q-Poincaré invariance of the AdS_3/CFT_2 R-matrix

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(Received 16 December 2017; published 7 March 2018)

We consider the exact R-matrix of AdS_3/CFT_2, which is the building block for describing the scattering of worldsheet excitations of the light-cone gauge-fixed backgrounds $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ with pure Ramond-Ramond fluxes. We show that R is invariant under a "deformed boost" symmetry, for which we write an explicit exact coproduct, i.e. its action on two-particle states. When we include the boost, the symmetries of the R-matrix close into a q-Poincaré superalgebra. Our findings suggest that the recently discovered boost invariance in AdS_5/CFT_4 may be a common feature of AdS/CFT systems that are treatable with the exact techniques of integrability. With the aim of going towards a universal formulation of the underlying Hopf algebra, we also propose a universal form of the AdS_3/CFT_2 classical r-matrix.

DOI: 10.1103/PhysRevD.97.066001

I. INTRODUCTION

A. Quantum group symmetries in AdS/CFT

The progress in our understanding of the algebraic structure behind the AdS/CFT correspondence, and the integrability of its most symmetric incarnation [1,2], seems to be continuing as more examples are being systematically explored. The core of the method defines an eigenvalue problem for the Hamiltonian of an effective two-dimensional integrable chain, and applies the Bethe ansatz to its exact S-matrix. Integrability is tied to a large algebra of non-Abelian symmetries which form a Hopf superalgebra, and this makes it possible to ultimately solve the system via the tools of the representation theory of quantum groups.

The path to such a solution is however not a straightforward one, as these Hopf superalgebras are rather exotic. They display an infinite tower of generators labeled by an integer, and are very close to Yangian algebras [3–10]. The level 0 typically coincides with the manifest superconformal symmetry of the theory, partially broken and centrally extended *à la* Beisert [11,12]. The central extension goes hand in hand with certain nonlinear constraints on the central charges, which, in turn, are linked to deformations appearing in the Hopf-algebra coproduct map [13,14]. Furthermore, the Yangian [9] displays extra generators [15,16] with no level-0 analog. These symmetries have been dubbed *secret* or *bonus*. They have also been observed in boundary scattering problems [17], n-point amplitudes [18], the pure-spinor formalism [19], in the quantum-affine deformations [20] and in the context of Wilson loops [21]. This makes it quite a significant feature of the system and not an isolated instance [22].

Some light was recently shed on the problem by applying the so-called RTT formulation [23]. In this approach, one starts from the S-matrix in the fundamental representation, and generates from it an algebra of symmetries of the integrable system at hand. In the process, the operator deforming the coproduct was reinterpreted as a particular Yangian generator of level -1. This has a correspondent in the classical r-matrix algebra [16] constructed in the spirit of Drinfeld's second realization of Yangians [24,25].

Even with this step, the accommodation of the full quantum-group tower of symmetries appears still out of reach, and the hope of finding the universal R-matrix and having full control of the representation theory [26,27], relies on further progress. It has very recently become clear in fact [28,29] that extra generators (automorphisms) are necessary. In Ref. [30] an entirely different generator was found for superstrings in $AdS_5 \times S^5$, as we will describe in a subsection below. This is the five-dimensional case, with a dual theory given by four-dimensional $\mathcal{N} = 4$ super Yang-Mills.

Analogous nonstandard quantum algebras and associated bonus generators have been found in lower-dimensional AdS/CFT as well. All these settings share

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peculiar algebraic features stemming from the vanishing of the Killing form of their superisometry [31,32], dictated by string coset integrability and σ-model scale invariance. This seems to tie in with the algebraic peculiarities we have been discussing, which, albeit with a richness of variants, appear to carry over to all cases. From a quantum-group viewpoint, the integrable structure behind the AdS_4 case [33–35] is reduced for the most part to the five-dimensional case (although the physics is very different).

