Gravitational entropy and the cosmological no-hair conjecture

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The gravitational entropy and no-hair conjectures seem to predict contradictory future states of our Universe. The growth of the gravitational entropy is associated with the growth of inhomogeneity, while the no-hair conjecture argues that a universe dominated by dark energy should asymptotically approach a homogeneous and isotropic de Sitter state. The aim of this paper is to study these two conjectures. The investigation is based on the Simsilun simulation, which simulates the universe using the approximation of the Silent Universe. The Silent Universe is a solution to the Einstein equations that assumes irrotational, nonviscous, and insulated dust, with vanishing magnetic part of the Weyl curvature. The initial conditions for the Simsilun simulation are sourced from the Millennium simulation, which results with a realistically appearing but relativistic at origin simulation of a universe. The Simsilun simulation is evolved from the early universe ($t = 25$ Myr) until far future ($t = 1000$ Gyr). The results of this investigation show that both conjectures are correct. On global scales, a universe with a positive cosmological constant and nonpositive spatial curvature does indeed approach the de Sitter state. At the same time it keeps generating the gravitational entropy.

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I. INTRODUCTION

Gravitational systems with their long range gravitational interactions have different properties than thermodynamic systems that we typically encounter on Earth. For example, a typical sequence of events for a gas injected into an empty box is to evolve from clumpiness towards homogeneity. However, for a system that is dominated by gravity, a reverse sequence of events is typically observed, and so the system evolves from homogeneity towards clumpiness [1]. It is still debatable, whether one can define a gravitational entropy, i.e. a quantity that is analogous to the thermodynamic entropy, which would encapsulate a typical behavior of all gravitational systems [2]. The issue of gravitational entropy is inevitably related to the issue of the cosmological arrow of time.

When in 1980s Penrose postulated the *Weyl curvature hypothesis* [1,3], the debate started whether this could serve as a meaningful measure of the cosmological arrow of time. The Weyl curvature hypothesis states that the universe starts with zero Weyl curvature and evolves to a state dominated by the Weyl curvature. The magnitude of the Weyl Curvature could thus be related to the arrow of time. However, the first attempt to link the Weyl curvature with the gravitational entropy was not fully successful, as such a definition of the gravitational entropy has problems with decaying modes [4]. Similarly, the definition of the gravitational entropy based on the ratio of the Weyl to Ricci curvatures has problems with radiation [5].

However, when Senovilla showed that the Bel-Robinson tensor can be used to construct a reasonable measure of the "energy" of the gravitation field [6], this promoted attempts to define the gravitational entropy based on the Bel-Robinson tensor [7,8]. Recently, Clifton, Ellis, and Tavakol showed how using the Bel-Robinson tensor one can construct an "effective energy-momentum tensor" of the gravitational field [2]. This allowed them to derive the formula for the gravitational entropy. The derivation was obtained in a similar way to a derivation of the thermodynamic entropy based on the energy-momentum tensor [2]. This new formula seems to meet requirements for the gravitational entropy, such as: (i) suitable limit for the Bekenstein-Hawking entropy of black holes [9,10], and (ii) for cosmological systems the increase of the gravitational entropy is associated with the growth of cosmic structures [2,11–15].

The procedure of defining the gravitational entropy based on the Bel-Robinson tensor requires a procedure of defining a *square root* of the Bel-Robinson tensor, which can only be done within spacetime of Petrov type D and N [6], and so this limits the applicability of such a procedure. However, this is not the only approach to the gravitational entropy that is being considered in the literature. Another approach, which seems quite promising for cosmological systems, is the one based on the Kullback-Leibler relative information entropy. Such a procedure has been suggested by Hosoya, Buchert, and Morita [16]. The gravitational

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entropy which is defined in this way is conjectured to grow in generic situations due to negative feedback of open gravitational systems, which is proved to hold for linear perturbations of an Einstein–de Sitter background model, and exact Lemaître-Tolman models [17]. It has also been shown that in the cosmological context the Kullback-Leibler relative information entropy can be well approximated by the Rényi relative entropy [18], and that it can be linked to Weyl curvature [19]. To distinguish this approach from the other, let us denote the entropy defined based on the information entropy as the *Hosoya-Buchert-Morita (HBM) gravitational entropy* and the gravitational entropy that is derived from the Bel-Robinson tensor as the *Clifton-Ellis-Tavakol (CET) gravitational entropy*.

In parallel to the gravitational entropy and Weyl curvature hypothesis, the idea of cosmic *inflation* was being developed [20–22]. During the cosmic inflation the dynamics of the universe is dominated by the scalar field and the universe rapidly evolves towards the de Sitter state. This observation, together with a number of other studies [23–25], formed foundations for the *cosmic no-hair conjecture*. The cosmic no-hair conjecture states that a universe with a positive cosmological constant evolves towards the de Sitter state. Although, for some configurations this does not occur [26,27], in general it is expected that our Universe will eventually evolve towards the de Sitter state. This is in contrast with what one expects based on the requirement of the growth of the gravitational entropy. This apparent contradiction motivates the research of this paper. The structure of this paper is as follows: Sec. II describes the Silent Universe; Sec. III sketches a derivation of the cosmological no-hair conjecture; Sec. IV derives the formula for the gravitational entropy of the silent universe; Sec. V presents and applies the Simsilun simulation to investigate the production rate of the gravitational entropy and future properties of the universe that is dominated by the cosmological constant; Sec. VI concludes the results.

