# Fresh look at the Abelian and non-Abelian Landau-Khalatnikov-Fradkin transformations

T. De Meerleer,[1,*] D. Dudal,[1,2,†] and S. P. Sorella[3,‡]

[1]*KU Leuven Campus Kortrijk—Kulak, Department of Physics, Etienne Sabbelaan 53 bus 7657, 8500 Kortrijk, Belgium*
[2]*Ghent University, Department of Physics and Astronomy, Krijgslaan 281-S9, 9000 Gent, Belgium*
[3]*Instituto de Física Teórica, Rua São Francisco Xavier 524, 20550-013 Maracanã, Rio de Janeiro, Brazil*

P. Dall'Olio[4,§] and A. Bashir[4,‖]

[4]*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, Morelia, Michoacán 58040, Mexico*

The Landau-Khalatnikov-Fradkin transformations (LKFTs) allow one to interpolate $n$-point functions between different gauges. We first offer an alternative derivation of these LKFTs for the gauge and fermions field in the Abelian (QED) case when working in the class of linear covariant gauges. Our derivation is based on the introduction of a gauge invariant transversal gauge field, which allows a natural generalization to the non-Abelian (QCD) case of the LKFTs. To our knowledge, within this rigorous formalism, this is the first construction of the LKFTs beyond QED. The renormalizability of our setup is guaranteed to all orders. We also offer a direct path integral derivation in the non-Abelian case, finding full consistency.

## I. INTRODUCTION

When we study strong color interaction, quantum chromodynamics (QCD), we start from the most basic fields, namely the quarks, gluons and also the Faddeev-Popov ghosts in covariant gauges. Due to the infrared enhancement of the strong coupling constant, perturbation theory alone is unable to provide a description of the observable hadronic world made up of quarks and gluons. Therefore, the need for nonperturbative approaches arises, requiring a radically different treatment of these interactions.

In the continuum formulation, gauge fixing is required to warrant computations, whatever nonperturbative scheme one has in mind. However, QCD remains a gauge theory, meaning physically observable quantities should not depend on what gauge is actually chosen to carry out the computation. In this article, we concern ourselves with linear covariant gauges, with the Landau gauge as a special case thereof.

Within the functional approach of Dyson-Schwinger equations (DSEs) [1–10] or functional renormalization group equations [11,12], one is confronted with an infinite tower of nonlinear coupled equations with an ever-increasing order of $n$-point correlation functions. This is of course unamenable to computation, so a sacrifice must be made: the tower is truncated and some of the necessary low order $n$-point correlation functions are introduced via a sensible *Ansätze* preserving some key features of a gauge field theory. Much care is generally taken as regards the low energy constraint of chiral symmetry, namely the axial vector Ward identity, and the pattern in which this symmetry is dynamically broken. The corresponding low energy Goldberger-Treiman relations provide an intimate connection between the quark propagator and the Bethe-Salpeter amplitudes of the corresponding bound state. It is of paramount importance, not only to get the correct QCD spectrum for low lying mesons, but also to study corresponding elastic and transition form factors which have come under immense experimental and theoretical scrutiny in the last few years [13–18].

However, the constraints of gauge covariance are not always fully implemented. These constraints manifest themselves not only in terms of Slavnov-Taylor identities but also as generalized Landau-Khalatnikov-Fradkin transformations (LKFTs) which are less studied, except for the

*tim.demeerleer@kuleuven.be
†david.dudal@kuleuven.be
‡silvio.sorella@gmail.com
§odiotin@hotmail.com
‖adnan@ifm.umich.mx

Abelian (QED) case; see e.g. [1,19–21] and recent works such as [22–24]. In principle, if gauge covariance is manifest, transforming the $n$-point functions in one gauge to those in another gauge has no consequence whatsoever on physical observables computed from these $n$-point functions. However, as soon as model-building is done in some particular gauge, there can be conflicts with gauge (BRST) invariance which can lead to uncontrollable gauge parameter dependence filtering into physical quantities. For example, one can expect such a thing to happen if models specific to Landau gauge were to be used in other gauges without appropriate gauge modifications.

In fact, most functional studies are restricted to the Landau gauge case, because of its interesting properties, such as it being a fixed point of the renormalization group and the fact that it is accessible also to gauge-fixed lattice QCD studies [25–35]. However, during the course of the past few years, we have witnessed an increased activity in extending both functional and numerical lattice efforts to general linear covariant gauges. In the long run, it will lead to a better understanding of how to extract truly gauge invariant physical information in a gauge-fixed context [36–49]. That such a goal is far from being trivial has been illustrated even in the case of QED, whose state-of-the-art is well captured by exhaustive studies of the fermion-photon vertex [50–54] to implement gauge invariance of physical observables. In principle, a sound *Ansatz* for the fermion-photon vertex should be made in one gauge, say the Landau gauge. The vertex in any other gauge can then be obtained as the LKFT of the Landau gauge *Ansatz*. The sensible implementation of this procedure guarantees gauge covariance and hence obviates any question about the gauge dependence of gauge invariant quantities.

We expect the same to be true for QCD, albeit with increasing complexity. The Landau gauge vertex models in QCD would transform under some generalized LKFTs to provide an appropriate model in other linear covariant gauges. An example which motivates current study takes into account the simple two-point Green function, more specifically, the Dyson-Schwinger output for the transversal projection of the gluon propagator for small values of the gauge parameter. From the available lattice data for the gluon propagator or its dressing function, it turns out that there is almost no gauge parameter dependence for the considered interval of gauge parameter variation [36–38]. This is in sharp contrast with the Dyson-Schwinger estimates presented in [39,40] which show sizeable variation with the gauge parameter; compare for example [38], Fig. 3 (right panel), with [39], Fig. 2 (right panel).

In the Abelian QED case, a frequently adopted strategy is based on the LKFTs [55,56] (see also [57]) which allow us to explicitly transform correlation functions from one linear covariant gauge with gauge parameter $\alpha$ to another gauge with parameter $\alpha'$. There is a large body of work which has used these transformations as a guiding principle toward an improved *Ansatz* for the three-point vertex and imposing gauge invariant chiral symmetry breaking; see for example [19,58–60]. More recently, these transformations have also been studied in the world line formalism, where LKFTs are generalized to arbitrary amplitudes in scalar QED [61].

Similar work in the non-Abelian QCD case has only just begun [62], the delay being mostly due to the complexities of the non-Abelian LKFTs. The purpose of the current article is to write down the formal and natural generalization of the LKFTs to the non-Abelian case of QCD without jeopardizing renormalizability.

We study LKFTs by using the gauge invariant fields $A^h$ and $\psi^h$ as introduced in [44,47–49,63]; see also [64,65]. We see that these fields, which correspond in fact to invariant nonlocal composite gluonic and fermionic operators, provide us with a rather natural setting to derive both the known Abelian and the novel non-Abelian LKFTs.

This article is organized as follows: in Sec. II the construction of the gauge invariant fields $A^h$ and $\psi^h$ [44,47,49] is briefly summarized, and then fully exploited in Sec. III to study the LKFTs for both gluon and fermion correlation functions. We take a closer look at the lowest order gluon propagator, retrieving the known LKFT for the photon propagator. Furthermore, we establish a relation for the LKFT for the fermion propagator. In Sec. IV the LKFTs are derived once more, but now from a different viewpoint, namely within the path integral formalism. Fully exploiting the gauge symmetry of the original classical action, the path integral allows us to recover the same LKFTs as in Sec. III. At last, Sec. V summarizes our conclusions and directions for future work.

## II. A SHORT SUMMARY TO THE GAUGE INVARIANT TRANSVERSAL GLUON FIELD $A_\mu^h$

We start from the action [47,49]

$$S = S_{\text{FP}} + S_f + S_h, \tag{1}$$

with the Faddeev-Popov term given by

$$S_{\text{FP}} = \int \mathrm{d}^4x \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{\alpha}{2} b^a b^a + i b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right), \tag{2}$$

the matter sector by

$$S_f = \int \mathrm{d}^4x (\bar{\psi}(i\slashed{D} + m_f)\psi), \tag{3}$$

and

$$S_h = \int \mathrm{d}^4x \left( \tau^a \partial_\mu A_\mu^{h,a} + \frac{m^2}{2} A_\mu^{h,a} A_\mu^{h,a} + \bar{\eta}^a \partial_\mu D_\mu^{ab}(A^h)\eta^b \right), \tag{4}$$

where $A_\mu^h$ is defined through

$$A_\mu^h = h^\dagger A_\mu h + \frac{i}{g} h^\dagger \partial_\mu h. \tag{5}$$

Here, we set

$$h = e^{ig\phi^a T^a}, \tag{6}$$

with $T^a$ being the adjoint generators of SU($N$). As it is apparent, the action $S_h$ contains a new field $\phi^a$, besides the Lagrange multiplier $\tau^a$ as well as the additional ghost fields $(\bar{\eta}^a, \eta^a)$.[1] All these fields belong to the adjoint representation.

