Anomalous decay $f_1(1285) \to \pi^+\pi^-\gamma$ in the Nambu–Jona-Lasinio model

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Using the Nambu–Jona-Lasinio model with the $U(2) \times U(2)$ chiral symmetric effective four-quark interactions, we derive the amplitude of the radiative decay $f_1(1285) \to \pi^+\pi^-\gamma$, find the decay width $\Gamma(f_1 \to \pi^+\pi^-\gamma) = 346$ keV and obtain the spectral dipion effective mass distribution. It is shown that in contrast to the majority of theoretical estimates (which consider the $a_1(1260)$ meson exchange as the dominant one), the most relevant contribution to this process is the ρ^0-resonance exchange related with the triangle $f_1\rho^0\gamma$ anomaly. The spectral function is obtained to be confronted with the future empirical data.

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I. INTRODUCTION

The anomalous radiative decay $f_1 \to \rho^0\gamma$ of the axial-vector $I^G(J^{PC}) = 0^+(1^{++})f_1(1285)$ meson has been described recently [1], with dynamics determined by the effective meson Lagrangian. For that the Nambu–Jona-Lasinio (NJL) model with $U(2) \times U(2)$ chiral symmetry spontaneously broken down to the diagonal $SU(2)_I \times U(1)_V$ subgroup [2–7] has been used (the quantum anomaly breaks the axial $U(1)_A$ symmetry). The model not only reproduces the nontrivial structure of the $f_1\rho\gamma$ vertex (which accumulates the most general restrictions imposed on it by gauge symmetry and statistics [8,9]), but also fixes fully the values of the coupling constants involved. The decay width that has been obtained, $\Gamma_{f_1\to\rho\gamma} = 311$ keV, is compatible with the recently published value $\Gamma_{f_1\to\rho^0\gamma} = 453 \pm 177$ keV [10] measured for the first time in photo-production from a proton target using the CLAS detector at Jefferson Laboratory. The calculated value, however, is four times smaller than the value $\Gamma_{f_1\to\rho^0\gamma} = 1326 \pm 313$ keV, quoted by the Particle Data Group [11]. Arguments based on the large N_c (number of colors) limit have been used to explain why the derivative expansion of the underlying triangle quark diagram is suitable for the phenomenological description of the $f_1 \to \rho^0\gamma$ decay amplitude.

In the present work we apply the same approach to describe the radiative decay $f_1(1285) \to \pi^+\pi^-\gamma$. We call attention to the fact that apart from the AVV triangle anomaly, which has been newly analyzed by the CLAS Collaboration [10], this process contains also information about the triangle AAA and the box $AAAV$ anomalies. Therefore, it would be interesting to analyze as well the role of these anomalies in the $f_1(1285) \to \pi^+\pi^-\gamma$ decay. Unfortunately, this transition has not been measured yet, although a clear signal of $f_1(1285)$ has been seen in the effective mass spectrum of the $\pi^+\pi^-\gamma$ system in the reaction $\pi^-N \to \pi^-\pi^+\pi^-\gamma N$ at a pion beam with the momentum $p_{\pi^-} = 37$ GeV/c studied at the VES spectrometer of IHEP [12]. The centrally produced exclusive final states formed in the reaction $pp \to p_f(\pi^+\pi^-\gamma)p_s$ at 300 GeV/c have been studied by the WA76 Collaboration at CERN Omega Spectrometer [13]. The $\pi^+\pi^-\gamma$ mass spectrum shows two enhancements, one at 0.96 GeV due to the $\eta'(958)$ and one at 1.27 GeV which could be due to the $f_1(1285)$. In particular, the WA76 Collaboration has measured the $\pi^+\pi^-$ mass spectrum from the reaction $f_1(1285) \to \pi^+\pi^-\gamma$, where a $\rho^0(770)$ signal can clearly be seen. Events $\gamma p \to pf_1(1285) \to p\pi^+\pi^-\gamma$ were also identified by CLAS [10] using kinematic fitting and time-of-flight selections.

