$D=6$, $\mathcal{N}=(2,0)$ and $\mathcal{N}=(4,0)$ theories

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Using a convolutive field-theoretic product, it is shown here that the "square" of an Abelian $D=6$, $\mathcal{N}=(2,0)$ theory yields the free $D=6$, $\mathcal{N}=(4,0)$ theory constructed by Hull, together with its generalized (super)gauge transformations. This offers a new perspective on the (4,0) theory and chiral theories of conformal gravity more generally, while at the same time extending the domain of the "gravity = gauge × gauge" paradigm.

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I. INTRODUCTION

It was at one time thought that nontrivial conformal quantum field theories exist in at most $D=4$ spacetime dimensions. This was somewhat at odds with Nahm's classification of admissible supersymmetries [1], which includes $D=6$ superconformal algebras. Indeed, a remarkable prediction of M-theory [2–4], anticipated in Refs. [5,6], is the existence of nontrivial $D=6$ quantum field theories with $\mathcal{N}=(2,0)$ supersymmetry and $\mathrm{OSp}^\star(8|4)$ superconformal symmetry, contradicting the received wisdom of the time while placing another feather in Nahm's cap. These "(2,0) theories" are not only central to our understanding of M-theory; they have fundamental implications for gauge theories more generally, from S-duality to the Alday-Gaiotto-Tachikawa (AGT) correspondence [7–9].

Of course, the consistency of a given superalgebra does not imply that a corresponding nontrivial quantum field theory necessarily exists. See for example Ref. [10]. However, taking confidence from the (2,0) story, it is tempting to speculate that the $D=6$, $\mathcal{N}=(4,0)$ multiplet with $\mathrm{OSp}^\star(8|8)$ superconformal symmetry, a longstanding and enticing outpost of Nahm's taxonomy, should also correspond to a nontrivial quantum theory. Indeed, drawing on a range of analogies with the (2,0) theories, Hull argued [11–13] that a nontrivial "(4,0) theory" may arise in the large $D=5$ Plank length, l_5, limit of M-theory compactified on a 6-torus, T^6. As emphasised by Hull, the (4,0) theory would constitute the maximally symmetric phase of M-theory. Moreover, it contains a self-dual "gravi-gerbe" field, suggestive of a $D=6$ chiral theory of conformal gravity. Note that a local variational principle, breaking *manifest* covariance, for the free gravi-gerbe field was recently developed in Ref. [14]. Consequently, just as for the (2,0) theories before it, establishing its existence would have profound implications for not only M-theory but also gravity more broadly understood. It should be stressed that, while there is a large body of strong evidence, originating from string/M-theory, for the (2,0) theories, there are at present no comparable arguments supporting the existence of the (4,0) theory, and it remains highly conjectural. For a more nuanced discussion of the various possibilities, and the associated difficulties, the reader is referred to Refs. [11–13,15,16].

Here, we reexamine the free (4,0) theory introduced in Refs. [11–13] from another, *a priori* unrelated, but equally provocative, perspective: "gravity = gauge × gauge." While on face value a radical proposal, this paradigm has been reinvigorated in recent years by the remarkable Bern-Carrasco-Johansson double-copy procedure [17–19]; the scattering amplitudes of (super)gravity are conjectured to be the double-copy of (super-)Yang-Mills amplitudes to all orders in perturbation theory. These fascinating amplitude relations are both computationally expedient and conceptually suggestive, facilitating previously intractable calculations while probing profound questions regarding the deep structure of perturbative quantum gravity [20,21].

In this context $D=5$, $\mathcal{N}=8$ supergravity, the low-energy limit of M-theory on a 6-torus, is the double-copy of $D=5$, $\mathcal{N}=4$ super Yang-Mills theory. Of course, $D=5$ Yang-Mills theory is nonrenormalizable, and we expect new physics to enter for energies $E\geq 1/g^2_{\mathrm{YM}}$. For instance, it can be regarded as the low-energy sector of the world volume theory of a stack of D4-branes in string theory. Taking the strong-coupling limit, the Yang-Mills theory uplifts to a (2,0) theory compactified on a circle of radius

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$R \propto g_{\mathrm{YM}}^2$, which in this setting constitutes the low-energy theory arising on a stack of M5-branes in M-theory. This raises a challenging question: what happens to the double-copy in this limit? Might we expect some relation of the type $(4,0) = (2,0) \times (2,0)$, morally the M-theory uplift of gravity = gauge × gauge?