By construction, the field $A_\mu^h$ turns out to be transverse, $\partial_\mu A_\mu^h = 0$, and gauge invariant [44,47]. The transversality of $A_\mu^h$ is precisely ensured by the presence of the Lagrange multiplier $\tau^a$. The gauge invariant character of $A_\mu^h$ can be nicely appreciated from the transformation laws

$$h \to U^\dagger h, \qquad h^\dagger \to h^\dagger U,$$

$$A_\mu \to A_\mu^U = U^\dagger A_\mu U + \frac{i}{g} U^\dagger \partial_\mu U, \tag{7}$$

with $U$ being a generic local SU($N$) transformation. From Eq. (5), it follows that

$$(A_\mu^h)^U = A_\mu^h. \tag{8}$$

At the quantum level, $A_\mu^h$ is a rather complicated composite operator, nonetheless via the Stueckelberg-like formulation of Eq. (5), the all order renormalizability of $A_\mu^h$, and thus of its correlation functions, was proven, thanks to the powerful Ward identities underlying the dynamics of the action (1) [47,49]. Upon solving the $\tau$-constraint (see also later in this paper), $A^h$ can be explicitly written as a formal power series in $g$ (or powers of the field $A$) given by

$$A_\mu^h = A_\mu - \frac{\partial_\mu}{\partial^2} \partial A + ig\left[A_\mu, \frac{1}{\partial^2}\partial A\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu \frac{1}{\partial^2}\partial A\right]$$
$$+ ig\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu}{\partial^2}\partial A, A_\nu\right] + i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + O(A^3). \tag{9}$$

From this expression, it is clear that in the Landau gauge, $\partial_\mu A_\mu = 0$, $A_\mu^h$ reduces to $A_\mu$, given the overall presence of $\partial_\mu A_\mu$. This is not a coincidence, as $A_\mu^h$ can be seen as the series solution to the stationary condition obtained from

---

[1]As underlined in [49], the additional ghosts $(\bar{\eta}^a, \eta^a)$ are needed to take into account the Jacobian arising from integration over the Lagrange multiplier $\tau^a$, which gives rise to a delta function of the type $\delta(\partial_\mu A_\mu^{h,a})$.

minimizing the functional $\int d^4x A_\mu^u A_\mu^u$ along the gauge orbit, $u \in$ SU($N$) [44,64,66,67]. This minimization procedure can be extended to the nonperturbative level, in particular, to provide a lattice formulation of the Landau gauge. Any solution to the stationary condition (first order variation) is a (local) minimum of the functional if the second order variation is positive. This amounts to requiring $-\partial_\mu D_\mu(A^h) > 0$, i.e. the Faddeev-Popov operator evaluated at $A = A^h$ should be a positive operator. This observation is closely linked to the issue of gauge (Gribov) copies [68]. Indeed, specifying to the Landau gauge, imposing $-\partial_\mu D_\mu(A) > 0$ eliminates the infinitesimal gauge copies related to zero modes of the Faddeev-Popov operator [68,69]. Recently, the generalization of this construction to a generic linear covariant gauge was developed in a series of papers [44–46,48,49,70], allowing one to put the Gribov-Zwanziger construction for the first time on a firm Bechi-Rouet-Stora-Tyutin (BRST) invariant footing.

The mass term, $\frac{m^2}{2}\int d^4x A_\mu^{h,a} A_\mu^{h,a}$, can be put to 0 in Eq. (1). The parameter $m^2$ rather serves to introduce the gauge invariant mass operator $\int d^4x A_\mu^h A_\mu^h$, well-known from phenomenology [71–75] and corresponding to the minimum of the functional $\int d^4x A_\mu^u A_\mu^u$. Though, for our current purposes, we set $m^2 = 0$, to restore full equivalence with the standard Yang-Mills QCD action. In that case, the quark-gluon-ghost dynamics of $(S_{\mathrm{FP}} + S_f + S_h)$ is equivalent to that $(S_{\mathrm{FP}} + S_f)$, as integrating over $\tau, \phi, \bar{\eta}, \eta$ gives no more than a unity. Let us explain in more detail. It is important to appreciate the role of the multipliers $\tau^a$ which impose the constraint $\partial_\mu A_\mu^{h,a} = 0$. The latter can be solved iteratively allowing one to express $\phi$ explicitly in terms of $A_\mu$. More precisely, one finds (see e.g. [46])

$$\phi = \frac{1}{\partial^2}\partial A + i\frac{g}{\partial^2}\left[\partial A, \frac{\partial A}{\partial^2}\right] + i\frac{g}{\partial^2}\left[A_\mu, \partial_\mu \frac{\partial A}{\partial^2}\right]$$
$$+ \frac{i}{2}\frac{g}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + \mathcal{O}(A^3). \tag{10}$$

In the expression above, we recognize that the fields $\phi^a$ are the SU($N$) gauge rotation angles we need to gauge transform a generic field configuration $A_\mu$ into its transversal, gauge equivalent, configuration $A_\mu^h$. When the $\tau$ is integrated over, i.e. we work with the on-shell $\phi$-formulation, the integration over the $\phi$ gives rise to a nontrivial Jacobian, which is lifted into the action via the Grassmann $\bar{\eta}, \eta$-fields, thereby giving an overall unity. This is discussed in more detail in other work. Notice that this procedure shares great similarity with the introduction of the unit factor corresponding to the Faddeev-Popov quantization procedure. Also here, both gauge condition and related Jacobian, i.e. the Faddeev-Popov determinant, are lifted into the action through the introduction of the Lagrange multiplier $b^a$ and of the Faddeev-Popov ghosts $(c^a, \bar{c}^a)$.

Let us also point out that the constraint $\partial_\mu A_\mu^{h,a} = 0$ is what discriminates between the standard Stueckelberg action and the formulation (1). In particular, as shown in [47,49], where a detailed comparison was made with the standard nonrenormalizable Stueckelberg theory, the condition $\partial_\mu A_\mu^{h,a} = 0$ plays a pivotal role in order to ensure the all order renormalizability of the action (1).

## III. DERIVATION OF THE LKFTS VIA $A^h$

In this section, we first rederive the Abelian LKFTs, followed by the non-Abelian generalization.

A Stueckelberg-based derivation of the LKFTs was already realized in [76], though this analysis is restricted to the Abelian case, with no clear generalization to the non-Abelian case.

Up to second order in the coupling constant $g$, we may write

$$
A_\mu^h = A_\mu - \partial_\mu\phi + ig[A_\mu,\phi] + \frac{ig}{2}[\phi,\partial_\mu\phi]
$$
$$
- \frac{g^2}{2}(A_\mu\phi^2 - 2\phi A_\mu\phi + \phi^2 A_\mu)
$$
$$
+ \frac{g^2}{3!}((\partial_\mu\phi)\phi^2 - 2\phi(\partial_\mu\phi)\phi + \phi^2(\partial_\mu\phi)) + \mathcal{O}(g^3),
$$

(11)

or, by denoting the color index explicitly

$$
A_\mu^{h,a} = A_\mu^a - \partial_\mu\phi^a - gf^{abc}A_\mu^b\phi^c - \frac{g}{2}f^{abc}\phi^b\partial_\mu\phi^c
$$
$$
- \frac{g^2}{2}D^{abcd}(A_\mu^b\phi^c\phi^d - 2\phi^b A_\mu^c\phi^d + \phi^b\phi^c A_\mu^d)
$$
$$
+ \frac{g^2}{3!}D^{abcd}((\partial_\mu\phi^b)\phi^c\phi^d - 2\phi^b(\partial_\mu\phi^c)\phi^d
$$
$$
+ \phi^b\phi^c(\partial_\mu\phi^d)) + \mathcal{O}(g^3),
$$

(12)

with $D^{abcd} = 2\mathrm{Tr}(T^a T^b T^c T^d)$.

### A. The LKFT for the photon propagator
### via $\langle A_\mu^h(p)A_\nu^h(-p)\rangle$

The expression (12) can be used to expand the two-point correlation function $\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle$. In the Abelian approximation, i.e. $A_\mu^h = A_\mu - \partial_\mu\phi$, one immediately obtains

$$
\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_\alpha
$$
$$
= \langle A_\mu^a(p)A_\nu^b(-p)\rangle_\alpha + \langle A_\mu^a(p)\partial_\nu\phi^b(-p)\rangle_\alpha
$$
$$
+ \langle\partial_\mu\phi^a(p)A_\nu^b(-p)\rangle_\alpha + \langle\partial_\mu\phi^a(p)\partial_\nu\phi^b(-p)\rangle_\alpha.
$$

(13)

The two-point correlation functions $\langle A_\mu^a(p)\phi^b(-p)\rangle_\alpha$ and $\langle\phi^a(p)\phi^b(-p)\rangle_\alpha$ are given by [47]

$$
\langle A_\mu^a(p)\phi^b(-p)\rangle_\alpha = i\alpha\frac{p_\mu}{p^4}\delta^{ab},
$$

(14)

$$
\langle\phi^a(p)\phi^b(-p)\rangle_\alpha = \frac{\alpha}{p^4}\delta^{ab}.
$$

(15)

So, Eq. (13) becomes[2]

$$
\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_\alpha = \langle A_\mu^a(p)A_\nu^b(-p)\rangle_\alpha - \alpha\frac{p_\mu p_\nu}{p^4}\delta^{ab},
$$

(16)

or, specifying to the Landau gauge, $\alpha = 0$,

$$
\langle A_\mu^{h,a}A_\nu^{h,b}\rangle_{\alpha=0} = \langle A_\mu^a A_\nu^b\rangle_{\alpha=0}.
$$

(17)

It is worth recalling that the transverse field $A_\mu^h$ is gauge invariant or, equivalently, BRST invariant; see [47,49]. From this important feature it follows that the correlation function $\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_\alpha$ is BRST invariant as well. As such, it does not depend on the gauge parameter $\alpha$ [47,49]. Therefore,

$$
\langle A_\mu^{h,a}A_\nu^{h,b}\rangle_\alpha = \langle A_\mu^{h,a}A_\nu^{h,b}\rangle_{\alpha=0}
$$

(18)

and we find

$$
\langle A_\mu^a(p)A_\nu^b(-p)\rangle_\alpha = \langle A_\mu^a(p)A_\nu^b(-p)\rangle_{\alpha=0} + \alpha\frac{p_\mu p_\nu}{p^4}\delta^{ab}.
$$

(19)

Said otherwise, we simply recover the LKFT for the photon. Of course, this result can also be easily derived using the underlying BRST invariance of the theory, which ensures that the longitudinal component of the gluon propagator does not receive any quantum correction, being given by its tree-level approximation.