An earlier attempt of the theoretical description of the $f_1(1285) \to \pi^+\pi^-\gamma$ decay width can be found in [14]. It is made in the framework of the $U(2)_L \times U(2)_R$ chiral theory of mesons. In this approach the amplitude gets two types of contributions: the direct coupling $f_1 \to \pi\pi\gamma$ and the $a_1(1260)$-meson exchange $f_1 \to a_1\pi \to \pi\pi\gamma$. The ρ exchange was not taken into account because the Bardeen's form of the non-Abelian anomaly [15–17] used there forbids the $f_1 \to \rho^0\gamma$ transition. As a result, the estimate $\Gamma_{f_1\to\pi\pi\gamma} = 18.5$ keV has been obtained. A consistent scheme [18,19] contains the anomalous $f_1\rho^0\gamma$ vertex. Its contribution through the channel $f_1 \to \rho^0\gamma \to \pi\pi\gamma$ can be more important than the a_1 exchange. The reason for

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this is very simple. The a_1 exchange is described by two propagators $(m_{a_1}^2 - t)^{-1}$, or $(m_{a_1}^2 - u)^{-1}$. The physical region for the kinematical variables t and u is given by $m_\pi \le \sqrt{t}, \sqrt{u} \le (m_{f_1} - m_\pi)$. It is clear that both propagators have no poles at physical values of meson masses. Contrary to this, the propagator of the ρ meson contributes as $(m_\rho^2 - s)^{-1}$, where $2m_\pi \le \sqrt{s} \le m_{f_1}$. Thus, it has a pole on the real axis (to lowest order in $1/N_c$). This certainly indicates that a contribution from the a_1 exchange alone is not sufficient.

The purpose of this paper is to examine the role of the vector $\rho(770)$ and axial-vector $a_1(1260)$ resonances in the radiative decay $f_1(1285) \to \pi^+\pi^-\gamma$ in more detail. On one hand, our treatment of the problem goes beyond the simplified version employed in [14]. On the other hand, our calculations of the dipion mass spectrum, may be further improved by taking into account the nonperturbative effects of final-state interactions. However, due to the absence of empirical data we postpone the calculations of these $1/N_c$ suppressed contributions.

The paper is organized as follows. In Sec. II we discuss the quark-meson Lagrangian of the NJL model and fix the coupling constants. In Sec. III we obtain the $f_1(1285) \to \pi^+\pi^-\gamma$ decay amplitude. Our estimates of the decay width and dipion spectral function are presented in Sec. IV. Finally, Sec. V contains our summary and conclusions.

II. EFFECTIVE LAGRANGIAN

The main tool of our study is the bosonized version of the NJL model with $U(2) \times U(2)$ chiral symmetric four-quark interactions. The effective meson vertices can be obtained through the derivative expansion of the underlying one-quark-loop diagrams. This can be done both in momentum space [2,3,6] or in position space [4,5,7]. In the latter case, the heat kernel technique adjusts the derivative expansion of the quark determinant in such a way that chiral symmetry is protected for each Seeley-DeWitt coefficient. The result is well known, thus we will write down here only the part which is responsible for quark-meson interactions governing the $f_1(1285) \to \pi^+\pi^-\gamma$ decay. The corresponding Lagrangian density is

$$\mathcal{L}_{\rm int} = \bar{q}\Big\{ i\gamma^\mu(\partial_\mu - ieA_\mu) - M + ig_\pi\gamma_5\vec{\pi}\vec{\tau} + \frac{g_\rho}{2}\gamma^\mu[\vec{\rho}_\mu\vec{\tau} + \gamma_5(\vec{a}'_{1\mu}\vec{\tau} + f_{1\mu}\tau_0)]\Big\} q. \quad (1)$$

Our notations are as follows: $\vec{\tau}$ are the standard $SU(2)$ Pauli matrices, τ_0 is a unit 2×2 matrix in the isospin space; γ^μ and γ_5 are Dirac matrices in a four dimensional Minkowski space. The light quark fields $q = (u, d)$ have color and 4-spinor indices which are suppressed. The diagonal matrix $M = m\tau_0$, where $m = 277$ MeV is a constituent quark mass, preserves isospin symmetry, i.e., $m = m_u = m_d$. $A_\mu = Q\mathcal{A}_\mu$, where $\mathcal{A}_\mu$ is the electromagnetic field, and Q is the matrix of the light quark's charges in relative units of the proton charge e