The $(4,0) = (2,0) \times (2,0)$ picture was proposed in Ref. [16], where the ultrashort (4,0) supermultiplet of the six-dimensional conformal superalgebra $\mathrm{OSp}^\star(8|8)$ was derived and shown to consistently factorize, with respect to the R-symmetry algebras $\mathrm{USp}(4) \times \mathrm{USp}(4) \subset \mathrm{USp}(8)$, into the product of two (2,0) tensor multiplets. However, as emphasized in Ref. [16], the intrinsically nonperturbative nature of the (2,0) theories makes amplitude relations hard to formulate, although there exist some limited tests [16,22,23]. Here, we avoid this hurdle altogether by appealing to a complementary and independent off-shell field-theoretic realization of gravity as the "square of Yang-Mills" developed in Refs. [24–33], which can be used to study the product of two gauge theories without reference to amplitudes, allowing one to derive various properties, such as curvatures, dynamics, off-shell local symmetries and duality relations, directly. For two gauge potentials belonging to two distinct Yang-Mills theories, referred to as the left (no tilde) and right (tilde) factors, with arbitrary gauge groups G and $\tilde{G}$, the product is given by [26]

$$A_\mu \circ \tilde{A}_\nu \coloneqq A_\mu^a \cdot \Phi_{a\tilde{a}} \cdot \tilde{A}_\nu^{\tilde{a}}, \tag{1}$$

where $[f \cdot g](x) = \int d^D y f(y) g(x-y)$. The biadjoint "spectator" scalar field Φ allows for arbitrary and independent G and $\tilde{G}$, while the convolution reflects the fact that the amplitude relations are multiplicative in momentum space. Crucially, together, they ensure that both the global and local symmetries of the two factors are consistently mapped into those of the corresponding gravitational theory, including general coordinate transformations [24–29]. To linear approximation, the equations of motion of the factors then imply those of the gravity theory, and classical solutions of the Yang-Mills factors are mapped into solutions of their product [26,30,31]. Extending this construction, it is shown here that, by defining a field (7) and ghost field (9) dictionary, the product of two arbitrary Abelian (2,0) theories generates, with no further input, the free (4,0) theory first constructed by Hull [11]. This represents a new perspective on the (4,0) theory that may be exploited to better understand its remarkable, as yet rather mysterious, properties, while at the same time extending the rapidly evolving domain [25–55] of the gravity = gauge × gauge paradigm.

II. STRONGLY COUPLED YANG-MILLS AND (2,0) THEORIES

The free (2,0) theory is described by the (2,0) tensor multiplet consisting of an Abelian two-form gauge potential $B_{\mu\nu}$ with self-dual three-form field strength $H = \star H$, four symplectic Majorana-Weyl spinors χ and five scalars Φ, transforming, respectively, as the **1**, **4** and **5** of the rigid $\mathrm{Spin}(5) \cong \mathrm{USp}(4)$ R-symmetry. The two-form gauge and gauge-for-gauge transforms are given by

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]}, \qquad \delta\lambda_\nu = \partial_\nu \lambda, \tag{2}$$

leaving $15 - 6 + 1 = 10$ off-shell degrees of freedom. The equation of motion $d \star H = 0$ leaves six on-shell degrees of freedom in the $(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$ representation of the spacetime little group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. The self-duality condition, which with the Bianchi identity $dH = 0$ implies the equation of motion, further reduces these to the chiral $(\mathbf{3},\mathbf{1})$ representation. Dimensionally reducing on a circle, S^1, with radius R yields the maximally supersymmetric Abelian $D = 5$, $\mathcal{N} = 4$ gauge theory, consisting of a one-form Abelian gauge potential A_m, four symplectic Majorana spinors ψ and five scalars ϕ, with coupling constant $g^2 \propto R$ and the same USp(4) R-symmetry.

