Self-dual Skyrmions on the spheres S^{2N+1}

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We construct self-dual sectors for scalar field theories on a $(2N+2)$-dimensional Minkowski space-time with the target space being the $2N+1$-dimensional sphere S^{2N+1}. The construction of such self-dual sectors is made possible by the introduction of an extra functional in the action that renders the static energy and the self-duality equations conformally invariant on the $(2N+1)$-dimensional spatial submanifold. The conformal and target-space symmetries are used to build an ansatz that leads to an infinite number of exact self-dual solutions with arbitrary values of the topological charge. The five-dimensional case is discussed in detail, where it is shown that two types of theories admit self-dual sectors. Our work generalizes the known results in the three-dimensional case that lead to an infinite set of self-dual Skyrmion solutions.

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I. INTRODUCTION

The beauty of self-duality is that it is characterized by first-order differential equations, such that their solutions also solve the second-order Euler-Lagrange equations of the full theory. The self-dual solutions—which in general can be constructed analytically—saturate a lower bound of the energy or Euclidean action for each sector characterized by the value of the topological charge. Examples include the instantons in Yang-Mills theories in four-dimensional Euclidean space [1], the Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles in three dimensions [2,3], the Belavin-Polyakov self-dual solutions of the $O(3)$ or CP^1 nonlinear sigma model in $(2+1)$ dimensions [4], the one-soliton solutions of integrable field theories in $(1+1)$ dimensions like the sine-Gordon model [5], field theories for d scalar fields in $(d+1)$ dimensions [6] (which for the case of $d=3$ include modifications of the Skyrme model [7–10]), and so on.

The interesting fact about the structures of self-duality that allow the construction of solutions by performing one less integration is not the use of dynamical conservation laws, but rather the existence in the theory of a topological charge that admits an integral representation. As explained in Sec. 2 of Ref. [6], one looks for a splitting of the density of topological charge Q as the product of two quantities, let us say

$$Q = \int \mathcal{A}_\alpha \tilde{\mathcal{A}}_\alpha, \tag{1.1}$$

where α may stand for a set of indices. Being a topological quantity means that it is invariant under any smooth variations of the fields, and so the relation $\delta Q = 0$ provides an identity for the fields that is bilinear in the quantities $\mathcal{A}_\alpha$ and $\tilde{\mathcal{A}}_\alpha$. One then introduces the self-duality equations as

$$\mathcal{A}_\alpha = \pm \tilde{\mathcal{A}}_\alpha. \tag{1.2}$$

It turns out that the bilinear identity coming from the topological charge together with the self-duality equations imply the Euler-Lagrange equation associated with the functional [6]

$$E = \frac{1}{2}\int (\mathcal{A}_\alpha^2 + \tilde{\mathcal{A}}_\alpha^2), \tag{1.3}$$

which can be the static energy or the Euclidean action of the theory. If the functional E is positive definite, this automatically provides a bound given by the topological charge, i.e.,

$$E = \frac{1}{2}\int (\mathcal{A}_\alpha \pm \tilde{\mathcal{A}}_\alpha)^2 + |Q| \geq |Q|. \tag{1.4}$$

Note that for a given splitting of the density of topological charge there is the freedom of transforming the quantities $\mathcal{A}_\alpha$ and $\tilde{\mathcal{A}}_\alpha$ as

$$\mathcal{A}_\alpha \to \mathcal{A}_\beta f_{\beta\alpha} \quad \text{and} \quad \tilde{\mathcal{A}}_\alpha \to f^{-1}_{\alpha\beta} \tilde{\mathcal{A}}_\beta, \tag{1.5}$$

where $f_{\beta\alpha}$ is an arbitrary invertible matrix. The possibility of introducing such a matrix is what allows the construction of nontrivial self-dual sectors for Skyrme-type models [8,9]. In fact, in order to preserve the Lorentz symmetry this quantity is a matrix in the internal indices only, contained in the set of indices α. In the cases considered in this paper α contains only spatial indices, and so f will be a scalar function.

Under the shift (1.5) the self-duality equations (1.2) become $\mathcal{A}_\beta h_{\beta\alpha} = \pm\tilde{\mathcal{A}}_\alpha$, with h being the symmetric invertible matrix $h \equiv f f^T$. The topological charge (1.1) remains unchanged, but the energy functional (1.3) becomes $E = \frac{1}{2}\int(\mathcal{A}_\alpha h_{\alpha\beta}\mathcal{A}_\beta + \tilde{\mathcal{A}}_\alpha h^{-1}_{\alpha\beta}\tilde{\mathcal{A}}_\beta)$. If one considers the entries of the matrix h as new extra fields, independent of those originally contained in $\mathcal{A}_\alpha$ and $\tilde{\mathcal{A}}_\alpha$, one observes a very interesting fact. If one varies E with respect to the fields h one gets that $\delta E = 0$, for any variation δh, if $\mathcal{A}\delta h\mathcal{A} = \tilde{\mathcal{A}}h^{-1}\delta h h^{-1}\tilde{\mathcal{A}}$. But that is guaranteed by the new self-duality equations. Therefore, the solutions of the self-duality equations are not only solutions of the Euler-Lagrange equations associated to the fields contained in $\mathcal{A}_\alpha$ and $\tilde{\mathcal{A}}_\alpha$, but also of the Euler-Lagrange equations associated to the fields h.

In the case of Euclidean Yang-Mills theory, for instance, one has that $\mathcal{A}_\alpha$ corresponds to the field tensor $F_{\mu\nu}$, and $\tilde{\mathcal{A}}_\alpha$ corresponds to its Hodge dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. Then the topological charge is the Pontryagin number $Q = \int d^4x \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$, E is the Euclidean action, i.e., $E = \frac{1}{4}\int d^4x \mathrm{Tr}(F^2_{\mu\nu}) = \frac{1}{8}\int d^4x \mathrm{Tr}(F^2_{\mu\nu} + \tilde{F}^2_{\mu\nu})$, and $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$: the well-known self-duality equations.

In this paper we apply the ideas of Ref. [6] (summarized above) to construct self-dual sectors for field theories in a $(2N+2)$-dimensional Minkowski space-time, with the target space being the $(2N+1)$-dimensional sphere S^{2N+1}. Therefore, our results will generalize in a rather simple way the results of Refs. [7,8] for self-dual Skyrmions on S^3. Self-duality equations in space-time dimensions higher than four have been considered extensively in the literature. Self-dual equations for the Yang-Mills theory in any dimension were constructed a long time ago (see, for instance, Refs. [11,12]) and are still a topic of interest [13,14]. In addition, self-dual and non-self-dual monopole solutions in higher dimensions have been constructed in Higgs-Yang-Mills systems (see, for instance, Refs. [15,16]). In the case of scalar field theories like Skyrmions and CP^N models, bounds relating energy and topological charge have been considered, even though self-duality equations were not constructed (see, for instance, Refs. [17,18]). However, it is worth mentioning that self-duality equations connecting gauge and scalar field theories have been obtained [19]. The solitons that we consider are static, and since there are no gauge symmetries, the finite-energy condition imposes that the fields should go to fixed constant values at spatial infinity. Therefore, as long as topology is concerned, one can compactify the space IR^{2N+1} into the sphere S^{2N+1}, and so the soliton solutions carry a topological charge given by the winding number of the map $S^{2N+1}_{\text{space}} \to S^{2N+1}_{\text{target}}$, which can be evaluated through the integral

$$Q_{2N+1} = \frac{2}{(4\pi)^{N+1}}\int d^{2N+1}x\,\varepsilon^{p_1p_2\cdots p_{2N+1}}A_{p_1}H_{p_2p_3}H_{p_4p_5}\cdots H_{p_{2N}p_{2N+1}}, \tag{1.6}$$

where we have parametrized the target space with $N+1$ complex fields Z_a, $a = 1, 2, \ldots N+1$, satisfying the constraint $Z^*_a Z_a = 1$, and have defined the quantities

$$A_\mu = iZ^\dagger \cdot \partial_\mu Z, \quad Z^\dagger \cdot Z = 1, \quad \mu,\nu = 0,1,2\ldots 2N+1, \tag{1.7}$$

and

$$H_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = i(\partial_\mu Z^\dagger \cdot \partial_\nu Z - \partial_\nu Z^\dagger \cdot \partial_\mu Z). \tag{1.8}$$

We shall use the metric with signature $(-)$ for the space coordinates and $(+)$ for the time coordinate, i.e., $ds^2 = dx_0^2 - dx_i^2$. In addition, we take $\varepsilon^{012\ldots 2N+1} = \varepsilon^{12\ldots 2N+1} = 1$. Note that even though the target space is S^{2N+1}, the target-space symmetry group of such theories is not $SO(2N+2)$. The quantities A_μ and $H_{\mu\nu}$ given above are invariant only under the subgroup $U(N+1)$, where the fields transform as $Z \to UZ$, $U \in U(N+1)$.

In Refs. [7,8] we considered the case $N = 1$, which led to an infinite number of exact self-dual Skyrmions on the three-dimensional space IR^3, with the fields taking values on the sphere S^3, or equivalently on the group $SU(2)$. In this paper we shall consider the case $N = 2$, which corresponds to theories in a Minkowski space-time IR^{5+1}, with target space S^5. As we show in Sec. III, there are basically two ways of splitting the density of topological charges, leading to two different theories. The static sectors of these theories are conformally invariant in IR^5, and (following the method of Ref. [20]) they lead to an ansatz based on a generalization of the toroidal coordinates for IR^5. The ansatz involves three integers associated with the angles of the toroidal coordinates, and the topological charge is the product of these three integers. For both theories we construct an infinite number of exact self-dual soliton solutions. However, for one of the theories these integers are arbitrary, and for the other they have to have equal moduli.