The $\mathrm{AdS}_3/\mathrm{CFT}_2$ integrability [36–38]—see also Refs. [39–41]—provides another fertile realization of these exotic group-theory structures [42]. This is the setup in which we will work in this paper. The program of integrability is carried out for superstrings on $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ and on $\mathrm{AdS}_3 \times S^3 \times T^4$. The former background contains a parameter α corresponding to the relative radii of the S^3's, reflected in the superisometry algebra $\mathfrak{d}(2,1;\alpha)_{\mathrm{L}} \oplus \mathfrak{d}(2,1;\alpha)_{\mathrm{R}}$. Here L and R label the two copies. An $\alpha \to 0$ contraction produces $\mathfrak{psu}(1,1|2)_{\mathrm{L}} \oplus \mathfrak{psu}(1,1|2)_{\mathrm{R}}$, the superisometry algebra of the latter background. The bonus symmetry was found in Ref. [43], cf. Ref. [42]. Before discussing the results in $\mathrm{AdS}_3/\mathrm{CFT}_2$ in more details, let us review the boost invariance that was identified in $\mathrm{AdS}_5/\mathrm{CFT}_4$.

B. Deformed Poincaré supersymmetry in $\mathrm{AdS}_5/\mathrm{CFT}_4$

In Ref. [30], a new symmetry of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ S-matrix was found, realizing the boost of a specific q-deformation of $1+1$-dimensional Poincaré superalgebra. Other q-deformations have appeared in Refs. [38,44–51]. In these parallel lines of investigation, however, the q-deformation is *superimposed* to the algebra, and deforms the theory. This is not what we study in this context, where the super q-Poincaré deformation is part of the ordinary superstring theory. Boost operators on spin chains have a long history [52–54]. See also Refs. [55,56] in the context of long-range spin chains, Refs. [57–59] in the study of sigma models, and Refs. [28,60,61] in the development of algebraic methods for AdS/CFT integrability.

The paper [62] was the first to investigate remnants of the Poincaré algebra in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ integrable problem. The exact dispersion relation of the excitations was interpreted as the Casimir of a q-Poincaré algebra:

$$C = \mathbf{H}^2 + g^2(\mathbf{K}^{\frac{1}{2}} - \mathbf{K}^{-\frac{1}{2}})^2, \qquad (1.1)$$

where $\mathbf{H}$ is the generator corresponding to the energy and $\mathbf{K} = \exp(i\mathbf{P})$ is the exponential of the worldsheet momentum. The coupling g, which is the tension of the string, plays the role of the deformation parameter of q-Poincaré. The *boost* generator $\mathbf{J}$ was introduced as producing shifts

$$\mathbf{J}\colon\ z \to z + c \qquad (1.2)$$

in the torus variable z that uniformizes the dispersion relation [63]. Immediately afterwards, the paper [64] generalized this construction to the full centrally extended $\mathfrak{psu}(2|2)$ algebra, under which the $\mathrm{AdS}_5/\mathrm{CFT}_4$ S-matrix is invariant. Nevertheless, a coproduct was given only for a carefully selected subalgebra of generators, and it turned out to be incompatible with the S-matrix—e.g. the energy was not cocommutative.

In Ref. [30], it was demonstrated that one can overcome these shortcomings by allowing a nonstandard coproduct for $\mathbf{J}$, in such a way that the boost is a symmetry of the S-matrix, as well as all other generators in the superalgebra. In Sec. II we adopt this strategy to extend these results to $\mathrm{AdS}_3/\mathrm{CFT}_2$.

C. Deformed Poincaré supersymmetry in $\mathrm{AdS}_3/\mathrm{CFT}_2$

The global superconformal symmetries of the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\mathrm{AdS}_3 \times S^3 \times T^4$ backgrounds are broken by a choice of vacuum. This corresponds to fixing light-cone gauge compatibly with the Berenstein-Maldacena-Nastase (BMN) ground state, or in the spin-chain picture to the choice of the reference state. The elementary excitations above the vacuum transform in the little group of residual symmetry which preserves the vacuum. These residual symmetries consist of two copies of the centrally extended $\mathfrak{su}(1|1)$ superalgebra in the case of $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ [65–67], while there are four copies in the case of $\mathrm{AdS}_3 \times S^3 \times T^4$ [68–71]; see also Refs. [72–80], and Sec. II for more details. The $\mathrm{AdS}_3/\mathrm{CFT}_2$ integrable problem contains not only massive but also massless excitations. This appears as a novel feature compared to the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case, and it offers both challenges (such as mismatches with perturbation theory waiting to be fully resolved [80]) and interesting physics [81]; see also Refs. [82–84].