Going beyond the free theories, it has been conjectured [56,57] that the strong-coupling limit of $D = 5$, $\mathcal{N} = 4$ Yang-Mills theory is given by an *interacting* (2,0) theory compactified on S^1 with $g_{\mathrm{YM}}^2 \propto R$. Crucial to this picture is the existence of 1/2-supersymmetric instantonic 0-branes in the $D = 5$, $\mathcal{N} = 4$ Yang-Mills theory, which preserve the full USp(4) R-symmetry. They have mass $\propto |n|/g_{\mathrm{YM}}^2$, where n is the instanton number, so that they become light in the strong-coupling limit and can be matched to the Kaluza-Klein modes of the (2,0) theory compactified on S^1, which have mass $\propto n/R$ [56].

III. STRONGLY COUPLED GRAVITY AND THE (4,0) THEORY

Maximally supersymmetric $D = 5$, $\mathcal{N} = 8$ supergravity has USp(8) R-symmetry and an exceptional noncompact global $E_{6(6)}(\mathbb{R})$-symmetry [58] that is broken by quantum effects to the discrete subgroup $E_{6(6)}(\mathbb{Z})$, corresponding to the U-duality group of M-theory compactified on T^6 [59]. Its massless fields include 27 one-form Abelian gauge potentials A_m, transforming in the fundamental **27** of $E_{6(6)}$. Hull [11–13] considered a large l_5 limit under the assumption that the $E_{6(6)}$-symmetry is preserved and all supersymmetric states are protected. Decomposing the $\mathcal{N} = 8$ multiplet with respect to an $\mathcal{N} = 4$ subalgebra, we obtain five $\mathcal{N} = 4$ Abelian gauge multiplets with coupling constant $g^2 = l_5$, each of which therefore lifts to an Abelian (2,0) theory as $l_5 \to \infty$, where $g^2 = l_5$ is identified with R as before. If the $E_{6(6)}$-symmetry is to be preserved, it follows that *all* 27 one-forms must lift to two-forms. Hence, if all supersymmetries survive the entire $\mathcal{N} = 8$ supergravity multiplet must lift to a $D = 6$ theory, where l_5 is identified with R such that the $l_5 \to \infty$ limit is conformal. We therefore require a superconformal gravitational theory in $D=6$ dimensions, consistent with a global $E_{6(6)}$-symmetry, that yields $D=5, \mathcal{N}=8$ supergravity when compactified on a circle. According to Nahm's classification, there is a unique candidate satisfying these criteria, the (4,0) theory.

As described in Refs. [11–13], the free (4,0) theory consists of eight two-form "gravitini" $\Psi_{\mu\nu}$, 27 Abelian self-dual two-forms $B_{\mu\nu}$, 48 symplectic Majorana-Weyl spinors λ and 42 scalars Φ, transforming, respectively, as the **8**, **27**, **48** and **42** of the USp(8) R-symmetry. Finally, rather than a graviton, there is a rank-4 tensor,

$$G_{\mu\nu\rho\sigma} = G_{[\mu\nu][\rho\sigma]} = G_{[\rho\sigma][\mu\nu]}, \qquad G_{[\mu\nu\rho]\sigma} = 0, \tag{3}$$

which might be thought of as a gravi-gerbe field [60,61]. It has a rank-6 field strength,

$$R_{\mu\nu\rho\sigma\tau\lambda} = 9\partial_{[\mu} G_{\nu\rho][\sigma\tau,\lambda]} = R_{\sigma\tau\lambda\mu\nu\rho}, \tag{4}$$

satisfying the first and second Bianchi identities,

$$R_{[\mu\nu\rho\sigma]\tau\lambda} = \partial_{[\kappa} R_{\mu\nu\rho]\sigma\tau\lambda} = 0. \tag{5}$$