We then consider the generic case of theories in $(2N+2)$-dimensional Minkowski space-time with target space S^{2N+1}. In such cases the number of possibilities of splitting the density of topological charge is very large, leading to theories which are conformally invariant in IR^{2N+1}. Again, this symmetry leads to a toroidal ansatz depending on $N+1$ integers. We consider the case where the splitting leads to a theory that admits an infinite number of self-dual soliton solutions for arbitrary values of these $N+1$ integers. It is worth mentioning that static Skyrmions in seven space dimensions have been obtained from self-dual Yang-Mills in eight Euclidean dimensions [18] following the Atiyah-Manton construction [21]. Even though the Skyrmion is obtained from a self-dual solution (instanton), it is not a self-dual Skyrmion in seven dimensions.

The paper is organized as follows. In Sec. II we review the results of Ref. [8] on the construction of self-dual Skyrmions on the three-dimensional space $\mathbb{R}^3$ with the target space S^3. In Sec. III we consider the case of theories in $(5+1)$ dimensions with the target space being the five-dimensional sphere S^5, and show in detail how to use the splitting of the topological charge to construct two types of theories admitting self-dual sectors. We then use the conformal and target-space symmetries of the self-duality equations to construct infinite sets of exact self-dual solutions for these two types of theories. We then generalize our results in Sec. IV to the case of theories in $(2N+2)$-dimensional Minkowski space-time with the target space being the $(2N+1)$-dimensional sphere S^{2N+1}. Again, we construct an infinite set of exact self-dual solutions for one type of theory coming from a particular choice of the splitting of the topological charge. In Sec. V we present our conclusions. In Appendix A we give the proof of the conformal symmetry of the self-duality equations, and in Appendix B we solve some integrals relevant for the calculation of the topological charges of the solutions.

II. SOLUTIONS ON S^3

We begin with a brief review of the work [8] on self-dual Skyrmions on the three-dimensional space $\mathbb{R}^3$ with the target space S^3. In this case field configurations are characterized by the topological charge $Q \in \pi_3(S^3) = \mathbb{Z}$ given by the integral formula

$$Q_3 = \frac{1}{8\pi^2}\int d^3x \varepsilon^{ijk} A_i H_{jk}, \tag{2.1}$$

where A_i and H_{ij} are defined in Eqs. (1.7) and (1.8) for $N=1$. We take the splitting of the topological charge density of the form [see Eq. (1.1)]

$$\mathcal{A}_i \equiv M f_1 A_i, \qquad \tilde{\mathcal{A}}_i \equiv \frac{1}{e f_1}\varepsilon_{ijk} H^{jk}, \qquad i,j,k = 1,2,3, \tag{2.2}$$

where M and e are coupling constants, and f_1 is an arbitrary function. The self-dual equations for such a splitting are

$$\lambda f_1^2 A^i = \varepsilon^{ijk} H_{jk}, \quad \text{with} \quad \lambda = \pm Me. \tag{2.3}$$

The solutions of the self-duality equations (2.3) solve the Euler-Lagrange equations associated to the following static energy functional:

$$E = \frac{1}{2}\int d^3x \left(M^2 f_1^2 A_i^2 + \frac{1}{e^2 f_1^2}(\varepsilon^{ijk} H_{jk})^2 \right). \tag{2.4}$$

The BPS bound for such a static energy is given by

$$\begin{aligned} E &= \frac{1}{2}\int d^3x \left(M f_1 A^i \pm \frac{1}{e f_1}\varepsilon^{ijk} H_{jk} \right)^2 \\ &\quad \mp \frac{M}{e}\int d^3x \varepsilon^{ijk} A_i H_{jk} \geq \frac{8M\pi^2}{e}|Q_3|. \end{aligned} \tag{2.5}$$

Using the methods of Ref. [20], in Ref. [8] an ansatz was constructed by exploring the conformal symmetry of the self-duality equations (2.3) in the three-dimensional space IR^3 (see Appendix A). The ansatz is given by

$$Z = (\sqrt{F(z)} e^{in\varphi}, \sqrt{1-F(z)} e^{im\xi}), \tag{2.6}$$

where m and n are integers, and (z, ξ, φ) are the toroidal coordinates on IR^3,

$$x_1 = \frac{a}{p}\sqrt{z}\cos\varphi, \quad x_2 = \frac{a}{p}\sqrt{z}\sin\varphi, \quad x_3 = \frac{a}{p}\sqrt{1-z}\sin\xi, \tag{2.7}$$

where

$$p = 1 - \sqrt{1-z}\cos\xi, \qquad z \in [0,1], \qquad \xi, \varphi \in [0, 2\pi]. \tag{2.8}$$

The infinite set of solutions found in Ref. [8] are given by

$$F = \frac{m^2 z}{m^2 z + n^2(1-z)}, \qquad f_1 = \sqrt{\frac{2p}{|\lambda| a}\frac{|mn|}{[m^2 z + n^2(1-z)]}}, \tag{2.9}$$

where the sign of λ is chosen to keep f_1 real, i.e., $\mathrm{sign}(\lambda) = -\mathrm{sign}(mn)$. The topological charge and static energy for such solutions are given by

$$Q_3 = -mn, \qquad E = \frac{8M\pi^2}{e}|mn|. \tag{2.10}$$

It turns out [8] that the solutions for the cases where $m^2 = n^2$ present a spherically symmetry energy density, and for the other cases the energy density has only an axial symmetry around the x_3 axis. In Fig. 1 we show the isosurfaces of the topological charge density (or, equivalently, the energy density) for the $Q_3 = -4$ cases, i.e., $(m=2, n=2)$, $(m=4, n=1)$, and $(m=1, n=4)$. It is

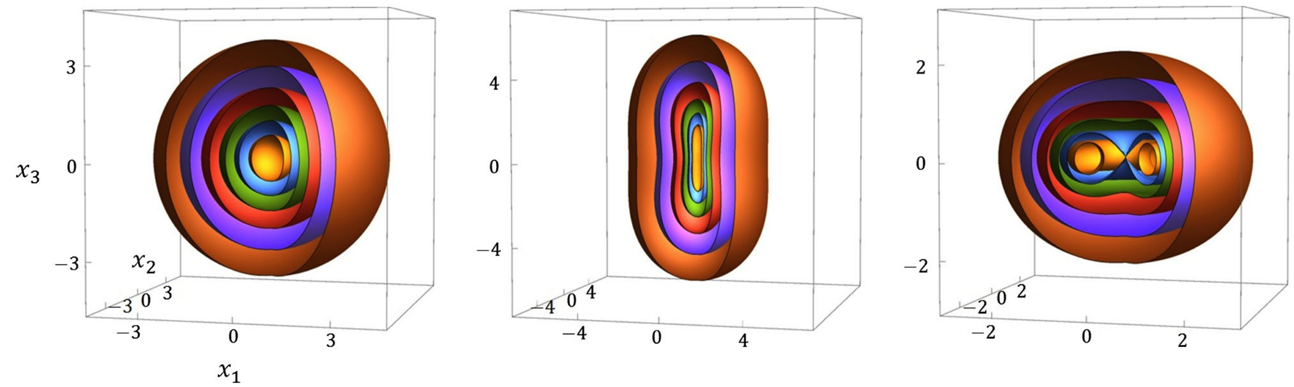

FIG. 1. The isosurfaces of the topological charge density (2.1) for the three-dimensional solutions (2.9), for $a = 1$. The densities correspond, from left to right, to $(m, n) = (2, 2), (m, n) = (4, 1)$, and $(m, n) = (1, 4)$. The layers from the core of the figure to its outside are colored in the order yellow (1), blue (2), green (3), red (4), violet (5), and brown (6), where the nth layer denotes the isosurface with the density $\mathcal{Q}_3 = 4^{2-n}/\pi^2$, i.e., the yellow surface corresponds to $\mathcal{Q}_3 = 4/\pi^2$, the green surface to $\mathcal{Q}_3 = (4\pi)^{-2}$, etc.

worth noting that for the cases where $m^2 \neq n^2$, the densities have a toroidal inner structure, which at large distances leads to an oblate ($n > m$) or prolate ($n < m$) shape. Indeed, in the $(m, n) = (4, 1)$ case, the outside looks prolate but the inside has a dumbbell-like form. In the $(m, n) = (1, 4)$ case, the outside looks oblate but there is a torus-shaped core. On the other hand, every isosurface is a sphere in the $(m, n) = (2, 2)$ case. Note that the energy density has the same profile as the topological charge density.

III. SOLUTIONS ON S^5

In this case the topological charge is the winding number of the map $S^5_{\rm space} \to S^5_{\rm target}$ and is given by

$$Q_5 = \frac{1}{32\pi^3} \int d^5x \varepsilon^{ijklm} A_i H_{jk} H_{lm}. \tag{3.1}$$

There are two basic ways of splitting the density of this topological charge [as in Eq. (1.1)] to construct theories with exact self-dual sectors, as we now explain.