II. SILENT UNIVERSE

A. Relativistic evolution of irrotational and insulated cosmic dust

We first assume that the gravitational field is sourced by the irrotational (no vorticity) and insulated (no heat transfer) dust with a cosmological constant. We then thread the spacetime with lines that are tangent to the flow of matter u^a, and slice the spacetime with surfaces that are orthogonal to u^a. This results with $1+3$ split and comoving coordinates [28,29]. Applying the energy-momentum conservation equations $T^{ab}{}_{;b}=0$, the Ricci identities $u_{a;d;c}-u_{a;c;d}=R_{abcd}u^b$, and the Bianchi identities $R_{ab[cd;e]}=0$, the evolution of the system is given by

$$\dot{\rho}+\Theta\rho=0, \tag{1}$$

$$\dot{\Theta}=-\frac{1}{3}\Theta^2-\frac{1}{2}\rho-2\sigma^2+\Lambda, \tag{2}$$

$$\dot{\sigma}_{\langle ab\rangle}=-\frac{2}{3}\Theta\sigma_{ab}-\sigma_{c\langle a}\sigma^c{}_{b\rangle}-E_{ab}, \tag{3}$$

$$\dot{E}_{\langle ab\rangle}=-\Theta E_{ab}-\frac{1}{2}\rho\sigma_{ab}+\mathrm{curl}H_{ab}+3\sigma_{\langle a}{}^cE_{b\rangle c}, \tag{4}$$

$$\dot{H}_{\langle ab\rangle}=-\Theta H_{ab}-\mathrm{curl}E_{ab}+3\sigma_{\langle a}{}^cH_{b\rangle c}. \tag{5}$$

In addition there are spatial constraints that follow from the spatial parts of the Ricci and Bianchi identities:

$$D^b\sigma_{ab}=\frac{2}{3}D_a\Theta, \tag{6}$$

$$H_{ab}=\mathrm{curl}\sigma_{ab}, \tag{7}$$

$$D^bE_{ab}=\frac{1}{3}D_a\rho+\epsilon_{abc}\sigma^b{}_dH^{cd}, \tag{8}$$

$$D^bH_{ab}=-\epsilon_{abc}\sigma^b{}_dE^{cd}. \tag{9}$$

The above set of equations is equivalent to the Einstein equations $G_{ab}-\Lambda g_{ab}=T_{ab}$. However, instead of solving the Einstein equations directly, we deal with equations that describe properties of dust (i.e. ρ), its velocity field (i.e. Θ, Σ), and spacetime geometry (i.e. E_{ab} and H_{ab}). Note that the constant $\kappa=8\pi G/c^4$ is assumed to be 1, which for a pressureless and insulated dust is equivalent to rescaling of density, i.e. $\kappa\rho\to\rho$.

B. Silent universe

The above equations describe a general relativistic evolution of irrotational and insulated cosmic dust. In the absence of pressure gradients, there are no sound waves making this universe almost "silent." In order to enforce strict "silence" and prevent any communication between the worldliness, we need to put $H_{ab}=0$, which will prevent propagation of gravitational waves. In such a case the above system of equations is expected to describe spacetimes that are Petrov D [30] with the shear and electric part of the Weyl tensors taking the form

$$\sigma_{ab}=\Sigma \mathrm{e}_{ab},\qquad E_{ab}=\mathcal{W}\mathrm{e}_{ab}, \tag{10}$$

where $\mathrm{e}_{ab}=h_{ab}-3z_az_b$, where z^a is a spacelike unit vector aligned with the Weyl principal tetrad. As a result the fluid equations (1)–(5) reduce only to four scalar equations [30,31]:

$$\dot{\rho}=-\rho\Theta, \tag{11}$$

$$\dot{\Theta}=-\frac{1}{3}\Theta^2-\frac{1}{2}\rho-6\Sigma^2+\Lambda, \tag{12}$$

$$\dot{\Sigma}=-\frac{2}{3}\Theta\Sigma+\Sigma^2-\mathcal{W}, \tag{13}$$

$$\dot{\mathcal{W}}=-\Theta\mathcal{W}-\frac{1}{2}\rho\Sigma-3\Sigma\mathcal{W}, \tag{14}$$

with the spatial constraints

$$D^b \sigma_{ab} = \frac{2}{3} D_a \Theta, \tag{15}$$

$$D^b E_{ab} = \frac{1}{3} D_a \rho. \tag{16}$$

III. COSMOLOGICAL NO-HAIR CONJECTURE

This section presents a heuristics derivation of the cosmological no-hair conjecture. This derivation should not be treated as a mathematically complete derivation, rather it should be treated as a point of reference for a further discussion. For a more strict derivation, the Reader is referred to Refs. [25,27,32,33].

Assuming a nonpositive spatial curvature

$$\mathcal{R} \leq 0, \tag{17}$$

it follows from Eq. (12) that

$$\dot{\Theta} \leq -\frac{1}{3}\Theta^2 + \Lambda \leq 0. \tag{18}$$

The first inequality follows from the fact that $\rho \geq 0$ and $\Sigma^2 \geq 0$, and the second follows from the Hamiltonian constraint,

$$-\frac{1}{3}\Theta^2 + \Lambda = -\rho - 3\Sigma^2 + \frac{1}{2}\mathcal{R}, \tag{19}$$

which shows that for $\mathcal{R} \leq 0$ the left-hand side of the above equation cannot be positive. Thus, from Eq. (18) it follows that the expansion rate decreases and

$$\Theta \to \sqrt{3\Lambda}. \tag{20}$$

If this happens then, as follows from the Hamiltonian constraint (19),

$$\rho + 3\Sigma^2 \to \frac{1}{2}\mathcal{R}.$$

However, since $\mathcal{R} \leq 0$, this implies that

$$\rho \to 0, \qquad \Sigma^2 \to 0, \qquad \mathcal{R} \to 0,$$

and from (14)

$$\mathcal{W} \to 0.$$

As a result, the universe asymptotically approaches the de Sitter space—spatially flat, homogeneous and isotropic FLRW model, with the expansion rate $\Theta = \sqrt{3\Lambda}$.

Conjecture 1 (cosmological no-hair conjecture).— A universe with a nonpositive spatial curvature and positive cosmological constant asymptotically evolves towards the de Sitter universe.