It is invariant under the gauge transformations,

$$\begin{aligned} \delta G_{\mu\nu\rho\sigma} &= \partial_{[\mu}\xi_{\nu]\rho\sigma} + \partial_{[\rho}\xi_{\sigma]\mu\nu} - 2\partial_{[\mu}\xi_{\nu\rho\sigma]} \\ &= \partial_{[\mu}\zeta_{\nu]\rho\sigma} + \partial_{[\rho}\zeta_{\sigma]\mu\nu}, \end{aligned} \tag{6}$$

where $\xi_{\rho\mu\nu} = \xi_{\rho[\mu\nu]}$ and $\zeta_{\nu\rho\sigma} \coloneqq \xi_{\rho\mu\nu} - \xi_{[\rho\mu\nu]}$. The natural free field equation, $R^{\mu}{}_{\nu\rho\mu\tau\lambda} = 0$, describes ten on-shell degrees of freedom in the $(\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5})$. This is reduced to the chiral $(\mathbf{5},\mathbf{1})$ representation by the self-duality relation $R = \star R = R \star$. It was shown in Ref. [12] that the free (4,0) theory compactified on a circle yields linearized $D=5$, $\mathcal{N}=8$ supergravity. The (4,0) theory is gravitational but does *not* contain a graviton.

As for the (2,0) theory, it is not possible to construct a conventional set of local covariant interactions, making the nonlinear theory difficult to probe. Nonetheless, an analysis of the Bogomol'nyi-Prasad-Sommerfield (BPS) spectrum analogous to that of the (2,0) theory suggests that the identification of the strong-coupling limit of $D=5, \mathcal{N}=8$ supergravity as the full interacting (4,0) theory compactified on S^1 is in principle consistent [11]. In particular, $D=5$, $\mathcal{N}=8$ supergravity admits 1/2-supersymmetric gravitational instantonic solutions, which preserve the $E_{6(6)}$-symmetry [62]. In analogy with the instantons appearing in the (2,0) story, these are the uplift of Euclidean $D=4$ self-dual gravitational instantons [62–64], which can be interpreted as 0-branes [62]. They carry mass $\propto |n|/l_5$ and so become light in the $l_5 \to \infty$ limit. The analyses of Refs. [2,56] indicate that these solutions may be regarded as the Kaluza-Klein modes of a $D=6$ theory on a circle of radius $R \propto l_5$ [11]. This proposal still requires many checks, but encouragingly, these 1/2-supersymmetric states sit in massive (4,0) multiplets that have precisely the correct content to have originated from an S^1 compactification of the $D=6$, (4,0) theory: 27 massive self-dual two-forms and 42 massive scalars. A more detailed analysis of the $D=5, \mathcal{N}=8$ and $D=6$, $\mathcal{N}=(4,0)$ supersymmetric multiplets paints a compelling picture. In particular, the 27 self-dual two-forms in $D=6$ couple to self-dual supersymmetric strings, which yield the required $D=5$ charged 0-branes and 1-branes transforming in the **27** and **27**$'$ of the global $E_{6(6)}$.

IV. (2,0) THEORY SQUARED

In direct analogy with (1), we apply the product to a pair of self-dual two-forms belonging to left and right Abelian (2,0) tensor multiplets,

$$\mathcal{G}_{\mu\nu\rho\sigma} \coloneqq B_{\mu\nu} \circ \tilde{B}_{\rho\sigma}. \tag{7}$$

Adopting this dictionary, we recover precisely the free (4,0) theory. In particular, the generalized gauge transformations of the gravi-gerbe field (3) are generated by the local symmetries of the left and right (2,0) factors. Since the supercharges of the left and right theories generate the supersymmetries of their product [26], the remaining fields of the (4,0) multiplet and their transformations then follow essentially automatically.

The field $\mathcal{G}$ has $15 \times 15 = 225$ components, reduced to $10 \times 10 = 100$ off-shell degrees of freedom by the generalized gauge transformations generated by (2). Explicitly, using $\partial(f \circ g) = \partial f \circ g = f \circ \partial g$, we obtain

$$\begin{aligned} \delta\mathcal{G}_{\mu\nu\rho\sigma} &= \delta B_{\mu\nu} \circ \tilde{B}_{\rho\sigma} + B_{\mu\nu} \circ \delta\tilde{B}_{\rho\sigma} \\ &= 2\partial_{[\mu} C_{\nu]} \circ \tilde{B}_{\rho\sigma} + B_{\mu\nu} \circ 2\partial_{[\rho}\tilde{C}_{\sigma]} \\ &= 2\partial_{[\mu} C^{(10)}_{\nu]\rho\sigma} + 2\partial_{[\rho} C^{(01)}_{\sigma]\mu\nu}, \end{aligned} \tag{8}$$