A. Type I theory on S^5

The first case corresponds to the following splitting of the topological charge density:

$$\mathcal{A}^I_i \equiv M f_I A_i, \qquad \tilde{\mathcal{A}}^I_i \equiv \frac{1}{e f_I} \varepsilon_{ijklm} H^{jk} H^{lm},$$
$$i, j, k, l, m = 1, 2, \ldots 5, \tag{3.2}$$

where A_i and H_{ij} are defined in Eqs. (1.7) and (1.8) for $N = 2$, f_I is an arbitrary functional of the complex fields Z_a, $a = 1, 2, 3$, and their derivatives, and M and e are coupling constants. Note that the topological charge density does not depend on the functional f_I, and this represents a freedom we have when we split it [6,8] [see Eq. (1.5)]. The self-duality equation in such a case is

$$\lambda f_I^2 A^i = \varepsilon^{ijklm} H_{jk} H_{lm}, \quad \text{with} \quad \lambda = \pm M e, \tag{3.3}$$

and solutions of it are solutions of the Euler-Lagrange equations associated to the static energy functional

$$E_I = \frac{1}{2} \int d^5x \left(M^2 f_I^2 A_i^2 + \frac{1}{e^2 f_I^2} (\varepsilon^{ijklm} H_{jk} H_{lm})^2 \right). \tag{3.4}$$

The corresponding action is therefore

$$S_I = \frac{1}{2} \int d^6x \left(M^2 f_I^2 A_\mu^2 - \frac{1}{2e^2 f_I^2} (\varepsilon^{\mu\nu\rho\sigma\alpha\beta} H_{\rho\sigma} H_{\alpha\beta})^2 \right). \tag{3.5}$$

The bound on the static energy is given by

$$\begin{aligned} E_I &= \frac{1}{2} \int d^5x \left(M f_I A^i \pm \frac{1}{e f_I} \varepsilon^{ijklm} H_{jk} H_{lm} \right)^2 \\ &\mp \frac{M}{e} \int d^5x \varepsilon^{ijklm} A_i H_{jk} H_{lm} \\ &\geq \frac{32 M \pi^3}{e} |Q_5|. \end{aligned} \tag{3.6}$$

In order to construct solutions we need an ansatz that explores the external (space) and internal (target) symmetries of the theory. We shall follow the methods described in Ref. [20]. As shown in Appendix A, the self-duality equations (3.3) are invariant under conformal transformations in five dimensions, i.e., it is invariant under the conformal group $SO(6, 1)$, which has rank 3. Therefore, the maximum number of commuting $U(1)$ subgroups is three, and they can be chosen to be generated by the following conformal transformations [20]:

$$\partial_{\varphi_1} \equiv x_1\partial_2 - x_2\partial_1,$$
$$\partial_{\varphi_2} \equiv x_3\partial_4 - x_4\partial_3,$$
$$\partial_{\xi} \equiv \frac{x_5}{a}(x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4) + \frac{1}{2a}(a^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)\partial_5. \quad (3.7)$$

The first two transformations are infinitesimal rotations in the planes $x_1 - x_2$ and $x_3 - x_4$, and φ_1 and φ_2 are the corresponding azimuthal angles. The third transformation is a linear combination of an infinitesimal special conformal transformation $V^{(c_5)} = x_5 x_i \partial_i - \frac{1}{2}x_i^2\partial_5$ and an infinitesimal translation $V^{(P_5)} = \partial_5$ along the x_5 axis, and a is a free length scale factor. In addition, ξ is the poloidal angle in five dimensions.

The target-space symmetries are given by the unitary group $U(3)$, a subgroup of $SO(6)$ which is the symmetry group of S^5. Indeed, the operators (1.7) and (1.8) are invariant under the transformations

$$Z_a \to U_{ab}Z_b, \quad Z_a^* Z_a = 1, \quad a, b = 1, 2, 3, \quad U^\dagger \cdot U = \mathbb{1}, \quad (3.8)$$

which also has rank 3. We shall choose the three (maximum) commuting $U(1)$ subgroups to be

$$\Omega_1 = \mathrm{diag}(e^{i\alpha_1}, 1, 1), \qquad \Omega_2 = \mathrm{diag}(1, e^{i\alpha_2}, 1),$$
$$\Omega_3 = \mathrm{diag}(1, 1, e^{i\alpha_3}). \quad (3.9)$$

Following Ref. [20], we choose an ansatz that is invariant under the joint action of the three external and three internal commuting $U(1)$'s given in Eqs. (3.7) and (3.9), respectively. The ansatz is

$$Z = (\sqrt{F_1(z,\theta)}e^{in_1\varphi_1}, \sqrt{F_2(z,\theta)}e^{in_2\varphi_2}, \sqrt{1 - F_1(z,\theta) - F_2(z,\theta)}e^{im\xi}), \quad (3.10)$$

where n_1, n_2, and m are winding numbers associated with the angles φ_1, φ_2 and ξ, respectively, and z and θ are the two coordinates on IR^5, orthogonal to the three angles φ_1, φ_2, and ξ, and defined as

$$z = \frac{4a^2(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{(a^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)^2},$$
$$\theta = \arctan\sqrt{\frac{x_3^2 + x_4^2}{x_1^2 + x_2^2}}. \quad (3.11)$$

One can check that, indeed, $\partial_\zeta z = \partial_\zeta \theta = 0$ for $\zeta = (\varphi_1, \varphi_2, \xi)$. The coordinates $(z, \theta, \xi, \varphi_1, \varphi_2)$ constitute a generalization to IR^5 of the toroidal coordinates on IR^3, and in terms of them the Cartesian coordinates are written as

$$x_1 = \frac{a}{p}\sqrt{z}\cos\theta\cos\varphi_1, \qquad x_2 = \frac{a}{p}\sqrt{z}\cos\theta\sin\varphi_1,$$
$$x_3 = \frac{a}{p}\sqrt{z}\sin\theta\cos\varphi_2, \qquad x_4 = \frac{a}{p}\sqrt{z}\sin\theta\sin\varphi_2,$$
$$x_5 = \frac{a}{p}\sqrt{1-z}\sin\xi, \quad (3.12)$$

with

$$p = 1 - \sqrt{1-z}\cos\xi, \quad (3.13)$$

where the domains of the variables are $z \in [0, 1]$, $\theta \in [0, \pi/2], \xi, \varphi_1, \varphi_2 \in [0, 2\pi]$. In terms of the new coordinates, the metric is written as

$$ds^2 = \frac{a^2}{p^2}\left(\frac{1}{4z(1-z)}dz^2 + zd\theta^2 + (1-z)d\xi^2 + z\cos^2\theta d\varphi_1^2 + z\sin^2\theta d\varphi_2^2\right). \quad (3.14)$$

From Eqs. (1.7) and (1.8) and the ansatz (3.10), one observes that $A_z = A_\theta = 0$, and also that $H_{z\theta} = H_{\varphi_1\varphi_2} = H_{\varphi_1\xi} = H_{\varphi_2\xi} = 0$. Therefore, the five equations in Eq. (3.3) reduce to only three, since two of them are automatically satisfied by the ansatz (3.10). In addition, the rhs of Eq. (3.3) for the three remaining equations are all proportional to the same function of z and θ, namely, $\partial_z F_1 \partial_\theta F_2 - \partial_z F_1 \partial_\theta F_2$. Therefore, substituting the ansatz (3.10) into the BPS equation (3.3) leads to the following three coupled first-order partial differential equations:

$$\lambda f_I^2 \frac{a^3}{p^3} n_1 F_1 \tan\theta = 16 m n_2 (\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1),$$
$$\lambda f_I^2 \frac{a^3}{p^3} n_2 F_2 \cot\theta = 16 m n_1 (\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1),$$
$$\lambda f_I^2 \frac{a^3}{p^3} \frac{z}{1-z} m(1 - F_1 - F_2)\sin\theta\cos\theta = 16 n_1 n_2 (\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1). \quad (3.15)$$

Since the right-hand sides of the equations in Eq. (3.15) are all proportional, they imply that

$$n_1^2 F_1 \tan\theta = n_2^2 F_2 \cot\theta = m^2 \frac{z}{1-z}(1 - F_1 - F_2)\sin\theta\cos\theta. \quad (3.16)$$

One can algebraically solve Eq. (3.16) for any nonzero integers m, n_1, and n_2, and the solutions are given by

$$F_1 = \frac{m^2 n_2^2 z \cos^2\theta}{n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)},$$
$$F_2 = \frac{m^2 n_1^2 z \sin^2\theta}{n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)}. \tag{3.17}$$

By substituting such solutions for F_1 and F_2 into Eq. (3.15), we obtain

$$f_I = \left(\frac{p^3}{|\lambda| a^3}\right)^{\frac{1}{2}} \frac{4\sqrt{2}|m n_1 n_2|^{3/2}}{[n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)]}. \tag{3.18}$$

Since f_I is a real function, the sign of λ and of the integers must satisfy

$$\mathrm{sign}\lambda = \mathrm{sign}(m n_1 n_2). \tag{3.19}$$

The density of the topological charge (3.1) is given by

$$\begin{aligned}\frac{1}{32\pi^3}&\varepsilon^{ijklm} A_i H_{jk} H_{lm}\\ &= \frac{1}{2\pi^3}\left(\frac{p}{a}\right)^5 \frac{m n_1 n_2}{z \sin\theta\cos\theta}[\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1]\\ &= \frac{1}{\pi^3}\left(\frac{p}{a}\right)^5 \frac{m^5 n_1^5 n_2^5}{[n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)]^3},\end{aligned} \tag{3.20}$$

where we have used the convention $\varepsilon^{12345} = 1$, and so

$$\varepsilon^{z\theta\varphi_1\varphi_2\xi} = \left(\frac{p}{a}\right)^5 \frac{2}{z \sin\theta\cos\theta}. \tag{3.21}$$

The volume element is

$$d^5x = \left(\frac{a}{p}\right)^5 \frac{1}{2} z \sin\theta\cos\theta dz d\theta d\varphi_1 d\varphi_2 d\xi. \tag{3.22}$$

We now use the fact that

$$\int_0^1 dz \int_0^{\frac{\pi}{2}} d\theta \frac{z \sin\theta\cos\theta}{[n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)]^3} = \frac{1}{4}\frac{1}{m^4 n_1^4 n_2^4} \tag{3.23}$$

to find that the topological charges of these solutions are

$$Q_5 = m n_1 n_2. \tag{3.24}$$

For the configurations satisfying the self-duality equations (3.3), the static energy (3.4) becomes

$$E_I = \int d^5x \mathcal{E}, \quad \text{with} \quad \mathcal{E} = M^2 f_I^2 A_i^2. \tag{3.25}$$

The energy density is given by

$$\mathcal{E} = 32\frac{M}{e}\left(\frac{p}{a}\right)^5 \frac{|m n_1 n_2|^5}{[n_1^2 n_2^2 (1-z) + m^2 z (n_1^2 \sin^2\theta + n_2^2 \cos^2\theta)]^3}. \tag{3.26}$$