### B. The LKFT for the gluon propagator
### via $\langle A_\mu^h(p)A_\nu^h(-p)\rangle$

As $A_\mu^h$ is defined also for the non-Abelian case, we can generalize the foregoing to get LKFTs for the gluon propagator via the expansion of

$$
\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_\alpha = \langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_{\alpha=0}
$$
$$
= \langle A_\mu^a(p)A_\nu^b(-p)\rangle_{\alpha=0},
$$

(20)

---

[2]Note that the partial derivations $\partial_\mu$ refer to coordinate space. A Fourier transformation of the relevant fields has been taken, under the convention $A_\mu(p) = \int A_\mu(x)e^{ip\cdot x}\mathrm{d}x$, so that $\partial_\mu A_\nu(p) = -ip_\mu A_\nu(p)$.

where, in the last step, we explicitly used that correlation functions of $A_\mu^h$ reduce to those of $A_\mu$ in the Landau gauge [48]. This property can be appreciated by realizing that the field $\phi^a$ decouples in the Landau gauge, as it becomes apparent from the vanishing of the correlation functions $\langle A_\mu^a(p)\phi^b(-p)\rangle_\alpha$, $\langle \phi^a(p)\phi^b(-p)\rangle_\alpha$, when $\alpha = 0$; see Eq. (15). Notice also that the leading order term of the expansion of $\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle_\alpha$ always contains the gluon propagator in the linear covariant gauge with gauge parameter $\alpha$.

In what follows, for the benefit of the reader, the next-to-leading order expansion of the lhs of Eq. (20) is given, but the contractions of the terms are left open, as this depends on the precise action one intends to use.[3] Also not included are the necessary vertex insertions in the lowest order terms to get the complete $\mathcal{O}(g^2)$ corrections.

Up to second order in the coupling constant, the expansion of the correlation function $\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle$ is found to be[4]

$$\langle A_\mu^{h,a}(p)A_\nu^{h,b}(-p)\rangle = \langle A_\mu^a(p)A_\nu^b(-p)\rangle + \langle A_\mu^a(p)\partial_\nu\phi^b(-p)\rangle + \langle \partial_\mu\phi^a(p)A_\nu^b(-p)\rangle + \langle \partial_\mu\phi^a(p)\partial_\nu\phi^b(-p)\rangle$$

$$+ gf^{bcd}\left[-\langle A_\mu^a(p)A_\nu^c(-p)\phi^d(-p)\rangle + \langle \partial_\mu\phi^a(p)A_\nu^c(-p)\phi^d(-p)\rangle\right.$$

$$\left. - \frac{1}{2}\langle A_\mu(p)^a\phi^c(-p)\partial_\nu\phi^d(-p)\rangle + \frac{1}{2}\langle \partial_\mu\phi^a(p)\phi^c(-p)\partial_\nu\phi^d(-p)\rangle\right]$$

$$+ gf^{acd}\left[-\langle A_\mu^c(p)\phi^d(p)A_\nu^b(-p)\rangle + \langle A_\mu^c(p)\phi^d(p)\partial_\nu\phi^b(-p)\rangle\right.$$

$$\left. - \frac{1}{2}\langle \phi^c(p)\partial_\mu\phi^d(p)A_\nu^b(-p)\rangle + \frac{1}{2}\langle \phi^c(p)\partial_\mu\phi^d(p)\partial_\nu\phi^b(-p)\rangle\right]$$

$$+ \frac{g^2}{3!}D^{bcde}[\langle A_\mu^a(p)\partial_\nu\phi^c(-p)\phi^d(-p)\phi^e(-p)\rangle - 2\langle A_\mu^a(p)\phi^c(-p)\partial_\nu\phi^d(-p)\phi^e(-p)\rangle$$

$$+ \langle A_\mu^a(p)\phi^c(-p)\phi^d(-p)\partial_\nu\phi^e(-p)\rangle - \langle \partial_\mu\phi^a(p)\partial_\nu\phi^c(-p)\phi^d(-p)\phi^e(-p)\rangle$$

$$+ 2\langle \partial_\mu\phi^a(p)\phi^c(-p)\partial_\nu\phi^d(-p)\phi^e(-p)\rangle - \langle \partial_\mu\phi^a(p)\phi^c(-p)\phi^d(-p)\partial_\nu\phi^e(-p)\rangle]$$

$$+ \frac{g^2}{3!}D^{acde}[\langle \partial_\mu\phi^c(p)\phi^d(p)\phi^e(p)A_\nu^b(-p)\rangle - 2\langle \phi^c(p)\partial_\mu\phi^d(p)\phi^e(p)A_\nu^b(-p)\rangle$$

$$+ \langle \phi^c(p)\phi^d(p)\partial_\mu\phi^e(p)A_\nu^b(-p)\rangle - \langle \partial_\mu\phi^c(p)\phi^d(p)\phi^e(p)\partial_\nu\phi^b(-p)\rangle$$

$$+ 2\langle \phi^c(p)\partial_\mu\phi^d(p)\phi^e(p)\partial_\nu\phi^b(-p)\rangle - \langle \phi^c(p)\phi^d(p)\partial_\mu\phi^e(p)\partial_\nu\phi^b(-p)\rangle]$$

$$- \frac{g^2}{2}D^{bcde}[\langle A_\mu^a(p)A_\nu^c(-p)\phi^d(-p)\phi^e(-p)\rangle - 2\langle A_\mu^a(p)\phi^c(-p)A_\nu^d(-p)\phi^e(-p)\rangle$$

$$+ \langle A_\mu^a(p)\phi^c(-p)\phi^d(-p)A_\nu^e(-p)\rangle - \langle \partial_\mu\phi^a(p)A_\nu^c(-p)\phi^d(-p)\phi^e(-p)\rangle$$

$$+ 2\langle \partial_\mu\phi^a(p)\phi^c(-p)A_\nu^d(-p)\phi^e(-p)\rangle - \langle \partial_\mu\phi^a(p)\phi^c(-p)\phi^d(-p)A_\nu^e(-p)\rangle]$$

$$- \frac{g^2}{2}D^{acde}[\langle A_\mu^c(p)\phi^d(p)\phi^e(p)A_\nu^b(-p)\rangle - 2\langle \phi^c(p)A_\mu^d(p)\phi^e(p)A_\nu^b(-p)\rangle$$

$$+ \langle \phi^c(p)\phi^d(p)A_\mu^e(p)A_\nu^b(-p)\rangle - \langle A_\mu^c(p)\phi^d(p)\phi^e(p)\partial_\nu\phi^b(-p)\rangle$$

$$+ 2\langle \phi^c(p)A_\mu^d(p)\phi^e(p)\partial_\nu\phi^b(-p)\rangle - \langle \phi^c(p)\phi^d(p)A_\mu^e(p)\partial_\nu\phi^b(-p)\rangle]$$

$$+ g^2 f^{acd}f^{bef}[\langle A_\mu^c(p)\phi^d(p)A_\nu^e(-p)\phi^f(-p)\rangle + \frac{1}{2}\langle A_\mu^c(p)\phi^d(p)\phi^e(-p)\partial_\nu\phi^f(-p)\rangle$$

$$+ \frac{1}{2}\langle \phi^c(p)\partial_\mu\phi^d(p)A_\nu^e(-p)\phi^f(-p)\rangle + \frac{1}{4}\langle \phi^c(p)\partial_\mu\phi^d(p)\phi^e(-p)\partial_\nu\phi^f(-p)\rangle] + \mathcal{O}(g^3). \quad (21)$$

As already remarked, the first term of the expansion, i.e. $\langle A_\mu^a(p)A_\nu^b(-p)\rangle$, is nothing but the gluon propagator in linear covariant gauge with gauge parameter $\alpha$.

---

[3]E.g. with or without the mass $m^2$ present.

[4]We suppress the index $\alpha$ from here on.