where δ is the Becchi-Rouet-Stora-Tyutin (BRST) transformation corresponding to (2) and we have introduced the ghost field dictionary,

$$C^{(10)}_{\nu\rho\sigma} = C_{\nu} \circ \tilde{B}_{\rho\sigma}, \qquad C^{(01)}_{\sigma\mu\nu} = B_{\mu\nu} \circ \tilde{C}_{\sigma}. \tag{9}$$

Here, the superscripts $(x\tilde{x})$ denote the ghost numbers of the left/right factors, which are additive so that the ghost number of $C^{(x\tilde{x})}$ is $x+\tilde{x}$. The ghosts $C^{(10)}_{\nu\rho\sigma}, C^{(01)}_{\nu\rho\sigma}$ have $6\times15+6\times15=180$ components. However, the left/right two-form ghost-for-ghost transformations, $\delta C_{\nu} = \partial_{\nu} C$, generate gravi-gerbe ghost-for-ghost transformations. Using $\delta(f^{(x)} \circ g^{(\tilde{x})}) = \delta f^{(x)} \circ g^{(\tilde{x})} + (-1)^{x} f^{(x)} \circ \delta g^{(\tilde{x})}$, the full set of BRST variations and ghost fields can be systematically determined by repeatedly varying the field (7) and ghost (9) dictionaries. This procedure yields

$$\delta C^{(10)}_{\nu\rho\sigma} = \partial_\nu C^{(20)}_{\rho\sigma} - 2\partial_{[\rho} C^{(11)}_{|\nu|\sigma]} \tag{10a}$$

$$\delta C^{(01)}_{\nu\rho\sigma} = \partial_\nu C^{(02)}_{\rho\sigma} + 2\partial_{[\rho} C^{(11)}_{\sigma]\nu} \tag{10b}$$

$$\delta C^{(11)}_{\rho\sigma} = \partial_\rho C^{(21)}_{\sigma} - \partial_\sigma C^{(12)}_{\rho} \tag{10c}$$

$$\delta C^{(20)}_{\rho\sigma} = 2\partial_{[\rho} C^{(21)}_{\sigma]} \tag{10d}$$

$$\delta C^{(02)}_{\rho\sigma} = 2\partial_{[\rho} C^{(12)}_{\sigma]} \tag{10e}$$

$$\delta C^{(21)}_{\rho} = \partial_\rho C^{(22)} \tag{10f}$$

$$\delta C^{(12)}_{\rho} = \partial_\rho C^{(22)}, \tag{10g}$$

where we have introduced the dictionary for the ghost-for-ghost fields,

$$C^{(20)}_{\rho\sigma} = C\circ\tilde{B}_{\rho\sigma}, \quad C^{(11)}_{\rho\sigma} = C_\rho\circ\tilde{C}_\sigma, \quad C^{(02)}_{\rho\sigma} = B_{\rho\sigma}\circ\tilde{C};$$
$$C^{(21)}_{\rho} = C\circ\tilde{C}_\rho, \quad C^{(12)}_{\rho} = C_\rho\circ\tilde{C}; \quad C^{(22)} = C\circ\tilde{C}. \tag{11}$$

The complete set of ghost fields removes a total of $125 = (90+90) - (15+15+36) + (6+6) - 1$ components from $\mathcal{G}$, leaving 100 off-shell degrees of freedom as expected. That the full set of generalized gauge transformations is generated directly by the left/right factors is a nice feature of the construction.