Therefore, using Eqs. (3.22) and (3.23), one gets

$$E_I = 32\pi^3 \frac{M}{e}|m n_1 n_2|. \tag{3.27}$$

From Eqs. (3.20) and (3.26), one observes that the densities of the topological charge and static energy are proportional. In order to visualize the shape of such densities, let us write the density of the topological charge [given in Eq. (3.20)] in terms of Cartesian coordinates as

$$\mathcal{Q} = \frac{32}{a^5 \pi^3} \frac{(1+\tilde{r}^2)(m n_1 n_2)^5}{[n_1^2 n_2^2 (1+\tilde{r}^2)^2 + 4 n_1^2 \tilde{\rho_2}^2 (m^2 - n_2^2) + 4 n_2^2 \tilde{\rho_1}^2 (m^2 - n_1^2)]^3}, \tag{3.28}$$

with

$$\tilde{\rho_1} = \sqrt{\tilde{x_1}^2 + \tilde{x_2}^2}, \qquad \tilde{\rho_2} = \sqrt{\tilde{x_3}^2 + \tilde{x_4}^2},$$
$$\tilde{r} = \sqrt{\tilde{\rho_1}^2 + \tilde{\rho_2}^2 + \tilde{x_5}^2}, \tag{3.29}$$

where $\tilde{x_i} = x_i/a$. Note that $\mathcal{Q}$ does not depend on the angles φ_1 and φ_2, and so the energy and topological charge densities are invariant under the group $SO(2)\times SO(2)$ of rotations in the $x_1 - x_2$ and $x_3 - x_4$ planes for any nonzero values of the integers m, n_1, and n_2. In addition, for the cases where $n_1^2 = n_2^2$, such densities only depend on $\tilde{r}^2$ and $\tilde{x_5}^2$, and so they are invariant under the group $SO(4)$ of rotations on the subspace IR^4 perpendicular to the x_5 axis. For the cases where $m^2 = n_1^2$ (or $m^2 = n_2^2$), the densities only depend on $\tilde{r}^2$ and $\tilde{\rho}_2{}^2$ (or $\tilde{r}^2$ and $\tilde{\rho}_1{}^2$), and so they are invariant under the group $SO(2)\times SO(3)$ of rotations in the $x_3 - x_4$ (or $x_1 - x_2$) plane, and on the subspace IR^3 perpendicular to the $x_3 - x_4$ (or $x_1 - x_2$) plane. Finally, for

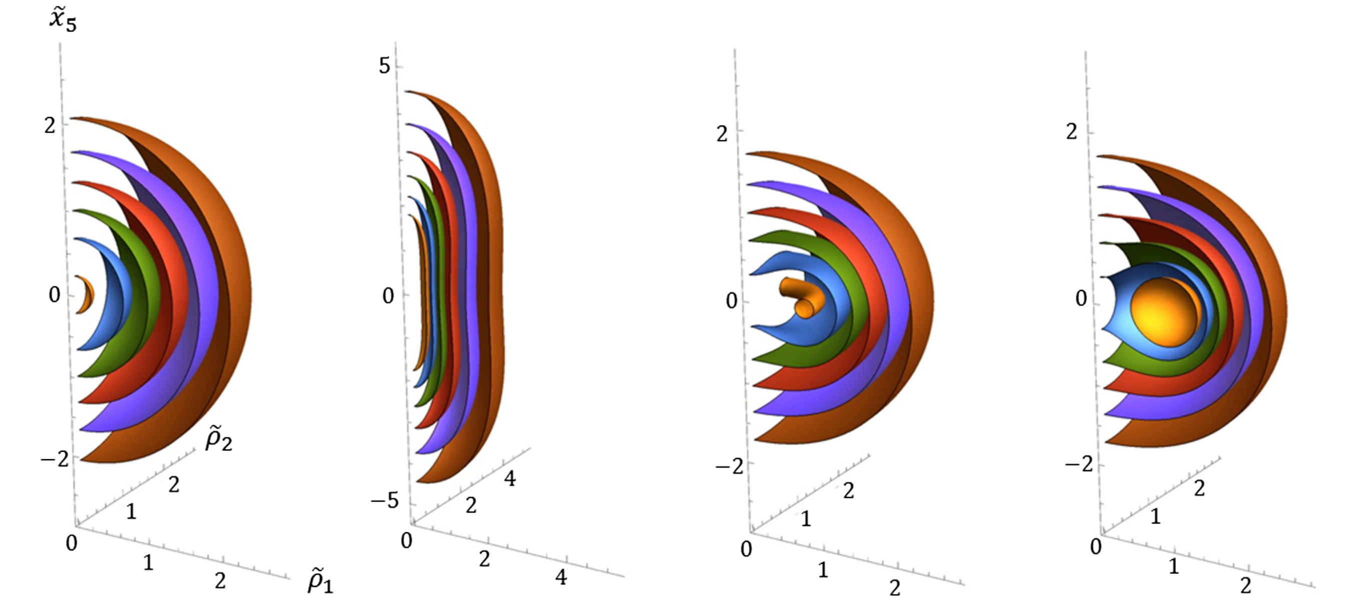

FIG. 2. The isosurfaces of the topological charge density (3.20) for the five-dimensional solutions (3.17), with $\tilde{\rho}_a$ and $\tilde{x}_5$ defined in Eq. (3.29). The densities correspond, from left to right, to $(m, n_1, n_2) = (1, 1, 1)$, $(m, n_1, n_2) = (4, 1, 1)$, $(m, n_1, n_2) = (1, 2, 2)$, and $(m, n_1, n_2) = (1, 4, 1)$. The layers from the core of the figure to its outside are colored in the order yellow (1), blue(2), green (3), red (4), violet (5), and brown (6), where the nth layer denotes the isosurface with the density $\mathcal{Q}_5 = 5^{3-n}/(a^5\pi^3)$, i.e., the yellow surface corresponds to $\mathcal{Q}_5 = 25/(a^5\pi^3)$, the green surface to $\mathcal{Q}_5 = 1/(a^5\pi^3)$, etc.

the cases where $m^2 = n_1^2 = n_2^2$, the densities only depend on $\tilde{r}^2$ and so they are invariant under the group $SO(5)$ of rotations on the whole space IR^5, i.e., the densities are spherically symmetric.

In Fig. 2 we show some examples of surfaces of constant topological charge density in terms of the three coordinates $\tilde{\rho_1}, \tilde{\rho_2}$, and $\tilde{x_5}$. Their structure is very similar to the three-dimensional case (see Fig. 1). When $m^2 = n_1^2 = n_2^2$, the isosurfaces are four-dimensional spheres, and thus they are $SO(5)$ invariant. For $n_1^2 = n_2^2$ the isosurfaces are indeed $SO(4)$ invariant, and we note that for $m^2 > n_1^2 = n_2^2$ the outer isosurfaces have a five-dimensional prolate shape, but the inside has a dumbbell-like structure. On the other hand, for $m^2 < n_1^2 = n_2^2$, the outer isosurfaces look oblate, but the inner shells have a five-dimensional torus shape.

B. Type II theory on S^5

The second field theory for the case $N = 2$ corresponds to the following splitting of the topological charge density:

$$\mathcal{A}_{ij}^{II} \equiv M f_{II} \varepsilon_{ijklm} A^k H^{lm}, \qquad \tilde{\mathcal{A}}_{ij}^{II} \equiv \frac{1}{e f_{II}} H_{ij},$$

$$i, j, k, l, m = 1, 2, \ldots 5, \tag{3.30}$$

with f_{II} having the same nature as f_I introduced above. The self-duality equations in this case are

$$\lambda f_{II}^2 \varepsilon_{ijklm} A^k H^{lm} = H_{ij}, \quad \text{with} \quad \lambda = \pm Me. \tag{3.31}$$

The solutions of Eq. (3.31) are also solutions of the Euler-Lagrange equations associated to the static energy functional

$$E_{II} = \frac{1}{2}\int d^5x \left(M^2 f_{II}^2 (\epsilon_{ijklm} A^k H^{lm})^2 + \frac{1}{e^2 f_{II}^2} H_{ij}^2 \right), \tag{3.32}$$

and the corresponding action is

$$S_{II} = -\frac{1}{2}\int d^6x \left(\frac{M^2}{3} f_{II}^2 (\epsilon_{\mu\nu\rho\sigma\alpha\beta} A^\sigma H^{\alpha\beta})^2 + \frac{1}{e^2 f_{II}^2} H_{\mu\nu}^2 \right). \tag{3.33}$$

The self-duality equations (3.31) is also invariant under conformal transformations in five dimensions, as shown in Appendix A. Therefore, we shall use the same ansatz [given in Eq. (3.10)] used to construct the solutions for the self-duality equations (3.3). When the ansatz (3.10) is inserted into the ten equations in Eq. (3.31), one finds that four of them are automatically satisfied. The remaining six equations are given by

$$\Lambda(F_1 \partial_\theta F_2 - F_2 \partial_\theta F_1) = -n_1^2 z(1-z)\tan\theta\, \partial_z F_1 + \Lambda \partial_\theta F_2, \tag{3.34}$$

$$\Lambda(F_1 \partial_\theta F_2 - F_2 \partial_\theta F_1) = -n_2^2 z(1-z)\cot\theta\, \partial_z F_2 - \Lambda \partial_\theta F_1, \tag{3.35}$$

$$\Lambda(F_1\partial_\theta F_2 - F_2\partial_\theta F_1) = m^2 z^2 \sin\theta\cos\theta\partial_z(F_1+F_2), \tag{3.36}$$

$$\Lambda(F_1\partial_z F_2 - F_2\partial_z F_1) = \frac{n_1^2}{4}\frac{\tan\theta}{z}\partial_\theta F_1 + \Lambda\partial_z F_2, \tag{3.37}$$

$$\Lambda(F_1\partial_z F_2 - F_2\partial_z F_1) = \frac{n_2^2}{4}\frac{\cot\theta}{z}\partial_\theta F_2 - \Lambda\partial_z F_1, \tag{3.38}$$

$$\Lambda(F_1\partial_z F_2 - F_2\partial_z F_1) = -\frac{m^2}{4}\frac{\sin\theta\cos\theta}{1-z}\partial_\theta(F_1+F_2), \tag{3.39}$$

where we have denoted

$$\Lambda \equiv \lambda f_{II}^2 \frac{p}{a} n_1 n_2 m. \tag{3.40}$$

The structure of the equations (3.34)–(3.39) is more complex than that of Eq. (3.15), and we have to analyze them more carefully. Subtracting Eq. (3.35) from Eq. (3.34) and then combining with Eq. (3.39) (multiplied by Λ), one gets

$$\left[\Lambda^2 F_2 - \frac{m^2}{4} z n_1^2 \sin^2\theta\right]\partial_z F_1 = \left[\Lambda^2 F_1 - \frac{m^2}{4} z n_2^2 \cos^2\theta\right]\partial_z F_2. \tag{3.41}$$

