Holographic Lifshitz superconductors: Analytic solution

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We construct an analytic solution for a one-parameter family of holographic superconductors in asymptotically Lifshitz spacetimes. We utilize this solution to explore various properties of the systems such as (1) the superfluid phase background and the grand canonical potential, (2) the order parameter response function or the susceptibility, (3) the London equation, and (4) the background with a superfluid flow or a magnetic field. From these results, we identify the dual Ginzburg-Landau theory including numerical coefficients. Also, the dynamic critical exponent z_D associated with the critical point is given by $z_D = 2$ irrespective of the value of the Lifshitz exponent z.

I. INTRODUCTION AND SUMMARY

The AdS/CFT duality [1–4] has been a useful tool to study realistic strongly coupled systems (see., e.g., Refs. [5–9] for textbooks). In condensed-matter applications, holographic superconductors provide particularly useful "theoretical laboratories" [10–12]. They are useful to explore not only standard aspects of a superconducting transition but also various related phenomena such as critical dynamics [13], defect formations [14–16], and superfluid turbulences [17].

The holographic superconductors arise in a broad range of gravitational theories with matter fields. From field theory point of view, this is natural since a superconductor is a robust phenomenon at low temperature. For example, it arises not only in asymptotically AdS spacetimes but also in asymptotically Lifshitz spacetimes [18,19], which is our main focus in this paper.

A holographic superconductor is typically an Einstein-Maxwell-complex scalar system. Such a system is hard to solve, in general. One often needs either a numerical computation or an approximation method, and there are only a few analytic solutions [20–22]. In this paper, we present an analytic solution for a one-parameter family of holographic Lifshitz superconductors.

A holographic Lifshitz superconductor has three parameters (p, z, Δ), where p is the number of boundary spatial dimensions, z is the Lifshitz exponent,[1] and Δ is the scaling dimension of the order parameter. In this paper, we consider the case where

(1) $p = 3z$, and
(2) $\Delta = (p + z)/2$, or the scalar mass m^2 saturates the Lifshitz Breitenlohner-Freedman (BF) bound [23].

These conditions still admit a one-parameter family of theories, and these parameters are related by

$$p = 3z, \qquad \Delta = 2z. \tag{1.1}$$

In this case, there is a simple analytic solution for the order parameter at the critical point,

$$\Psi = \frac{u^{2z}}{1 + u^{2z}}, \tag{1.2}$$

where u is the inverse of the radial coordinate. The $z = 1$ case, namely the AdS case, has been discussed in Ref. [22]. Below we utilize this solution to explore various properties of these systems, e.g.,

(1) The background solution in the superfluid phase (Sec. V A).

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[1]The parameter z is known as "dynamic critical exponent," but we call it "Lifshitz exponent" to avoid confusion with z_D below.

(2) The grand canonical potential, and the phase transition is second order (Sec. V B).

(3) The order parameter response function or the susceptibility. The dynamic response is obtained in the normal phase (Sec. IV), and the thermodynamic response is obtained in the superfluid phase (Sec. V C).

(4) The London equation. From the equation, one gets the London penetration depth and the imaginary part of the conductivity has the $1/\omega$-pole which implies the diverging DC conductivity (Sec. V D).

(5) The background solution with a vector potential. This gives the critical superfluid velocity and the critical magnetic field (Sec. VI).

(6) From these results, all static critical exponents $(\alpha, \beta, \gamma, \delta, \nu, \eta)$, the dynamic critical exponent z_D, and the ratio of critical amplitudes. These results are consistent with the standard Ginzburg-Landau (GL) theory or the ϕ^4 mean-field theory. We identify the dual GL theory including numerical coefficients (Sec. VII A).

The holographic Lifshitz superconductors have been studied previously, but it is still nice to analyze these properties all at once analytically for an infinite number of theories. First, in previous works, the system was studied mostly using numerical methods. Second, the system was studied only for some specific values of (p, z, Δ).[2] Third, some of the above properties were studied but not all were studied.

In particular, previous works typically have shown that (i) there exists a $\Psi \neq 0$ solution at low temperatures, (ii) the solution is favorable from the free energy or from the grand canonical potential, (iii) the spontaneous condensate has the standard ϕ^4 mean-field exponent $\beta = 1/2$, and (iv) the diverging DC conductivity.

On the other hand, the other properties are newly investigated, e.g., the other critical exponents as well as exact expressions for various numerical coefficients including critical amplitudes. Also, the critical dynamics of a holographic Lifshitz superconductor has never been investigated.[3] At a finite-temperature critical point, the correlation length ξ and the relaxation time τ of the order parameter obey a scaling law,

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[2]For example, Ref. [18] considers the $(p, z) = (2, 3/2)$ case and obtains a charged Lifshitz black hole with scalar hair. The other works typically take the probe limit to study the system. Reference [19] considers the $(p, z, \Delta) = (2, 2, 3)$ case in a Lifshitz black hole background [24]. Reference [25] considers the $(p, z, \Delta) = (2, 2, 3), (2, 2, 4)$ cases in the same Lifshitz black hole background as ours. Reference [26] considers $(p, z, \Delta) = (2, 2, 2), (2,2,3), (3,2,3), (3,3,3)$ cases in the same Lifshitz black hole background as ours. As far as we are aware, our $(p, z, \Delta) = (3z, z, 2z)$ case was not studied before.

[3]Dynamics in Lifshitz geometry has been studied, e.g., in Refs. [27,28]. Critical dynamics has been studied in holography, e.g., in Refs. [13,29–31].

$$\tau \propto \xi^{z_D}. \tag{1.3}$$

We obtain $z_D = 2$ irrespective of the value of the Lifshitz exponent z. We discuss the relation between z and z_D in Sec. VII B.

II. PRELIMINARIES

A. Lifshitz black hole

The Lifshitz geometry [32] is given by

$$ds^2_{p+2} = -\left(\frac{r}{L}\right)^{2z} dt^2 + \left(\frac{r}{L}\right)^2 dx_i^2 + L^2 \frac{dr^2}{r^2} \tag{2.1}$$

(see, e.g., Ref. [33] for a review). The geometry is invariant under an anisotropic scaling

$$t \to a^z t, \qquad x^i \to a x^i, \qquad r \to r/a. \tag{2.2}$$

There are various Lifshitz black hole solutions known in the literature, both analytically and numerically, depending on bulk theories. We use the solution in Refs. [34,35]. The metric is given by

$$ds^2 = -\left(\frac{r}{L}\right)^{2z} h\, dt^2 + \left(\frac{r}{L}\right)^2 dx_i^2 + L^2 \frac{dr^2}{r^2 h}, \tag{2.3a}$$

$$= -\left(\frac{r_h}{L}\right)^{2z} \frac{h}{u^{2z}} dt^2 + \left(\frac{r_h}{L}\right)^2 \frac{dx_i^2}{u^2} + L^2 \frac{du^2}{u^2 h}, \tag{2.3b}$$

$$h = 1 - \left(\frac{r_h}{r}\right)^{p+z} = 1 - u^{p+z}, \tag{2.3c}$$

where $u := r_h/r$, and r_h is the horizon radius. The metric is invariant under the Lifshitz scaling (2.2) with the scaled horizon radius $r_h \to r_h/a$. The Hawking temperature is given by

$$T = \frac{p+z}{4\pi L} \left(\frac{r_h}{L}\right)^z. \tag{2.4}$$

The metric can be obtained as a solution of an Einstein-Maxwell-dilaton system,[4]

$$S = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \Bigg\{ R - 2\Lambda$$
$$- \frac{1}{2}(\partial_M \phi)^2 - \frac{1}{4} e^{\lambda\phi} \mathcal{F}^2_{MN} \Bigg\}, \tag{2.5}$$

where G_{p+2} is the $(p + 2)$-dimensional Newton's constant and

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[4]We use capital Latin indices $M, N, \ldots$ for the $(p + 2)$-dimensional bulk spacetime coordinates and use Greek indices $\mu, \nu, \ldots$ for the $(p + 1)$-dimensional boundary coordinates. The boundary coordinates are written as $x^\mu = (t, x^i) = (t, x, y, \cdots)$.

$$\Lambda = -\frac{(p+z-1)(p+z)}{2L^2}, \qquad \lambda^2 = \frac{2p}{z-1}, \qquad \text{(2.6a)}$$

$$\mathcal{F}_{MN} = 2\partial_{[M}\mathcal{A}_{N]}. \qquad \text{(2.6b)}$$

The matter field solutions are given by

$$e^{\lambda\phi} = u^{2p}, \qquad \text{(2.7a)}$$

$$\mathcal{A}_t = -\sqrt{\frac{2(z-1)}{p+z}}\frac{1}{u^{p+z}}. \qquad \text{(2.7b)}$$

But for our purpose, the point using this solution is that (i) it provides an analytic Lifshitz black hole solution, and (ii) a class of holographic superconductors in this background admits an analytic solution.

B. Holographic Lifshitz superconductors

We couple an additional matter system, a Maxwell-complex scalar system, in addition to the above system [18,19],

$$S = -\frac{1}{e^2}\int d^{p+2}x\sqrt{-g}\left\{\frac{1}{4}F_{MN}^2 + |D_M\Psi|^2 + V(\Psi)\right\}, \quad \text{(2.8)}$$

where

$$F_{MN} = 2\partial_{[M}A_{N]}, \quad D_M := \nabla_M - iA_M, \quad V = m^2|\Psi|^2. \qquad \text{(2.9)}$$

The $U(1)$-field A_M is different from $\mathcal{A}_M$ in Eq. (2.5).