Let us now define the irreducible $GL(6,\mathbb{R})$ representations,

$$G_{\mu\nu\rho\sigma} = \frac{1}{2}(\mathcal{G}_{\mu\nu\rho\sigma} + \mathcal{G}_{\rho\sigma\mu\nu}) - \mathcal{G}_{[\mu\nu\rho\sigma]}, \tag{12a}$$

$$\Phi_{\mu\nu\rho\sigma} = \mathcal{G}_{[\mu\nu\rho\sigma]}, \tag{12b}$$

$$\mathcal{B}_{\mu\nu\rho\sigma} = \frac{1}{2}(\mathcal{G}_{\mu\nu\rho\sigma} - \mathcal{G}_{\rho\sigma\mu\nu}), \tag{12c}$$

which transform as the $\mathbf{1}+\mathbf{20}+\mathbf{84}$, $\mathbf{15}$ and $\mathbf{15}+\mathbf{45}+\bar{\mathbf{45}}$ of Spin(1, 5), respectively.

First, $G_{\mu\nu\rho\sigma}$ has the symmetries of (3) and, directly from (8), the generalized gauge transformations given in (6), where we have identified the ghost field,

$$\xi_{\nu\rho\sigma} \coloneqq C^{(10)}_{\nu\rho\sigma} + C^{(01)}_{\nu\rho\sigma}. \tag{13}$$

Hence, it is naturally identified with the gravi-gerbe field (3) of the (4,0) multiplet. Note that G has a total of $50 = 105 - 70 + 15$ off-shell degrees of freedom sitting in the $\mathbf{1}+\mathbf{14}+\mathbf{35}$ of Spin(5). This follows directly from the generalized ghost and ghost-for-ghost transformations generated by (10) through the dictionary (11),

$$\delta\zeta_{\nu\rho\sigma} = \partial_\nu\zeta_{\rho\sigma} + \partial_{[\sigma}\zeta_{\rho]\mu}, \qquad \delta\zeta_{\rho\sigma} = 0, \tag{14}$$

where $\zeta_{\rho\sigma} \coloneqq 3(\xi_{\rho\sigma} - C^{(11)}_{[\rho\sigma]})/4$ and $2\xi_{\rho\sigma} \coloneqq C^{(20)}_{\rho\sigma} + C^{(02)}_{\rho\sigma}$.

Similarly, it is straightforward to show that the left/right two-form gauge symmetries imply that $\Phi_{\mu\nu\rho\sigma}$ has four-form gauge transformations given by

$$\delta\Phi_{\mu\nu\rho\sigma} = 4\partial_{[\mu}\Lambda_{\nu\rho\sigma]}, \qquad \delta\Lambda_{\nu\rho\sigma} = 3\partial_{[\nu}\Lambda_{\rho\sigma]}, \tag{15a}$$

$$\delta\Lambda_{\rho\sigma} = 2\partial_{[\rho}\Lambda_{\sigma]}, \qquad \delta\Lambda_\sigma = \partial_\sigma\Lambda, \tag{15b}$$

where $\Lambda_{\nu\rho\sigma} = \xi_{[\nu\rho\sigma]}$, $\Lambda_{\rho\sigma} = \xi_{[\rho\sigma]} + 2C^{(11)}_{[\rho\sigma]}$, $\Lambda_\sigma = 3(C^{(21)}_\sigma + C^{(12)}_\sigma)/2$, and $\Lambda = 3C^{(22)}/2$, leaving $5 = 15–20 + 15 - 6 + 1$ off-shell degrees of freedom in the $\mathbf{5}$ of Spin(5).

Finally, Eq. (12c) transforms as

$$\delta\mathcal{B}_{\mu\nu\rho\sigma} = \partial_{[\mu}\alpha_{\nu]\rho\sigma} - \partial_{[\rho}\alpha_{\sigma]\mu\nu}, \tag{16a}$$

$$\delta\alpha_{\nu\rho\sigma} = \partial_\nu\alpha_{\rho\sigma} - 2\partial_{[\rho}\beta_{\sigma]\nu}, \tag{16b}$$

$$\delta\alpha_{\rho\sigma} = 2\partial_{[\rho}\alpha_{\sigma]}, \qquad \delta\beta_{\sigma\nu} = 2\partial_{(\sigma}\alpha_{\nu)}, \tag{16c}$$