In work in progress, expression (21) will be used to verify the just derived gluonic LKFT in perturbation theory, thereby extending the results of [62]. Comparison can be made with the known perturbative results in generic linear covariant gauge [77]. We refrain from including this explicit verification here, since there are actually a few dozen Feynman diagrams involved. The main reason for this large number of diagrams is twofold: first, there is the increasing number of vertices when the nonlocal field $A^h$ is expanded in $g$ up to the required order in the action (1), and, secondly, there are various mixed propagators that increase the number of possible contractions giving nontrivial contributions. We kindly refer the interested reader to Appendix A of [46] for an exhaustive list of all propagators in the $(A, b, \phi, \tau)$-sector.[5] Albeit complicated, this loop computation does not require any further assumption or approximation. A resummation into exponential form might be necessary. We come back to this issue in our concluding Sec. VI where we survey the connection with the Nielsen identities. Depending on the employed underlying action, the number of vertices and diagrams can further grow. Let us mention here that the operators $A_\mu^h$ and $A_\mu^h A_\mu^h$ have also been shown to be fully renormalizable when working with the (refined) Gribov-Zwanziger effective action [49]. This ensures that the operator $-\partial_\mu D_\mu(A^h)$, see also the discussion below Eq. (9), is positive when performing the path integration over the gauge configurations. This fact guarantees that one would actually be considering, at the level of expectation values, those gauge fields $A^h$ which correspond to a minimum of the functional $\int d^4 x A_\mu^u A_\mu^u$. Taking all these considerations into account, it should be clear that any formal LKF relation in this paper remains fully valid if we prefer to work with the (refined) Gribov-Zwanziger action when explicitly evaluating the necessary expectation values, of course at the cost of introducing further computational complexity.

## C. The gauge invariant fermion fields and associated LKFT

In the matter sector, the fermion fields also have a gauge invariant counterpart, namely [46,64]

$$\psi^h = h^\dagger \psi, \tag{22}$$

with $h$ being still defined via Eq. (6), using the same $\phi^a$, but now coupled to the generators of the fundamental representation. Clearly, $\psi^h$ is gauge invariant per construction. This feature can be explicitly verified by combining the gauge transformation of the fermion field

$$\psi \to U^\dagger \psi \tag{23}$$

and Eq. (7).

The renormalizability of the composite operator $\psi^h$, although not yet fully established, can be achieved along the same lines of the proof of the renormalizability of the operator $A^h$ [47,49]. Even if this is technically challenging, we do not expect major conceptual issues to occur in the fermion sector when compared to the gluon $A_\mu^h$ case.

In principle, $\psi^h$ can be expanded in powers of the $\phi$-field as before, yielding

$$\psi^h = \psi - ig\phi\psi - \frac{g^2}{2}\phi^2\psi + \mathcal{O}(g^3). \tag{24}$$

## D. The LKFT for general *n*-point functions

Overall, when the gauge invariance is translated into the corresponding BRST symmetry [47,49], it turns out that the correlation functions of gauge invariant quantities like, for instance, $\langle A_\mu^h ... \psi^h ... \bar\psi^h ... \rangle_\alpha$, are independent from the gauge parameters. Therefore, for a general *n*-point function, it must hold that

$$\langle A_\mu^h ... \psi^h ... \bar\psi^h ... \rangle_\alpha = \langle A_\mu^h ... \psi^h ... \bar\psi^h ... \rangle_{\alpha'} \tag{25}$$

as all entering variables are explicitly gauge invariant. At first order this becomes

$$\left\langle A_\mu - \partial_\mu\phi + ig[A_\mu, \phi] + \frac{ig}{2}[\phi, \partial_\mu\phi]...\psi - ig\phi\psi...\bar\psi + ig\phi\bar\psi... \right\rangle_\alpha$$
$$= \left\langle A_\mu - \partial_\mu\phi + ig[A_\mu, \phi] + \frac{ig}{2}[\phi, \partial_\mu\phi]...\psi - ig\phi\psi...\bar\psi + ig\phi\bar\psi... \right\rangle_{\alpha'}.$$

Proceeding as before, in the gluon sector, we can always connect to $\alpha' = 0$ (i.e., the Landau gauge), thereby replacing $A_\mu^h$ by $A_\mu$.

Let us have a closer look at the fermion sector to illustrate what happens there. We specify to the fermion propagator. Using the gauge symmetry,

$$\langle \bar\psi^h(x)\psi^h(y) \rangle_\alpha = \langle \bar\psi^h(x)\psi^h(y) \rangle_{\alpha=0}, \tag{26}$$

the transformation of the $\bar\psi\psi$-propagator can be expressed as

$$\langle \bar\psi(x)e^{ig\phi(x)}e^{-ig\phi(y)}\psi(y) \rangle_\alpha = \langle \bar\psi(x)e^{ig\phi(x)}e^{-ig\phi(y)}\psi(y) \rangle_{\alpha=0}$$
$$= \langle \bar\psi(x)\psi(y) \rangle_{\alpha=0}, \tag{27}$$

---

[5]The field $\phi$ is denoted by $\xi$ in the latter reference.

where we used that $\partial A = 0$ in the Landau gauge. The relation (27) can be equivalently written as

$$\langle \bar{\psi}(x)\psi(y) \rangle_\alpha = \langle \bar{\psi}(x)e^{-ig\phi(x)}e^{ig\phi(y)}\psi(y) \rangle_{\alpha=0} \quad (28)$$

which is nothing else than the conventional LKFT for the fermion propagator; see for instance [55–57,62].

In the standard Abelian works on LKFTs, the rhs of Eq. (28) is usually factorized into

$$\langle \bar{\psi}(x)\psi(y) \rangle_\alpha = \langle \bar{\psi}(x)\psi(y) \rangle_{\alpha=0} \langle e^{-ig\phi(x)}e^{ig\phi(y)} \rangle_{\alpha=0}, \quad (29)$$

with

$$\langle \phi(p)\phi(-p) \rangle = -\alpha \frac{1}{p^4}. \quad (30)$$

We come back in detail to this issue in the next section.

## IV. LFKTS FROM THE PATH INTEGRAL: THE ABELIAN CASE

In what follows, we refresh the direct path integral derivation of the Abelian LKFT, which is a kind of rewriting of the original argument provided in [55,56] in a more modern language. In the next section, we generalize this derivation to the non-Abelian case, at the cost of adding several complications of course.

Consider for now the QED action

$$S = \int d^4x \left( \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}D\!\!\!/\psi + ib\partial_\mu A_\mu \right. $$
$$\left. + \frac{\alpha}{2}b^2 + \bar{c}\partial^2 c + \bar{J}_\psi \psi + \bar{\psi}J_{\bar{\psi}} \right), \quad (31)$$

where we included sources for $\psi$ and $\bar{\psi}$ to define the generating functional of Green functions, $Z(J)$, via the path integral[6]

$$Z(J) = \int [d\mu]e^{-S}. \quad (32)$$

Next, we transform the path integral variables $A$, $\psi$, and $\bar{\psi}$ using the gauge transformation

$$U = e^{ie\phi}, \quad (33)$$

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \phi, \quad (34)$$

$$\psi \to \psi' = U^\dagger \psi \quad (35)$$

---

[6]We consider here the complete Green functions obtained by differentiating $Z(J)$ with respect to the source $J$, though the conclusions immediately go through for the connected Green functions as well when $Z(J)$ is replaced by the corresponding generator $Z^c(J)$ via the usual identification $Z(J) = e^{-Z^c(J)}$.

and we select

$$\phi = -X \frac{1}{\partial^2} \partial_\mu A_\mu, \quad (36)$$

where the constant $X$ can still be chosen appropriately; see later. The gluon field transforms as

$$A_\mu \to A'_\mu = A_\mu + X \frac{1}{\partial^2} \partial_\mu \partial_\nu A_\nu \quad (37)$$

and so

$$\partial_\mu A'_\mu = (1 + X)\partial_\mu A_\mu. \quad (38)$$

When we perform the following transformation on the Lagrange multiplier $b$,

$$b \to b' = \frac{1}{1 + X}b, \quad (39)$$

and redefine the gauge parameter via

$$\alpha \to \alpha' = (1 + X)^2 \alpha, \quad (40)$$

the action, up to its source part, is transformed into itself, except that the gauge parameter $\alpha$ gets replaced by $\alpha'$. Importantly, also the source terms vary; more precisely we end up with

$$S' = \int d^4x \left( \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}'D\!\!\!/\psi' + ib'\partial_\mu A'_\mu + \frac{\alpha'}{2}b'^2 \right.$$
$$\left. + \bar{c}\partial^2 c + \bar{J}_\psi U\psi' + \bar{\psi}'U^\dagger J_{\bar{\psi}} \right). \quad (41)$$

It is consequently found that the $\phi$-propagator has the expected form [56,57,62]

$$\langle \phi(p)\phi(-p) \rangle_{\alpha'} = -\frac{X^2}{(1+X)^2} \frac{p_\mu}{p^2} \frac{p_\nu}{p^2} \langle A'_\mu(p)A'_\nu(-p) \rangle_{\alpha'} \quad (42)$$

$$= -\frac{X^2}{(1+X)^2} \frac{p_\mu}{p^2} \frac{p_\nu}{p^2} \alpha' \frac{p_\mu p_\nu}{p^4} \quad (43)$$

$$= -\frac{X^2}{(1+X)^2} \alpha' \frac{1}{p^4} \quad (44)$$

$$= -X^2 \alpha \frac{1}{p^4}. \quad (45)$$

Starting from any gauge $\alpha$, if we take the limit $X \to -1$ we arrive at the Landau gauge $\alpha' = 0$, while the $\phi$-propagator remains proportional to $\frac{1}{p^4}$. This rather singular behaviour is

fundamental to correctly transform the longitudinal projection of the gluon propagator. We recall here that the latter projection is uniquely fixed by means of the underlying Ward identities, i.e. the BRST invariance as well as other additional Ward identities defining the class of linear covariant gauges at the quantum level; see for instance [2,47].