By adopting the spirit of Refs. [62,64], in Ref. [85] it was shown that for massless excitations of the above AdS_3 backgrounds the corresponding residual symmetries can be extended to a q-Poincaré superalgebra analogous to that of $\mathrm{AdS}_5/\mathrm{CFT}_4$. Due to the massless dispersion relation, the q-deformed energy coproduct turns out to be cocommutative, and hence an exact symmetry of the S-matrix. This new interpretation of the magnon supersymmetry in the massless case also allows a very concise reformulation of the comultiplication map, and connections with the scattering of phonons [86]. Matching more closely with the relativistic theory might be of significance to describe certain limits of the putative dual field theories [87,88]. In this setting, however, the boost coproduct is not a symmetry of the S-matrix, but rather it annihilates it. This was shown in Ref. [89], where an associated differential-geometric framework was then proposed based on a flat would-be connection. It seems that the construction of Refs. [85,89] is limited to the case of massless excitations, and it is not clear how to extend it to representations of generic mass.

In this paper we adopt a point of view close to Ref. [30], and our discussion of the q-Poincaré supersymmetry is valid for generic values of the mass. In Sec. II we construct a coproduct for the boost, and we check that it is a symmetry of the S-matrix in the relevant two-particle representations. We also study how the boost transforms under crossing transformations. In Sec. III we take the semiclassical limit of the deformed superalgebra, which yields a classical Lie superalgebra that may be obtained also as a contraction of $\mathfrak{sl}(1|2)$. In Sec. IV we write down a proposal for a universal classical r-matrix that matches the known results in the fundamental representation, and we check that it satisfies the classical Yang-Baxter equation in universal form. We use this result in Sec. V to compute the cobrackets of the generators, including the boost.

II. SYMMETRY ALGEBRA AND THE BOOST

In this section we will review in more detail the superalgebra of symmetries of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ integrable models, and its short fundamental representations. The formulation we employ here differs from that of Refs. [65,68] in how we treat the off-shell central extension; in particular, instead of introducing the central elements $\mathbf{C}, \bar{\mathbf{C}}$, we write the results of the corresponding anticommutators just in terms of the momentum generator $\mathbf{P}$, or more conveniently in terms of[1] $\mathbf{K} \equiv \exp(i\mathbf{P})$. As done in Refs. [30,64] for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case, we therefore formulate the symmetry algebra as a deformation of the universal enveloping algebra. In this formulation g, which at large values of the tension of the string is the tension itself, plays the role of the deformation parameter.

As recalled in the Introduction, after fixing the light-cone gauge on the worldsheet for the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ background one ends up with a centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ superalgebra, where the labels left (L) and right (R) distinguish the two copies. Worldsheet excitations are organized in two-dimensional irreducible representations of this superalgebra. They carry labels L or R (see below) which remind us that on shell ($\mathbf{P} = 0$) only the L (R) copy of the superalgebra acts nontrivially on L (R) excitations. Their masses can take only the values $m = 0, \alpha, 1-\alpha, 1$. The construction carried out for the $\mathrm{AdS}_3 \times S^3 \times T^4$ background leaves instead a larger symmetry algebra. This may be obtained by considering two copies of the above centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ superalgebra, where we mod out half of the central elements to leave only their symmetric combinations, and the odd generators are organized in (anti) fundamental representations of an additional $\mathfrak{su}(2)$ symmetry. The worldsheet excitations are still labeled by L and R, and their masses can be just $m = 0$, 1. To keep the discussion as general as possible, in the following we will consider just one copy of the centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ superalgebra, and we will consider L and R representations of generic mass m. Therefore, in order to obtain the results for the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ background it will be enough to set the masses to the desired values. The results for the $\mathrm{AdS}_3 \times S^3 \times T^4$ background are instead obtained by constructing the bifundamental representations as explained in Refs. [68,70]; see in particular Sec. 3.1 of Ref. [70].

The centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ superalgebra is spanned by the supercharges $\mathbf{Q}_{\mathrm{I}}$, $\bar{\mathbf{Q}}_{\mathrm{I}}$ and the central elements $\mathbf{H}_{\mathrm{I}}$, $\mathbf{P}$ (here the subscript $\mathrm{I} = \mathrm{L}, \mathrm{R}$ denotes the two copies), which close into the anticommutation relations

$$
\begin{aligned}
\{\mathbf{Q}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{L}}\} &= \mathbf{H}_{\mathrm{L}}, \\
\{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} &= \frac{ig}{2}(\mathbf{K}^{\frac{1}{2}} - \mathbf{K}^{-\frac{1}{2}}), \\
\{\mathbf{Q}_{\mathrm{R}}, \bar{\mathbf{Q}}_{\mathrm{R}}\} &= \mathbf{H}_{\mathrm{R}}, \\
\{\bar{\mathbf{Q}}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{R}}\} &= \frac{ig}{2}(\mathbf{K}^{\frac{1}{2}} - \mathbf{K}^{-\frac{1}{2}}).
\end{aligned} \tag{2.1}
$$

Useful combinations are the Hamiltonian $\mathbf{H} = \mathbf{H}_{\mathrm{L}} + \mathbf{H}_{\mathrm{R}}$, and the central charge $\mathbf{M} = \mathbf{H}_{\mathrm{L}} - \mathbf{H}_{\mathrm{R}}$ which, as we will recall later, is related to the mass. The two copies of $\mathfrak{su}(1|1)$ decouple on shell, i.e. when $\mathbf{P} = 0$.

We now introduce a boost generator $\mathbf{J}$ such that

$$
\begin{aligned}
[\mathbf{J}, \mathbf{P}] &= i\mathbf{H}, \\
[\mathbf{J}, \mathbf{Q}_{\mathrm{I}}] &= -\frac{ig}{4}(\mathbf{K}^{\frac{1}{2}} + \mathbf{K}^{-\frac{1}{2}})\bar{\mathbf{Q}}_{\bar{\mathrm{I}}}, \\
[\mathbf{J}, \mathbf{H}] &= \frac{g^2}{2}(\mathbf{K} - \mathbf{K}^{-1}), \\
[\mathbf{J}, \bar{\mathbf{Q}}_{\mathrm{I}}] &= -\frac{ig}{4}(\mathbf{K}^{\frac{1}{2}} + \mathbf{K}^{-\frac{1}{2}})\mathbf{Q}_{\bar{\mathrm{I}}},
\end{aligned} \tag{2.2}
$$

where $\mathrm{I} = \mathrm{L}$, R and $\bar{\mathrm{L}} = \mathrm{R}, \bar{\mathrm{R}} = \mathrm{L}$. A difference with respect to the construction of Refs. [85,89] is that here we do not introduce a boost generator for each copy L and R; rather we have one common boost relating the two copies. The boost also breaks the centrality of $\mathbf{H}$ and $\mathbf{P}$. The above commutation relations are also compatible with the automorphism $\mathbf{b}$ acting only on the supercharges as

$$
\begin{aligned}
[\mathbf{b}, \mathbf{Q}_{\mathrm{L}}] &= -2\mathbf{Q}_{\mathrm{L}}, \qquad & [\mathbf{b}, \bar{\mathbf{Q}}_{\mathrm{L}}] &= +2\bar{\mathbf{Q}}_{\mathrm{L}}, \\
[\mathbf{b}, \mathbf{Q}_{\mathrm{R}}] &= +2\mathbf{Q}_{\mathrm{R}}, \qquad & [\mathbf{b}, \bar{\mathbf{Q}}_{\mathrm{R}}] &= -2\bar{\mathbf{Q}}_{\mathrm{R}}.
\end{aligned} \tag{2.3}
$$

The generator $\mathbf{b}$ is the only combination of the two $\mathbf{b}_{\mathrm{I}}$ outer automorphisms of $\mathfrak{su}(1|1)_{\mathrm{I}}$ that survive after introducing the central extension ($\mathbf{P} \neq 0$). For convenience, we summarize our conventions for the generators we shall use and their fermionic degree in the following table:

[1] To avoid confusion coming from different notations used in the literature, we stress that $\mathbf{P}$, $\mathbf{K}$ are respectively the worldsheet momentum and its exponential (multiplied by i).