We take the probe limit $e \gg 1$, where the backreaction of these matter fields onto the geometry is ignored. Namely, we solve the system (2.8) in the background (2.3). The equations of motion are given by

$$(D^2 - m^2)\Psi = 0, \qquad \text{(2.10a)}$$

$$\nabla_N F^{MN} = j^M \qquad \text{(2.10b)}$$

$$:= ig^{MN}[(D_N\Psi)^\dagger\Psi - \Psi^\dagger(D_N\Psi)]. \qquad \text{(2.10c)}$$

In the $A_u = 0$ gauge, the asymptotic behaviors of the matter fields are given by

$$A_t \sim A_t^{(0)} + A_t^{(1)}\tilde{u}^{p-z}, \qquad (p > z), \qquad \text{(2.11a)}$$

$$A_i \sim A_i^{(0)} + A_i^{(1)}\tilde{u}^{p+z-2}, \qquad \text{(2.11b)}$$

$$\Psi \sim \Psi^{(0)}\tilde{u}^{\Delta_-} + \Psi^{(1)}\tilde{u}^{\Delta_+}, \qquad \text{(2.11c)}$$

$$\tilde{u} := \frac{L}{r}, \qquad \text{(2.11d)}$$

$$\Delta_{\pm} := \frac{p+z}{2} \pm \sqrt{\frac{(p+z)^2}{4} + L^2m^2}, \qquad \text{(2.11e)}$$

where $A_t^{(1)}$ represents the charge density ρ, and $A_t^{(0)} = \mu$ is the chemical potential. Similarly, $A_i^{(1)}$ represents the current density J^i, and $A_i^{(0)}$ is the vector potential. For Ψ, $\Psi^{(1)}$ represents the order parameter $\mathcal{O}$, and $\Psi^{(0)}$ is the external source for $\mathcal{O}$. (See Appendix A for the precise dictionary.) Then, the BF bound in the asymptotically Lifshitz geometry is given by

$$m_{\rm BF}^2 = -\frac{(p+z)^2}{4L^2}. \qquad \text{(2.12)}$$

When the BF bound is saturated, the asymptotic behavior is replaced by

$$\Psi \sim \Psi^{(0)}\tilde{u}^{\Delta}\ln\tilde{u} + \Psi^{(1)}\tilde{u}^{\Delta}, \qquad \Delta := \frac{p+z}{2}. \qquad \text{(2.13)}$$

The equations of motion (2.10) admit a solution

$$A_t = \mu(1 - u^{p-z}), \qquad (p > z), \qquad \text{(2.14a)}$$

$$A_i = A_u = 0, \qquad \text{(2.14b)}$$

$$\Psi = 0, \qquad \text{(2.14c)}$$

where boldface letters indicate background values. But, at the critical point, the $\Psi = 0$ solution becomes unstable and is replaced by a $\Psi \neq 0$ solution. We see this in detail below.

III. CRITICAL POINT

Below we consider the case $p = 3z$. It is convenient to introduce a new coordinate $s := u^{2z}$. The metric then becomes[5]

$$ds^2 = -\left(\frac{r_h}{L}\right)^{2z}\frac{h}{s}dt^2 + \left(\frac{r_h}{L}\right)^2\frac{dx_i^2}{s^{1/z}} + \left(\frac{L}{2z}\right)^2\frac{ds^2}{s^2h}, \qquad \text{(3.1a)}$$

$$h = 1 - s^2. \qquad \text{(3.1b)}$$

We consider the scalar which saturates the Lifshitz BF bound $m^2 = -(2z/L)^2$. The scaling dimension Δ is given by $\Delta = 2z$. First, consider the static homogeneous solution $\Psi = \Psi(s)$, and approach the critical point from high temperature. Near the critical point, the scalar field Ψ remains small, and one can ignore the backreaction of Ψ onto the Maxwell field. In this region, one can use

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[5]It is well known that the $z = \infty$ limit of the Lifshitz geometry is $AdS_2 \times \mathbb{R}^p$ asymptotically. But this is the case for a finite p. In our case, $p = 3z$, and the $z = \infty$ limit does not reduce to AdS_2.

Eq. (2.14) for the Maxwell field, and it is enough to solve the Ψ-equation.[6] The Ψ-equation becomes

$$\partial_s\left(\frac{h}{s}\partial_s\Psi\right)+\left\{\left(\frac{\mu}{2\pi T}\right)^2\frac{(1-s)^2}{hs^2}+\frac{1}{s^3}\right\}\Psi=0, \qquad (3.2)$$

$$T=\frac{z}{\pi L}\left(\frac{r_h}{L}\right)^z. \qquad (3.3)$$

Thus, the solution is parametrized by a dimensionless parameter μ/T. The equation admits a solution

$$\Psi\propto\frac{s}{1+s}, \quad \text{at}\left(\frac{\mu}{T}\right)_c=2\pi,$$

$$\text{or}\quad \frac{\mu}{\frac{1}{L}\left(\frac{r_h}{L}\right)^z}=2z. \qquad (3.4)$$

This is the solution at the critical point.

The z-dependence disappears in Eq. (3.2), and it only appears implicitly in the definition of T. One can understand this as follows. For the static homogeneous solution, the Laplacian becomes

$$\nabla^2\Psi=\frac{1}{\sqrt{-g}}\partial_s(\sqrt{-g}g^{ss}\partial_s\Psi). \qquad (3.5)$$

The z-dependence appears in the boundary spatial metric g_{ij}, and it appears only through $\det g$. But in our case,

$$-\det g\propto s^{-3-p/z}=s^{-6}, \qquad (3.6)$$

so p and z disappear. The metric g^{ss} is also proportional to z^2, but it is factored out in the Ψ-equation since $m^2\propto-z^2$ and $1/T^2\propto 1/z^2$. It then follows that the Ψ-equation formally reduces to the same equation for all z.

Note that Eq. (3.4) is the solution directly at the critical point. As one lowers temperature further, the solution is modified, and we construct the background solution Ψ, A_t in Sec. V. The z-dependence can also be eliminated from the A_t-equation [by redefining Ψ and A_t as in Eq. (3.9).] Thus, the static homogeneous solution is essentially the same as the $z=1$ case apart from various factors of $(2z)$. Then, from the analysis of Ref. [22], Eq. (3.4) is the solution at the critical point, and the solution has a lower grand canonical potential than the $\Psi=0$ solution at low temperature.

However, the full equations of motion do not reduce to the same equations as the $z=1$ case. In general, more nontrivial z-dependence appears. For example, they appear when one considers

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[6]We later use a perturbative expansion for a systematic analysis (Sec. V A).

(i) inhomogeneous perturbations in the boundary spatial directions, e.g., $\delta\phi\propto e^{iqx}$ (Sec. IV), or
(ii) perturbations or solutions with a vector potential A_i (Secs. V D and VI).

Below we construct the background solution Ψ, A_M. We also consider the linear perturbations from the background,

$$\Psi=\Psi+\delta\Psi, \qquad (3.7a)$$

$$A_M=A_M+\delta A_M. \qquad (3.7b)$$

We take the gauge $A_s=\delta A_s=0$. We consider the perturbations of the form

$$\delta\phi(k)\sim e^{-i\omega t+iqx}, \qquad (3.8)$$

where $k^\mu=(\omega,q,0,\ldots)$. Then, the Maxwell perturbations are decomposed as
(i) vector modes, e.g., δA_y, and
(ii) scalar modes δA_t, δA_x which can couple to $\delta\Psi$, in general.
For simplicity, we set $e=L=r_h=1$ below. In this unit, $\mu_c=2z$, and we vary the chemical potential μ. Also, we often use quantities with "$^-$". All quantities with "$^-$" are defined by

$$\bar\phi:=\frac{\phi}{2z}, \qquad (3.9)$$

when $r_h=1$. For example, $\bar\mu_c=1$. We restore units for some of our main results in Appendix C.

IV. HIGH-TEMPERATURE PHASE

At high temperature, the background solution is given by

$$A_t=\mu(1-s), \qquad (4.1a)$$

$$A_i=0, \qquad (4.1b)$$

$$\Psi=0. \qquad (4.1c)$$

The interesting quantity in the high-temperature phase is the "order parameter response function," the susceptibility, or the correlation function of the order parameter. We show that the response function takes the form

$$\chi_k=\frac{\delta\mathcal{O}(k)}{\delta\Psi^{(0)}(k)}$$

$$\propto\frac{1}{-\frac{2i}{c_K\Gamma}\omega+q^2+\frac{1}{\xi^2}}, \qquad (4.2a)$$

$$\xi^2\propto|\epsilon_\mu|^{-1}, \qquad (4.2b)$$

for a small (ω,q,ϵ_μ), where $\epsilon_\mu:=\mu-\mu_c$, and c_K and Γ are parameters we use to compare with GL theory (Sec. VII A). The function gives the following information:

(i) The $\omega = q = 0$ limit is the thermodynamic response function

$$\chi_T = A/|\epsilon_\mu|, \tag{4.3a}$$

where the coefficient A is known as the critical amplitude. Then, the exponent γ defined by $\chi_T \propto |\epsilon_\mu|^{-\gamma}$ is $\gamma = 1$.

(ii) The $\omega = 0$ limit is the static response

$$\chi_{\omega=0,q} \propto (q^2 + \xi^{-2})^{-1}. \tag{4.3b}$$

Then, ξ is the correlation length, and the exponent ν defined by $\xi \propto |\epsilon_\mu|^{-\nu}$ is $\nu = 1/2$ from Eq. (4.2b). Also, the anomalous exponent η defined by $\chi_{\omega=0,q}|_{\mu_c} \propto q^{-2+\eta}$ is $\eta = 0$.

(iii) The $\omega \neq 0$ case is the dynamic response. Then, the relaxation time behaves as

$$\tau_{q=0} \propto \xi^2, \tag{4.3c}$$

and the dynamic critical exponent z_D defined by $\tau_{q=0} \propto \xi^{z_D}$ is $z_D = 2$.

Thus, the computation determines the exponents (γ, ν, η, z_D) as well as the critical amplitude A. An explicit solution is not really necessary to compute critical exponents, and analytic arguments are possible [13,16]. On the other hand, an explicit solution is useful to obtain various numerical coefficients such as A.

The response function can be obtained from the bulk scalar field Ψ. Consider the linear perturbation from the background $\Psi = \Psi + \delta\Psi$. From the bulk point of view, the response function pole corresponds to a quasinormal pole of $\delta\Psi$. When $\Psi = 0$, Maxwell scalar modes δA_t and δA_x decouple from the $\delta\Psi$-equation.[7] Thus, to determine the order parameter response, it is enough to consider the $\delta\Psi$-equation,

$$\partial_s\left(\frac{h}{s}\partial_s\delta\Psi\right) + \left\{\frac{(\bar{A}_t + \bar{\omega})^2}{hs^2} - \frac{\bar{q}^2}{s^{3-1/z}} + \frac{1}{s^3}\right\}\delta\Psi = 0. \tag{4.4}$$

Asymptotically, we impose the boundary condition $\delta\Psi(s \to 0) = \delta\Psi^{(0)} s \ln s/(2z)$. At the horizon, we impose the incoming-wave boundary condition.