### A. Application to the fermion propagator

The relevant partition function is given by

$$Z_\alpha = \int [\mathrm{d}\mu] e^{-S} \tag{46}$$

and, after the earlier described path integral variable transformation, by

$$Z_{\alpha'} = \int [\mathrm{d}\mu] e^{-S'}, \tag{47}$$

with $S$ and $S'$ given by Eqs. (31) and (41). We assume that the measure remains invariant, a feature which is proven explicitly in the next section. Let us emphasize that

$$Z_\alpha \equiv Z_{\alpha'}. \tag{48}$$

The fermion propagator is found by deriving $Z_\alpha$ with respect to $J_{\bar\psi}$ and $\bar J_\psi$,

$$\langle \bar\psi(x)\psi(y)\rangle_\alpha = \frac{\delta^2 Z_\alpha}{\delta \bar J_\psi(y)\delta J_{\bar\psi}(x)} \tag{49}$$

$$= \int [\mathrm{d}\mu]\bar\psi(x)\psi(y)e^{-S}. \tag{50}$$

Moreover, from Eq. (48), it is also given by

$$\langle \bar\psi(x)\psi(y)\rangle_\alpha = \frac{\delta^2 Z_{\alpha'}}{\delta \bar J_\psi(y)\delta J_{\bar\psi}(x)} \tag{51}$$

$$= \int [\mathrm{d}\mu]\bar\psi'(x)U^\dagger(x)U(y)\psi'(y)e^{-S'}. \tag{52}$$

When $\phi$ is a free field, it is evidently possible to factorize

$$\langle \bar\psi(x)\psi(y)\rangle_\alpha = \langle \bar\psi'(x)\psi'(y)\rangle_{\alpha'}\langle U^\dagger(x)U(y)\rangle_{\alpha'}. \tag{53}$$

We may ask ourselves if this is still the case when the $\phi$-field couples to $A$. The propagator in the new gauge becomes

$$\langle \bar\psi'(x)e^{\frac{-ieX}{1+X}\frac{1}{\partial^2}\partial_\mu A'_\mu(x)} e^{\frac{ieX}{1+X}\frac{1}{\partial^2}\partial_\nu A'_\nu(y)}\psi'(y)\rangle_{\alpha'}. \tag{54}$$

In the next step we expand the exponentials. In first order, this becomes the $\langle\bar\psi\psi\rangle$-propagator. In second order we

obtain, upon inclusion of a single fermion-gauge boson vertex,

$$\int \mathrm{d}^4 z \frac{-ieX}{1+X}(ie\gamma_\nu)\langle\bar\psi'(z)\psi'(y)\rangle_{\alpha'}\langle\bar\psi'(x)\psi'(z)\rangle_{\alpha'}$$
$$\times \left\langle \frac{1}{\partial^2}\partial_\mu A'_\mu(x)A'_\nu(z)\right\rangle_{\alpha'} \tag{55}$$

which might spoil the above-mentioned factorization. Notice, however, that expression (55) is proportional to

$$\frac{X}{1+X}\alpha' = X\sqrt{\alpha\alpha'}. \tag{56}$$

The $\alpha'$ in the lhs of Eq. (56) arises from the longitudinal part of the gauge boson propagator, hidden in the last factor of Eq. (55). In the Landau gauge, i.e. $\alpha' = 0$ from $X \to -1$, this term disappears and we are effectively able to factorize this expectation value as in Eq. (29). This also holds at higher orders, since any contraction of a gauge field $A$ from a vertex with a field $A$ lurking in the exponential of $\phi$ always vanishes in the Landau gauge, similarly to what was just illustrated.

### B. Application to the photon propagator

We can also investigate the photon propagator. Therefore, we add the term $\int \mathrm{d}^4 x J_\mu A_\mu$ to the action, so that

$$\langle A_\mu(x)A_\nu(y)\rangle_\alpha = \frac{\delta^2 Z_\alpha}{\delta J_\nu(y)\delta J_\mu(x)} \tag{57}$$

in the original gauge.

This extra source term in the Lagrangian transforms as

$$J_\mu A_\mu \to J_\mu(A'_\mu + \partial_\mu\phi) \tag{58}$$

$$= J_\mu\left(A'_\mu - \frac{X}{1+X}\frac{1}{\partial^2}\partial_\mu\partial_\nu A'_\nu\right). \tag{59}$$

From this, we find for the photon propagator

$$D_{\mu\nu}^{(\alpha)}(p^2) = \langle A_\mu(p)A_\nu(-p)\rangle_\alpha \tag{60}$$

$$= \left\langle \left(A'_\mu(p) - \frac{X}{1+X}\frac{1}{\partial^2}\partial_\mu\partial_\alpha A'_\alpha(p)\right) \right.$$
$$\times \left.\left(A'_\nu(-p) - \frac{X}{1+X}\frac{1}{\partial^2}\partial_\nu\partial_\beta A'_\beta(-p)\right)\right\rangle_{\alpha'} \tag{61}$$

$$= D_{\mu\nu}^{(\alpha')}(p^2) + \left(-2\alpha'\frac{X}{1+X} + \alpha'\frac{X^2}{(1+X)^2}\right)\frac{L_{\mu\nu}}{p^2} \tag{62}$$

$$= D_{\mu\nu}^{(\alpha')}(p^2) - \alpha\left(\frac{\alpha'}{\alpha} - 1\right)\frac{p_\mu p_\nu}{p^4}. \tag{63}$$

We used $\alpha' = \alpha(1 + X)^2$ and the standard photon propagator decomposition in a general linear gauge,

$$\langle A_\mu(p)A_\nu(-p)\rangle_\alpha = D_{\mu\nu}^{(\alpha)}(p^2) = \Delta(p^2)P_{\mu\nu} + \frac{\alpha}{p^2}L_{\mu\nu}, \quad (64)$$

with the transversal and longitudinal projectors

$$P_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad (65)$$

$$L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}. \qquad (66)$$

Clearly, Eq. (64) expresses that only the longitudinal part of the photon propagator is affected by shifting $\alpha \to \alpha'$, as it is well known in the Abelian case.

## V. LFKT FROM THE PATH INTEGRAL: THE NON-ABELIAN CASE

We now generalize the foregoing path integral derivation of the LKFTs to a non-Abelian gauge theory supplemented with fermion matter.

We must first establish a general SU($N$) transformation with matrix $U = e^{ig\phi}$ for all fields, namely, gauge, matter and Faddeev-Popov ghosts, while maintaining the property that $\partial_\mu A'_\mu = (1 + X)\partial_\mu A_\mu$. This is a necessary requirement, as it precisely allows for the rescaling of the Lagrange multiplier $b$, and thereby for that of the gauge parameter $\alpha$. As before, $\phi = \phi^a T^a$.

The gauge and matter fields transform as

$$A_\mu \to A'_\mu = U^\dagger A_\mu U + \frac{i}{g}U^\dagger \partial_\mu U, \qquad (67)$$

$$\psi \to \psi' = U^\dagger \psi, \qquad (68)$$

$$U = e^{ig\phi} = 1 + ig\phi - \frac{g^2}{2}\phi^2 + \mathcal{O}(\phi^3). \quad (69)$$

Now, if we let the Faddeev-Popov ghosts transform in the adjoint representation,

$$c \to U^\dagger c U, \qquad (70)$$

$$\bar{c} \to U^\dagger \bar{c} U, \qquad (71)$$

we obtain

$$\bar{c}\partial_\mu D_\mu c \to \bar{c}\partial_\mu D_\mu c + \bar{c}U(\partial_\mu U^\dagger)D_\mu c + \bar{c}D_\mu c(\partial_\mu U)U^\dagger. \qquad (72)$$

This variation can be reabsorbed by means of the shift

$$c \to c' = c + \frac{1}{\partial_\mu D_\mu}U(\partial_\mu U^\dagger)D_\mu c + \frac{1}{\partial_\mu D_\mu}D_\mu c(\partial_\mu U)U^\dagger. \qquad (73)$$

Doing so, we obtain the original action, but now with the changed gauge parameter $\alpha'$.