IV. GRAVITATIONAL ENTROPY

A. CET gravitational entropy

In analogy to thermodynamic and relativistic systems, one can define an "effective" energy-momentum tensor of the free gravitational field [2]

$$\mathcal{T}^{ab} = \rho_{\rm grav} u^a u^b + p_{\rm grav} h^{ab} + \Pi^{ab}_{\rm grav} + 2q^{(a}_{\rm grav} u^{b)}, \tag{21}$$

which for Petrov D spacetimes is [2]

$$\begin{aligned}
q^a_{\rm grav} &= 0,\\
p_{\rm grav} &= 0,\\
\Pi^{ab}_{\rm grav} &= \frac{\alpha}{4\pi}|\Psi_2|(x_a x_b + y_a y_b - z_a z_b + u^a u^b),\\
\rho_{\rm grav} &= \frac{\alpha}{4\pi}|\Psi_2| = \frac{\alpha}{4\pi}\mathcal{W}{\rm sgn}(\mathcal{W}),\\
T_{\rm grav} &= \frac{1}{2\pi}\left(\frac{1}{3}\Theta - 2\Sigma\right),
\end{aligned} \tag{22}$$

where Ψ_2 is the conformal Newman-Penrose invariant, α is a constant, and ${\rm sgn}(\mathcal{W})$ is the sign of $\mathcal{W}$, i.e. ${\rm sgn}(\mathcal{W}) = |\mathcal{W}|/\mathcal{W}$.

The growth of the gravitational entropy is thus

$$\dot{s}_{\rm CET} = \frac{1}{T_{\rm grav}}(\rho_{\rm grav}\delta v)^{\dot{}}, \tag{23}$$

where δv is the local volume element. Since the rate of change of volume is proportional to the expansion rate

$$\delta\dot{v} = \delta v \Theta, \tag{24}$$

thus

$$\dot{s}_{\rm CET} = \frac{\delta v}{T_{\rm grav}}(\dot{\rho}_{\rm grav} + \rho_{\rm grav}\Theta). \tag{25}$$

Finally using Eq. (14),

$$\dot{s}_{\rm CET} = -\alpha\frac{3}{4}\frac{\rho\Sigma + 6\Sigma\mathcal{W}}{|\Theta - 6\Sigma|}{\rm sgn}(\mathcal{W})\delta v. \tag{26}$$

Below, out of convenience, the arbitrary constant α is set to

$$\alpha = \frac{4}{3H_0^2},$$

where H_0 is the Hubble constant (H_0 has the same units as Θ and Σ and $\rho^{1/2}$). Thus, for the silent universe, the growth of the gravitational entropy is

$$\dot{s}_{\rm CET} = -\frac{\Sigma}{H_0^2}\frac{\rho + 6\mathcal{W}}{|\Theta - 6\Sigma|}\frac{\mathcal{W}}{|\mathcal{W}|}\delta v. \tag{27}$$

Integrating over the whole domain $\mathcal{D}$, the change of rate of the gravitational entropy of the silent universe is

$$\dot{S}_{\rm CET} = -\int_{\mathcal{D}} \delta v \frac{\Sigma}{H_0^2}\frac{\rho + 6\mathcal{W}}{|\Theta - 6\Sigma|}\frac{\mathcal{W}}{|\mathcal{W}|}. \tag{28}$$

B. HBM gravitational entropy

In analogy to information entropy when the relative entropy measures how one distribution diverges from the other, Hosoya, Buchert, and Morita suggested to define the gravitational entropy as a measure of divergence of the matter density field from its global average [16],

$$S_{\rm HBM} = \int_{\mathcal{D}} \delta v \rho \ln\frac{\rho}{\langle\rho\rangle_{\mathcal{D}}}, \tag{29}$$

where $\langle\rho\rangle_{\mathcal{D}}$ is the volume average density

$$\langle\rho\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \delta v \rho. \tag{30}$$

To make the units of $S_{\rm HBM}$ the same as of $S_{\rm CET}$ we scale it by H_0^3 and so the rate of change of the HBM gravitational entropy can be written as [16]

$$\dot{S}_{\rm HBM} = -\frac{1}{H_0^3}\left(\int_{\mathcal{D}} \delta v \rho \Theta\right) + \frac{1}{H_0^3 V_{\mathcal{D}}}\left(\int_{\mathcal{D}} \delta v \rho\right)\left(\int_{\mathcal{D}} \delta v \Theta\right). \tag{31}$$

C. Gravitational entropy in the early universe, i.e. small perturbations around the Einstein–de Sitter model

The early universe is often described using the Einstein–de Sitter model. The reason for that is that the contribution for spatial curvature $\mathcal{R}$ and the cosmological constant Λ is negligible small compared to the contribution from matter energy density ρ. In addition, if the distribution of matter is sufficiently uniform (standard assumption in cosmology) then it seems that the application of the Einstein–de Sitter model to describe the properties of the early universe is justified. In such a case, the Hamiltonian constraint (19) reduces to

$$3\bar{\rho} = \bar{\Theta}^2, \tag{32}$$

where the bar is used to denote the Einstein–de Sitter model. The early universe is not strictly spatially homogeneous and isotropic, but there are perturbations around the Einstein–de Sitter background,

$$\rho = \bar{\rho} + \Delta\rho \quad \text{and} \quad \Theta = \bar{\Theta} + \Delta\Theta.$$

If the perturbations are small and dominated by the growing mode then [34]

$$\Delta\rho = \bar{\rho}\delta \quad \text{and} \quad \Delta\Theta = -\frac{1}{3}\bar{\Theta}\delta.$$

Inserting (10) to (15) and (16),

$$\mathrm{e}_{ab} D^b \Sigma + \Sigma D^b \mathrm{e}_{ab} = \frac{2}{3} D_a \Theta = -\frac{2}{9}\bar{\Theta}^2 D_a \delta, \tag{33}$$

$$\mathrm{e}_{ab} D^b \mathcal{W} + \mathcal{W} D^b \mathrm{e}_{ab} = \frac{1}{3} D_a \rho = \frac{1}{3}\bar{\rho} D_a \delta_i = \frac{1}{9}\bar{\Theta}^2 D_a \delta. \tag{34}$$