The $\delta\Psi$ perturbation cannot be solved for a generic μ, so we set $\bar{\epsilon}_\mu = \bar{\mu} - 1 < 0$ and employ the $\bar{\epsilon}_\mu$-expansion as well as the (ω, q)-expansion:

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[7]The Maxwell scalar modes give a diffusion pole, and one can determine the diffusion constant. But at high temperature, the computation is not unique to holographic superconductors. It is just a Maxwell field problem in the Lifshitz background. The Maxwell vector mode can determine the conductivity. Again, at high temperature, the computation is not unique to holographic superconductors. But we compute the vector mode in the low-temperature phase. The $O(\omega)$-coefficient is common both to the high-temperature and the low-temperature phases.

$$\delta\Psi(s, k) = (1 - s)^{-i\bar{\omega}/2}$$
$$\times ((\psi_c + \bar{\epsilon}_\mu\psi_\epsilon + \cdots) + \bar{\omega}\psi_\omega + \bar{q}^2\psi_q + \cdots). \tag{4.5}$$

This form is taken to implement the incoming-wave boundary condition. Then, the boundary condition reduces to the regularity condition for ψ_c and so on. The equation of motion reduces to

$$\mathcal{L}_\psi\psi_c = 0, \tag{4.6a}$$

$$\mathcal{L}_\psi\psi_i = \mathsf{j}_i(\psi_c), \tag{4.6b}$$

where

$$\mathcal{L}_\psi = \partial_s\left(\frac{h}{s}\partial_s\right) + \left\{\frac{(1-s)^2}{hs^2} + \frac{1}{s^3}\right\},$$

and the index i collectively represents ϵ, ω, and q. The homogeneous equation $\mathcal{L}_\psi\psi_c = 0$ can be solved as

$$\psi_c = c_1\frac{s}{1+s} + c_2\frac{s}{1+s}\ln\left[\frac{s}{(1-s)^2}\right]. \tag{4.7}$$

From the regularity at the horizon, $c_2 = 0$.

The source terms of inhomogeneous equations then become

$$\mathsf{j}_\epsilon = -\frac{2(1-s)}{s(1+s)^2}c_1, \tag{4.8a}$$

$$\mathsf{j}_\omega = \frac{-4 - i(1+s)}{2s(1+s)^2}c_1, \tag{4.8b}$$

$$\mathsf{j}_q = \frac{1}{s^{2-1/z}(1+s)}c_1. \tag{4.8c}$$

The ψ_ϵ and ψ_ω solutions are

$$\psi_\epsilon = -c_1\frac{s}{1+s}\left\{-\frac{1}{2}\ln s + \ln(1+s)\right\} \tag{4.9a}$$

$$\sim c_1\frac{1}{2}s\ln s \quad (s \to 0), \tag{4.9b}$$

$$\psi_\omega = c_1\frac{s}{4(1+s)}\{(3+i)\ln s - 2\ln(1+s)\} \tag{4.9c}$$

$$\sim c_1\frac{3+i}{4}s\ln s \quad (s \to 0). \tag{4.9d}$$

For ψ_q, we discuss the $z = 1$ and $z \neq 1$ cases separately.

A. $z = 1$

The ψ_q solution is given by

$$\psi_q = -c_1 \frac{1}{2(1+s)} s \ln s \qquad (4.10)$$

$$\sim -c_1 \frac{1}{2} s \ln s \qquad (s \to 0). \qquad (4.11)$$

The asymptotic behavior then becomes

$$\delta\Psi \sim c_1 (1-s)^{-i\bar{\omega}/2}$$
$$\times \left\{ -\frac{1}{2}\left(\bar{q}^2 - \frac{3+i}{2}\bar{\omega} + |\bar{\epsilon}_\mu| + \cdots\right) s \ln s + s + \cdots \right\}. \qquad (4.12)$$

The asymptotic boundary condition determines c_1. Then, the order parameter response function becomes

$$\chi^>_{\omega,q} = \frac{\delta\mathcal{O}}{\delta\Psi^{(0)}} = \frac{2}{\frac{1}{2}q^2 - \frac{3+i}{2}\omega + |\epsilon_\mu|}, \qquad (4.13)$$

where we use the dictionary in Appendix A. The response function indeed takes the form of Eqs. (4.2), and

$$\xi^2_> = \frac{1}{2|\epsilon_\mu|}, \qquad (4.14a)$$

$$A_> = 2. \qquad (4.14b)$$

The dispersion relation is given by

$$\omega = \frac{3-i}{5}\left(|\epsilon_\mu| + \frac{1}{2}q^2 + \cdots\right). \qquad (4.15)$$

The relaxation time τ then becomes

$$\tau^{-1} = \frac{1}{5}|\epsilon_\mu| = \frac{1}{10}\xi^{-2}_>. \qquad (4.16)$$

B. $z > 1$

In this case, we are not able to obtain the generic expression for ψ_q. Besides, even when the analytic expression is available, it is too cumbersome to write here. However, the slow falloff has a simple expression,

$$\psi^{(0)}_q = -c_1 I(z), \qquad (4.17a)$$

$$I(z) := \int_0^1 \frac{ds}{s^{1-1/z}(1+s)^2} \qquad (4.17b)$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{z}\right)$$
$$\times \left\{\psi_0\left(\frac{1}{2z} + \frac{1}{2}\right) - \psi_0\left(\frac{1}{2z}\right)\right\}, \qquad (4.17c)$$

where $\psi_0(x)$ is the digamma function:

$$\psi_0(x) = \frac{d}{dx}\ln\Gamma(x). \qquad (4.18)$$

A few examples of $I(z)$ are

$$I(1) = \frac{1}{2},$$

$$I(2) = \frac{1}{2} + \frac{\pi}{4} \approx 1.285,$$

$$I(3) = \frac{1}{2} + \frac{2}{3}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right) \approx 2.171.$$

The combination $I(z)/z$ monotonically increases with z and reaches 1 for $z \to \infty$. In order to obtain the falloff, we essentially used the standard method to solve an inhomogeneous differential equation (Appendix B).

Then, the order parameter response function becomes

$$\chi^>_{\omega,q} = \frac{2}{\frac{I(z)}{z}q^2 - \frac{3+i}{2}\omega + |\epsilon_\mu|}, \qquad (4.19)$$

which gives

$$\xi^2_> = \frac{I(z)/z}{|\epsilon_\mu|}, \qquad (4.20a)$$

$$A_> = 2. \qquad (4.20b)$$

The dispersion relation is given by

$$\omega = \frac{3-i}{5}\left(|\epsilon_\mu| + \frac{I(z)}{z}q^2 + \cdots\right). \qquad (4.21)$$

The relaxation time is given by

$$\tau^{-1} = \frac{1}{5}|\epsilon_\mu| = \frac{I(z)}{5z}\xi^{-2}_>. \qquad (4.22)$$

C. $z \gg 1$

For large z,

$$I(z) \approx z, \qquad (4.23)$$

so

$$\chi^>_{\omega,q} \approx \frac{2}{q^2 - \frac{3+i}{2}\omega + |\epsilon_\mu|}, \qquad (4.24a)$$

$$\omega \approx \frac{3-i}{5}(|\epsilon_\mu| + q^2 + \cdots), \qquad (4.24b)$$

$$\xi^2_> \approx \frac{1}{|\epsilon_\mu|}, \qquad \tau^{-1} = \frac{1}{5}|\epsilon_\mu| \approx \frac{1}{5}\xi^{-2}_>. \qquad (4.24c)$$

V. LOW-TEMPERATURE PHASE

In the low-temperature phase, our task is
(1) to construct the background,
(2) to show that the $\Psi \neq 0$ solution has a lower grand canonical potential, and
(3) to derive the London equation. (This establishes that the $\Psi \neq 0$ phase is a superconducting phase.)

A. Background solution

The solution (1.2) is the solution only at the critical point, and we first construct the background solution in the low-temperature phase. As mentioned in Sec. III, the construction is essentially the same as the $z = 1$ case [22].

Consider the solution of the form

$$\Psi = \Psi(s), \qquad A_t = A_t(s), \qquad A_i = A_s = 0. \quad (5.1)$$

The equations of motion are given by

$$\partial_s \left(\frac{h}{s} \partial_s \bar{\Psi} \right) + \left\{ \frac{\bar{A}_t^2}{hs^2} + \frac{1}{s^3} \right\} \bar{\Psi} = 0, \qquad (5.2a)$$

$$\partial_s^2 \bar{A}_t = \frac{2}{hs^2} \bar{\Psi}^2 \bar{A}_t, \qquad (5.2b)$$

$$\Psi^{\dagger} \Psi' - \Psi^{\dagger'} \Psi = 0. \qquad (5.2c)$$

One can set Ψ to be real. We construct the background perturbatively,

$$\bar{\Psi} = \epsilon^{1/2} (\Psi_1 + \epsilon \Psi_2 + \cdots), \qquad (5.3a)$$

$$\bar{A}_t = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \cdots, \qquad (5.3b)$$

where ϵ is a small parameter whose meaning will be clear in a moment. From Sec. III, we already know

$$\Phi_0 = 1 - s, \qquad (5.4a)$$

$$\Psi_1 = \frac{s}{1 + s}. \qquad (5.4b)$$

To proceed to higher orders in ϵ, we impose the boundary conditions following Ref. [22]:
(i) Ψ_n: Asymptotically, no slow falloff and no fast falloff, or $\Psi_n^{(0)} = \Psi_n^{(1)} = 0$ (for $n \geq 2$). The former means the condition for a spontaneous condensate. The latter means that $\mathcal{O}$ comes only from Ψ_1. At the horizon, we impose the regularity condition.
(ii) Φ_n: $\Phi_n(s = 1) = 0$ at the horizon.
Namely, we fix the fast falloff $\mathcal{O}$, but the chemical potential is corrected as

$$\bar{\mu} = 1 + \epsilon \delta \bar{\mu}_1 + \epsilon^2 \delta \bar{\mu}_2 + \cdots. \qquad (5.5)$$

Under these boundary conditions,

$$\bar{\Psi} \sim \epsilon^{1/2} s, \qquad (5.6)$$

so $\epsilon^{1/2}$ represents the order parameter $\mathcal{O}$. We impose four boundary conditions in total, which completely fixes the solution. For example, Φ_1 and Ψ_2 have four integration constants, and they are determined by the four conditions.