$$Q = \frac{1}{2}\left(\tau_3 + \frac{1}{3}\right). \quad (2)$$

The $\vec{\pi}$ and $\vec{\rho}_\mu$ are the field operators associated with the isotriplet of pions $\pi(140)$ and vector $\rho(770)$-mesons; $f_{1\mu}$ describes the isosinglet axial-vector $f_1(1285)$-meson, and $\vec{a}'_{1\mu}$ stands for the unphysical axial-vector field that should be redefined to avoid the $\vec{\pi} - \vec{a}'_{1\mu}$ mixing

$$\vec{a}'_{1\mu} = \vec{a}_{1\mu} + \sqrt{\frac{2Z}{3}}\kappa m\partial_\mu\vec{\pi}. \quad (3)$$

Here $Z = (1 - 6m^2/m_{a_1}^2)^{-1} = 1.4$, and a dimensionful parameter κ is fixed by requiring that the Lagrangian does not contain the $\vec{\pi} - \vec{a}_{1\mu}$ transitions, it gives $\kappa = 3/m_{a_1}^2$. The $\vec{a}_{1\mu}$ field represents a physical axial-vector state $a_1(1260)$ of mass $m_{a_1} = 1230 \pm 40$ MeV. In our estimates we take the value $m_{a_1} = 1260$ MeV.

Since the free part of the meson Lagrangian following from evaluation of the one-quark-loop self-energy diagrams must preserve its canonical form, one should renormalize the bare meson fields by introducing the Yukawa coupling constants g_π and g_ρ in Eq. (1). To absorb infinities of self-energy graphs, these couplings depend on the divergent integral which is regularized in a standard way [7]

$$J_1(m^2) = \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2}. \quad (4)$$

A finite ultraviolet cutoff $\Lambda = 1240$ MeV restricts the region of integration in the quark-loop integrals and characterizes the energy scale where the NJL model is applicable. Thus, we have

$$g_\pi = \sqrt{Z}g, \qquad g_\rho = \sqrt{6}g, \qquad g^2 = \frac{4\pi^2}{N_c J_1}. \quad (5)$$

The coupling g_π satisfies the quark analog of the Goldberger-Treiman relation, which is $m = f_\pi g_\pi$, where $f_\pi = 93$ MeV is the weak pion decay constant. g_ρ is the coupling of the $\rho \to \pi\pi$ decay ($g_\rho^2/4\pi = \alpha_\rho = 3$).

To summarize the above input data, we can state that, on the whole, the phenomenological value of $\alpha_\rho = 3$ leads [with the use of Eq. (5)] to the estimate $N_c J_1 = 2\pi$. That gives the ratio $\Lambda/m = 4.48$, and determines the coupling $g = \sqrt{2\pi}$. The Goldberger-Treiman relation contains the second input value f_π. However, this does not give us the value of m due to its dependence on factor Z. Namely, $m = f_\pi\sqrt{2\pi Z}$. To find Z we use the mass of the $a_1(1260)$ axial-vector meson. In this case, the Goldberger-Treiman relation leads to the quadratic equation with respect to $x = 6m^2/m_{a_1}^2 = 2\kappa m^2$, $x(1-x) = 12\pi(f_\pi/m_{a_1})^2$. Solving it and taking the root $x = 0.289$, one obtains that $Z = (1-x)^{-1} = 1.4$, $m = m_{a_1}\sqrt{x/6} = 277$ MeV, and $\Lambda = 4.48m = 1240$ MeV.

Let us consider now the effective meson vertices generated by the Lagrangian density (1) in the one-quark-loop approximation (large N_c limit) which describe the $f_1(1285) \to \pi^+\pi^-\gamma$ decay.