Comparing the right-hand sides of the above equations and neglecting higher order terms, such as $\Sigma\delta$ we arrive at

$$\mathcal{W} = -\frac{1}{2}\bar{\Theta}\Sigma = -\frac{3}{2}\frac{\Sigma}{\bar{\Theta}}\bar{\rho}. \tag{35}$$

Inserting above to (28),

$$\dot{S}_{\rm CET} = \int_{\mathcal{D}} \delta v \frac{3}{2}\frac{\Sigma^2}{H_0^2}\frac{\bar{\rho} + 6\mathcal{W}}{|\bar{\Theta} - 6\Sigma|}\frac{\bar{\rho}}{|\mathcal{W}|\bar{\Theta}} \approx \int_{\mathcal{D}} \delta v \frac{3}{2}\frac{\Sigma^2}{H_0^2}\frac{\bar{\rho}^2}{\bar{\Theta}^2|\mathcal{W}|}, \tag{36}$$

where the higher order terms have been dropped. The above formula, despite appearance of a second order quantity (i.e. Σ^2), is first order in perturbations ($\mathcal{W} \sim \Sigma$), however, unlike a first-order quantity it does not vanish after averaging over the whole domain, as the integrand is positive.

In the case of the HBM gravitational entropy, for small and compensated perturbations, the integral (31) reduces to

$$\dot{S}_{\rm HBM} = -\frac{1}{H_0^3}\int_{\mathcal{D}} \delta v \Delta\rho\Delta\Theta \approx \frac{1}{9}\frac{\bar{\Theta}^3}{H_0^3}\int_{\mathcal{D}} \delta v \delta^2. \tag{37}$$

Unlike the CET, this is truly the second-order quantity (the first order quantities have been integrated out) and within the applicability of the above assumptions, the growth rate of the HBM gravitational entropy is positive.

D. Gravitational entropy conjecture

In both cases (CET and HBM), the growth of the gravitational entropy vanishes in the FLRW case. In the FLRW case the shear Σ and Weyl curvature $\mathcal{W}$ vanish and so the integrand (28) vanishes leading to $\dot{S}_{\rm CET} = 0$. For the HBM case, in the FLRW regime, the first term in (31) is equal to the second one and so $\dot{S}_{\rm HBM} = 0$. Thus, as expected, *the*

FLRW models do not produce the gravitational entropy. Treating this as a logical proposition, the negation of the reverse is also a logically correct statement.

Proposition 1.—Any universe that generates gravitational entropy cannot belong to a family of spatially homogeneous and isotropic FLRW models.

As shown in Sec. IV C, for small perturbations around the Einstein–de Sitter model the production rate of the gravitational entropy is positive. Thus, it seems that it is reasonable to expect that a realistic model of a universe can be characterized with a positive rate of change of the gravitational entropy. Therefore, the following conjecture is postulated.

Conjecture 2 (cosmological gravitational entropy conjecture).—The evolution of the universe proceeds in such a way that it keeps generating the gravitational entropy.

The above is just a conjecture as it is based on properties of small perturbations. In the next section we will test this conjecture by performing a simulation that will allow us to trace the evolution far into the nonlinear regime.

V. RESULTS

Conjecture 2 (the cosmological gravitational entropy conjecture) together with Proposition 1 seem to be in contradiction with Conjecture 1 (the cosmological no-hair conjecture), which postulates that a universe with a positive cosmological constant will end up as a homogeneous and isotropic de Sitter model. In this section we will test these conjectures using the Simsilun simulation [35].

A. The Simsilun simulation

The Simsilun simulation is based on the code SIMSILUN [36]. The description of the code, equations, and its applications are described in the "Methods Paper" [35]. The Methods Paper describes how one can use the Millennium simulation [37–39] to set up the initial conditions for the code SIMSILUN. The Simsilun simulation is based on solving Eqs. (11)–(14), with the initial conditions given by

$$\rho_i = \bar{\rho} + \Delta\rho = \bar{\rho}(1 + \delta_i), \tag{38}$$

$$\Theta_i = \bar{\Theta} + \Delta\Theta = \bar{\Theta}\left(1 - \frac{1}{3}\delta_i\right), \tag{39}$$

$$\Sigma_i = -\frac{1}{3}\Delta\Theta = \frac{1}{9}\bar{\Theta}\delta_i, \tag{40}$$

$$\mathcal{W}_i = -\frac{1}{6}\bar{\rho}\delta_i, \tag{41}$$

where the subscript i denotes the initial values, and δ_i is the initial density contrast sourced from the Millennium Simulation [37–39]. Here we use the MFIELD, which stores the matter distribution smoothed with a Gaussian kernel of radius $2.5h^{-1}$ Mpc. Since the MFIELD consists of 256^3 cells thus the resulted simulation, referred to as the Simsilun simulation, consists of 16 777 216 worldlines. In addition, the virialization mechanism number 1 is implemented, whose technical details are as described in Sec. III in the Methods Paper [35]. The Millennium Simulation is based on the ΛCDM model with $\Omega_M = 0.25$, $\Omega_\Lambda = 0.75$, and $H_0 = 73.0\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. This background model meets all the requirements for the applicably of the cosmic no-hair conjecture: it contains a positive cosmological constant and nonpositive spatial curvature [25], and it should asymptotically approach the de Sitter solution.