At $O(\epsilon)$,

$$\Phi_1 = \delta \bar{\mu}_1 (1 - s) - \frac{s(1 - s)}{2(1 + s)} \qquad (5.7)$$

$$\sim \delta \bar{\mu}_1 + (-1/2 - \delta \bar{\mu}_1) s + \cdots, \qquad (5.8)$$

where we imposed the boundary condition $\Phi_1(s = 1) = 0$, and $\delta \bar{\mu}_1$ is the remaining integration constant. It is fixed at $O(\epsilon^{3/2})$ from the condition that $\Psi_2^{(0)} = 0$.

At $O(\epsilon^{3/2})$, there are two more integration constants and $\delta \bar{\mu}_1$. After imposing the boundary condition at the horizon and the $\Psi_2^{(1)} = 0$ condition, one obtains

$$\Psi_2 = -\frac{s^2}{3(1 + s)^2} \qquad (5.9)$$

$$+ \left(\delta \bar{\mu}_1 - \frac{1}{12} \right) \frac{s \ln s}{2(1 + s)} + \left(\frac{1}{4} - \delta \bar{\mu}_1 \right) \frac{s \ln(1 + s)}{1 + s}$$

$$\sim \frac{1}{2} \left(\delta \bar{\mu}_1 - \frac{1}{12} \right) s \ln s, \qquad (5.10)$$

so the remaining no slow falloff condition $\Psi_2^{(0)} = 0$ gives $\delta \bar{\mu}_1 = 1/12$. Then, at $O(\epsilon)$, the chemical potential becomes

$$\bar{\mu} = \bar{A}_t|_{s=0} = 1 + \frac{1}{12} \epsilon + \cdots, \qquad (5.11)$$

so ϵ is determined as

$$\epsilon = 12 \bar{\epsilon}_\mu = 12 \epsilon_\mu / (2z). \qquad (5.12)$$

Thus,

$$\mathcal{O} = -(2z) \epsilon^{1/2} = -(24 z \epsilon_\mu)^{1/2}, \qquad (5.13)$$

and the critical exponent $\beta = 1/2$. More generally, $A_t^{(0)}$ gives the GL equation of motion (Secs. V C and VII A).

At $O(\epsilon^2)$,

$$\Phi_2 = -\frac{(1-s)}{1728(1+s)^2}(253 + 842s + 253s^2)$$

$$+ \frac{1}{36}(7 - 13s)\ln 2 + \frac{\ln(1+s)}{3(1+s)} \tag{5.14}$$

$$\sim \frac{-253 + 336\ln 2}{1728} + \frac{493 - 624\ln 2}{1728}s + \cdots \tag{5.15}$$

$$= \delta\bar{\mu}_2 + \left(\frac{373}{864} - \frac{5}{9}\ln 2 + \delta\bar{\mu}_2\right)s + \cdots, \tag{5.16}$$

$$\delta\bar{\mu}_2 = \frac{-253 + 336\ln 2}{1728}. \tag{5.17}$$

Again, we determine an integration constant $\delta\bar{\mu}_2$ at $O(\epsilon^{5/2})$ from the condition $\Psi_3^{(0)} = 0$. The expression for Ψ_3 is too cumbersome to write here.

B. Grand canonical potential

We use the Lorentzian formalism to evaluate the grand canonical potential Ω. (Note $S_E = \beta\Omega = -S_L$.) The matter on-shell action is given by

$$S_{OS} = \int d^{p+1}x \left\{ -\frac{p-z}{2}A_t^{(0)}A_t^{(1)} \right.$$

$$\left. + \int_0^1 du\sqrt{-g}g^{tt}A_t^2|\Psi|^2 \right\}. \tag{5.18}$$

We are interested in the grand canonical potential of the spontaneous condensate, or the solution with $\Psi^{(0)} = 0$, so the boundary term from Ψ vanishes.

We evaluate the difference of the grand canonical potential between the $\Psi = 0$ solution and the $\Psi \neq 0$ solution. We fix the chemical potential as $\bar{\mu} = 1 + \epsilon\delta\bar{\mu}_1 + \epsilon^2\delta\bar{\mu}_2 + \cdots$, where $\delta\bar{\mu}_1$ and $\delta\bar{\mu}_2$ are obtained in the previous subsection. It turns out that $\delta S_{OS} = 0$ at $O(\epsilon)$, so we evaluate the difference at $O(\epsilon^2)$. This implies that one has to take into account up to $O(\epsilon^2)$ of A_t in order to evaluate the above boundary action.

For the $\Psi = 0$ solution,

$$\bar{A}_t \sim (1 + \epsilon\delta\bar{\mu}_1 + \epsilon^2\delta\bar{\mu}_2 + \cdots)(1 - s). \tag{5.19}$$

In this case, only the boundary action contributes since $\Psi = 0$. The on-shell action becomes

$$S_{\Psi=0} = \beta V_p(2z)^3 \left\{ \frac{1}{2} + \delta\bar{\mu}_1\epsilon + \frac{1}{2}(\delta\bar{\mu}_1^2 + 2\delta\bar{\mu}_2)\epsilon^2 + \cdots \right\} \tag{5.20}$$

$$= \beta V_p(2z)^3 \left\{ \frac{1}{2} + \frac{\epsilon}{12} \right.$$

$$\left. + \left(-\frac{247}{1728} + \frac{7}{36}\ln 2\right)\epsilon^2 + \cdots \right\}, \tag{5.21}$$

where β is the inverse temperature, and V_p is the boundary spatial volume. For the $\Psi \neq 0$ solution,

$$S_{\Psi\neq 0} = \beta V_p(2z)^3 \left\{ \frac{1}{2} + \delta\bar{\mu}_1\epsilon \right. \tag{5.22}$$

$$\left. + \left(-\frac{181}{1728} - \frac{1}{4}\delta\bar{\mu}_1 + \frac{1}{2}\delta\bar{\mu}_1^2 + \frac{7}{36}\ln 2\right)\epsilon^2 + \cdots \right\}$$

$$= \beta V_p(2z)^3 \left\{ \frac{1}{2} + \frac{\epsilon}{12} \right.$$

$$\left. + \left(-\frac{211}{1728} + \frac{7}{36}\ln 2\right)\epsilon^2 + \cdots \right\}. \tag{5.23}$$

Thus, the difference is

$$\delta S_{OS} = S_{\Psi\neq 0} - S_{\Psi=0} = \beta V_p \frac{z^3}{6}\epsilon^2 = -\beta\delta\Omega, \tag{5.24}$$

$$\Rightarrow \frac{\delta\Omega}{V_p} = -6z\epsilon_\mu^2. \tag{5.25}$$

$\delta\Omega < 0$, so the $\Psi \neq 0$ solution is favorable. The difference is proportional to $\epsilon_\mu^2 = (\mu - \mu_c)^2 \propto (T_c - T)^2$, which implies the second-order phase transition. (The difference and its first derivative are continuous, but the second derivative is discontinuous.) The specific heat C_μ behaves as $C_\mu = -T\partial^2\Omega/\partial T^2 \propto T$, which determines the critical exponent $\alpha = 0$, where α is defined by $C_\mu \propto (T_c - T)^{-\alpha}$.

C. Background with source

We construct the background without the source of the order parameter, but it is straightforward to extend the construction to the background with the source. Going back to Eq. (5.27), we obtained

$$\Psi_2 = -\frac{s^2}{3(1+s)^2} \tag{5.26}$$

$$+ \left(\delta\bar{\mu}_1 - \frac{1}{12}\right)\frac{s\ln s}{2(1+s)} + \left(\frac{1}{4} - \delta\bar{\mu}_1\right)\frac{s\ln(1+s)}{1+s}$$

$$\sim \frac{1}{2}\left(\delta\bar{\mu}_1 - \frac{1}{12}\right)s\ln s, \tag{5.27}$$

so the asymptotic behavior becomes

$$\bar{\Psi} \sim \frac{1}{2}\left(\delta\bar{\mu}_1 - \frac{1}{12}\right)\epsilon^{3/2}s\ln s + \epsilon^{1/2}s. \tag{5.28}$$

Previously, we imposed the source-free condition $\Psi^{(0)} = 0$, which gives $\delta\bar{\mu}_1 = 1/12$. We now allow $\Psi^{(0)} \neq 0$. The chemical potential is given by $\bar{\mu} = 1 + \epsilon\delta\bar{\mu}_1$. At the critical point, $\bar{\mu} = 1$, so $\delta\bar{\mu}_1 = 0$. From the asymptotic behavior (5.28), $\mathcal{O} \propto \epsilon^{1/2}$ and $\Psi^{(0)} \propto \epsilon^{3/2}$. Then, the exponent δ defined by $\mathcal{O} \propto (\Psi^{(0)})^{1/\delta}$ (at $\mu = \mu_c$) is $\delta = 3$.

One can evaluate the thermodynamic response function at low temperature. By imposing our boundary conditions,

$$\bar{\Psi}^{(0)} = z\left(\delta\bar{\mu}_1 - \frac{1}{12}\right)\epsilon^{3/2}. \qquad (5.29)$$

The chemical potential is then determined as

$$\bar{\mu} = 1 + \epsilon\delta\bar{\mu}_1 = 1 + \frac{1}{12}\epsilon + \frac{\bar{\Psi}^{(0)}}{z\epsilon^{1/2}}, \qquad (5.30)$$

which is rewritten as

$$\bar{\epsilon}_\mu = \frac{1}{12}\epsilon + \frac{\bar{\Psi}^{(0)}}{z\epsilon^{1/2}}. \qquad (5.31)$$

This is essentially the GL equation of motion (Sec. VII A). For a fixed μ, this gives

$$d\epsilon = -\frac{12}{z\epsilon^{1/2}}d\bar{\Psi}^{(0)} + O(\bar{\Psi}^{(0)}d\epsilon). \qquad (5.32)$$

Thus,

$$\begin{aligned}\chi_T^< &= \left.\frac{\partial\mathcal{O}}{\partial\Psi^{(0)}}\right|_{\Psi^{(0)}=0} \\ &= \left.\frac{d\mathcal{O}/d\epsilon}{d\Psi^{(0)}/d\epsilon}\right|_{\Psi^{(0)}=0} = \frac{12}{2z\epsilon} = \frac{1}{\epsilon_\mu},\end{aligned} \qquad (5.33)$$

$$\Rightarrow A_< = 1. \qquad (5.34)$$

(Recall $\mathcal{O} = -2z\epsilon^{1/2}$.)