The two nonanomalous vertices needed for our calculations are [7]

$$\mathcal{L}_{\rho\pi\pi} = -i\frac{g_\rho}{4}\mathrm{tr}\left(\rho_\mu[\pi, \partial^\mu\pi] - \frac{Z-1}{2m_{a_1}^2}\rho_{\mu\nu}[\partial^\mu\pi, \partial^\nu\pi]\right), \qquad (6)$$

$$\mathcal{L}_{a_1\pi\gamma} = \frac{i}{2}f_\pi e g_\rho Z\mathrm{tr}\left\{a_1^\mu[A_\mu, \pi] + \frac{1}{m_{a_1}^2}(F_{\mu\nu}[a_1^\mu, \partial^\nu\pi] + a_1^{\mu\nu}[A_\mu, \partial_\nu\pi])\right\}, \qquad (7)$$

where $\rho_\mu = \vec{\rho}_\mu\vec{\tau}$, $a_{1\mu} = \vec{a}_{1\mu}\vec{\tau}$, $\pi = \vec{\pi}\vec{\tau}$; the quantities $\rho_{\mu\nu}, a_{1\mu\nu}, F_{\mu\nu}$ stand for the field strengths $\rho_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu$, $a_{1\mu\nu} = \partial_\mu a_{1\nu} - \partial_\nu a_{1\mu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the following we neglect the second term in (6). On the ρ-meson mass shell it has a small factor $(Z-1)(m_\rho/m_{a_1})^2/2 = 0.076$ (compared with the factor 1 of the first term).

Following [1], we may write down the effective Lagrangian density, which describes the $f_1\rho^0\gamma$ anomalous transition

$$\mathcal{L}_{f_1\rho^0\gamma} = -\frac{e\alpha_\rho}{8\pi m^2}e^{\mu\nu\alpha\beta}\left(\rho^0_{\mu\nu}F_{\alpha\sigma}\partial^\sigma f_{1\beta} + \frac{1}{2}f_{1\mu\nu}F_\alpha{}^\sigma\rho^0_{\sigma\beta} + F_{\mu\nu}\partial^\sigma\rho^0_{\sigma\alpha}f_{1\beta}\right) \qquad (8)$$

with the strength tensor $f_{1\mu\nu} = \partial_\mu f_{1\nu} - \partial_\nu f_{1\mu}$.

The other anomalous vertex describes the $f_1a_1\pi$ interaction

$$\mathcal{L}_{f_1a_1\pi} = g_{f_1a_1\pi}e^{\alpha\beta\mu\nu}f_{1\alpha}\partial_\mu\vec{a}_{1\beta}\partial_\nu\vec{\pi}, \qquad (9)$$

where

$$g_{f_1a_1\pi} = \frac{\alpha_\rho}{2\pi f_\pi}\left[1 + (1-3a)\frac{Z-1}{2Z}\right]. \qquad (10)$$

The second term in the brackets is due to the replacement (3). The derivative coupling $\bar{q}\gamma^\mu\gamma_5\partial_\mu\pi q$ makes the corresponding triangle quark diagram linearly divergent. A superficial linear divergence appears in the course of evaluation of the overall finite integral. Shifts in the internal momentum variable of the closed fermion loop integrals induce an arbitrary finite surface term contribution proportional to $(1-3a)$. Here a is a dimensionless constant, controlling the magnitude of an arbitrary local part [20,21]. Hereinafter it will be fixed by the requirement of gauge invariance of the $f_1(1285) \to \pi^+\pi^-\gamma$ decay amplitude.

III. AMPLITUDE OF THE $f_1(1285) \to \pi^+\pi^-\gamma$ DECAY

The Lagrangian density (1) generates three different types of contributions to the amplitude of the reaction $f_1(l) \to \pi^+(p_+) + \pi^-(p_-) + \gamma(p)$. These are the contributions through the intermediate ρ^0 and $a_1^\pm$ mesons and a contact interaction induced by the quark box diagram (Figs. 1–3). The corresponding amplitude can be written as follows

$$T = ie_{\mu\nu\alpha\beta}\epsilon^\beta(l)\epsilon^*_\gamma(p)[g^{\alpha\gamma}(F_1l^\mu p_+^\nu + F_2l^\mu p_-^\nu + F_3p_+^\mu p_-^\nu) + F_4p^\alpha l^\gamma p_+^\mu p_-^\nu] \qquad (11)$$

where $\epsilon_\beta(l)$, $\epsilon_\gamma(p)$ are the polarization vectors of the f_1-meson and the photon; l, p, p_+, p_- are the 4-momenta of f_1-meson, photon and charge pions. For the respective form factors $F_i = F_i^{(\rho)} + F_i^{(a_1)} + F_i^{(b)}$ ($i = 1, 2, 3, 4$) the diagram with the exchange of the ρ^0-meson (Fig. 1) gives

$$F_1^{(\rho)} = \frac{3eg_\rho}{Z(4\pi f_\pi)^2}\frac{m_{f_1}^2 - 2m_{f_1}(\varepsilon - \varepsilon_-)}{m_\rho^2 - s - im_\rho\Gamma_\rho(s)}, \qquad (12)$$