Now, subtracting Eq. (3.38) from Eq. (3.37) and then combining with Eq. (3.36) (multiplied by Λ), one gets

$$\left[\Lambda^2 F_2 - \frac{m^2}{4} z n_1^2 \sin^2\theta\right]\partial_\theta F_1 = \left[\Lambda^2 F_1 - \frac{m^2}{4} z n_2^2 \cos^2\theta\right]\partial_\theta F_2. \tag{3.42}$$

Equations (3.41) and (3.42) imply that

$$\left[\Lambda^2 F_1 - \frac{m^2}{4} z n_2^2 \cos^2\theta\right]\left[\Lambda^2 F_2 - \frac{m^2}{4} z n_1^2 \sin^2\theta\right] \times [\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1] = 0. \tag{3.43}$$

If we impose $\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1 = 0$, then it follows that the density of topological charge vanishes [see Eq. (3.20)], and so the solution will be topologically trivial. Therefore, we have to take

$$F_1 = \frac{m^2 z}{4\Lambda^2} n_2^2 \cos^2\theta, \qquad F_2 = \frac{m^2 z}{4\Lambda^2} n_1^2 \sin^2\theta. \tag{3.44}$$

But Eq. (3.44) implies that both F_1 and F_2 have the same z dependence, and so it follows that $F_1\partial_z F_2 - F_2\partial_z F_1 = 0$. But from Eq. (3.39) that implies that $\partial_\theta(F_1+F_2) = 0$, and consequently [using Eq. (3.44)] one has that

$$\Lambda^2 = \frac{m^2 z}{4\eta(z)^2}[n_1^2\sin^2\theta + n_2^2\cos^2\theta] \tag{3.45}$$

for some function $\eta(z)$. Therefore,

$$F_1 = \eta(z)^2 \frac{n_2^2\cos^2\theta}{n_1^2\sin^2\theta + n_2^2\cos^2\theta}, \qquad F_2 = \eta(z)^2 \frac{n_1^2\sin^2\theta}{n_1^2\sin^2\theta + n_2^2\cos^2\theta}. \tag{3.46}$$

Subtracting Eq. (3.35) from Eq. (3.34) and using the relations above, one gets an equation that can only be satisfied if $n_1^2 = n_2^2 \equiv n^2$. Now, we multiply Eq. (3.34) by $\cos^2\theta$, add it to Eq. (3.35) multiplied by $\sin^2\theta$, and subtract that from Eq. (3.36) to get

$$\partial_z \eta^2 [m^2 z^2 + n^2 z(1-z)] - 2\eta^2\Lambda = 0. \tag{3.47}$$

Subtracting Eq. (3.37) from Eq. (3.38), one gets

$$2z\Lambda\partial_z\eta^2 - n^2\eta^2 = 0. \tag{3.48}$$

Multiplying Eq. (3.48) by 2Λ and subtracting Eq. (3.47) (multiplied by n^2), one gets

$$\partial_z\eta^2[4\Lambda^2 - n^2(m^2 z + n^2(1-z))] = 0. \tag{3.49}$$

If we take η to be constant, then F_1 and F_2 do not depend on z, and so $\partial_z F_1 \partial_\theta F_2 - \partial_z F_2 \partial_\theta F_1 = 0$, which means that the density of topological charge vanishes [see Eq. (3.20)] and we do not want that because the solutions would be topologically trivial. We then have to take $\Lambda^2 = n^2(m^2 z + n^2(1-z))/4$. But to make this compatible with Eq. (3.45), we need to take $\eta^2 = m^2 z/[m^2 z + n^2(1-z)]$. But inserting that into Eq. (3.48), with $\Lambda = \pm|n|\sqrt{m^2 z + n^2(1-z)}/2$, one gets that we need $|n| = \pm\sqrt{m^2 z + n^2(1-z)}$. The only possible solution is $m^2 = n^2$ and to take Λ to be positive, and thus from Eq. (3.40) one gets the restriction

$$\mathrm{sign}(\lambda) = \mathrm{sign}(n), \tag{3.50}$$

where we have denoted

$$n_1^2 = n_2^2 = m^2 \equiv n^2. \tag{3.51}$$

Summarizing, the self-dual solutions are

$$F_1 = z\cos^2\theta, \qquad F_2 = z\sin^2\theta, \qquad f_{II} = \sqrt{\frac{1}{2|\lambda n|}\frac{a}{p}}. \tag{3.52}$$

Note that the solutions (3.52) for F_1 and F_2 are the same as the solutions (3.17) for the cases where $m^2 = n_1^2 = n_2^2$.

Consequently, the solutions for the fields Z_a, and thus for the vector A_i and tensor H_{ij}, are the same for the type II theory (3.33) as for the type I theory (3.4) for the cases $m^2 = n_1^2 = n_2^2$. The solutions for the functions f_I and f_{II}, however, are different even if $m^2 = n_1^2 = n_2^2$. Since the topological charge density does not depend on the functions f_I and f_{II}, it is the same for those two classes of solutions of these two types of theories. Therefore, the topological charge for the solutions (3.52) is given by

$$Q_5 = \mathrm{sign}(mn_1n_2)|n|^3, \tag{3.53}$$

where the sign of the charge comes from the choice of relative signs between n and the integers n_1, n_2, and m in Eq. (3.51).

The energy densities of these solutions are also the same due to their self-dual character. Indeed, from Eqs. (3.3) and (3.4) one obtains that for self-dual solutions one has $E_I \sim \int d^5x f_I^2 A_i^2$. Similarly, from Eqs. (3.31) and (3.33) one gets that $E_{II} \sim \int d^5x \frac{1}{f_{II}^2} H_{ij}^2$ for self-dual solutions. But Eq. (3.3) implies $f_I^2 A_i^2 \sim \varepsilon^{ijklm} A_i H_{jk} H_{lm}$, and Eq. (3.31) implies $\frac{1}{f_{II}^2} H_{ij}^2 \sim \varepsilon^{ijklm} A_i H_{jk} H_{lm}$. Consequently, for the solutions (3.52), the topological charge density and energy density are proportional and spherically symmetric, like the solutions (3.17) for the cases where $m^2 = n_1^2 = n_2^2$ [see discussion below Eq. (3.29)]. In fact, we have that the energy of the solutions (3.52) is given by

$$E_{II} = \frac{32\pi^3 M}{e} |n|^3. \tag{3.54}$$

IV. SOLUTIONS ON S^{2N+1}

For the case of self-dual models defined on $\mathbb{R}^{2N+1}$, with generic values of N, there are many possibilities for the splitting of the density of topological charge (1.6). We shall consider only the case where the splitting is such that

$$\begin{aligned} \mathcal{A}_{p_1}^N &\equiv M f_N A_{p_1}, \\ \tilde{\mathcal{A}}_{p_1}^N &\equiv \frac{1}{e f_N} \varepsilon_{p_1 p_2 \cdots p_{2N+1}} H^{p_2 p_3} H^{p_4 p_5} \cdots H^{p_{2N} p_{2N+1}}, \end{aligned} \tag{4.1}$$

and the self-duality equation is

$$\lambda f_N^2 A_{p_1} = \varepsilon_{p_1 p_2 \cdots p_{2N+1}} H^{p_2 p_3} H^{p_4 p_5} \cdots H^{p_{2N} p_{2N+1}}, \quad \text{with} \quad \lambda = \pm M e. \tag{4.2}$$

Therefore, according to the reasoning explained in the Introduction, solutions of Eq. (4.2) are solutions of the Euler-Lagrange equations following from the static energy functional given by

$$E_N = \frac{1}{2} \int d^{2N+1}x \left[M^2 f_N^2 A_i^2 + \frac{1}{e^2 f_N^2} (\varepsilon_{p_1 p_2 \cdots p_{2N+1}} H^{p_2 p_3} H^{p_4 p_5} \cdots H^{p_{2N} p_{2N+1}})^2 \right]. \tag{4.3}$$

The corresponding action in the $(2N+2)$-dimensional Minkowski space-time is

$$S_N = \frac{1}{2} \int d^{2N+2}x \left[M^2 f_N^2 A_\mu^2 - \frac{1}{2e^2 f_N^2} (\varepsilon_{\mu_0 \mu_1 \mu_2 \cdots \mu_{2N+1}} H^{\mu_2 \mu_3} H^{\mu_4 \mu_5} \cdots H^{\mu_{2N} \mu_{2N+1}})^2 \right]. \tag{4.4}$$

The bound on the static energy is given by

$$\begin{aligned} E_N &= \frac{1}{2} \int d^{2N+1}x \left[M f_N A_{p_1} \pm \frac{1}{e f_N} \varepsilon_{p_1 p_2 \cdots p_{2N+1}} H^{p_2 p_3} H^{p_4 p_5} \cdots H^{p_{2N} p_{2N+1}} \right]^2 \\ &\mp \frac{M}{e} \int d^{2N+1}x \varepsilon_{p_1 p_2 \cdots p_{2N+1}} A^{p_1} H^{p_2 p_3} H^{p_4 p_5} \cdots H^{p_{2N} p_{2N+1}} \\ &\geq \frac{(4\pi)^{N+1} M}{2e} |Q_{2N+1}|, \end{aligned} \tag{4.5}$$

where Q_{2N+1} was given in Eq. (1.6). Clearly the bound is saturated by solutions of the self-duality equations (4.2).