B. Gravitational entropy and the cosmological "no-hair" conjectures

We calculate the evolution of the universe as described in the Methods Paper [35], but instead of stopping at the present day, the evolution of the system is followed until $t = 1000$ Gyr. Also, in addition to the evolution equations (11)–(14), the volume of each element (cell) δv is evolved using

$$\delta\dot{v} = \delta v\Theta. \tag{42}$$

Finally, the average properties of the Simsilun simulation are evaluated using the volume averages,

$$\rho_{\mathcal{D}} = \langle\rho\rangle_{\mathcal{D}} = \frac{\sum_j \delta v_j \rho_j}{\sum_j \delta v_j}, \tag{43}$$

$$\Theta_{\mathcal{D}} = \langle\Theta\rangle_{\mathcal{D}} = \frac{\sum_j \delta v_j \Theta_j}{\sum_j \delta v_j}, \tag{44}$$

$$\Sigma_{\mathcal{D}} = \langle\Sigma\rangle_{\mathcal{D}} = \frac{\sum_j \delta v_j \Sigma_j}{\sum_j \delta v_j}, \tag{45}$$

$$\mathcal{W}_{\mathcal{D}} = \langle\mathcal{W}\rangle_{\mathcal{D}} = \frac{\sum_j \delta v_j \mathcal{W}_j}{\sum_j \delta v_j}, \tag{46}$$

where ρ_j, Θ_j, Σ_j, and $\mathcal{W}_j$ are quantities evaluated at each cell, whose volume is δv_j. The volume of the domain of averaging $\mathcal{D}$ is the total volume of the Simsilun simulation and is evaluated as

$$V_{\mathcal{D}} = \sum_j \delta v_j. \tag{47}$$

The volume averaged properties of the Simsilun simulation and their evolution is presented in Fig. 1. The evolution has been evaluated until $t = 1000$ Gyr (for clarity of presentation, Fig. 1 presents only evolution until $t = 100$ Gyr). Also, the evolution of shear Σ is multiplied by -1 so that it can be presented in the log-y plot. The presented results show that the volume averaged properties of the Simsilun simulation asymptotically approach the

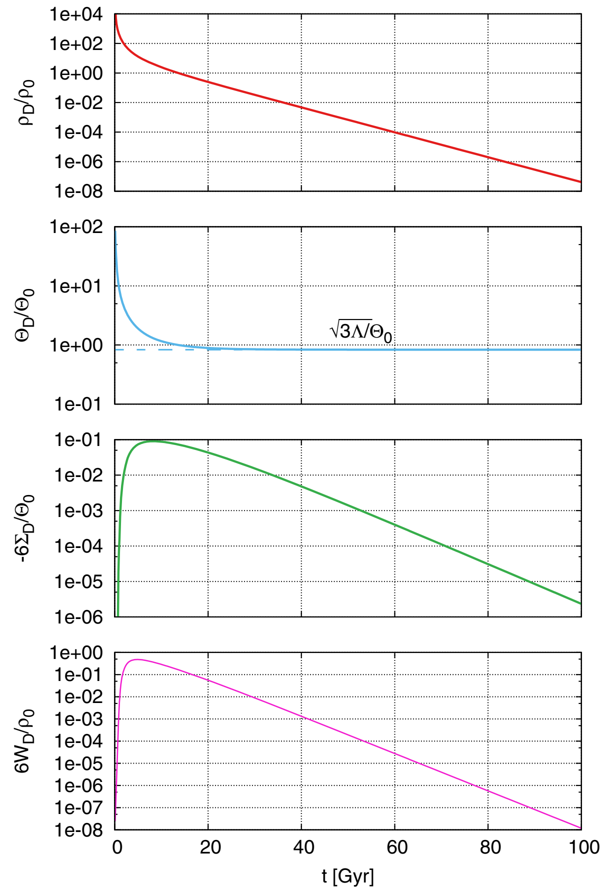

FIG. 1. The evolution of the volume averages of the density field normalized by the present-day density (upper most panel); expansion rate normalized by the present-day expansion rate (second upper panel); shear normalized by the one-sixth of the present-day expansion rate [cf. (27)] and multiplied by −1 so that is can be presented in the log-y plot (second lower panel); the Weyl curvature normalized by the one-sixth of the present-day density (lower most panel). As seen, asymptotically the system approaches the de Sitter state, i.e. $\rho_{\mathcal{D}} \to 0$, $\Sigma_{\mathcal{D}} \to 0$, $\mathcal{W}_{\mathcal{D}} \to 0$, and $\Theta \to \sqrt{3\Lambda}$. Also, the product of the shear and Weyl curvature is negative $\Sigma\mathcal{W} < 0$ which as follows from (27) should imply a non-negative rate of change of the gravitational entropy.

de Sitter state, i.e. $\rho_{\mathcal{D}} \to 0$, $\Sigma_{\mathcal{D}} \to 0$, $\mathcal{W}_{\mathcal{D}} \to 0$, and $\Theta \to \sqrt{3\Lambda}$. Thus, these results seem to confirm the no-hair conjecture.

However, the results presented in Fig. 2 also confirm the gravitational entropy conjecture as they show the positive rate of change of both CET and HBM gravitational entropies. The rate of change of the gravitational entropy peaks a few billion years after the big bang, at the similar time scale when the shear and Weyl curvature reach their maximum amplitude (cf. Fig. 1). Thus, both formulas for the gravitational entropy (CET and HBM) provide a similar picture. This is not surprising as it has been shown that these two formulas are correlated [19]. However, what is surprising is that when the universe approaches the de Sitter stage, the rate of change decreases, but does not vanish and after approximately 100 Gyr remains constant.

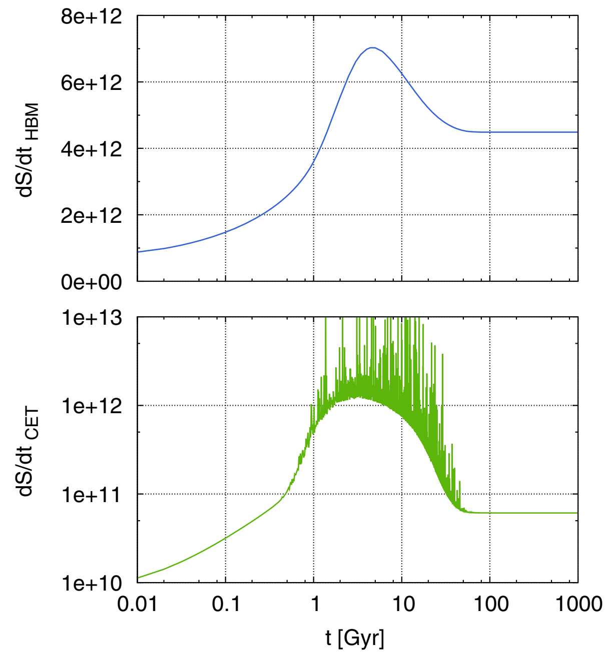

FIG. 2. The rate of change of the gravitational entropy within the whole domain of the Simsilun simulation. Upper panel: change of rate of the HBM gravitational entropy; lower panel: change of rate of the CET gravitational entropy.