Concretely, let us expand the transformation (67) to second order in the fields,

$$A'_\mu = A_\mu - \partial_\mu\phi - ig\phi A_\mu + igA_\mu\phi + ig\phi\partial_\mu\phi - \frac{ig}{2}\partial_\mu\phi^2$$
$$+ \mathcal{O}(\text{fields}^3) \qquad (74)$$

$$= A_\mu - \partial_\mu\phi + ig[A_\mu, \phi] + \frac{ig}{2}[\phi, \partial_\mu\phi] + \mathcal{O}(\text{fields}^3); \quad (75)$$

thence we impose that

$$\partial_\mu A'_\mu = \partial_\mu A_\mu - \partial^2\phi + ig[A_\mu, \partial_\mu\phi] + ig[\partial_\mu A_\mu, \phi]$$
$$+ \frac{ig}{2}[\phi, \partial^2\phi] + \mathcal{O}(\text{fields}^3) \qquad (76)$$

$$\equiv (1 + X)\partial_\mu A_\mu \qquad (77)$$

and so we require

$$\partial^2\phi = -X\partial_\mu A_\mu + ig[A_\mu, \partial_\mu\phi] + ig[\partial_\mu A_\mu, \phi]$$
$$+ \frac{ig}{2}[\phi, \partial^2\phi] + \mathcal{O}(\text{fields}^3). \qquad (78)$$

At first order this gives

$$\phi = -X\frac{1}{\partial^2}\partial_\mu A_\mu, \qquad (79)$$

which is nothing but the Abelian result. Solving iteratively for $\phi$ in powers of $A_\mu$, we get

$$\phi = -X\frac{1}{\partial^2}\partial_\mu A_\mu - igX\frac{1}{\partial^2}\left[A_\mu, \frac{1}{\partial^2}\partial_\mu\partial_\nu A_\nu\right]$$
$$- igX\frac{1}{\partial^2}\left[\partial_\nu A_\nu, \frac{1}{\partial^2}\partial_\mu A_\mu\right] + \frac{igX^2}{2}\frac{1}{\partial^2}\left[\frac{1}{\partial^2}\partial_\mu A_\mu, \partial_\nu A_\nu\right]$$
$$+ \mathcal{O}(A^3). \qquad (80)$$

Using this solution, we can calculate $A'_\mu$ as a function of the original $A_\mu$,

$$A'_\mu = A_\mu + X\frac{\partial_\mu\partial A}{\partial^2} - igX\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu\partial A}{\partial^2}, A_\nu\right]$$
$$- igX\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] - \frac{igX^2}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right]$$
$$+ igX\left[\frac{\partial A}{\partial^2}, A_\mu\right] + \frac{igX^2}{2}\left[\frac{\partial A}{\partial^2}, \frac{\partial_\mu\partial A}{\partial^2}\right] + \mathcal{O}(A^3). \quad (81)$$

Note that for the Landau gauge, $X = -1$, this expression coincides with the gauge invariant transversal field $A_\mu^h$; see e.g. [47]. In general, $A_\mu'$ is not transversal though.

Denoting the color dependence explicitly, this equation becomes

$$A_\mu'^a = A_\mu^a + X\frac{\partial_\mu \partial A^a}{\partial^2} - igXf^{abc}\frac{\partial_\mu}{\partial^2}\left(\frac{\partial_\nu \partial A^b}{\partial^2}A_\nu^c\right)$$
$$- igXf^{abc}\frac{\partial_\mu}{\partial^2}\left(\frac{\partial A^b}{\partial^2}\partial A^c\right) - \frac{igX^2}{2}f^{abc}\frac{\partial_\mu}{\partial^2}\left(\frac{\partial A^b}{\partial^2}\partial A^c\right)$$
$$+ igXf^{abc}\frac{\partial A^b}{\partial^2}A_\mu^c + \frac{igX^2}{2}f^{abc}\frac{\partial A^b}{\partial^2}\frac{\partial_\mu \partial A^c}{\partial^2} + \mathcal{O}(A^3).$$
$$(82)$$

This functional relation can be inverted to find the old fields in terms of the new. Using $\partial A' = (1 + X)\partial A$ we obtain

$$A_\mu = A_\mu' - \frac{X}{1+X}\frac{\partial_\mu \partial A'}{\partial^2} + \frac{igX}{1+X}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu \partial A'}{\partial^2}, A_\nu\right]$$
$$+ \frac{igX}{(1+X)^2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A'}{\partial^2}, \partial A'\right] + \frac{igX^2}{2(1+X)^2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A'}{\partial^2}, \partial A'\right]$$
$$- \frac{igX}{1+X}\left[\frac{\partial A'}{\partial^2}, A_\mu\right] - \frac{igX^2}{2(1+X)^2}\left[\frac{\partial A'}{\partial^2}, \frac{\partial_\mu \partial A'}{\partial^2}\right]$$
$$+ \mathcal{O}(A^3).$$
$$(83)$$

Again, in first order we find the Abelian result

$$A_\mu = A_\mu' - \frac{X}{1+X}\frac{\partial_\mu \partial A'}{\partial^2}.$$
$$(84)$$

Up to second order we find the old $A$-field as a function of the new

$$A_\mu = A_\mu' - \frac{X}{1+X}\frac{\partial_\mu \partial A'}{\partial^2} + \frac{igX}{1+X}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu \partial A'}{\partial^2}, A_\nu'\right] - \frac{igX^2}{(1+X)^2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu \partial A'}{\partial^2}, \frac{\partial_\nu \partial A'}{\partial^2}\right] + \frac{igX}{(1+X)^2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A'}{\partial^2}, \partial A'\right]$$
$$+ \frac{igX^2}{2(1+X)^2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A'}{\partial^2}, \partial A'\right] - \frac{igX}{1+X}\left[\frac{\partial A'}{\partial^2}, A_\mu'\right] + \frac{igX^2}{(1+X)^2}\left[\frac{\partial A'}{\partial^2}, \frac{\partial_\mu \partial A'}{\partial^2}\right] - \frac{igX^2}{2(1+X)^2}\left[\frac{\partial A'}{\partial^2}, \frac{\partial_\mu \partial A'}{\partial^2}\right] + \mathcal{O}(A^3)$$
$$(85)$$

which constitutes the generalization of Eq. (58). When applied to the source term, we can perform a similar derivation to get an explicit connection between the gluon propagator in two different linear covariant gauges, parametrized by $\alpha$ and $\alpha'$. Since the expression in the rhs of Eq. (85) is not restricted to terms containing a space-time derivative $\partial_\mu$ beyond leading order, this implies that the transformation (85) is also affecting the transversal component of the gluon propagator. In particular, when transforming to the Landau gauge, it is clear that we recover the same transformation law as obtained in Sec. III A.

Given that we have constructed $\phi$ in full generality, we can also easily construct the non-Abelian transformation law for the fermion propagator. In fact, the analysis leading to Eq. (51) can be mostly taken over; thus we find

$$\langle\bar{\psi}(x)\psi(y)\rangle_\alpha = \langle\bar{\psi}'(x)U^\dagger(x)U(y)\psi'(y)\rangle_{\alpha=0}$$
$$(86)$$

$$= \langle\bar{\psi}'(x)e^{-ig\phi(x)}e^{ig\phi(y)}\psi'(y)\rangle_{\alpha=0}.$$
$$(87)$$

As expected, this non-Abelian LKFT law is in perfect agreement with the alternative derivation with the gauge invariant fermion field $\psi^h$ that resulted in Eq. (28). For completeness, in the Landau gauge, the factorization into

$$\langle\bar{\psi}(x)\psi(y)\rangle_\alpha = \langle\bar{\psi}'(x)\psi'(y)\rangle_{\alpha=0}\langle e^{-ig\phi(x)}e^{ig\phi(y)}\rangle_{\alpha=0}$$
$$(88)$$

still holds, following the same logic as in the Abelian case. It is important to realize here the inherent complication compared to the Abelian case: the LKF field $\phi$ is now an infinite series, represented by Eq. (80). This is important, in particular, for the renormalizability of the whole construction; see also the comments in Sec. II and [47,49]. As such, our construction is more general than that explored in [62], where higher order corrections to the quark propagator LKFT were explored, though keeping the Abelian approximation for the fields $\phi^a$. We have now unraveled that a self-consistent approach requires adding more and more terms to $\phi^a$ as the perturbative order increases.

Before turning to our conclusions and giving an outlook to follow-up work, there is a subtle point we did not address so far. In order that $Z_\alpha = Z_{\alpha'}$, we used that the action remains invariant under the applied transformations. At the level of the path integral, in order for our derivation to be correct, we also need that the integration measure remains invariant,

$$[\mathrm{d}\mu] = [\mathrm{d}\mu'],$$
$$(89)$$

i.e., that there is no Jacobian. We do not expect a nontrivial Jacobian, since we already derived the transformations using the gauge invariant $h$-fields without encountering any differences with the rederivation via path integral tools and deliberate omittance of the Jacobian. Though, to be sure, let

us also verify this explicitly. More precisely, we show that the super-Jacobian[7] of the transformation,

$$\mathfrak{J} = \left( \frac{A'^a_\mu, c'^a, \bar{c}'^a}{A^b_\nu, c^b, \bar{c}^b} \right) = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \qquad (90)$$

is trivial. The superdeterminant is given by

$$\text{sdet} \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} = \det \mathfrak{A} + \mathfrak{B}\mathfrak{D}^{-1}\mathfrak{C} \det \mathfrak{D}^{-1}. \quad (91)$$

The transformation of the gluon fields $A_\mu$ is independent of $c$ and $\bar{c}$, resulting in $\mathfrak{C} = 0$ and we see that the superdeterminant collapses to the product of the individual determinants.

For the gluons, using Eq. (82), the argument of this determinant can be calculated (in what follows only the color dependence concerns us),

$$\frac{\delta A'^a_\mu}{\delta A^b_\nu} = \delta^{ab} C + f^{abc} D. \qquad (92)$$

The infinitesimal transformation in the ghost sector, Eq. (70), is found to be

$$\delta c^a = -igf^{abc}\phi^b c^c, \qquad (93)$$

$$\delta \bar{c}^a = -igf^{abc}\phi^b \bar{c}^c, \qquad (94)$$

and thence the matrix $\mathfrak{D}$ becomes

$$\begin{pmatrix} \delta^{ab} - igf^{acb}\phi^c & 0 \\ 0 & \delta^{ab} - igf^{acb}\phi^c \end{pmatrix}, \qquad (95)$$

which also leads to trivial 1 when taking the determinant. Finally, the shift of Eq. (73) evidently comes with a trivial Jacobian.