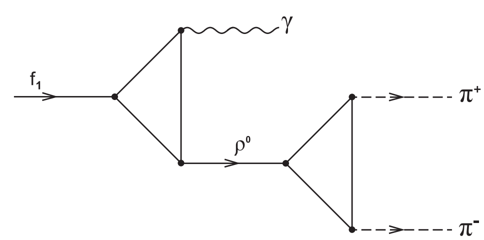

FIG. 1. The contribution through the intermediate ρ^0 meson to the radiative decay amplitude $f_1 \to \pi^+\pi^-\gamma$. The first triangle diagram is described by the effective Lagrangian density (8), the second one by the Lagrangian density (6).

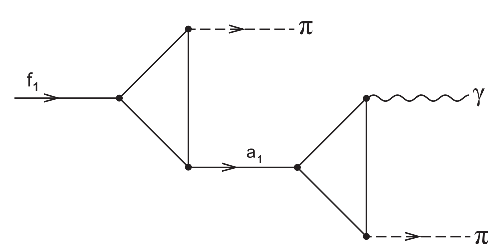

FIG. 2. The contribution through the intermediate $a_1(1260)$ meson to the radiative decay amplitude $f_1 \to \pi^+\pi^-\gamma$. The first triangle diagram is described by the effective Lagrangian density (9), the second one by the Lagrangian density (7).

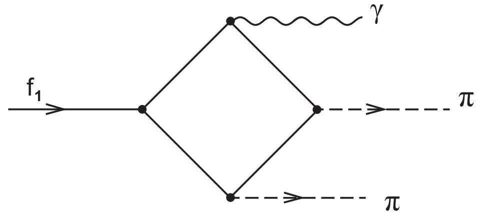

FIG. 3. The contribution of the contact interaction to the radiative decay amplitude $f_1 \to \pi^+\pi^-\gamma$.

$$F_2^{(\rho)} = \frac{-3eg_\rho}{Z(4\pi f_\pi)^2}\frac{m_{f_1}^2 - 2m_{f_1}(\varepsilon - \varepsilon_+)}{m_\rho^2 - s - im_\rho\Gamma_\rho(s)}, \quad (13)$$

$$F_3^{(\rho)} = \frac{3eg_\rho}{Z(2\pi f_\pi)^2}\frac{m_{f_1}^2 - m_{f_1}\varepsilon}{m_\rho^2 - s - im_\rho\Gamma_\rho(s)}, \quad (14)$$

$$F_4^{(\rho)} = \frac{-3eg_\rho}{2Z(2\pi f_\pi)^2}\frac{1}{m_\rho^2 - s - im_\rho\Gamma_\rho(s)}, \quad (15)$$

where $s = (l-p)^2$, $\varepsilon, \varepsilon_\pm$ are the energies of photon and charged pions in the rest frame of the f_1 meson, and $\Gamma_\rho(s)$ is the hadronic off-shell width of the $\rho(770)$ resonance [22]

$$\Gamma_\rho(s) = \frac{m_\rho s}{96\pi f_\pi^2}\left[\sigma_\pi^3\theta(s - 4m_\pi^2) + \frac{1}{2}\sigma_K^3(s - 4m_K^2)\right], \quad (16)$$

with $\sigma_P = \sqrt{1 - 4m_P^2/s}$, $m_\pi = 138$ MeV and $m_K = 494$ MeV. We find that the contribution of this channel is $\Gamma(f_1 \to \rho^0\gamma \to \pi^+\pi^-\gamma) = 276$ keV. One can use the sequential decay formula to do a cross check

$$\Gamma(f_1 \to \rho^0\gamma \to \pi^+\pi^-\gamma) = \frac{\Gamma(f_1 \to \rho^0\gamma)\Gamma(\rho^0 \to \pi^+\pi^-)}{\Gamma_\rho}. \quad (17)$$

Since ρ decays into $\pi\pi$ to hundred percent, and it is known from our previous estimates that $\Gamma(f_1 \to \rho^0\gamma) = 311$ keV [1] we conclude that both results are in good agreement. Notice that for simplicity we could ignore the energy dependence in $\Gamma_\rho(s)$ taking its empirical value $\Gamma_\rho = 149.1 \pm 0.8$ MeV. This would diminish the result only on 1%.