The CET gravitational entropy features a number of spikes. The origin of these spikes is explained in Fig. 3, which presents the rate of change of the CET gravitational entropy for a single underdense and a single overdense cell. The initial conditions for the underdense and overdense regions are $\delta_i = -0.02$ and $\delta_i = 0.02$ respectively [cf. (38)–(41)] and the rate of change of their gravitational entropies follows from (27).

For the overdense region (upper panel in Fig. 3), approximately after 3 Gyr of evolution the expansion rate slows down and $\Theta - 6\Sigma \to 0$. This results with a spike (due to a finite numerical step of integration, the spike does not reach ∞, however in reality it does). After the overdensity becomes virialized, the production rate of its gravitational entropy becomes constant. This is in contrast with the no-hair conjecture, but as stated in Sec. III, the cosmological no-hair conjecture does not apply if the spatial curvature is positive. For the region with a positive spatial curvature, which undergoes collapse and eventually virialization, the future asymptotic state is not the de Sitter state and the production rate of the gravitational entropy does not asymptomatically vanish. The rate of change of the gravitational entropy does asymptotically vanish for the underdense region (lower panel in Fig. 3), where the spatial curvature is negative and where the cosmological no-hair conjecture does apply, which is also in agreement with results presented in Ref. [14] where the evolution of cosmic voids and their gravitational entropy was investigated using the Lemaître–Tolman model.

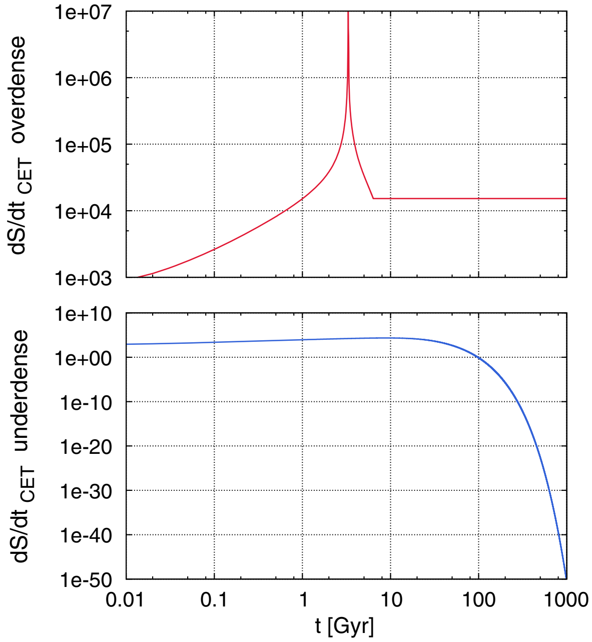

FIG. 3. The rate of change of the CET gravitational entropy for a single overdense cell (upper panel), and a single underdense cell (lower panel).

The results of Fig. 3 allow one to understand the results of Fig. 2, which shows that the rate of change of the gravitational entropy does not asymptotically vanish even though (as seen from Fig. 1) the Simsilun simulation asymptotically approaches the de Sitter state. The reason for this is following: the virialized overdense regions occupy little volume, and since they do not expand, thus with time, their contribution to the total volume is negligibly small, and therefore their contribution to the volume averages (cf. Fig. 1) is negligibly small. The Simsilun simulation consists of overdense and underdense regions. As a result, the volume of the Simsilun simulation is asymptotically dominated by underdense regions, but the production of the gravitational entropy is asymptotically dominated by overdense regions. The reason why overdense regions produce the gravitational entropy is linked to the fact that in the expanding universe virialized regions have nonzero shear [which is the source of the CET gravitational entropy, Eq. (28)] and their density does not asymptotically approach the de Sitter limit [i.e. $\rho/\langle\rho\rangle_{\mathcal{D}} \neq 1$ and asymptotically diverges, which sources the HBM entropy, Eq. (29)]. Therefore, even though the volume averaged properties of the Simsilun simulation asymptotically approach the de Sitter state, the rate of change of the gravitational entropy does not asymptotically vanish.

VI. CONCLUSIONS

This paper investigated the cosmological no-hair and gravitational entropy conjectures. The investigation was based on the Simsilun simulation [35]. The Simsilun simulation simulates the universe using the approximation to the Einstein equations, which is based on the silent universes [30,31]. In addition, the Simsilun simulation uses the initial data sourced from the Millennium simulation [37–39].

The obtained results show that the global properties of the Simsilun universe asymptotically approach the de Sitter state (cf. Fig. 1). This result confirms the cosmological no-hair conjecture, which stipulates that a universe with a nonpositive spatial curvature and positive cosmological constant asymptotically approaches the de Sitter state. On the other hand, the results obtained within the Simsilun simulation also confirm the gravitational entropy conjecture (cf. Fig. 2), which states that the evolution of the universe should be associated with the production on the gravitational entropy.

Within the Simsilun simulation, the production of the gravitational entropy is related to the evolution of cosmic structures and the presence of virialized objects (cf. Fig. 3). For underdense regions the gravitational entropy saturates (i.e. the production rate asymptomatically vanishes, cf. Ref. [14] which studied the evolution of cosmic voids and their gravitational entropy). The Simsilun simulation consists of 16 777 216 cells with the average cell's size (at the present-day instant) of a few Mpc. Increasing the resolution and decreasing the size of the cells would require inclusion of several phenomena, which are not included in the Simsilun simulation but are non-negligible on sub-Mpc scales such as rotation and pressure gradients.

