > 0$) and unable to tunnel deep into negative values of z (hence, spending little time in the antigravity region I or III in Fig. 10), then passing through $z = 0$ again (a cosmological big bang) and moving on to the positive region $z > 0$ (an expanding universe in gravity region II in Fig. 10). Thus, the exact wave function for the universe, which consists of $\Psi_{1,2,3,4}(z,\sigma,\alpha_1,\alpha_2)$ as described above, should have appropriate boundary conditions that restrict the coefficients $a_{1\pm}$ etc. to fit this physical behavior.

The exact analytic solution for the wave function $z^{1/2}\psi_p(z)$ confirms this expected behavior [3]. It should be emphasized that this quantum behavior of a general wave packet is in complete agreement with the classical solution displayed in [21] that featured an attractor behavior for a cosmological bounce consisting of Crunch-Bang transition with an antigravity region in between. As should be expected, due to the fuzziness introduced by quantum mechanics, the passage through the singularity in the E-frame at $a_E = 0$, is much softer in the quantum version as compared to the classical version in [21]. This transition

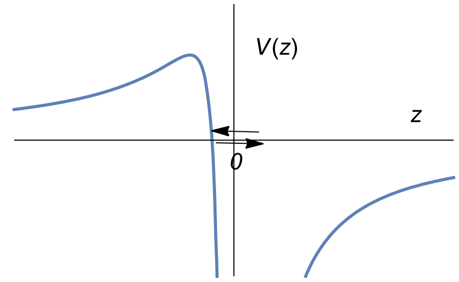

FIG. 11. Cosmological crunch and bang with antigravity in between.

was managed in [21] by using Weyl symmetry, while in the quantum case here, it amounts to the continuity of the wave function at the horizons just as discussed for the $\varphi_{1,2,3,4}$ in Sec. III.

Note that Fig. 11 is the same as Fig. 8 after replacing $z = -2y$, and the effective potential $V(z)$ is the same as $V_{\text{eff}}(y)$ in Eq. (45) after renaming the parameters, $p_1^2 + p_2^2 + p_3^2 = \omega^2$ and $\rho_r = \mu^2/2$. Therefore, the analytic solutions for the geodesically complete cosmological wave function $z^{1/2}\psi_p(z)$ have exactly the same analyticity behavior as the Rindler field $\varphi_{1,2,3,4}(u,v)$ given in Eq. (27). The physical boundary conditions (dying off wave function in asymptotic antigravity regions I&III) are reproduced by the horizon boundary conditions (35), (44) employed for the $\varphi_{1,2,3,4}$ and can again be used here. We find that near $z = 0$, or equivalently at the $u = 0$ or $v = 0$ horizons in Fig. 10, there are branch points and associated branch cuts that lead to the same multiverse behavior discussed in the main body of this paper.

What makes up a multiverse is the analytic properties of the wave function that, via monodromy transformations, automatically contains different coefficients on different levels of the multiverse resulting from the canonical transformations like those in Eqs. (33), (34). This implies “discretized jumps” in probability for certain phenomena at different levels of the multiverse. Further progress will be reported in [3].

In this way, we have demonstrated that there is the possibility of a new cosmological multiverse in a geodesically complete cyclic-type cosmology. Now there are interactions, so there remains to figure out if transitions between the various levels of the cosmological multiverse can occur. In the context of trying to determine the wavefunction for the universe, as in this appendix and in [3], the multiverse concept discussed in the main body of the paper is more fitting and it is quite intriguing.

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