In order to construct solutions to the self-duality equations (4.2) we explore their symmetries. As discussed below Eq. (1.8), the quantities A_i and H_{ij} are invariant under the transformations $Z \to UZ$, with $U \in U(N+1)$, and so Eq. (4.2) are invariant under such $U(N+1)$ symmetry. On the other hand, as shown in Appendix A, the self-duality equations (4.2) are invariant under the conformal group $SO(2N+2,1)$. It turns out that both $U(N+1)$ and $SO(2N+2,1)$ have $N+1$ commuting $U(1)$ subgroups. For the case of $U(N+1)$ these subgroups can be taken to form the Cartan subgroup of diagonal matrices, i.e., $U = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \ldots e^{i\alpha_{N+1}})$. For the conformal group $SO(2N+2,1)$ we shall take these commuting $U(1)$ subgroups to be generated by N commuting spatial rotations plus a linear combination of a special conformal transformation and a translation along the x_{N+1} axis, as follows (see Ref. [20] for details):

$$\partial_{\varphi_i} \equiv x_{2i-1}\partial_{x_{2i}} - x_{2i}\partial_{x_{2i-1}}, \quad i = 1, 2, \ldots N,$$
$$\partial_\xi \equiv \frac{x_{2N+1}}{a}\sum_{i\neq 2N+1} x_i\partial_{x_i} + \frac{1}{2a}\left(a^2 + x_{2N+1}^2 - \sum_{i\neq 2N+1} x_i^2\right)\partial_{x_{2N+1}}, \tag{4.6}$$

where a is an arbitrary parameter with dimension of length. We shall construct an ansatz that is invariant under the diagonal action of the internal and external $N+1$ commuting $U(1)$ subgroups, i.e., $e^{i\alpha_i} \otimes \partial_{\varphi_i}$, $i = 1, 2, \ldots N$, and $e^{i\alpha_{2N+1}} \otimes \partial_\xi$. The appropriate coordinates for such an ansatz are a generalization of the toroidal coordinates to $\mathbb{R}^{2N+1}$, made of the angles φ_i, $i = 1, 2, \ldots N$, and ξ, together with coordinates z, $0 \leq z \leq 1$, and y_α, $\alpha = 1, 2, \ldots N-1$, with $0 \leq y_\alpha \leq 1$, where the Cartesian coordinates are written as follows:

$$\begin{aligned}
x_1 &= \frac{a}{p}\sqrt{z}\sqrt{1-y_1}\cos\varphi_1, \qquad x_2 = \frac{a}{p}\sqrt{z}\sqrt{1-y_1}\sin\varphi_1,\\
x_3 &= \frac{a}{p}\sqrt{z}\sqrt{y_1(1-y_2)}\cos\varphi_2, \qquad x_4 = \frac{a}{p}\sqrt{z}\sqrt{y_1(1-y_2)}\sin\varphi_2,\\
x_5 &= \frac{a}{p}\sqrt{z}\sqrt{y_1 y_2(1-y_3)}\cos\varphi_3, \qquad x_6 = \frac{a}{p}\sqrt{z}\sqrt{y_1 y_2(1-y_3)}\sin\varphi_3,\\
&\vdots\\
x_{2\alpha-1} &= \frac{a}{p}\sqrt{z}\sqrt{1-y_\alpha}\prod_{\beta=1}^{\alpha-1}\sqrt{y_\beta}\cos\varphi_\alpha, \qquad x_{2\alpha} = \frac{a}{p}\sqrt{z}\sqrt{1-y_\alpha}\prod_{\beta=1}^{\alpha-1}\sqrt{y_\beta}\sin\varphi_\alpha,\\
&\vdots\\
x_{2N-1} &= \frac{a}{p}\sqrt{z}\prod_{\alpha=1}^{N-1}\sqrt{y_\alpha}\cos\varphi_N, \qquad x_{2N} = \frac{a}{p}\sqrt{z}\prod_{\alpha=1}^{N-1}\sqrt{y_\alpha}\sin\varphi_N,\\
x_{2N+1} &= \frac{a}{p}\sqrt{1-z}\sin\xi,
\end{aligned} \tag{4.7}$$

with $z \in [0, 1]$, $y_\alpha \in [0, 1]$, and $\xi, \varphi_i \in [0, 2\pi]$, with $\alpha = 1, 2, \ldots N-1$, $i = 1, 2, \ldots N$, and where we have introduced

$$p = 1 - \sqrt{1-z}\cos\xi. \tag{4.8}$$

The metric in $\mathbb{R}^{2N+1}$ is given by

$$ds^2 = h_z^2 dz^2 + \sum_{\alpha=1}^{N-1} h_{y_\alpha}^2 dy_\alpha^2 + \sum_{i=1}^{N} h_{\varphi_i}^2 d\varphi_i^2 + h_\xi^2 d\xi^2, \tag{4.9}$$

where the scaling factors are

$$\begin{aligned}
h_z &= \frac{a}{p}\frac{1}{2\sqrt{z(1-z)}}, \qquad h_{y_\alpha} = \frac{a}{p}\sqrt{z}\frac{\prod_{\beta=1}^{\alpha-1}\sqrt{y_\beta}}{2\sqrt{y_\alpha(1-y_\alpha)}}, \qquad h_\xi = \frac{a}{p}\sqrt{1-z},\\
h_{\varphi_\alpha} &= \frac{a}{p}\sqrt{z}\sqrt{1-y_\alpha}\prod_{\beta=1}^{\alpha-1}\sqrt{y_\beta}, \qquad h_{\varphi_N} = \frac{a}{p}\sqrt{z}\prod_{\alpha=1}^{N-1}\sqrt{y_\alpha},
\end{aligned} \tag{4.10}$$

with $\alpha = 1, 2, \ldots N-1$.

The ansatz that is invariant under the diagonal action of $U(1)$'s internal and external commuting subgroups (described above) is given by

$$Z = \left(\sqrt{F_1(z, y_\alpha)}e^{in_1\varphi_1}, \sqrt{F_2(z, y_\alpha)}e^{in_2\varphi_2}, \ldots, \sqrt{F_N(z, y_\alpha)}e^{in_N\varphi_N}, \sqrt{1 - \sum_{k=1}^{N} F_k}\, e^{im\xi}\right), \tag{4.11}$$

where n_i and m are integers. Inserting the ansatz (4.11) into the quantities A_i and H_{ij}, introduced in Eqs. (1.7) and (1.8), one obtains that

$$A_z = A_{y_\alpha} = 0, \qquad A_\xi = -m\left(1 - \sum_{k=1}^{N} F_k\right), \qquad A_{\varphi_i} = -n_i F_i, \tag{4.12}$$

and

$$\begin{aligned} &H_{\varphi_i z} = n_i \partial_z F_i, \qquad H_{\varphi_i y_\alpha} = n_i \partial_{y_\alpha} F_i, \qquad H_{\varphi_i \varphi_j} = H_{\varphi_i \xi} = H_{z y_\alpha} = H_{y_\alpha y_\beta} = 0, \\ &H_{\xi z} = -m \sum_{k=1}^{N} \partial_z F_k, \qquad H_{\xi y_\alpha} = -m \sum_{k=1}^{N} \partial_{y_\alpha} F_k. \end{aligned} \tag{4.13}$$

Therefore, from Eq. (4.12) one observes that the lhs of the self-duality equations (4.2) will be nonzero only when the index p_1 corresponds to one of the variables in the set of $N+1$ variables (φ_i, ξ). On the other hand, the rhs of Eq. (4.2) contains the product of N components of the tensor H_{ij}, and so if p_1 does not belong to the set (φ_i, ξ), the set of indices $p_2 p_3 \ldots p_{2N+1}$ will contain all the indices of that set, and so at least one of the components of the tensor H_{ij} in that product will have its two indices in the set (φ_i, ξ), and so it vanishes. Therefore, both sides of Eq. (4.2) vanish when the index p_1 does not belong to the set (φ_i, ξ). It turns out that when the index p_1 belongs to the set (φ_i, ξ), the rhs of Eq. (4.2) will be proportional to $\varepsilon_{p_1 r_1 r_2 \ldots r_N z y_1 \ldots y_{N-1}} H^{r_1 z} H^{r_2 y_1} \ldots H^{r_N y_{N-1}}$, with the indices r_i taking values in the set (φ_i, ξ), but different from p_1. But that is proportional to the determinant of the $N \times N$ matrix $\partial_i F_j$, with the index i belonging to the set of N variables (z, y_α). Consequently, the self-duality equations (4.2) reduce to a set of $N+1$ equations where their left-hand sides are linear in the functions F_i, and do not involve their derivatives. On the other hand, their right-hand sides are all proportional to the determinant of the matrix $\partial_i F_j$. Choosing the sign of the ε symbol such that

$$\varepsilon^{\xi \varphi_1 \ldots \varphi_N z y_1 \ldots y_{N-1}} = \frac{1}{h_\xi h_{\varphi_1} \ldots h_{\varphi_N} h_z h_{y_1} \ldots h_{y_{N-1}}}, \tag{4.14}$$

one then gets that the self-duality equations (4.2) imply the following relations:

$$\begin{aligned} \frac{n_1^2 F_1}{h_{\varphi_1}^2} = \frac{n_2^2 F_2}{h_{\varphi_2}^2} = \ldots = \frac{n_N^2 F_N}{h_{\varphi_N}^2} &= \frac{m^2}{h_\xi^2}\left(1 - \sum_{k=1}^{N} F_k\right) \\ &= -\frac{(-1)^{N(N-1)/2} 2^N N!}{h_\xi h_{\varphi_1} \ldots h_{\varphi_N} h_z h_{y_1} \ldots h_{y_{N-1}}} \left(m \prod_{k=1}^{N} n_k\right) \frac{\det(\partial F)}{\lambda f_N^2}, \end{aligned} \tag{4.15}$$

where

$$\det(\partial F) \equiv \varepsilon_{i_1 i_2 \ldots i_N} \partial_z F_{i_1} \partial_{y_1} F_{i_2} \partial_{y_2} F_{i_3} \ldots \partial_{y_{N-1}} F_{i_N}, \tag{4.16}$$

with $\varepsilon_{123\ldots N} = 1$. We are interested in those cases where all of the integers m and n_i, $i = 1, 2, \ldots N$ are nonzero since otherwise (as we show below) the topological charge vanishes. Then, in such cases one can easily solve these algebraic equations to get the F_i's as

$$\begin{aligned} F_i &= \frac{h_{\varphi_i}^2 / n_i^2}{h_\xi^2 / m^2 + \sum_{j=1}^{N} h_{\varphi_j}^2 / n_j^2} \equiv \frac{\kappa_i / n_i^2}{\Delta}, \\ \Delta &= \frac{\kappa_\xi}{m^2} + \sum_{j=1}^{N} \frac{\kappa_j}{n_j^2}, \qquad i = 1, 2, \ldots N, \end{aligned} \tag{4.17}$$

with

$$\kappa_\xi = \frac{1 - z}{z}, \qquad \kappa_N = \prod_{\alpha=1}^{N-1} y_\alpha, \qquad \kappa_\alpha = (1 - y_\alpha) \prod_{\beta=1}^{\alpha-1} y_\beta, \qquad \alpha = 1, 2, \ldots N-1. \tag{4.18}$$