Generator	Degree
$\mathbf{Q}_{\mathrm{L}}$	1
$\bar{\mathbf{Q}}_{\mathrm{L}}$	1
$\mathbf{Q}_{\mathrm{R}}$	1
$\bar{\mathbf{Q}}_{\mathrm{R}}$	1
$\mathbf{P}$	0
$\mathbf{K}$	0
$\mathbf{H}$	0
$\mathbf{M}$	0
$\mathbf{J}$	0
$\mathbf{b}$	0
$\hat{\mathbf{B}}$	0

The generator $\hat{\mathbf{B}}$ will appear later, cf. Eq. (2.16). The Casimir of the q-Poincaré subalgebra (generated by $\mathbf{H}$, $\mathbf{P}$, $\mathbf{J}$) is denoted by $C = \mathbf{H}^2 + g^2(\mathbf{K}^{\frac{1}{2}} - \mathbf{K}^{-\frac{1}{2}})^2$. When comparing it to the shortening condition $\mathbf{H}^2 = \mathbf{M}^2 - g^2(\mathbf{K}^{\frac{1}{2}} - \mathbf{K}^{-\frac{1}{2}})^2$ given in Refs. [65,68] we see that we should set $C = \mathbf{M}^2$.

The short irreducible representations of the centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ are two-dimensional. They are labeled by three parameters[2]: the mass m, the momentum p and the coupling g. We will be interested in the L and the R representations[3] ϱ_{L} and ϱ_{R}, each spanned by a boson ϕ^{I} and a fermion ψ^{I}. On the (reducible) representation $\varrho_{\mathrm{L}} \oplus \varrho_{\mathrm{R}} = \mathrm{span}\{\phi^{\mathrm{L}}, \psi^{\mathrm{L}}, \phi^{\mathrm{R}}, \psi^{\mathrm{R}}\}$ the above generators may be realized as explicit 4×4 matrices

$$\mathbf{Q}_{\mathrm{L}} = a_p\sigma_- \oplus b_p\sigma_+, \qquad \bar{\mathbf{Q}}_{\mathrm{L}} = \bar{a}_p\sigma_+ \oplus \bar{b}_p\sigma_-,$$
$$\mathbf{Q}_{\mathrm{R}} = b_p\sigma_+ \oplus a_p\sigma_-, \qquad \bar{\mathbf{Q}}_{\mathrm{R}} = \bar{b}_p\sigma_- \oplus \bar{a}_p\sigma_+, \tag{2.4}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The L and R representations may be mapped to each other by swapping the labels L $\leftrightarrow$ R on the charges and on the states. We take

$$a_p = \bar{a}_p = \sqrt{\frac{g}{2}}\gamma_p,$$
$$b_p = \bar{b}_p = i\sqrt{\frac{g}{2}}\gamma_p^{-1}\left(\left(\frac{x^+}{x^-}\right)^{1/2} - \left(\frac{x^+}{x^-}\right)^{-1/2}\right),$$
$$\gamma_p = \sqrt{i(x_p^- - x_p^+)}, \tag{2.5}$$

and we make use of the Zhukovski variables $x_p^\pm$ which satisfy

$$x_p^+ + \frac{1}{x_p^+} - x_p^- - \frac{1}{x_p^-} = \frac{2im}{g}, \qquad \frac{x_p^+}{x_p^-} = e^{ip}. \tag{2.6}$$

Notice the dependence on the mass m in the first of the above constraints. One also finds

$$\mathbf{H} = h_p[\mathbf{1}_2 \oplus \mathbf{1}_2],$$
$$\mathbf{M} = m[\mathbf{1}_2 \oplus (-\mathbf{1}_2)],$$
$$\mathbf{b} = \sigma_3 \oplus (-\sigma_3), \tag{2.7}$$

with

$$h_p = \frac{ig}{2}\left(x_p^- - x_p^+ + \frac{1}{x_p^+} - \frac{1}{x_p^-}\right) = \sqrt{m^2 + 4g^2\sin^2\frac{p}{2}}. \tag{2.8}$$

The sign of the eigenvalue of $\mathbf{M}$ allows us to distinguish between the L and R representations. The action of the generator $\mathbf{b}$ also differs on the two representations by a sign. Finally, the boost is realized as $\mathbf{J} = i\mathbf{H}\partial_p$.