Let us consider now the amplitudes corresponding to the diagrams plotted on Figs. 2 and 3. The diagram with the intermediate a_1 meson contributes as a contact interaction. The terms with a_1 propagators have a factor p^2 which is equal zero for real photons. Actually, this is a common feature of the NJL, massive Yang-Mills and hidden local symmetry approaches [23,24]. In all these models the decay amplitude $a_1 \to \pi\gamma$ is zero on the a_1 mass shell. The experimental situation is not settled yet (the PDG [11] does not give a definite number for the $a_1 \to \pi\gamma$ decay width). For this reason we are not going into details here and postponing such analyses for the future, when we will have enough experimental information on this mode. Thus, we get

$$T^{(a_1)} = -i\frac{eg_\rho}{8\pi^2 f_\pi^2}e^{\mu\nu\alpha\beta}\epsilon_\beta(l)\epsilon_\alpha^*(p) \times 2\kappa m^2[1 + (1-3a)\kappa m^2]l_\mu q_\nu \quad (18)$$

where the 4-vector $q = p_+ - p_-$. Notice that only contributions due to $\vec{\pi} - \vec{a}_1$ transitions survived in (18). Observing that

$$e_{\mu\nu\alpha\beta}(l^\mu q^\nu) = e_{\mu\nu\alpha\beta}(p^\mu q^\nu - 2p_+^\mu p_-^\nu) \quad (19)$$

one sees that the term $\propto p_+^\mu p_-^\nu$ brakes gauge invariance. Thus there must be other diagrams to restore the symmetry.

These diagrams are shown in Fig. 3 (we do not show there, but it is assumed, that each pion line represents the direct creation of a pion by the quark-antiquark pair and the indirect one through the $\vec{\pi} - \vec{a}_1$ transition). At leading order of the derivative expansion we obtain the amplitude

$$T^{(b)} = i\frac{eg_\rho}{8\pi^2 f_\pi^2}e^{\mu\nu\alpha\beta}\epsilon_\beta(l)\epsilon_\alpha^*(p) \times \left[\frac{p_\mu q_\nu}{Z} - \kappa m^2(4 - \kappa m^2)p_+^\mu p_-^\nu\right]. \quad (20)$$

Now, one can restore the gauge symmetry of the whole amplitude by fixing the parameter a. The requirement is to cancel the unwanted $p_+^\mu p_-^\nu$ term of the sum $T^{(a_1)} + T^{(b)}$. It gives $a = 5/12$. The rest of the sum

$$T^{(a_1+b)} = T^{(a_1)} + T^{(b)} = iAe^{\mu\nu\alpha\beta}\epsilon_\beta(l)\epsilon_\alpha^*(p)p_\mu q_\nu, \quad (21)$$

where

$$A = \frac{eg_\rho}{8\pi^2 f_\pi^2}\left[\frac{2-Z}{Z} + \frac{(Z-1)^2}{8Z^2}\right], \quad (22)$$

will contribute to the amplitude (11) in the form

$$F_1^{(a_1+b)} = -F_2^{(a_1+b)} = \frac{1}{2}F_3^{(a_1+b)} = A. \quad (23)$$

The latter follows from Eq. (19). This contribution numerically is not large $\Gamma^{(a_1+b)}(f_1 \to \pi^+\pi^-\gamma) = 5.5$ keV. Nonetheless, in the next section we will show that the interference of this amplitude with the pure ρ exchange channel is positive and relatively large. It enhances noticeably the final result.