## VI. CONCLUSIONS AND OUTLOOK

We have employed the gauge invariant fields $A^h_\mu$ and $\psi^h$ to provide an alternative way to derive the LKFTs for general $n$-point correlators. This derivation was first performed for the Abelian LKFT for the photon and fermion fields. It reproduced the correct relations as already known from the literature. The extension to non-Abelian theories was then presented. To our knowledge, this is the first time in which the non-Abelian LKFTs have been derived for arbitrary $n$-point correlators without any approximation.

To lend further credit to the validity of our non-Abelian LKFTs, we also presented an independent derivation of the

---

[7]As we have a mix of commuting and anticommuting variables, we must consider the super-Jacobian (Berezinian).

LKFTs, from the viewpoint of the path integral formalism, leading to exactly the same transformations.

Specifically, considering the gluon and quark propagators for an SU($N$) non-Abelian gauge theory, such as QCD, leads to the relations (20), (21) and (88). Although these non-Abelian LKFTs do look (and are) nonlocal in nature, we stress here that our framework can be cast in a fully local, and even renormalizable formulation. This claim follows from the observation that the gauge invariant composite operator $A^h_\mu$ is renormalizable, as discussed in [47,49]. The key is using the algebraic renormalization formalism based on a Stueckelberg-like reformulation of $A^h_\mu$, in which case the field $\phi$ is kept as a basic field with its corresponding propagator given by Eq. (30). A delicate point is the potential occurrence of infrared singularities when such a propagator is explicitly used in $d = 4$. However, this can be overcome in a BRST consistent fashion by introducing a regulating mass in the $\phi$-sector that is to be sent to 0 at the end of any calculation [47–49]; see also [78]. Details on this are presented in a work in progress, where the one loop explicit check in terms of Feynman diagrams will be worked out.

Let us discuss the prospect of applying these non-Abelian LKFTs to nonperturbative functional studies of QCD, in particular, related to its constituent gluon and quark dynamics, followed by their role in the Bethe-Salpeter and Faddeev equations, usually employed to study the bound state spectrum of QCD. In the short run, a perturbative verification of our formalism is planned. The gauge invariance of chiral quark condensate, associated primarily with the quark propagator, may be a next relatively more involved problem.

In the long run, a comprehensive study of hadronic observables through DSEs and establishing their strict gauge invariance would be highly desirable, thus raising this formalism to a higher level of credibility and acceptance. In this connection, as already mentioned in the introduction, till now such efforts have been mostly restricted to the Landau gauge. Several (constrained) *Ansätze* have been put forward, with increasing complexity, each time making improved contact with phenomenology [79] and also with underlying QCD dynamics [80]. The validity of such *Ansätze* at the level of gauge covariance, and ultimately gauge invariance, is crucial. An ideal goal is to construct *Ansätze* in a generic linear covariant gauge parametrized in terms an arbitrary value of $\alpha$, either explicitly or through the defining entities of different Green functions. Not only should such *Ansätze* abide by the key symmetries of QCD but they should also stand firm against any explicit check to what extent physical quantities are effectively gauge invariant. This is a prohibitively daunting task, as is evident in the much simpler QED studies as well; see [50–54]. However, we must realize that even if we now have access to the non-Abelian LKFTs, our construction has been cast in a perturbative form, viz.

determined by the infinite series expansion of the gauge invariant variables $A_\mu^h$ (or $\psi^h$) in the field $\phi$. An intrinsically nonperturbative setup would require one to work with the matrix field $h$ introduced in Eq. (6), and its exact quantum behavior. This does not appear feasible at the moment within our approach. So we may have to resort to an approximate framework. Given that e.g. Dyson-Schwinger equations anyhow require a truncation at finite order (i.e. finite number of $n$-point interactions), one could restrict to the four-point level expansion, which includes the three-point and four-point gluon vertices, the three-point ghost-gluon vertex and three-point quark-gluon vertex. For each of these vertices in Landau gauge, several results are available in the literature from a variety of sources; see [81–87] for a small and thus incomplete selection. The rigorous formalism developed in the current paper can be applied to get corresponding vertices in another gauge. At first instance, this can be done in perturbation theory, thereby extending the work of [24,62]. This might be more realistic than *a priori* expected, as it is conceivable that the relevant nonperturbative infrared physics, hiding in gauge variant interaction vertices and resulting in gauge-invariant physical observables, may have the gauge dependent pieces of a perturbative nature, not necessarily or easily summable in a closed form. Though seemingly an interesting viewpoint, it needs closer scrutiny and further exploration.

This being said, it is well known that the contemporary way to deal with issues related to the gauge covariance is via the powerful BRST invariance [88–91], or more precisely via its functional representation, the Slavnov-Taylor identity. From the latter, it is not only possible to derive various relations between different correlation functions in a fixed gauge, but also how $n$-point functions vary in terms of the gauge parameter. The latter relations are encoded in the Nielsen identities [92–94], which follow directly from the Slavnov-Taylor identity. Given that the original LKFTs predate the BRST construction by about two decades, one cannot help but wonder if it would not be possible to construct such transformations directly from the Nielsen identities, which after all have the same goal as the LKFTs: a mathematical way to write down how $n$-point functions change under a changing gauge parameter. In recent work [48], the Nielsen identity and its

consequences in relation to the gauge invariant propagator $\langle A_\mu^h(p) A_\nu^h(-p) \rangle$ were already discussed. As a corollary, we derived the Abelian LKFT for the photon from the integrated version of the photon propagator's Nielsen identity. Moreover, in a recent work, LKFT have been employed to show the gauge invariance of the electron pole mass in QED, something that was proved through the corresponding Nielsen identities earlier [95]. We are now aiming at exploring how this can be generalized to the non-Abelian case, thereby hopefully uncovering new powerful uses of the Nielsen identities. For example, when integrated with respect to the gauge parameter, the Nielsen identity automatically leads to an exponential factor connecting the propagators in different linear gauges. Such a relation was hinted at in [62] but not yet proven in the non-Abelian case. The Nielsen identities also naturally provide a factorization between the propagator in Landau gauge and that of any other linear covariant gauge. An illustrative representation of this fundamental property can be found in [48], Sec. 4.4. In the case of the fermion propagator, this property is already clear from the result (88), but in the gauge sector, this is not immediately seen from Eq. (21). These and other matters are discussed in a forthcoming work, including the introduction of a unifying framework to discuss LKFTs and Nielsen identities. In this context, it is also interesting to point out that the Nielsen identities were also explored in [40] in relation to a dynamical mass generation in linear covariant gauges in a Dyson-Schwinger framework, thereby uncloaking certain subtleties that deserve further attention.

Overall, it should be clear that the current article is a first preliminary step in our rigorous formalism that can be extended in several directions, as sketched above.

[1] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994).

[2] R. Alkofer and L. von Smekal, Phys. Rep. **353**, 281 (2001).

[3] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **78**, 025010 (2008).

[4] D. Binosi and J. Papavassiliou, Phys. Rep. **479**, 1 (2009).

[5] C. S. Fischer, A. Maas, and J. M. Pawlowski, Ann. Phys. (Amsterdam) **324**, 2408 (2009).

[6] P. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, J. High Energy Phys. 06 (2008) 099.

[7] P. Boucaud, J. P. Leroy, A. L. Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, Few Body Syst. **53**, 387 (2012).

[8] A. Bashir, L. Chang, I. C. Cloet, B. El-Bennich, Y. X. Liu, C. D. Roberts, and P. C. Tandy, Commun. Theor. Phys. **58,** 79 (2012).

[9] A. Maas, Phys. Rep. **524,** 203 (2013).

[10] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, Prog. Part. Nucl. Phys. **91,** 1 (2016).

[11] J. M. Pawlowski, Ann. Phys. (Amsterdam) **322,** 2831 (2007).

[12] A. K. Cyrol, M. Mitter, J. M. Pawlowski, and N. Strodthoff, Phys. Rev. D **97,** 054006 (2018).

[13] K. Raya, M. A. Bedolla, J. J. Cobos-Martínez, and A. Bashir, arXiv:1711.00383.

[14] G. M. Huber *et al.* (Jefferson Lab Collaboration), Phys. Rev. C **78,** 045203 (2008).

[15] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. D **80,** 052002 (2009).

[16] S. Uehara *et al.* (Belle Collaboration), Phys. Rev. D **86,** 092007 (2012).

[17] L. Chang, I. C. Cloët, C. D. Roberts, S. M. Schmidt, and P. C. Tandy, Phys. Rev. Lett. **111,** 141802 (2013).

[18] K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martinez, L. X. Gutiérrez-Guerrero, C. D. Roberts, and P. C. Tandy, Phys. Rev. D **93,** 074017 (2016).

[19] C. J. Burden and C. D. Roberts, Phys. Rev. D **47,** 5581 (1993).

[20] A. Bashir and A. Raya, Phys. Rev. D **66,** 105005 (2002).

[21] A. Bashir, A. Raya, S. Sanchez-Madrigal, and C. D. Roberts, Few Body Syst. **46,** 229 (2009).

[22] S. Jia and M. R. Pennington, Phys. Lett. B **769,** 146 (2017).

[23] S. Jia and M. R. Pennington, Phys. Rev. D **95,** 076007 (2017).

[24] R. Bermudez, L. Albino, L. X. Gutiérrez-Guerrero, M. E. Tejeda-Yeomans, and A. Bashir, Phys. Rev. D **95,** 034041 (2017).