We obtained $\chi_T^<$ from the background solution, but it should also be possible to obtain it from the scalar perturbation as in Sec. IV. One would also obtain the full response function $\chi_{\omega,q}^<$ using the (ϵ, ω, q)-expansion. But, in the low-temperature phase, $\delta\Psi$ couple with δA_t and δA_x, and the computation is more involved, so we leave it to a future work.

D. Vector modes

From the vector mode, one can show the London equation and compute the conductivity. The δA_y-equation is given by

$$\partial_s\left(\frac{h}{s^{1-1/z}}\partial_s\delta A_y\right)$$

$$+ \left\{\frac{\bar{\omega}^2}{hs^{2-1/z}} - \frac{\bar{q}^2}{s^{3-2/z}} - \frac{2\bar{\Psi}^2}{s^{3-1/z}}\right\}\delta A_y = 0, \qquad (5.35)$$

where $\bar{\Psi}$ was constructed in Eq. (5.3). We impose the incoming-wave boundary condition at the horizon and $\delta A_y|_{s=0} = A_y^{(0)}$ asymptotically. We again employ the (ϵ, ω)-expansion,

$$\begin{aligned}\delta A_y = (1-s)^{-i\bar{\omega}/2} \\ \times ((a_c + \epsilon a_\epsilon + \cdots) + \bar{\omega}a_\omega + \cdots).\end{aligned} \qquad (5.36)$$

The equation of motion reduces to

$$\mathcal{L}_a a_c = 0, \qquad (5.37a)$$

$$\mathcal{L}_a a_i = j_i(a_c), \qquad (5.37b)$$

where

$$\mathcal{L}_a = \partial_s\left(\frac{h}{s^{1-1/z}}\partial_s\right).$$

The homogeneous equation $\mathcal{L}_a a_c = 0$ can be solved as

$$\begin{aligned}a_c = c_1 \\ + c_2 s^{2-1/z}\frac{2z}{2z-1}{}_2F_1\left(1, \frac{2z-1}{2z}, \frac{4z-1}{2z}; s^2\right).\end{aligned} \qquad (5.38)$$

From the regularity at the horizon, $c_2 = 0$.

The source terms of inhomogeneous equations then become

$$j_\epsilon = \frac{2}{s^{1-1/z}(1+s)^2}c_1, \qquad (5.39a)$$

$$j_\omega = -\frac{i}{2}\left(\frac{1+s}{s^{1-1/z}}\right)' c_1. \qquad (5.39b)$$

Again, we discuss the $z = 1$ and $z \neq 1$ cases separately.

1. z = 1

The solution is

$$a_\epsilon = c_1\frac{1}{1+s} \sim c_1(1-s), \qquad (5.40a)$$

$$a_\omega = \frac{1}{2}ic_1\ln(1+s) \sim ic_1s/2. \qquad (5.40b)$$

The asymptotic behavior then becomes

$$\delta A_y = \frac{A_y^{(0)}(k)}{1+\epsilon+\cdots}(1-s)^{-i\bar{\omega}/2}$$

$$\times\left\{1+\frac{\epsilon}{1+s}+\frac{1}{2}i\bar{\omega}\ln(1+s)+\cdots\right\}\quad(5.41)$$

$$\sim A_y^{(0)}\{1+(-\epsilon+i\bar{\omega}+\cdots)s\}.\quad(5.42)$$

We determine the constant c_1 from the asymptotic boundary condition $\delta A_y|_{s=0} = A_y^{(0)}$. So,

$$J^y = (4z-2)A_y^{(1)}\quad(5.43)$$

$$= 2(-\epsilon+i\bar{\omega}+\cdots)A_y^{(0)}.\quad(5.44)$$

The $\omega \to 0$ limit gives the London equation,

$$J^y = -(1/\lambda^2)A_y^{(0)},\quad(5.45)$$

with the London penetration depth $\lambda^{-2} = 2\epsilon$. The conductivity is then given by

$$\sigma(\omega) = \frac{J^y}{i\omega A_y^{(0)}} = \frac{2i\epsilon}{\omega}+1+\cdots.\quad(5.46)$$

$\text{Im}(\sigma)$ has the $1/\omega$-pole which implies the diverging DC conductivity.

A superconductor has singular behaviors in the current, but its essence is not in the diverging DC conductivity but in the London equation. A diverging DC conductivity also appears in a perfect conductor, but the London equation is unique to superconductors.

When one combines the London equation with the Maxwell equation, one obtains the Meissner effect. However, for usual holographic superconductors, the boundary Maxwell field is added just as an external source and is not dynamical in the boundary theory, so the Meissner effect does not arise; the magnetic field can always penetrate into the material. In this sense, a holographic superconductor may be regarded as a superfluid. (In low spatial dimensions $p \le 2$, one can obtain a boundary theory with a dynamical Maxwell field. See, e.g., Ref. [36].)

However, the London equation must hold even in this case if the system is really a superconductor or a superfluid. The London equation is the response of the current under the external source, and whether the source is dynamical or not is irrelevant to the issue.

2. $z > 1$

For a_ω, one can get the generic expression[8]

$$a_\omega = \frac{1}{2}ic_1\left\{s^{2-1/z}\frac{2z}{2z-1}{}_2F_1\left(1,\frac{2z-1}{2z},\frac{4z-1}{2z};s^2\right)\right.$$
$$\left.+\ln(1-s)\right\}\quad(5.47)$$

$$\sim O(s)+ic_1\frac{z}{2z-1}s^{2-1/z}.\quad(5.48)$$

For a_ϵ, the generic expression is either difficult to obtain or too cumbersome, but again the fast falloff has a simple expression,

$$a_\epsilon^{(1)} = -\frac{2z}{2z-1}c_1 I(z).\quad(5.49)$$

The asymptotic behavior then becomes

$$\delta A_y \sim A_y^{(0)}\left[1+\cdots+\frac{1}{4z-2}\{-4zI(z)\epsilon\right.$$
$$\left.+i\omega+\cdots\}s^{2-1/z}+\cdots\right].\quad(5.50)$$

Thus,

$$J^y = \{-4zI(z)\epsilon+i\omega+\cdots\}A_y^{(0)}.\quad(5.51)$$

Again, the $\omega \to 0$ limit gives the London equation $J^y = -(1/\lambda^2)A_y^{(0)}$ with the London penetration depth $\lambda^{-2} = 4zI(z)\epsilon$. The conductivity is then given by

$$\sigma = \frac{J^y}{i\omega A_y^{(0)}} = \frac{i}{\omega}\frac{I(z)}{z}\mathcal{O}^2+1+\cdots.\quad(5.52)$$

For large z,

$$\sigma \approx \frac{i}{\omega}\mathcal{O}^2+1+\cdots.\quad(5.53)$$

The GL parameter κ is defined by

$$\kappa^2 := \left(\frac{\lambda}{\xi_>}\right)^2 = \frac{z}{24I(z)^2}.\quad(5.54)$$

In conventional superconductors, $\kappa^2 < 1/2$ for type I and $\kappa^2 > 1/2$ for type II superconductors. For $z = 1$, $\kappa^2 = 1/6$, so one may conclude that our system is type I (in the sense of κ), but whether our system is type I or II is more subtle. Physically, $1/\lambda$ represents the Maxwell field mass, so we

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[8]We set $p = 3z$, but the $O(\omega, q^2)$-equations can actually be solved for a generic (p, z).

should determine the normalization of λ by comparing with normalization of the boundary Maxwell action. However, as mentioned above, the boundary Maxwell field is added as an external source here and is not dynamical in the boundary theory, so the normalization cannot be determined.[9] (Holographic superconductors are type II superconductors in the sense that there is no Meissner effect.)

VI. BACKGROUND WITH VECTOR POTENTIAL

In this section, we add a vector potential A_i as a background. We again consider the perturbative expansion:

$$\bar{\Psi} = \epsilon^{1/2}(\Psi_0' + \cdots), \tag{6.1a}$$

$$\bar{A}_t = \Phi_0' + \cdots, \tag{6.1b}$$

$$\bar{A}_y = A_{y,0} + \epsilon A_{y,1} + \cdots. \tag{6.1c}$$

Note that we take into account (1) A_y as a background and (2) the backreaction of A_y onto the other fields. (That is why we use variables with primes.) The former is the difference from the perturbative expansion in Sec. V A, and the latter is difference from the vector mode computation in Sec. V D.

At $O(\epsilon^0)$, the Maxwell equation becomes

$$\nabla_N F_{,0}^{MN} = 0, \tag{6.2}$$

where $F_{MN,0} = \partial_M A_{N,0} - \partial_N A_{M,0}$. The equation has simple solutions. Namely, $\Phi_0' = \bar{\mu}(1 - s)$ and two interesting solutions for $A_{y,0}$:

$$A_{y,0} = \bar{a}_y = (\text{constant}), \tag{6.3a}$$

$$A_{y,0} = \bar{B}x. \tag{6.3b}$$

The former corresponds to adding a constant superfluid flow a_y, and the latter corresponds to adding a constant magnetic field B.

A. Superfluid flow

For the superfluid flow, it is enough to consider homogeneous perturbations. The equations of motion are given by

$$\partial_s\left(\frac{h}{s}\partial_s\bar{\Psi}\right) + \left\{\frac{\bar{A}_t^2}{hs^2} - \frac{\bar{A}_y^2}{s^{3-1/z}} + \frac{1}{s^3}\right\}\bar{\Psi} = 0, \tag{6.4a}$$

$$\partial_s^2\bar{A}_t = \frac{2}{hs^2}\bar{\Psi}^2\bar{A}_t, \tag{6.4b}$$

$$\partial_s\left(\frac{h}{s^{1-1/z}}\partial_s\bar{A}_y\right) - \frac{2\bar{\Psi}^2}{s^{3-1/z}}\bar{A}_y = 0. \tag{6.4c}$$

We impose the same boundary conditions as Sec. V A.

Our main interest is the phase diagram, i.e., the deviation of the critical point by the vector potential. Then, we evaluate how A_y at $O(\epsilon^0)$ affects Ψ_1' at $O(\epsilon^{1/2})$. This in turn affects the value of μ_c. We employ the a_y-expansion as well as the ϵ-expansion [22]. Namely,

$$\Psi_1' = \Psi_1 + \bar{a}_y^2\Psi_a + \cdots, \tag{6.5a}$$

$$\Phi_0' = (1 + \bar{a}_y^2 c_0 + \cdots)(1 - s), \tag{6.5b}$$

where c_0 is a constant. This expansion is consistent with the above equations of motion.