IV. $f_1(1285) \to \pi^+\pi^-\gamma$ DECAY WIDTH

The rate of the three-body decay $f_1(1285) \to \pi^+\pi^-\gamma$ can be obtained from the standard formula

$$d\Gamma = \frac{|T|^2}{24 m_{f_1} (2\pi)^3} d\varepsilon d\varepsilon_+ \tag{24}$$

where

$$|T|^2 = \sum_{i \le j} \mathrm{Re}(F_i F_j^*) T_{ij}, \tag{25}$$

$F_i = F_i^{(\rho)} + F_i^{(a_1+b)}$ and the coefficients T_{ij} are

$$\begin{aligned}
T_{11} &= 2m_{f_1}^2 \vec{p}_+^{\,2} \\
T_{22} &= 2m_{f_1}^2 \vec{p}_-^{\,2} \\
T_{33} &= 2[(p_+p_-)^2 - m_\pi^4] + (\vec{p}_+ \times \vec{p})^2 \\
T_{44} &= -m_{f_1}^4 (\vec{p}_+ \times \vec{p})^2 \\
T_{12} &= 4m_{f_1}^2 \vec{p}_+ \vec{p}_- \\
T_{13} &= 4m_{f_1} [m_\pi^2 \varepsilon_- - (p_+p_-)\varepsilon_+] \\
T_{23} &= -4m_{f_1} [m_\pi^2 \varepsilon_+ - (p_+p_-)\varepsilon_-] \\
T_{14} &= T_{24} = 0 \\
T_{34} &= -2m_{f_1}^2 (\vec{p}_+ \times \vec{p})^2.
\end{aligned} \tag{26}$$

Notice that

$$\begin{aligned}
(\vec{p}_+ \times \vec{p})^2 &= (\vec{p}_- \times \vec{p})^2 = (\vec{p}_+ \times \vec{p}_-)^2 \\
&= \vec{p}_+^{\,2} \vec{p}^{\,2} - (\vec{p}_+ \vec{p})^2.
\end{aligned} \tag{27}$$

Here all kinematical variables are given in the rest frame of the f_1 meson. With the use of the kinematic invariants $s = (l - p)^2$, $t = (l - p_+)^2$ and $u = (l - p_-)^2$ one can find the boundary of the physical region in the Mandelstam plane. For a given value of s from the closed interval $4m_\pi^2 \le s \le m_{f_1}^2$ the boundary is given by the two roots of the quadratic equation. They are

$$t_\pm(s) = \frac{1}{2}\left(m_{f_1}^2 + 2m_\pi^2 - s \pm \sqrt{D(s)}\right), \tag{28}$$

where $m_\pi^2 \le t_\pm \le (m_{f_1} - m_\pi)^2$, and

$$D(s) = (m_{f_1}^2 + 2m_\pi^2 - s)^2 - 4m_\pi^2 \left(\frac{m_{f_1}^4}{s} - m_{f_1}^2 + m_\pi^2\right). \tag{29}$$

Collecting above formulas we come to the following result

$$\Gamma = \frac{1}{24 m_{f_1} (2\pi)^3} \int_0^{\varepsilon^{\max}} d\varepsilon \int_{\varepsilon_+^{\min}}^{\varepsilon_+^{\max}} d\varepsilon_+ |T|^2 \tag{30}$$

where

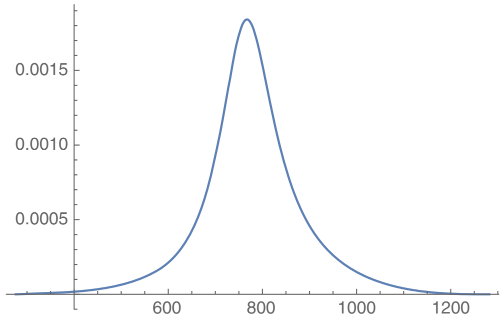

FIG. 4. The spectral effective mass distribution for the system of two charged pions, $d\Gamma/d\sqrt{s}$, as a function of $\sqrt{s}$ for the radiative decay $f_1 \to \pi^+\pi^-\gamma$.

$$\begin{aligned}
\varepsilon^{\max} &= \frac{m_{f_1}^2 - 4m_\pi^2}{2m_{f_1}}, \\
\varepsilon_+^{\min} &= \frac{1}{2}\left[m_{f_1} - \varepsilon\left(1 + \sqrt{\Omega(\varepsilon)}\right)\right], \\
\varepsilon_+^{\max} &= \frac{1}{2}\left[m_{f_1} - \varepsilon\left(1 - \sqrt{\Omega(\varepsilon)}\right)\right], \\
\Omega(\varepsilon) &= 1 - \frac{4m_\pi^2}{m_{f_1}(m_{f_1} - 2\varepsilon)}.
\end{aligned} \tag{31}$$