Therefore, we have that $\partial_z F_i = -F_i \frac{\partial_z \Delta}{\Delta}$ and $\partial_{y_\alpha} F_i = \frac{\partial_{y_\alpha} \kappa_i}{n_i^2 \Delta} - F_i \frac{\partial_{y_\alpha} \Delta}{\Delta}$, where $\partial_z \Delta = -1/z^2 m^2$. Consequently, Eq. (4.16) becomes

$$\det(\partial F) = \frac{\det M}{z^2 \Delta^{N+1} m^2 \prod_{j=1}^{N} n_j^2}, \tag{4.19}$$

where the matrix M has the entries $M_{1j} = \kappa_j$ and $M_{ij} = \partial_{y_{i-1}} \kappa_j$ for $i \geq 2$, and so $\det M = \varepsilon_{i_1 i_2 \ldots i_N} \kappa_{i_1} \times \partial_{y_1} \kappa_{i_2} \partial_{y_2} \kappa_{i_3} \ldots \partial_{y_{N-1}} \kappa_{i_N}$. We now introduce the quantities

$$\iota_N \equiv \sum_{j=1}^{N} \kappa_j = 1, \qquad \iota_\alpha \equiv \sum_{j=1}^{\alpha} \kappa_j = 1 - \prod_{\beta=1}^{\alpha} y_\beta, \qquad \alpha = 1, 2, \ldots N-1. \tag{4.20}$$

Consider a matrix Λ with entries $\Lambda_{ij} = 1$ for $i \leq j$, and $\Lambda_{ij} = 0$ for $i > j$, and so $\det \Lambda = 1$. Therefore, the matrix $N \equiv M\Lambda$ has entries $N_{1j} = \iota_j$ and $N_{ij} = \partial_{y_{i-1}} \iota_j$ for $i \geq 2$, and so $\det M = \det N = \varepsilon_{i_1 i_2 \ldots i_N} \iota_{i_1} \partial_{y_1} \iota_{i_2} \partial_{y_2} \iota_{i_3} \ldots \partial_{y_{N-1}} \iota_{i_N}$.

Since $\iota_N = 1$, the only possibility for i_1 in this expression is $i_1 = N$, and since only ι_{N-1} depends on y_{N-1}, it follows that the only possibility for i_N is $i_N = N-1$. It then follows that the only possibility for i_{N-1} is $i_{N-1} = N-2$, and so on. Therefore,

$$\det M = \varepsilon_{N12\ldots N-1}\partial_{y_1}\iota_1\partial_{y_2}\iota_2\ldots\partial_{y_{N-1}}\iota_{N-1} = y_1^{N-2}y_2^{N-3}\ldots y_{N-3}^2 y_{N-2}, \tag{4.21}$$

and so

$$\det(\partial F) = \frac{\prod_{\beta=1}^{N-2} y_\beta^{N-\beta-1}}{z^2\Delta^{N+1}m^2\prod_{j=1}^N n_j^2}. \tag{4.22}$$

From Eqs. (4.15) and (4.22) one can determine f_N as

$$f_N^2 = -\left(\frac{a}{p}\right)^2 \frac{(-1)^{N(N-1)/2}2^N N!}{h_\xi h_{\varphi_1}\ldots h_{\varphi_N}h_z h_{y_1}\ldots h_{y_{N-1}}} \frac{\prod_{\beta=1}^{N-2} y_\beta^{N-\beta-1}}{z\Delta^N\lambda m\prod_{j=1}^N n_j}. \tag{4.23}$$

Since f_N is real, one needs

$$\mathrm{sign}\left[\lambda m\prod_{j=1}^N n_j\right] = -(-1)^{N(N-1)/2}. \tag{4.24}$$

Using Eq. (4.10) one gets that $h_\xi h_{\varphi_1}\ldots h_{\varphi_N}h_z h_{y_1}\ldots h_{y_{N-1}} = (\frac{a}{p})^{2N+1}\frac{z^{N-1}}{2^N}y_1^{N-2}y_2^{N-3}\ldots y_{N-2}$, and so

$$f_N = \sqrt{\left(\frac{p}{a}\right)^{2N-1}\frac{2^{2N}N!}{z^N\Delta^N|\lambda m\prod_{j=1}^N n_j|}}. \tag{4.25}$$

Let us now evaluate the topological charge Q_{2N+1} given in Eq. (1.6). Due to the self-duality equations (4.2), one can write it as

$$Q_{2N+1} = \int d^{2N+1}x\,\mathcal{Q}_{2N+1}, \qquad \mathcal{Q}_{2N+1} = \frac{2\lambda}{(4\pi)^{N+1}}f_N^2 A_p^2. \tag{4.26}$$

Using the solutions given in Eqs. (4.12) and (4.17), one gets that

$$A_p^2 = \frac{A_\xi^2}{h_\xi^2} + \sum_{i=1}^N \frac{A_{\varphi_i}^2}{h_{\varphi_i}^2} = \left(\frac{p}{a}\right)^2\frac{1}{z\Delta}. \tag{4.27}$$

Therefore, the density of topological charge is given by

$$\mathcal{Q}_{2N+1} = \frac{\mathrm{sign}(\lambda)2^{2N+1}N!}{(4\pi)^{N+1}|m\prod_{j=1}^N n_j|}\left(\frac{p}{a}\right)^{2N+1}\frac{1}{z^{N+1}\Delta^{N+1}}. \tag{4.28}$$

On the other hand, the volume element is

$$d^{2N+1}x = \left(\frac{a}{p}\right)^{2N+1}\frac{z^{N-1}}{2^N} \times y_1^{N-2}y_2^{N-3}\ldots y_{N-2}d\xi d\varphi_1\ldots d\varphi_N dz dy_1\ldots dy_{N-1}. \tag{4.29}$$

Integrating in the angles ξ and φ_i, $i = 1, 2, \ldots N$, one gets that

$$Q_{2N+1} = \frac{\mathrm{sign}(\lambda)N!}{|m\prod_{j=1}^N n_j|}\int dz dy_1\ldots dy_{N-1}\frac{y_1^{N-2}y_2^{N-3}\ldots y_{N-2}}{z^2\Delta^{N+1}} \tag{4.30}$$

Using the results of Appendix B [see Eq. (B8)] one gets that

$$Q_{2N+1} = \mathrm{sign}(\lambda)|m\prod_{j=1}^N n_j| = -(-1)^{N(N-1)/2}m\prod_{j=1}^N n_j, \tag{4.31}$$

where we have used Eq. (4.24).

V. CONCLUSIONS

In this paper, we have introduced Skyrme-type models in $(2N+2)$-dimensional Minkowski space-time with the target space being the spheres S^{2N+1}. The models do not have a gauge symmetry, and consequently in order to have finite-energy static solutions the fields must go to a constant at spatial infinity. Therefore, as long as topological considerations are concerned, the space submanifold $\mathbb{R}^{2N+1}$ can be compactified into $S^{2N+1}_{\rm space}$, and the static solutions define maps $S^{2N+1}_{\rm space} \to S^{2N+1}_{\rm target}$. The topological charge (winding number) associated to such maps has an integral representation, and therefore can be used to construct field theories with self-dual sectors as explained in the Introduction. We have used the freedom described in Eq. (1.5) to introduce an extra functional f that makes the theories conformally invariant in the space submanifold $\mathbb{R}^{2N+1}$. Using the methods of Ref. [20], we used the conformal group $SO(2N+2,1)$ and the target-space symmetry group $U(N+1)$ to construct a static ansatz based on a generalization of the toroidal coordinates to a space of $(2N+1)$ dimensions. The ansatz was then used to obtain an infinite number of solutions of the self-duality equations carrying nontrivial topological charges. Our construction generalizes the results obtained in Ref. [8] for the three-dimensional case ($N=1$). As shown in Refs. [7,8] the three-dimensional models do not present finite-energy solutions when the functional f is constant. This is a consequence of a theorem due to Chandrasekhar in the context of plasma and solar physics [22]. We believe the same happens for the models in $(2N+1)$ dimensions considered in this paper, and it would be interesting to generalize that theorem in a more general context.

As explained in the text, the number of possible ways to split the density of the topological charge grows substantially as N increases. Each one of these possibilities leads to a new model. For the five-dimensional case ($N = 2$) we have considered in detail the two possible models and constructed the topological self-dual Skyrmions for them. For the higher-dimensional cases ($N > 2$), we considered only one possibility corresponding to the case where the self-duality equations impose the vector A_i [defined in Eq. (1.7)] multiplied by the functional f_N^2, to be proportional to the Hodge dual of the exterior product of N tensors H_{ij} [defined in Eq. (1.8)]. This case is physically more interesting because the corresponding theory has a kinetic term quadratic in space-time derivatives of the fields. In addition, it does present restrictions on the possible values of the topological charges of the solutions.