In the Simsilun simulation there is no rotation, nor gradients of pressure which could prevent the collapse [40], and so the virialization needs to be externally implemented (cf. [41–43]). This is a weak part of the Simsilun simulation and thus the nonzero production rate of the gravitational entropy of the virialized structures should be treated qualitatively. For quantitative results, more realistic simulations are needed, for example the one based on the relativistic Zeldovich approximation (RZA) [44–49]. The RZA is a general-relativistic approximation that extends the standard perturbation theory. Recently, it has been shown that the RZA can successfully describe collapsing structures and is comparable with Newtonian simulations but includes the relativistic effects [50].

In addition, it should be noted that the Simsilun simulation is based on the Silent Universes which are Petrov type D. This means that there are no gravitational waves within the Simsilun simulation. Since recent detections of gravitational radiation [51–54] we know that our Universe should have a large number of sources of gravitational radiation. For gravitational waves the formula (28) (the CET case) which was derived for Petrov D does not apply [2], however formula (31) (the HBM case) should still hold. In addition the gravitational waves deform the spacetime producing the so-called *memory* *effect* [55–60], which will also contribute to the gravitational entropy. Thus the presence of gravitational waves does affect the rate of change of the gravitational entropy. For example, in the case of Petrov D spacetimes, inside cosmic voids the production rate of the gravitational entropy asymptomatically vanishes. Yet, with the inclusion of the gravitational waves and the memory effect this may change and lead to a nonzero production rate of the gravitational entropy inside cosmic voids. Thus more work is required in the context of the gravitational entropy generated by the gravitational waves.

In summary, even though the cosmological no-hair and gravitational entropy conjectures appear, at first sight, in contradiction, they both correctly capture properties on a universe with a positive cosmological constant and nonpositive spatial curvature. Therefore, we should expect that our own universe will keep producing the gravitational entropy, even though in the far future its global properties will approach the de Sitter state.

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[1] R. Penrose, Difficulties with inflationary cosmology, Ann. N.Y. Acad. Sci. **571**, 249 (1989).
[2] T. Clifton, G. F. R. Ellis, and R. Tavakol, A gravitational entropy proposal, Classical Quantum Gravity **30**, 125009 (2013).
[3] R. Penrose, *The emperor's new mind. Concerning computers, minds and laws of physics* (Oxford University Press, Oxford, England, 1989).
[4] T. Rothman and P. Anninos, Phase space approach to the gravitational arrow of time, Phys. Rev. D **55**, 1948 (1997).
[5] W. B. Bonnor, Arrow of time for a collapsing and radiating sphere, Phys. Lett. A **122**, 305 (1987).
[6] J. M. M. Senovilla, Super-energy tensors, Classical Quantum Gravity **17**, 2799 (2000).
[7] N. Pelavas and K. Lake, Measures of gravitational entropy: Self-similar spacetimes, Phys. Rev. D **62**, 044009 (2000).
[8] N. Pelavas and A. Coley, Gravitational entropy in cosmological models, Int. J. Theor. Phys. **45**, 1258 (2006).
[9] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[10] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).
[11] K. Bolejko and W. R. Stoeger, Intermediate homogenization of the Universe and the problem of gravitational entropy, Phys. Rev. D **88**, 063529 (2013).
[12] R. A. Sussman and J. Larena, Gravitational entropies in LTB dust models, Classical Quantum Gravity **31**, 075021 (2014).
[13] R. A. Sussman, Gravitational entropy of cosmic expansion, Astron. Nachr. **335**, 587 (2014).
[14] R. A. Sussman and J. Larena, Gravitational entropy of local cosmic voids, Classical Quantum Gravity **32**, 165012 (2015).
[15] G. Marozzi, J.-P. Uzan, O. Umeh, and C. Clarkson, Cosmological evolution of the gravitational entropy of the large-scale structure, Gen. Relativ. Gravit. **47**, 114 (2015).
[16] A. Hosoya, T. Buchert, and M. Morita, Information Entropy in Cosmology, Phys. Rev. Lett. **92**, 141302 (2004).
[17] M. Morita, T. Buchert, A. Hosoya, and N. Li, Relative information entropy of an inhomogeneous universe, edited by J.-M. Alimi and A. Fuözfa, *AIP Conference Proceedings* (AIP, New York, 2010), Vol. 1241, pp. 1074–1082.
[18] V. G. Czinner and F. C. Mena, Relative information entropy in cosmology: The problem of information entanglement, Phys. Lett. B **758**, 9 (2016).
[19] N. Li, T. Buchert, A. Hosoya, M. Morita, and D. J. Schwarz, Relative information entropy and Weyl curvature of the inhomogeneous Universe, Phys. Rev. D **86**, 083539 (2012).
[20] A. A. Starobinskii, Spectrum of relict gravitational radiation and the early state of the universe, JETP Lett. **30**, 682 (1979).
[21] A. H. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D **23**, 347 (1981).
[22] A. D. Linde, A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems, Phys. Lett. B **108**, 389 (1982).
[23] R. M. Wald, Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant, Phys. Rev. D **28**, 2118 (1983).
[24] J. D. Barrow and J. Stein-Schabes, Inhomogeneous cosmologies with cosmological constant, Phys. Lett. A **103**, 315 (1984).
[25] T. Pacher and J. A. Stein-Schabes, On the locality of the no hair conjecture and the measure of the Universe, Ann. Phys. (N.Y.) **503**, 518 (1991).
[26] J. D. Barrow, The deflationary universe: An instability of the de Sitter universe, Phys. Lett. B **180**, 335 (1986).

[27] G. Götz, On the cosmological no-hair conjecture, Phys. Lett. A **128**, 129 (1988).