At $O(\epsilon^{1/2})$, $\Psi_1 = s/(1 + s)$. The Ψ_a-equation becomes

$$\mathcal{L}_\psi \Psi_a = \mathrm{j}_a, \tag{6.6a}$$

$$\mathrm{j}_a = -2c_0\frac{1 - s}{s(1 + s)^2} + \frac{1}{s^{2-1/z}(1 + s)}. \tag{6.6b}$$

The equation is hard to solve, in general. However, to determine the $\bar{a}_y$-dependence on the chemical potential, it is enough to obtain the slow falloff of Ψ_a. The slow falloff has a simple expression:

$$\Psi_a^{(0)} = -\int_0^1 ds\, \mathrm{j}_a\frac{s}{1 + s} \tag{6.7}$$

$$= \frac{c_0}{2} - I(z). \tag{6.8}$$

We impose the boundary condition $\Psi_a^{(0)} = 0$, which gives $c_0 = 2I(z)$.

At the critical point, the order parameter vanishes, so $\epsilon = 0$. Then, to determine the critical point, set $\epsilon = 0$, and $A_t^{(0)}$ gives the critical chemical potential,

$$\mu_c = \mu_{c,0} + a_y^2\frac{I(z)}{z} + \cdots \tag{6.9}$$

$$\sim \mu_{c,0} + a_y^2 \quad (z \gg 1), \tag{6.10}$$

where $\mu_{c,0} = 2z$ is the critical point without superfluid flow.

To obtain J^μ, one needs to obtain A_μ. This is necessary to derive the second sound c_2 [22,39]:

$$c_2^2 = -\frac{\partial J^y/\partial a_y}{\partial J^t/\partial \mu}\bigg|_{a_y=0}. \tag{6.11}$$

To derive c_2, it is enough to use the results obtained in the previous section. The $A_{y,1}$-equation is the same as the vector mode perturbation a_ϵ. So, J^y is given by

$$\bar{A}_y \sim \bar{a}_y \left[1 - \frac{1}{4z-2} 4zI(z)\epsilon s^{2-1/z} + \cdots \right],$$

$$\Rightarrow J^y = (4z-2)A_y^{(1)} = -4zI(z)\epsilon a_y. \qquad (6.12)$$

J^t is given by

$$\bar{A}_t \sim (1-s) + \epsilon\{\delta\bar{\mu}_1 + (-1/2 - \delta\bar{\mu}_1)s\} + \cdots$$

$$= \bar{\mu} + (6 - 7\bar{\mu})s + \cdots,$$

$$\Rightarrow J^t = -2zA_t^{(1)} = 2z(7\mu - 12z). \qquad (6.13)$$

Thus,

$$c_2^2 = \frac{I(z)}{14z^2}\mathcal{O}^2 \qquad (6.14)$$

$$\sim \frac{1}{14z}\mathcal{O}^2 \qquad (z \gg 1). \qquad (6.15)$$

B. Magnetic field

We follow Ref. [40] to obtain the critical magnetic field. The Ψ-equation is given by

$$\partial_s\left(\frac{h}{s}\partial_s\bar{\Psi}\right) + \left\{\frac{\bar{A}_t^2}{hs^2} + \frac{1}{s^{3-1/z}}(\bar{\partial}_i - i\bar{A}_i)^2 + \frac{1}{s^3}\right\}\bar{\Psi} = 0. \qquad (6.16)$$

Here, $\bar{\partial}_i = \partial_i/(2z)$, where $\underline{x}^i := (2z)x^i$. The vector potential is given by $\bar{A}_y = \bar{B}x = \bar{\bar{B}}\underline{x}$, where $\bar{\bar{B}} := B/(2z)^2$.

This problem can be solved as a Landau-level problem after separation of variables. First, set[10]

$$\bar{\Psi}(x, y, s) = e^{ik_y y}\varphi(x, s; k_y).$$

The equation then takes the form

$$s^{3-1/z}\left[\partial_s\left(\frac{h}{s}\partial_s\right) + \frac{\bar{A}_t^2}{hs^2} + \frac{1}{s^3}\right]\varphi$$
$$= [-\partial_{\underline{x}}^2 + (\bar{k}_y - \bar{\bar{B}}\underline{x})^2]\varphi, \qquad (6.17)$$

so setting

$$\varphi_n(x, s; k_y) = \rho_n(s)\gamma_n(x; k_y),$$

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[10]For simplicity, we set the other momenta as $k_3 = k_4 = \cdots = 0$.

one obtains

$$(-\partial_X^2 + X^2)\gamma_n = \lambda_n\gamma_n, \qquad (6.18a)$$

$$\left[\partial_s\left(\frac{h}{s}\partial_s\right) + \frac{\bar{A}_t^2}{hs^2} + \frac{1}{s^3}\right]\rho_n = \bar{\bar{B}}\lambda_n\frac{\rho_n}{s^{3-1/z}}, \qquad (6.18b)$$

where $X := \sqrt{\bar{\bar{B}}}(\underline{x} - \bar{k}_y/\bar{\bar{B}})$, and λ_n is a separation constant. The γ_n-equation is solved by the Hermite function H_n as

$$\gamma_n(X) = e^{-X^2/2}H_n(X), \qquad (6.19)$$

with eigenvalue λ_n as

$$\lambda_n = 2n + 1 \quad (n \geq 0). \qquad (6.20)$$

The solution is parametrized by $\bar{\bar{B}}\lambda_n$, so one has the largest magnetic field B_{c2} when λ_n takes the minimal value, namely the $n = 0$ solution.

The ρ_0-equation is given by

$$\partial_s\left(\frac{h}{s}\partial_s\rho_0\right) + \left\{\frac{\bar{A}_t^2}{hs^2} - \frac{\bar{\bar{B}}_{c2}}{s^{3-1/z}} + \frac{1}{s^3}\right\}\rho_0 = 0. \qquad (6.21)$$

Then, the problem formally reduces to the same problem as the superfluid flow one with the replacement $\bar{a}_y^2$ by $\bar{\bar{B}}_{c2}$. Thus, the critical point is given by

$$\mu_c = \mu_{c,0} + B_{c2}\frac{I(z)}{z}, \qquad (6.22)$$

Using the result of $\xi_>^2$ in Sec. IV, we get

$$B_{c2} = 1/\xi_>^2. \qquad (6.23)$$

VII. THE DUAL GL THEORY

A. Identifying the dual GL theory

We thus obtained all critical exponents and critical amplitudes

$$(\alpha, \beta, \gamma, \delta, \nu, \eta, z_D) = \left(0, \frac{1}{2}, 1, 3, \frac{1}{2}, 0, 2\right), \qquad (7.1a)$$

$$A_> = 2A_<. \qquad (7.1b)$$

The results are consistent with the standard GL theory or the ϕ^4 mean-field theory. In fact, the following GL theory reproduces all our results[11]:

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[11]As always, presumably the dual theory is some large-N_c gauge theory. This is the effective GL theory at low energy and momentum.

$$H_{GL} = \int d^p x \left\{ \frac{c_K}{2} |D_i \phi|^2 - \frac{c_2}{2} \epsilon_\mu |\phi|^2 + \frac{c_4}{4} |\phi|^4 + \cdots \right.$$

$$\left. - c_\phi (\phi J^\dagger + \phi^\dagger J) \right\}, \tag{7.2a}$$

$$D_i := \partial_i - i c_A A_i^{(0)}. \tag{7.2b}$$

In the dynamic case, consider the time-dependent GL equation (for Model A dynamic universality class):

$$\Gamma^{-1} \partial_t \phi = -\frac{\delta H_{GL}}{\delta \phi} \tag{7.3}$$

$$= -\frac{c_K}{2} D_i^2 \phi - \frac{c_2}{2} \epsilon_\mu \phi + \frac{c_4}{2} \phi |\phi|^2 - c_\phi J. \tag{7.4}$$

We determine the GL parameters $(c_2, c_4, c_\phi, c_K, c_A, \Gamma)$ to reproduce our holographic results.

In the static homogeneous case, the ϕ-equation becomes

$$c_2 \epsilon_\mu \phi - c_4 \phi |\phi|^2 + 2 c_\phi J = 0. \tag{7.5}$$

Substituting the $J = 0$ solution $|\phi|^2 = c_2 \epsilon_\mu / c_4$ into H_{GL}, one obtains the grand canonical potential:

$$\frac{\Omega}{V} = -\frac{c_2^2}{4 c_4} \epsilon_\mu^2. \tag{7.6}$$

The current is given by

$$J^i := -\frac{\delta H_{GL}}{\delta A_i^{(0)}} = -c_K c_A^2 |\phi|^2 A_i^{(0)}. \tag{7.7}$$

In the high-temperature phase, the response function is given by

$$\chi_k^> = \frac{c_\phi \Gamma}{-i\omega + \frac{\Gamma}{2}(c_K q^2 + c_2 |\epsilon_\mu|)}, \tag{7.8}$$

which implies

$$\omega = -i \frac{\Gamma}{2} c_2 \left(|\epsilon_\mu| + \frac{c_K}{c_2} q^2 \right), \tag{7.9}$$

$$\xi_>^2 = \frac{c_K}{c_2} |\epsilon_\mu|^{-1}. \tag{7.10}$$

Add a background vector potential. When a constant $A_y = a_y$ is added, the critical point is shifted as

$$\mu_c = \mu_{c,0} + \frac{c_K}{c_2} c_A^2 a_y^2. \tag{7.11}$$

When a magnetic field is added, the critical magnetic field is given by

$$B_{c2} = \frac{c_2}{c_K c_A} \epsilon_\mu = \frac{1}{c_A} \xi_>^{-2}, \tag{7.12}$$

by solving the Landau-level problem.

Returning to our holographic results, Eq. (5.31) is rewritten as

$$\epsilon_\mu \mathcal{O} - \frac{1}{24z} \mathcal{O}^3 + 2\Psi^{(0)} = 0, \tag{7.13}$$

which takes the form of the GL theory equation of motion (7.5). The grand canonical potential, the current, and the response function are obtained in Eqs. (5.24), (5.51), and (4.19), respectively. These determine the GL parameters as

$$\mathcal{H} = \frac{I(z)}{2z} |(\partial_i - iA_i^{(0)})\phi|^2 - \frac{\epsilon_\mu}{2} |\phi|^2 + \frac{1}{96z} |\phi|^4 + \cdots$$
$$- (\phi J^\dagger + \phi^\dagger J), \tag{7.14a}$$

$$\frac{\Gamma}{2} = \frac{1 + 3i}{5}. \tag{7.14b}$$

In the presence of a background vector potential, Eqs. (6.9) and (6.23) agree with Eqs. (7.11) and (7.12), respectively.