where $\alpha_{\nu\rho\sigma} \coloneqq C^{(10)}_{\nu\rho\sigma} - C^{(01)}_{\nu\rho\sigma}$, $\alpha_{\rho\sigma} \coloneqq C^{(20)}_{\rho\sigma} - C^{(02)}_{\rho\sigma}$, $\alpha_\sigma \coloneqq C^{(21)}_\sigma - C^{(12)}_\sigma$ and $\beta_{\rho\sigma} \coloneqq 2C^{(11)}_{(\rho\sigma)}$. This leaves $45 = 105 - 90 + 36 - 6$ off-shell degrees of freedom in the $\mathbf{10}+\mathbf{35}$ of Spin(5). In total, we have 100 off-shell degrees of freedom in the $\mathbf{10}\times\mathbf{10} = \mathbf{1}_s + \mathbf{14}_s + \mathbf{35}'_s + \mathbf{5}_s + \mathbf{10}_a + \mathbf{35}_a$, as expected since each two-form represents a $\mathbf{10}$ of Spin(5). While (12a) and (12b) are immediately recognizable as the off-shell potentials for the gravi-gerbe (3) and a scalar field (in its dual form), respectively, Eq. (12c) is perhaps less familiar. It describes the same on-shell degrees of freedom as a self-dual two-form, as is most easily seen by going to physical gauge [65] using the gauge transformations given in (16). In this case, we have $\mathcal{B}_{ijkl} = \mathcal{B}_{[ij][kl]} = -\mathcal{B}_{klij}$, $i, j = 1, \ldots, 4$, where the self-duality relations $\mathcal{B} = \star\,\mathcal{B} = \mathcal{B}\,\star$ (which follow directly from the left and right self-duality relations in physical gauge $B_{ij} = \star\, B_{ij}$) leave three independent degrees of freedom in the $(\mathbf{3}, \mathbf{1})$ of $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$.

Applying global supersymmetries to the factors, the rest of the (4,0) multiplet follows. For example, the eight two-form gravitini $\Psi_{\mu\nu}$ are identified with the eight products, $\chi\circ\tilde{B}_{\mu\nu}$ and $B_{\mu\nu}\circ\tilde{\chi}$ [66]. The super-BRST variation $\delta\Psi_{\mu\nu} = 2\partial_{[\mu}\eta_{\nu]}$ is generated by the left/right two-form transformations, where the *bosonic* spinor-vector ghosts η_ν are identified with $\chi\circ\tilde{C}_\nu$ and $C_\nu\circ\tilde{\chi}$. The complete details will be presented elsewhere.

Before concluding, we note, briefly, that by going first to physical gauge the equations of motion, Bianchi identities and self-dualities relations for the free (4,0) theory follow straightforwardly from those of the (2,0) factors. Recall that the on-shell degrees of freedom of a self-dual two-form are given by a symmetric bispinor B_{AB}, $A, B = 1, 2$, in the $(\mathbf{3}, \mathbf{1})$ of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, where $\Box B_{AB} = 0$. Hence, for example, the symmetrized product $G_{(ABCD)} = B_{(AB} \circ \tilde{B}_{CD)}$ yields the $(\mathbf{5}, \mathbf{1})$ representation satisfying $\Box G_{(ABCD)} = 0$, which corresponds to the gravi-gerbe field (3) in physical gauge,

$$G_{ijkl} = G_{[ij][kl]} = G_{klij}, \qquad G_{[ijk]l} = 0, \tag{17}$$

where $G_{ijkl} = \star\, G_{ijkl} = G \star_{ijkl}$ [11].

V. CONCLUSIONS

We have shown that the linear (4,0) theory and its local symmetries follow from the square of Abelian (2,0) theories. This leaves a number of directions for future work. Perhaps most obvious is the need to understand the (4,0) theory beyond the linear approximation. A natural setting for such a question is higher gauge theory [67]. For example, a number of higher gauge (2,0) models were developed in Refs. [68–71] using superconformal twistors. However, the (4,0) theory will require new structures, gravitational analogs of the (2,0) models, and it is not *a priori* clear how to proceed. Here, however, we have an extra input to guide our considerations: the (4,0) higher gauge theory will be *required* to be consistent with the square of the (2,0) theory.