The introduction of the functionals f_N in the splitting of the topological charges has lead to the conformal symmetry of the models in the space submanifold, and made possible the existence of finite-energy self-dual solutions of nontrivial topological charges. As we mentioned above, we believe that there cannot exist finite-energy solutions for such theories when these functionals are constants. Despite the important role played by such functionals, their physical nature is not well understood yet, and further studies are necessary to understand them. In addition, it would be interesting to investigate the breaking of the conformal symmetry and its effects on the soliton solutions.

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APPENDIX A: CONFORMAL SYMMETRY OF THE BPS EQUATIONS

In this appendix, we prove that the BPS equations are invariant under the conformal group on $\mathbb{R}^{2N+1}$. Let us consider infinitesimal coordinate transformations $\delta x_i = \zeta_i$, $i = 1, 2, \ldots 2N + 1$. We take the vector of complex fields Z_a, $a = 1, 2, \ldots N + 1$ [introduced in Eq. (1.7)] as scalar fields under such space transformations. Therefore, one has [20]

$$\delta Z = 0, \quad \delta A_i = -\partial_i \zeta_j A_j, \quad \delta H_{ij} = -\partial_i \zeta_k H_{kj} - \partial_j \zeta_k H_{ik}. \tag{A1}$$

The splitting of the topological charge (1.6) [see Eq. (1.1)] leads to self-duality equations of the form $\lambda f^2 w = \tilde{v}$, where w and v are differential forms constructed out of the vector and tensor fields A_i and H_{ij} [introduced in Eqs. (1.7) and (1.8)]. One has that w is a $(2p + 1)$-form as $w = A \wedge H \wedge H \ldots \wedge H$ (with p H's) and v is a $2(N - p)$-form as $v = H \wedge H \ldots \wedge H$ [with $(N - p)$ H's]. In addition, $\tilde{v}$ is the Hodge dual of v. In components, the self-duality equations read

$$\lambda f^2 w_{i_1 i_2 \ldots i_{2p+1}} = \frac{1}{2(N-p)!} \varepsilon_{i_1 i_2 \ldots i_{2p+1} j_1 j_2 \ldots j_{2(N-p)}} v_{j_1 j_2 \ldots j_{2(N-p)}}, \tag{A2}$$

with $\lambda = \pm Me$ [see Eq. (4.2)]. Using Eq. (A1), one then gets that the self-duality equations (A2) transform as

$$\begin{aligned}
&\lambda f^2 \left[2\frac{\delta f}{f} w_{i_1 i_2 \ldots i_{2p+1}} - \partial_{i_1} \zeta_k w_{k i_2 \ldots i_{2p+1}} - \partial_{i_2} \zeta_k w_{i_1 k \ldots i_{2p+1}} \ldots - \partial_{i_{2p+1}} \zeta_k w_{i_1 i_2 \ldots k} \right] \\
&= -\frac{\varepsilon_{i_1 i_2 \ldots i_{2p+1} j_1 j_2 \ldots j_{2(N-p)}}}{2(N-p)!} [\partial_{j_1} \zeta_k v_{k j_2 \ldots j_{2(N-p)}} + \partial_{j_2} \zeta_k v_{j_1 k \ldots j_{2(N-p)}} + \partial_{j_{2(N-p)}} \zeta_k v_{j_1 j_2 \ldots k}] \\
&= -\frac{\varepsilon_{i_1 i_2 \ldots i_{2p+1} j_1 j_2 \ldots j_{2(N-p)}}}{[2(N-p)-1]!} \partial_{j_1} \zeta_k v_{k j_2 \ldots j_{2(N-p)}} \\
&= -\frac{\varepsilon_{i_1 i_2 \ldots i_{2p+1} j_1 j_2 \ldots j_{2(N-p)}} \varepsilon_{l_1 l_2 \ldots l_{2p+1} k j_2 \ldots j_{2(N-p)}}}{[2(N-p)-1]!(2p+1)!} \lambda f^2 \partial_{j_1} \zeta_k w_{l_1 l_2 \ldots l_{2p+1}},
\end{aligned} \tag{A3}$$

where in the last equality we have used the Hodge dual of Eq. (A2). Therefore, in order for the self-duality equations to be invariant one needs that

$$\left[2\frac{\delta f}{f} + \partial_k \zeta_k \right] w_{i_1 i_2 \ldots i_{2p+1}} - (\partial_{i_1} \zeta_k + \partial_k \zeta_{i_1}) w_{k i_2 \ldots i_{2p+1}} \ldots - (\partial_{i_{2p+1}} \zeta_k + \partial_k \zeta_{i_{2p+1}}) w_{i_1 i_2 \ldots k} = 0.$$

Such a relation holds true if the space transformations are conformal, i.e., if the functions ζ_i satisfy

$$\partial_i\zeta_j + \partial_j\zeta_i = 2D\delta_{ij} \tag{A4}$$

for some function D, and if the transformation of the function f satisfies

$$\delta f = \frac{1}{2}[4p - 2N + 1]Df. \tag{A5}$$

As was shown in Ref. [20], the equations in Eq. (A4) are actually the equations that define the conformal transformations. Indeed, if D is a linear function of x_i, ζ_i corresponds to the special conformal transformations; if D is a constant, then ζ_i leads to the dilatations; and if $D = 0$, then ζ_i defines the translations and rotations.

In addition, one can check that

$$\delta(d^{2N+1}x) = (2N+1)Dd^{2N+1}x, \tag{A6}$$

$$\delta(f^2w^2) = -(2N+1)Df^2w^2, \tag{A7}$$

$$\delta(f^{-2}\tilde{v}^2) = -(2N+1)Df^{-2}\tilde{v}^2, \tag{A8}$$

$$\delta(w\tilde{v}) = -(2N+1)Dw\tilde{v}. \tag{A9}$$

Therefore, the topological charge $Q \sim \int d^{2N+1}xw\tilde{v}$ and the static energy given by $E \sim \int d^{2N+1}x[M^2f^2w^2 + \frac{\tilde{v}^2}{e^2f^2}]$ (with $\lambda = \pm Me$) are invariant under the conformal transformations in $\mathbb{R}^{2N+1}$.

Note that the functions f_1, f_I, and f_N, introduced in Eqs. (2.2), (3.2), and (4.1) respectively, correspond to the cases $p = 0$ and $N = 1$, $N = 2$, and $N = N$ respectively. The function f_{II}, introduced in Eq. (3.30), corresponds to the case $p = 1$ and $N = 2$. Therefore, from Eq. (A5), one has that such functions transform under the conformal group as

$$\frac{\delta f_1}{f_1} = -\frac{D}{2}, \qquad \frac{\delta f_I}{f_I} = -\frac{3}{2}D, \qquad \frac{\delta f_{II}}{f_{II}} = \frac{D}{2},$$
$$\frac{\delta f_N}{f_N} = -\frac{(2N-1)}{2}D. \tag{A10}$$

APPENDIX B: THE TOPOLOGICAL CHARGE INTEGRAL

In this appendix we evaluate the integral appearing in Eq. (4.30) for the topological charge. In fact, instead of evaluating it directly we find a recursive relation for such integrals. We start with the first one, corresponding to the case $N = 1$ and given by

$$I_1(m, n_1) \equiv \int_0^1 \frac{dz}{z^2}\frac{1}{\Delta_{(1)}^2} = m^2n_1^2, \quad \text{with}$$
$$\Delta_{(1)} \equiv \frac{1-z}{zm^2} + \frac{1}{n_1^2}. \tag{B1}$$

The second integral is

$$I_2(m, n_1, n_2) \equiv \int_0^1 \frac{dz}{z^2}\int_0^1 dy_1 \frac{1}{\Delta_{(2)}^3}, \quad \text{with}$$
$$\Delta_{(2)} \equiv \frac{1-z}{zm^2} + \frac{1-y_1}{n_1^2} + \frac{y_1}{n_2^2}. \tag{B2}$$

The quantity $\Delta_{(2)}$ in the denominator is linear in y_1 and so the y_1 integration can be easily performed to give

$$I_2(m, n_1, n_2) = \frac{1}{2}\frac{n_1^2n_2^2}{(n_1^2 - n_2^2)}[I_1(m, n_1) - I_1(m, n_2)]$$
$$= \frac{1}{2}m^2n_1^2n_2^2. \tag{B3}$$

We now consider the integral appearing in Eq. (4.30) for $N \geq 3$, which is given by

$$I_N(m, n_1, n_2 \ldots, n_N) \equiv \int_0^1 \frac{dz}{z^2}\int_0^1 dy_1 \ldots \int_0^1 dy_{N-1}$$
$$\times \frac{y_1^{N-2}y_2^{N-3}\ldots y_{N-2}}{\Delta_{(N)}^{N+1}}, \tag{B4}$$

where $\Delta_{(N)}$ is the same as Δ defined in Eq. (4.17), which we write here as

$$\Delta_{(N)} \equiv \Delta_{(N-1)} + b_{(N)}y_{N-1}, \tag{B5}$$

with

$$\Delta_{(N-1)} \equiv \frac{1-z}{zm^2} + \sum_{j=1}^{N-2}\frac{\kappa_j}{n_j^2} + \frac{1}{n_{N-1}^2}\prod_{\beta=1}^{N-2}y_\beta,$$
$$b_{(N)} \equiv \left[\frac{1}{n_N^2} - \frac{1}{n_{N-1}^2}\right]\prod_{\beta=1}^{N-2}y_\beta, \tag{B6}$$

where κ_j is defined in Eq. (4.18). Again, $\Delta_{(N)}$ is linear in y_{N-1} and the y_{N-1} integration leads to the recursion relation

$$I_N(m, n_1, n_2 \ldots, n_N) = \frac{1}{N}\frac{(n_{N-1}^2 - n_N^2)}{n_{N-1}^2n_N^2}$$
$$\times [I_{N-1}(m, n_1, n_2 \ldots, n_{N-2}, n_{N-1})$$
$$- I_{N-1}(m, n_1, n_2 \ldots, n_{N-2}, n_N)]. \tag{B7}$$

Using such a recursion relation, one gets that

$$I_N(m, n_1, n_2 \ldots, n_N) = \frac{1}{N!}m^2n_1^2n_2^2\ldots n_N^2. \tag{B8}$$

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