[28] G. F. R. Ellis, Relativistic cosmology, in *Proceedings of the International School of Physics "Enrico Fermi", Course 47: General relativity and cosmology*, edited by R. K. Sachs (Academic Press, New York, 1971), pp. 104–182.

[29] G. F. R. Ellis, Republication of: Relativistic cosmology, General Relativity and Gravitation **41**, 581 (2009).

[30] H. van Elst, C. Uggla, W. M. Lesame, G. F. R. Ellis, and R. Maartens, Integrability of irrotational silent cosmological models, Classical Quantum Gravity **14**, 1151 (1997).

[31] M. Bruni, S. Matarrese, and O. Pantano, Dynamics of silent universes, Astrophys. J. **445**, 958 (1995).

[32] M. Hadžić and J. Speck, The global future stability of the flrw solutions to the dust-Einstein system with a positive cosmological constant, J. Hyper. Differ. Equ. **12**, 87 (2015).

[33] T. A. Oliynyk, Future stability of the flrw fluid solutions in the presence of a positive cosmological constant, Commun. Math. Phys. **346**, 293 (2016).

[34] P. J. E. Peebles, *The Large-Scale Structure of the Universe* (Princeton University Press, Princeton, NJ, 1980).

[35] K. Bolejko, Relativistic numerical cosmology with silent universes, Classical Quantum Gravity **35**, 024003 (2018).

[36] https://bitbucket.org/bolejko/simsilun.

[37] V. Springel, S. D. M. White, A. Jenkins, C. S. Frenk, N. Yoshida, L. Gao, J. Navarro, R. Thacker, D. Croton, J. Helly, J. A. Peacock, S. Cole, P. Thomas, H. Couchman, A. Evrard, J. Colberg, and F. Pearce, Simulations of the formation, evolution and clustering of galaxies and quasars, Nature (London) **435**, 629 (2005).

[38] M. Boylan-Kolchin, V. Springel, S. D. M. White, A. Jenkins, and G. Lemson, Resolving cosmic structure formation with the Millennium-II Simulation, Mon. Not. R. Astron. Soc. **398**, 1150 (2009).

[39] Q. Guo, S. White, R. E. Angulo, B. Henriques, G. Lemson, M. Boylan-Kolchin, P. Thomas, and C. Short, Galaxy formation in WMAP1 and WMAP7 cosmologies, Mon. Not. R. Astron. Soc. **428**, 1351 (2013).

[40] K. Bolejko and P. D. Lasky, Pressure gradients, shell-crossing singularities and acoustic oscillations—Application to inhomogeneous cosmological models, Mon. Not. R. Astron. Soc. **391**, L59 (2008).

[41] K. Bolejko and P. G. Ferreira, Ricci focusing, shearing, and the expansion rate in an almost homogeneous Universe, J. Cosmol. Astropart. Phys. 05 (**2012**) 003.

[42] B. F. Roukema, J. J. Ostrowski, and T. Buchert, Virialisation-induced curvature as a physical explanation for dark energy, J. Cosmol. Astropart. Phys. 10 (2013) 043.

[43] B. F. Roukema, Replacing dark energy by silent virialisation, arXiv:1706.06179.

[44] T. Buchert and M. Ostermann, Lagrangian theory of structure formation in relativistic cosmology: Lagrangian framework and definition of a nonperturbative approximation, Phys. Rev. D **86**, 023520 (2012).

[45] T. Buchert, C. Nayet, and A. Wiegand, Lagrangian theory of structure formation in relativistic cosmology. II. Average properties of a generic evolution model, Phys. Rev. D **87**, 123503 (2013).

[46] A. Alles, T. Buchert, F. Al Roumi, and A. Wiegand, Lagrangian theory of structure formation in relativistic cosmology. III. Gravitoelectric perturbation and solution schemes at any order, Phys. Rev. D **92**, 023512 (2015).

[47] F. A. Roumi, T. Buchert, and A. Wiegand, Lagrangian theory of structure formation in relativistic cosmology IV: Lagrangian approach to gravitational waves, arXiv:1711.01597.

[48] R. A. Sussman and I. Delgado Gaspar, Multiple nonspherical structures from the extrema of Szekeres scalars, Phys. Rev. D **92**, 083533 (2015).

[49] R. A. Sussman, I. Delgado Gaspar, and J. C. Hidalgo, Coarse-grained description of cosmic structure from Szekeres models, J. Cosmol. Astropart. Phys. 03 (2016) 012.

[50] J. J. Ostrowski, T. Buchert, and B. F. Roukema, Mass function of galaxy clusters in relativistic inhomogeneous cosmology, Acta Polytech. B., Proc. Suppl. **10**, 407 (2017).

[51] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari *et al.*, Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[52] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari *et al.*, GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[53] B. P. Abbott, R. Abbott, T. D. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, V. B. Adya *et al.*, GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017).

[54] B. P. Abbott, R. Abbott, T. D. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, V. B. Adya *et al.*, GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[55] Y. B. Zel'dovich and A. G. Polnarev, Radiation of gravitational waves by a cluster of superdense stars, Sov. Astron. **18**, 17 (1974).

[56] V. B. Braginskii and K. S. Thorne, Gravitational-wave bursts with memory and experimental prospects, Nature (London) **327**, 123 (1987).

[57] D. Christodoulou, Nonlinear Nature of Gravitation and Gravitational-Wave Experiments, Phys. Rev. Lett. **67**, 1486 (1991).

[58] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman, and Y. Chen, Detecting Gravitational-Wave Memory with LIGO: Implications of GW150914, Phys. Rev. Lett. **117**, 061102 (2016).

[59] L. O. McNeill, E. Thrane, and P. D. Lasky, Detecting Gravitational Wave Memory without Parent Signals, Phys. Rev. Lett. **118**, 181103 (2017).

[60] W. Kulczycki and E. Malec, Axial gravitational waves in FLRW cosmology and memory effects, Phys. Rev. D **96**, 063523 (2017).