One would be tempted to ask how various results change as we vary z. But to make such a comparison, one must keep in mind that (1) we consider a special class of Lifshitz theories and (2) we must specify what quantities to fix as we vary z.

We consider a special class of theories where $p = 3z$ and $m^2 = -4z^2$. Even the spatial dimensionality p is different for a different z, and it is unclear if the comparison with a different z is physically meaningful. Also, some results may be generic for Lifshitz theories in general but some are not. As a simple example, in our case, $(\mu/T)_c$ is independent of z. This is so by construction of our theories as discussed in Sec. III and is certainly not a generic behavior. It simply means that holographic Lifshitz superconductors have enough parameters to fix $(\mu/T)_c$ as we vary z.

We also have to specify what quantities to fix. One natural candidate is μ_c (or T_c) since $(\mu/T)_c$ is z-independent, but it is unclear if this is really appropriate. We do not have the answer to this question. So far we set $r_h = 1$ just for simplicity, so here we simply fix r_h (and ϵ_μ) and how various results change as we vary z. Again, we do not mean that fixing r_h is natural from the boundary point of view. Rather, the following comparison should be regarded as a handy way to understand the z-dependence of our holographic results or the dual GL theory.

(1) In the dual GL theory, the coefficient of the ϕ^4-term becomes smaller as we increase z. So, the condensate increases as $|\phi|^2 \propto z$.

(2) The z-dependence appears only in the kinetic term and the ϕ^4-term. Thus, the relaxation time $\tau_{q=0}$ of the order parameter does not depend on z.

(3) The correlation length ξ depends on the kinetic term so does depend on z. It monotonically increases as $I(z)/z$ but increases slowly and reaches a constant value for $z \gg 1$.

(4) On the other hand, the London penetration depth λ decreases. This is because $J^i = -(1/\lambda^2)A_i^{(0)} \propto -|\phi|^2 A_i^{(0)}$ and because ϕ increases. (λ also depends on the kinetic term so has the factor $I(z)/z$, but it is not a dominant factor.) Then, the GL parameter κ decreases.

(5) As usual, the presence of a background vector potential $A_i^{(0)}$ increases the critical chemical potential μ_c. From the gravity point of view, this is because A_i increases the effective mass of Ψ. μ_c monotonically increases as $I(z)/z$ since $A_i^{(0)}$ comes from the covariant derivative in the kinetic term.

B. Lifshitz exponent and dynamic critical exponent

We already mentioned that some results are not generic to Lifshitz theories, in general. Then, what results are expected to be generic? An obvious answer is critical exponents and the ratio of critical amplitudes. The ϕ^4 mean-field critical exponents are likely to hold for theories of Eq. (2.8). The dynamic critical exponent $z_D = 2$ is also likely to hold.

For a Lifshitz geometry, one would expect a dispersion relation of the form

$$\omega \propto q^z. \tag{7.15}$$

This form is expected from the Lifshitz scaling (2.2). But from the analysis of the high-temperature phase, the order parameter obeys the dispersion relation

$$\omega \propto q^2, \tag{7.16}$$

i.e, the dynamic critical exponent $z_D = 2$ irrespective of the value of the Lifshitz exponent z. This does not contradict with the Lifshitz scaling. If we restore the horizon radius r_h,

$$\omega \propto r_h^{z-2} q^2. \tag{7.17}$$

Namely, at finite temperature, there are two length scales r_h and $1/q$. They combine to give the scaling dimension z. In other words, the Lifshitz scaling alone does not determine the dynamic critical exponent.

Then, what determines z_D? We obtain $z_D = 2$ because the order parameter is not a conserved charge. According to the classification of Hohenberg and Halperin [41], all our models belong to Model A universality class.

The dynamic universality class is classified based on
(1) whether the order parameter is conserved or not,

(2) whether there are the other hydrodynamic modes which couple to the order parameter (none for Model A and B).

Conservation laws play important roles to determine the dynamic universality class since a conservation law forces the relaxation to proceed more slowly. When only the order parameter matters in critical dynamics, a nonconserved order parameter gives Model A, and a conserved order parameter gives Model B.

The Lifshitz geometry is conjectured to describe a quantum critical point. Using holographic Lifshitz superconductors, one prepares a new finite-temperature critical point in addition to the Lifshitz critical point. What we have shown is that the dynamic critical exponent z_D associated with the new critical point can differ from z. Instead, the value of z_D is determined by the critical dynamics of the new critical point.

VIII. DISCUSSION

A. Lifshitz geometry and holographic superconductors

The Lifshitz geometry appears even in the context of the standard $z = 1$ holographic superconductor [42,43]. Consider the backreaction of matter fields onto the geometry. In the high-temperature phase, $\Psi = 0$, so the geometry is the Reissner-Nordström AdS black hole. In the low-temperature phase, $\Psi \neq 0$, but one may expect that the geometry is somewhat similar to the Reissner-Nordström AdS black hole. However, the $T = 0$ geometry is conjectured to be a Lifshitz geometry in the IR and the AdS geometry in the UV. The solution in IR has been constructed, but the full geometry remains an open question.

It is unclear what happens at low temperature, but it is natural to expect that a Lifshitz-like black hole appears at low enough temperature. (Unfortunately, the Lifshitz black hole used in this paper is not a solution of the Einstein-Maxwell-complex scalar system.) Then, one should consider the Einstein-Maxwell-complex scalar system in a Lifshitz (IR)-AdS (UV) black hole. This is not an easy task however. First, the full geometry is not constructed even in the $T = 0$ limit. Second, the stability of the geometry is an different issue. Finally, one has to solve perturbations in the full geometry to explore various properties.

What we have done in this paper is one small step towards this program; we solved matter fields in a simple Lifshitz black hole background. As we have seen in this paper, qualitative behaviors of holographic Lifshitz superconductors are the same as the ones of the standard holographic superconductors. In particular, static and dynamic critical exponents are the same. One would expect those behaviors are common even in the full problem. Critical dynamics is governed by dynamics of the critical point itself (such as criteria 1 and 2 in previous subsection) and is not governed by the Lifshitz exponent z in the underlying geometry.

B. Implications to quantum criticality

We briefly discuss the implications of our result on quantum criticality. The Lifshitz geometry is conjectured to describe a quantum critical point. In this sense, our system has two critical points:

(i) One is the $T = 0$ quantum critical point. Its dynamic scaling is determined by z.

(ii) The other is the $T \neq 0$ superconducting critical point explored in this paper. Its dynamic scaling is determined by z_D as we have shown in this paper.

It has been proposed that quantum criticality explains strange metallic behaviors of high-T_c superconductors. According to the proposal, a quantum critical point is "hidden" inside the superconducting dome, and the quantum criticality explains scaling behaviors of various transport coefficients even in the normal phase.

Our model is far from real materials, but roughly speaking, the quantum critical point could correspond to the $T = 0$ Lifshitz geometry, and the superconducting dome could correspond to the holographic Lifshitz superconductor. The Lifshitz scaling may determine the scaling behaviors in the normal phase. But our result implies that the Lifshitz scaling does not determine the scaling behavior of the order parameter near T_c. Rather, the $T \neq 0$ critical point has its own scaling. Whatever the value of z a quantum critical point has, the $T \neq 0$ critical point is likely to have $z_D = 2$ at the mean-field level.

ACKNOWLEDGMENTS

APPENDIX A: FIELD/OPERATOR CORRESPONDENCE AND HOLOGRAPHIC RENORMALIZATION

The asymptotic behaviors of matter fields are given by

$$A_t \sim A_t^{(0)} + A_t^{(1)} \tilde{u}^{p-z}$$
$$\Rightarrow A_t^{(0)} + A_t^{(1)} \tilde{s}, \tag{A1a}$$

$$A_i \sim A_i^{(0)} + A_i^{(1)} \tilde{u}^{p+z-2}$$
$$\Rightarrow A_i^{(0)} + A_i^{(1)} \tilde{s}^{2-1/z}, \tag{A1b}$$

$$\Psi \sim \Psi^{(0)} \tilde{u}^{\Delta} \ln u + \Psi^{(1)} \tilde{u}^{\Delta}$$
$$\Rightarrow \frac{\Psi^{(0)}}{2z} \tilde{s} \ln s + \Psi^{(1)} \tilde{s}. \tag{A1c}$$

(In expressions after "⇒", we set $p = 3z = 3\Delta/2$ and used $\tilde{s} := \tilde{u}^{2z}$.)

The field/operator correspondence is derived by evaluating the on-shell action. The bulk on-shell action, in general, diverges, and one needs to add counterterm actions. We take the probe limit, so we discuss counterterm actions for matter fields only. We use the Lorentzian formalism.

In the static homogeneous case, or at the leading order in the (ω, q)-expansion, the scalar action diverges, and the counterterm action is

$$S_{CT} = \frac{1}{e^2} \int_{\partial\mathcal{M}} d^{p+1}x \ L_1, \tag{A2a}$$

$$L_1 = -\frac{1}{L} \sqrt{-\gamma} \left(\Delta + \frac{1}{\ln \delta} \right) |\Psi|^2$$
$$\Rightarrow -\frac{2z}{L} \sqrt{-\gamma} \left(1 + \frac{1}{\ln \delta_s} \right) |\Psi|^2, \tag{A2b}$$

where $\gamma_{\mu\nu}$ is the $(p+1)$-dimensional boundary metric and $u = \delta$ (or $s = \delta_s := \delta^{2z}$) is the UV cutoff. As usual, the second term is necessary for the scalar which saturates the BF bound.

Using the standard holographic technique, one then gets

$$\rho = -\frac{p-z}{e^2 L} A_t^{(1)} \Rightarrow -\frac{2z}{e^2 L} A_t^{(1)}, \tag{A3a}$$

$$J^i = \frac{p+z-2}{e^2 L} A_i^{(1)} \Rightarrow \frac{4z-2}{e^2 L} A_i^{(1)}, \tag{A3b}$$

$$\mathcal{O} = -\frac{1}{e^2 L} \Psi^{(1)} \Rightarrow -\frac{1}{e^2 L} \Psi^{(1)}. \tag{A3c}$$

(More precisely, left-hand sides represent expectation values such as $\langle \mathcal{O} \rangle$.)