In Ref. [65] an R-matrix in the fundamental representation was found by demanding that it should be invariant under the symmetries, with the exception of the boost; the reader is also referred to Refs. [90–92]. In our conventions, when scattering the tensor-product representation $\varrho \otimes \chi$, the symmetry invariance of the R-matrix is imposed as

$$\Delta^{op}_{\chi\otimes\varrho}(\mathbf{q})R = R\Delta_{\varrho\otimes\chi}(\mathbf{q}), \tag{2.9}$$

where we use the subscript to specify the tensor-product representation on which we should evaluate the coproduct, and we define $\Delta^{op}_{\chi\otimes\varrho} \equiv \Pi_g\Delta_{\chi\otimes\varrho}\Pi_g$ where Π_g is the graded permutation.[4] The coproduct that we use here is the one in the *most symmetric frame*

$$\Delta(\mathbf{Q}_{\mathrm{I}}) = \mathbf{Q}_{\mathrm{I}} \otimes \mathbf{K}^{-\frac{1}{4}} + \mathbf{K}^{\frac{1}{4}} \otimes \mathbf{Q}_{\mathrm{I}},$$
$$\Delta(\mathbf{H}) = \mathbf{H} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H},$$
$$\Delta(\bar{\mathbf{Q}}_{\mathrm{I}}) = \bar{\mathbf{Q}}_{\mathrm{I}} \otimes \mathbf{K}^{\frac{1}{4}} + \mathbf{K}^{-\frac{1}{4}} \otimes \bar{\mathbf{Q}}_{\mathrm{I}},$$
$$\Delta(\mathbf{M}) = \mathbf{M} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{M},$$
$$\Delta(\mathbf{b}) = \mathbf{b} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{b},$$
$$\Delta(\mathbf{P}) = \mathbf{P} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{P}. \tag{2.10}$$

The R-matrix is decomposed into blocks related by LR symmetry; see Refs. [65,68]. The two independent blocks are LL and LR, and one finds

[2]In the spirit of the original paper [65] we prefer to denote by m the mass of the excitations, so that $m \geq 0$. In Ref. [67] m was used for the eigenvalue of $\mathbf{M}$ (at $q = 0$), which is positive/negative on L/R representations; in that case the mass of the excitations would be $|m|$.

[3]The two additional representations denoted by $\tilde{\varrho}_{\mathrm{L}}, \tilde{\varrho}_{\mathrm{R}}$ in Ref. [70] are simply obtained from the above ones by exchanging the roles of the bosons and the fermions.

[4]In an explicit matrix realization, when defining the "op" of a coproduct one should also take care to swap the labels $\{p_1, m_1\} \leftrightarrow \{p_2, m_2\}$ everywhere. With these conventions, the states are ordered as $\{(p_1, m_1), (p_2, m_2)\}$ both before and after the action of the R-matrix.

$$
\begin{aligned}
&R|\phi^{\rm L}\phi^{\rm L}\rangle = |\phi^{\rm L}\phi^{\rm L}\rangle,\\
&R|\phi^{\rm L}\psi^{\rm L}\rangle = a_{12}^{\rm LL}|\phi^{\rm L}\psi^{\rm L}\rangle + b_{12}^{\rm LL}|\psi^{\rm L}\phi^{\rm L}\rangle,\\
&R|\psi^{\rm L}\psi^{\rm L}\rangle = c_{12}^{\rm LL}|\psi^{\rm L}\psi^{\rm L}\rangle,\\
&R|\psi^{\rm L}\phi^{\rm L}\rangle = (a_{21}^{\rm LL})^*|\psi^{\rm L}\phi^{\rm L}\rangle + (b_{21}^{\rm LL})^*|\phi^{\rm L}\psi^{\rm L}\rangle, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
&R|\phi^{\rm L}\phi^{\rm R}\rangle = a_{12}^{\rm LR}|\phi^{\rm L}\phi^{\rm R}\rangle + b_{12}^{\rm LR}|\psi^{\rm L}\psi^{\rm R}\rangle,\\
&R|\phi^{\rm L}\psi^{\rm R}\rangle = |\phi^{\rm L}\psi^{\rm R}\rangle,\\
&R|\psi^{\rm L}\psi^{\rm R}\rangle = a_{21}^{\rm LR}|\psi^{\rm L}\psi^{\rm R}\rangle + b_{21}^{\rm LR}|\phi^{\rm L}\phi^{\rm R}\rangle,\\
&R|\psi^{\rm L}\phi^{\rm R}\rangle = c_{12}^{\rm LR}|\psi^{\rm L}\phi^{\rm R}\rangle, \qquad (2.12)
\end{aligned}
$$