Irrespectively, we can still test $(4,0) = (2,0) \times (2,0)$ by considering its compatification, in the first instance, on a circle. Besides testing the expected amplitude relations [23], we anticipate a matching of classical solutions, at least in a weak-field approximation, using the methodology developed in Refs. [30,31]. In particular, it is natural to expect that the $1/2$-supersymmetric gravitational instantonic solutions of $D = 5$, $\mathcal{N} = 8$ supergravity, which must be identified with Kaluza-Klein modes of the would-be (4,0) theory, are related to the "square" of the $1/2$-supersymmetric instantonic 0-branes in the $D = 5$, $\mathcal{N} = 4$ Yang-Mills theory, which are the Kaluza-Klein modes of the (2,0) factors.

We conclude with some rather speculative comments regarding the strong/weak gravitational S-duality suggested by the (4,0) theory [11–13]. First, note that the generalized gauge invariant curvature, self-duality relations and Bianchi identities for $\mathcal{G}$ follow directly from those of $B_{\mu\nu}$ and $\tilde{B}_{\rho\sigma}$. In particular, the generalized gauge invariant curvature is the product of the left and right three-form curvatures,

$$\mathcal{R}_{\mu\nu\rho\sigma\tau\lambda} = 9\partial_{[\mu}\mathcal{G}_{\nu\rho][\tau\lambda,\sigma]} = H_{\mu\nu\rho} \circ \tilde{H}_{\sigma\tau\lambda}. \tag{18}$$

It then follows immediately that the left/right two-form self-duality conditions, $H = \star\, H$, $\tilde{H} = \star\, \tilde{H}$, and Bianchi identities, $dH = d\tilde{H} = 0$, imply the self-duality relations, $\mathcal{R} = \star\, \mathcal{R} = \mathcal{R}\, \star$, and the Bianchi identities, $\partial_{[\mu}\mathcal{R}_{\nu\rho\sigma]\tau\lambda\kappa} = \partial_{[\kappa}\mathcal{R}_{|\mu\nu\rho|\sigma\tau\lambda]} = 0$, respectively. Now, recall that a $D = 6$ Abelian two-form with self-dual field strength, $H = \star\, H$, compactified on T^2 yields an $SL(2, \mathbb{Z})$ doublet of $D = 4$ one-forms A^i, $i = 1, 2$, which are related through $F^i = \star\, F^j \varepsilon_{jk}\gamma^{ki}$, where γ^{ki} is the constant metric on T^2. Since the gravi-gerbe field strength originates from $H \circ \tilde{H}$, feeding this observation into the $(2,0) \times (2,0)$ construction, we anticipate an $\mathrm{SL}(2, \mathbb{Z})$ triplet of $D = 4$ linearized Riemann tensors,

$$\mathcal{R}^{(ij)} \sim F^{(i} \circ \tilde{F}^{j)}, \tag{19}$$

obeying the duality constraint $\mathcal{R}^{(ij)} = \star\, \mathcal{R}^{(kj)}\varepsilon_{jk}\gamma^{ki}$. This is indeed the case; the free (4,0) theory compactified on T^2 yields linear $\mathcal{N} = 8$ supergravity, with an $\mathrm{SL}(2, \mathbb{Z})$-symmetry acting on a triplet of duality related gravitational field strengths [11–13]. Here, it is shown to be the square of the familiar $\mathrm{SL}(2, \mathbb{Z})$ of the Abelian (2,0) multiplet compactified on T^2. Of course, this symmetry is broken by interactions. This is not, however, necessarily an argument against its existence; it simply tells us that it is not a symmetry of classical $\mathcal{N} = 8$ supergravity, just as S-duality is not a symmetry of classical $\mathcal{N} = 4$ super-Yang-Mills theory. While this picture is suggestive, it is highly speculative and will depend crucially on the nonlinear structure of the complete $(4, 0)$ theory. Clearly, it may fail to materialize, and a strong degree of scepticism is advised, but the lessons in gauge theory and gravity learned on the journey will regardless return much insight. Even more speculatively, if the (4,0) theory on $M^6 = X \times C$, where C is a punctured Riemann surface, admits quantities that are protected as we vary the size of X or C, then one might expect a gravitational analog, or square, of the AGT correspondence.

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