The Lifshitz scaling (2.2) is just a coordinate transformation from the bulk point of view. The Maxwell field is a one-form, and Ψ is a scalar, so they transform as

$$A_t \rightarrow A_t/a^z, \qquad A_i \rightarrow A_i/a, \qquad \Psi \rightarrow \Psi, \tag{A4}$$

under the scaling. Then, the scaling dimensions are

$$[\mu]_s = z, \qquad [\rho]_s = p, \tag{A5a}$$

$$[A_i^{(0)}]_s = 1, \qquad [J^i]_s = p + z - 1, \tag{A5b}$$

$$[\Psi^{(0)}]_s = [\mathcal{O}]_s = \Delta. \tag{A5c}$$

On the other hand, the mass dimensions are

$$[\mu] = [A_i^{(0)}] = [\Psi^{(0)}] = M, \tag{A6a}$$

$$[\rho] = [J^i] = [\mathcal{O}] = \mathrm{M}^p. \tag{A6b}$$

(We choose mass dimensions as $[e^2] = \mathrm{M}^{2-p}$ and $[A_M] = [\Psi] = \mathrm{M}$.)

Continuing higher orders in the derivative expansion, one has additional counterterms:

$$L_2 = \frac{1}{4} M_F(\Psi) \sqrt{-\gamma} \gamma^{\mu\nu} \gamma^{\rho\sigma} F_{\mu\sigma} F_{\nu\sigma}, \tag{A7a}$$

$$L_3 = M_\Psi(\Psi) \sqrt{-\gamma} \gamma^{\mu\nu} (D_\mu \Psi)^\dagger D_\nu \Psi. \tag{A7b}$$

M_F and M_Ψ are power series in Ψ whose explicit forms are not necessary in the discussion below. In the text, we take into account $O(\omega, q^2)$ terms in the scalar perturbation and $O(\omega)$ term in the vector perturbation, so it is enough to consider L_2 and L_3, but they make no contribution. For the scalar perturbation in the high-temperature phase, $F_{\mu\nu} = 0$, so $L_2 = 0$, and

$$L_3 = \left(\frac{2z}{L}\right)^2 M_\Psi h^{1/2} \left\{ -\frac{(\bar\omega + \bar A_t)^2}{hs} + \frac{\bar q^2}{s^{2-1/z}} \right\} |\delta\Psi(k)|^2$$
$$\sim O(s(\ln s)^2) + O(s^{1/z}(\ln s)^2), \tag{A8}$$

so L_3 makes no contribution as $s \to 0$ (for a finite z). For the vector perturbation, $L_2 \sim O(\omega^2, q^2)$, so L_2 makes no contribution,[12] and

$$L_3 = \left(\frac{2z}{L}\right)^2 M_\Psi \frac{h^{1/2}}{s^{2-1/z}} |\bar\Psi|^2 \delta A_y(-k) \delta A_y(k)$$
$$\sim \epsilon O(s^{1/z}), \tag{A9}$$

where we consider the case of the spontaneous condensate for $\bar\Psi$.

APPENDIX B: EXTRACTING FALLOFFS

We solve the following differential equation:

$$\mathcal{L}\varphi = \mathbf{j}, \tag{B1a}$$

$$\mathcal{L} = \partial_s(p(s)\partial_s). \tag{B1b}$$

Denote two independent solutions of the homogeneous equation $\mathcal{L}\varphi = 0$ as φ_1 and φ_2. We assume that φ_1 satisfies the boundary condition at the horizon $s = 1$. The solution of the inhomogeneous equation (B1a) which is regular at the horizon is given by

$$\varphi(s) = -\varphi_1(s) \int_0^s ds' \frac{\mathbf{j}(s')\varphi_2(s')}{p(s')W(s')}$$
$$- \varphi_2(s) \int_s^1 ds' \frac{\mathbf{j}(s')\varphi_1(s')}{p(s')W(s')}, \tag{B2}$$

where W is the Wronskian $W(s) := \varphi_1\varphi_2' - \varphi_1'\varphi_2$.

For example, for $\delta\Psi$ and δA_y,

$$\delta\Psi: \quad \varphi_1 = \frac{s}{1+s},$$
$$\varphi_2 = \frac{s}{1+s} \ln\left[\frac{s}{(1-s)^2}\right] \sim s\ln s,$$
$$p(s) = \frac{h}{s}.$$
$$\delta A_y: \quad \varphi_1 = \frac{z}{2z-1},$$
$$\varphi_2 = s^{2-1/z} {}_2F_1\left(1, \frac{2z-1}{2z}, \frac{4z-1}{2z}; s^2\right)$$
$$\sim s^{2-1/z},$$
$$p(s) = \frac{h}{s^{1-1/z}}.$$

For both cases, $pW = 1$.

Even if the integral (B2) is difficult to evaluate or has a cumbersome expression, one can extract a falloff. Suppose that φ_2 has the appropriate falloff. Then, near the AdS boundary $s \to \delta_s$,

$$\varphi(\delta_s) \sim -\varphi_2(\delta_s) \int_{\delta_s}^1 ds\,\mathbf{j}(s)\varphi_1(s). \tag{B3}$$

This integral essentially gives the falloff coefficient we want.

The δ_s-dependence in the integral essentially has no contribution from the following reason. First, the integral may or may not converge:

(1) When it converges, one can take the $\delta_s \to 0$ limit since the δ_s-dependence in the integral does not produce an appropriate falloff when it is combined with $\varphi_2(\delta_s)$; it gives a subleading falloff.
(2) When it diverges, simply discard the δ_s-dependence in the integral since again it does not produce an appropriate falloff.[13] Even if it diverges as $\delta_s \to 0$, the expression (B2) itself does not.

For example, the slow falloff of ψ_q becomes

$$\psi_q^{(0)} = -\int_0^1 ds\,\mathbf{j}_q \frac{s}{1+s} = -c_1 \int_0^1 \frac{ds}{s^{1-1/z}(1+s)^2}$$
$$= -c_1 I(z). \tag{B4}$$

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[12]Since L_2 is a relevant operator for $z \geq 1$ in our theories, it should be taken into account at $O(\omega^2, q^2)$.

[13]There may be an exception. The δ_s-dependence in the integral may produce an appropriate falloff when it is combined with the *subleading* term of $\varphi_2(\delta_s)$.

Similarly, the fast falloff of a_ϵ becomes

$$a_\epsilon^{(0)} = -\int_0^1 ds\, \mathbf{j}_\epsilon \frac{z}{2z-1} = -\frac{2z}{2z-1} c_1 \int_0^1 \frac{ds}{s^{1-1/z}(1+s)^2}$$

$$= -\frac{2z}{2z-1} c_1 I(z). \tag{B5}$$

APPENDIX C: RESTORING UNITS

We set $e = L = r_h = 1$, but here we present some of our main results by restoring units.

(i) The scalar mode (high-temperature phase): the dispersion relation, the relaxation time, and the correlation length are given by

$$\omega = \frac{3-i}{5}\left\{|\epsilon_\mu| + \frac{I(z)}{z}L\left(\frac{r_h}{L}\right)^{z-2}q^2 + \cdots\right\}, \tag{C1a}$$

$$\tau^{-1} = \frac{1}{5}|\epsilon_\mu| = \frac{I(z)}{5z}L\left(\frac{r_h}{L}\right)^{z-2}\xi_>^{-2}, \tag{C1b}$$

$$\xi_>^2 = \frac{I(z)}{z}L\left(\frac{r_h}{L}\right)^{z-2}\frac{1}{|\epsilon_\mu|}. \tag{C1c}$$

(ii) The order parameter:

$$\mathcal{O} = -\frac{1}{e^2 L}\left(\frac{r_h}{L}\right)^{2z}\frac{2z}{L}\epsilon^{1/2} \tag{C2a}$$

$$= -\frac{1}{e^2 L}\left(\frac{r_h}{L}\right)^{3z/2}\left(\frac{24z}{L}\epsilon_\mu\right)^{1/2} \tag{C2b}$$

$$= -\frac{\sqrt{48}\pi^2}{e^2}T_c^{3/2}(T_c - T)^{1/2} \qquad (z=1). \tag{C2c}$$

(iii) The current (low-temperature phase):

$$J^y = \frac{1}{e^2 L}\left(\frac{r_h}{L}\right)^{4z-2}\left\{-\frac{I(z)}{z}(4z^2\epsilon) + \frac{i\omega L}{(\frac{r_h}{L})^z} + \cdots\right\}A_y^{(0)} \tag{C3a}$$

$$= -\frac{L}{e^2}\left\{\frac{1}{L}\left(\frac{r_h}{L}\right)^{3z-2}24I(z)\epsilon_\mu\right\}A_y^{(0)} \quad (\omega = 0) \tag{C3b}$$

$$= -\frac{I(z)}{z}\frac{e^2 L^3}{(\frac{r_h}{L})^2}\mathcal{O}^2 A_y^{(0)} \qquad (\omega = 0). \tag{C3c}$$

In our conventions, it is natural to define λ as

$$J^y = -\frac{L}{e^2}\frac{1}{\lambda^2}A_y^{(0)}. \tag{C4}$$

Then, λ and κ are given by

$$\lambda^2 = L\left(\frac{r_h}{L}\right)^{-3z+2}\frac{1}{24I(z)\epsilon_\mu}, \tag{C5a}$$

$$\kappa^2 := \left(\frac{\lambda}{\xi_>}\right)^2 = \frac{z}{24I(z)^2}\left(\frac{r_h}{L}\right)^{-4z+4}. \tag{C5b}$$

(iv) The dual GL theory:

$$\mathcal{H} = \frac{I(z)}{2z}L\frac{(eL)^2}{(\frac{r_h}{L})^2}|(\partial_i - iA_i^{(0)})\phi|^2$$

$$- \frac{\epsilon_\mu}{2}\frac{(eL)^2}{(\frac{r_h}{L})^z}|\phi|^2 + \frac{1}{96z}\frac{(eL)^6}{(\frac{r_h}{L})^{4z}L}|\phi|^4 + \cdots$$

$$- (\phi J^\dagger + \phi^\dagger J), \tag{C6a}$$

$$\frac{\Gamma}{2} = \frac{1+3i}{5}\frac{(\frac{r_h}{L})^z}{(eL)^2}. \tag{C6b}$$

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