Integrating over energies ε_+ and ε in (30) we obtain the radiative decay width $\Gamma(f_1 \to \pi^+\pi^-\gamma) = 346$ keV. This includes the contributions of the ρ^0 and $a_1^\pm$ exchanges, contact interaction and all possible interferences among these amplitudes. The vector ρ^0 resonance gives the major contribution. To be precise, we have found that $\Gamma_{\text{tot}} = (346{,}4 = 275.6 + 5.5 + 65.3)$ keV where in the sum we present the contributions of ρ^0 (first term), a_1 plus contact interaction (second term), and interferences (third term).

From Eq. (30) one can also derive the spectral effective mass distribution for the system of two charged pions, i.e., a derivative $d\Gamma/d\sqrt{s}$ as a function of $\sqrt{s}$. We plot this function in Fig. 4. The curve has a clear peak located at $\sqrt{s} = 767$ MeV. If one would neglect a contact interaction the maximum would shift to the point $\sqrt{s} = 768$ MeV. We conclude that the shift due to a contact interaction is rather small, or, in other words, the AAA and AAAV anomalies do not affect much the position of the extremum.

On the other hand, a new empirical information is hoped to give more insight on the structure of the spectral function. In particular, this can shed some light on the axial-vector decay amplitude $a_1(1260) \to \pi\gamma$. There are experimental data indicating that the decay width of $a_1(1260) \to \pi\gamma$ is not zero, $\Gamma(a_1^+ \to \pi^+\gamma) = 640 \pm 246$ keV [25]. This may modify the spectral function of Fig. 4 at energies $\sqrt{s} \sim m_{a_1}$. Such theoretical study is possible, however, without precise empirical data this analysis would be too speculative.

V. CONCLUSIONS

In this work, we calculate the radiative decay width of the $f_1(1285)$ axial-vector meson into two charged pions. Our main assumptions are that the $f_1(1285)$ state is mostly made of u and d quarks; the decay is governed by the quark triangle AVV anomaly; and the transition $a_1 \to \pi\gamma$ is suppressed on the mass shell of the a_1 meson. We also neglect the effects of the final-states interaction, which can be essential for the $\pi\pi$ system. The latter can be a subject of more refined amplitude analysis, as soon as high statistics empirical data will be available.

We estimate the contribution of the box AAAV anomaly and conclude that it is rather small. Nonetheless it affects the result through the positive interference with an intermediate ρ resonance amplitude, enhancing the total value of the decay width on 25%.

We also derive the spectral function of the two pion system and show that it has a clear signal of the $\rho(770)$ state. We recommend the experimental study of this spectral function in the future. Our reasoning is that in this way one can extract new important information on the anomaly structure of the amplitude. In particular, to clarify the role of AAA and AAAV anomalies, by studying two pions, and photon spectral functions. It would be also interesting to measure the peak position in the dipion mass spectrum. A deviation would indicate that a contribution from the ρ exchange alone is not sufficient.

The decay width found, $\Gamma(f_1 \to \pi^+\pi^-\gamma) = 346$ keV, agrees with our previous estimate, $\Gamma_{f_1\to\rho\gamma} = 311$ keV, and is compatible with the recently measured value $\Gamma_{f_1\to\rho^0\gamma} = 453 \pm 177$ keV [10] within errors. The contributions to $\Gamma(f_1 \to \pi^+\pi^-\gamma)$ are quantified as follows: $\Gamma(f_1 \to \rho^0\gamma \to \pi^+\pi^-\gamma) = 275.6$ keV, $\Gamma_A(f_1 \to \pi^+\pi^-\gamma) = 5.5$ keV, $\Gamma_I(f_1 \to \pi^+\pi^-\gamma) = 65.3$ keV, where the subscript A marks the box AAAV anomaly, and I is used for the interference effects. The ρ resonance dominates the amplitude whereas the box anomaly is important through its positive interference with the ρ exchange channel.

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