[25] A. Cucchieri and T. Mendes, *Proc. Sci.*, LATTICE2007 (2007) 297 [arXiv:0710.0412].

[26] A. Cucchieri and T. Mendes, Phys. Rev. Lett. **100,** 241601 (2008).

[27] A. Cucchieri and T. Mendes, Phys. Rev. D **78,** 094503 (2008).

[28] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Phys. Lett. B **676,** 69 (2009).

[29] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, A. Sternbeck, L. von Smekal, A. G. Williams, and J. Zhang, Phys. Rev. D **76,** 094505 (2007).

[30] A. Maas, Phys. Rev. D **79,** 014505 (2009).

[31] P. J. Silva and O. Oliveira, Nucl. Phys. **B690,** 177 (2004).

[32] O. Oliveira and P. J. Silva, Phys. Rev. D **86,** 114513 (2012).

[33] D. Dudal, O. Oliveira, and P. J. Silva, Phys. Rev. D **89,** 014010 (2014).

[34] S. Furui and H. Nakajima, Phys. Rev. D **69,** 074505 (2004).

[35] L. von Smekal, K. Maltman, and A. Sternbeck, Phys. Lett. B **681,** 336 (2009).

[36] A. Cucchieri, T. Mendes, and E. M. S. Santos, Phys. Rev. Lett. **103,** 141602 (2009).

[37] A. Cucchieri, T. Mendes, G. M. Nakamura, and E. M. S. Santos, AIP Conf. Proc. **1354,** 45 (2011).

[38] P. Bicudo, D. Binosi, N. Cardoso, O. Oliveira, and P. J. Silva, Phys. Rev. D **92,** 114514 (2015).

[39] M. Q. Huber, Phys. Rev. D **91,** 085018 (2015).

[40] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **91,** 085014 (2015).

[41] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **95,** 034017 (2017).

[42] F. Siringo, Phys. Rev. D **90,** 094021 (2014).

[43] F. Siringo, Phys. Rev. D **92,** 074034 (2015).

[44] M. A. L. Capri, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, S. P. Sorella, D. Dudal, I. F. Justo, A. D. Pereira, and R. F. Sobreiro, Phys. Rev. D **92,** 045039 (2015).

[45] M. A. L. Capri, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, S. P. Sorella, D. Dudal, I. F. Justo, A. D. Pereira, and R. F. Sobreiro, Phys. Rev. D **93,** 065019 (2016).

[46] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimaraes, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P. Sorella, Phys. Rev. D **94,** 025035 (2016).

[47] M. A. L. Capri, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, and S. P. Sorella, Phys. Rev. D **94,** 065009 (2016).

[48] M. A. L. Capri, D. Dudal, A. D. Pereira, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, and S. P. Sorella, Phys. Rev. D **95,** 045011 (2017).

[49] M. A. L. Capri, D. Fiorentini, A. D. Pereira, and S. P. Sorella, Phys. Rev. D **96,** 054022 (2017).

[50] A. Bashir and M. R. Pennington, Phys. Rev. D **50,** 7679 (1994).

[51] A. Bashir, Y. Concha-Sanchez, and R. Delbourgo, Phys. Rev. D **76,** 065009 (2007).

[52] A. Bashir, A. Raya, and S. Sanchez-Madrigal, Phys. Rev. D **84,** 036013 (2011).

[53] A. Bashir, R. Bermudez, L. Chang, and C. D. Roberts, Phys. Rev. C **85,** 045205 (2012).

[54] A. Kızılersü, T. Sizer, M. R. Pennington, A. G. Williams, and R. Williams, Phys. Rev. D **91,** 065015 (2015).

[55] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **89,** 89 (1955) [Sov. Phys. JETP **2,** 69 (1956)].

[56] E. S. Fradkin, Zh. Eksp. Teor. Fiz. **29,** 258 (1955) [Sov. Phys. JETP **2,** 361 (1956)].

[57] K. Johnson and B. Zumino, Phys. Rev. Lett. **3,** 351 (1959).

[58] A. Bashir and R. Delbourgo, J. Phys. A **37,** 6587 (2004).

[59] A. Bashir and A. Raya, Few Body Syst. **41,** 185 (2007).

[60] L. A. Fernandez-Rangel, A. Bashir, L. X. Gutierrez-Guerrero, and Y. Concha-Sanchez, Phys. Rev. D **93,** 065022 (2016).

[61] N. Ahmadiniaz, A. Bashir, and C. Schubert, Phys. Rev. D **93,** 045023 (2016).

[62] M. J. Aslam, A. Bashir, and L. X. Gutierrez-Guerrero, Phys. Rev. D **93,** 076001 (2016).

[63] D. Dudal and D. Vercauteren, Phys. Lett. B **779,** 275 (2018).

[64] M. Lavelle and D. McMullan, Phys. Rep. **279,** 1 (1997).

[65] R. Horan, M. Lavelle, and D. McMullan, Pramana **51,** 317 (1998).

[66] M. Semenov-Tyan-Shanskii and V. Franke, *A Variational Principle for the Lorentz Condition and Restriction of the Domain of Path Integration in non-Abelian Gauge Theory* (Zap. Nauch. Sem. Leningrad. Otdeleniya Matematicheskogo Instituta im V. A. Steklov, AN SSSR 120, 1982), p. 159; English translation, M. Semenov-Tyan-Shanskii and V. Franke (Plenum Press, New York, 1986).

[67] G. Dell'Antonio and D. Zwanziger, Commun. Math. Phys. **138**, 291 (1991).

[68] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).

[69] N. Vandersickel and D. Zwanziger, Phys. Rep. **520**, 175 (2012).

[70] M. A. L. Capri, D. Dudal, M. S. Guimaraes, A. D. Pereira, B. W. Mintz, L. F. Palhares, and S. P. Sorella, arXiv:1802 .04582.

[71] K. G. Chetyrkin, S. Narison, and V. I. Zakharov, Nucl. Phys. **B550**, 353 (1999).

[72] F. V. Gubarev, L. Stodolsky, and V. I. Zakharov, Phys. Rev. Lett. **86**, 2220 (2001).

[73] F. V. Gubarev and V. I. Zakharov, Phys. Lett. B **501**, 28 (2001).

[74] P. Boucaud, A. Le Yaouanc, J. P. Leroy, J. Micheli, O. Pene, and J. Rodriguez-Quintero, Phys. Rev. D **63**, 114003 (2001).

[75] H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen, Phys. Lett. B **516**, 307 (2001).

[76] H. Sonoda, Phys. Lett. B **499**, 253 (2001).

[77] A. I. Davydychev, P. Osland, and O. V. Tarasov, Phys. Rev. D **54**, 4087 (1996); Phys. Rev. D **59**, 109901(E) (1999).

[78] M. A. L. Capri, D. M. van Egmond, G. Peruzzo, M. S. Guimaraes, O. Holanda, S. P. Sorella, R. C. Terin, and H. C. Toledo, Ann. Phys. (Amsterdam) **390**, 214 (2018).

[79] I. C. Cloet, C. D. Roberts, and A. W. Thomas, Phys. Rev. Lett. **111**, 101803 (2013).

[80] D. Binosi, L. Chang, J. Papavassiliou, and C. D. Roberts, Phys. Lett. B **742**, 183 (2015).

[81] A. Cucchieri, A. Maas, and T. Mendes, Phys. Rev. D **77**, 094510 (2008).

[82] A. L. Blum, M. Q. Huber, M. Mitter, and L. von Smekal, Phys. Rev. D **89**, 061703 (2014).

[83] D. Binosi, L. Chang, J. Papavassiliou, S. X. Qin, and C. D. Roberts, Phys. Rev. D **95**, 031501 (2017).

[84] E. Rojas, J. P. B. C. de Melo, B. El-Bennich, O. Oliveira, and T. Frederico, J. High Energy Phys. 10 (2013) 193.

[85] A. Sternbeck, P. H. Balduf, A. Kızılersü, O. Oliveira, P. J. Silva, J. I. Skullerud, and A. G. Williams, *Proc. Sci.*, LATTICE20162017 (2017) 349 [arXiv:1702.00612].

[86] M. Peláez, M. Tissier, and N. Wschebor, Phys. Rev. D **92**, 045012 (2015).

[87] A. K. Cyrol, L. Fister, M. Mitter, J. M. Pawlowski, and N. Strodthoff, Phys. Rev. D **94**, 054005 (2016).

[88] C. Becchi, A. Rouet, and R. Stora, Phys. Lett. **52B**, 344 (1974).

[89] C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. **42**, 127 (1975).

[90] O. Piguet and S. P. Sorella, Lect. Notes Phys., M: Monogr. **28**, 1 (1995).

[91] G. Barnich, F. Brandt, and M. Henneaux, Phys. Rep. **338**, 439 (2000).

[92] N. K. Nielsen, Nucl. Phys. **B101**, 173 (1975).

[93] J. C. Breckenridge, M. J. Lavelle, and T. G. Steele, Z. Phys. C **65**, 155 (1995).

[94] O. Piguet and K. Sibold, Nucl. Phys. **B253**, 517 (1985).

[95] P. Dall'Olio and A. Bashir, in *Proceedings of the Workshop to Pay Homeage to Late Dr. Victor Villanueva, December 6, 2013, Morelia, Mexico* [Institute of Physics, Journal of Physics (to be published)].