where $*$ denotes complex conjugation—under which $(x^\pm)^* = x^\mp$. Here we have chosen an arbitrary normalization by setting one element in each block to 1; the remaining coefficients are

$$
\begin{aligned}
a_{12}^{\rm LL} &= \left(\frac{x_1^+}{x_1^-}\right)^{-1/2}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-},\\
b_{12}^{\rm LL} &= \left(\frac{x_1^+}{x_1^-}\right)^{-1/4}\left(\frac{x_2^+}{x_2^-}\right)^{1/4}\frac{x_2^+ - x_2^-}{x_2^+ - x_1^-}\frac{\gamma_1}{\gamma_2},\\
c_{12}^{\rm LL} &= \left(\frac{x_1^+}{x_1^-}\right)^{-1/2}\left(\frac{x_2^+}{x_2^-}\right)^{1/2}\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}, \qquad (2.13)
\end{aligned}
$$

and

$$
\begin{aligned}
a_{12}^{\rm LR} &= \left(\frac{x_1^+}{x_1^-}\right)^{-1/2}\frac{x_2^- x_1^+ - 1}{x_1^- x_2^- - 1},\\
b_{12}^{\rm LR} &= \left(\frac{x_2^+}{x_2^-}\right)^{-1/4}\left(\frac{x_1^+}{x_1^-}\right)^{-1/4}\frac{i\gamma_1\gamma_2}{x_1^- x_2^- - 1},\\
c_{12}^{\rm LR} &= \left(\frac{x_1^+}{x_1^-}\right)^{-1/2}\left(\frac{x_2^+}{x_2^-}\right)^{-1/2}\frac{x_1^+ x_2^+ - 1}{x_1^- x_2^- - 1}. \qquad (2.14)
\end{aligned}
$$

Braiding unitarity is written as $R^{op}R = 1$, and one may check that the Yang-Baxter equation is satisfied; a convenient way to check it is done by introducing the S-matrix $S = \Pi_g R$ so that

$$
\begin{aligned}
&S_{12}(p_2,p_3)S_{23}(p_1,p_3)S_{12}(p_1,p_2)\\
&\quad = S_{23}(p_1,p_2)S_{12}(p_1,p_3)S_{23}(p_2,p_3). \qquad (2.15)
\end{aligned}
$$

The subscripts denote the subspaces on which the S-matrix is acting, e.g. $S_{12} = S \otimes \mathbf{1}$, and one should take care to evaluate the S-matrix in the relevant representation.

As discovered in Ref. [43], one may identify a secret symmetry similar to the one appearing in the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$. The antisymmetric combination of the L and R secret symmetries of Ref. [43] (at level 0) should be identified with our automorphism $\mathbf{b}$. The symmetric combination instead may be identified with $\hat{\mathbf{B}}$, the counterpart of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ secret symmetry. See also Eq. (6.1) for the explicit relation to generators used in the literature. In the $\varrho_{\rm L} \oplus \varrho_{\rm R}$ fundamental representation we write $\hat{\mathbf{B}}$ as

$$
\hat{\mathbf{B}} = \frac{1}{4}\left(x_p^+ + x_p^- - \frac{1}{x_p^+} - \frac{1}{x_p^-}\right)(\sigma_3 \oplus \sigma_3), \qquad (2.16)
$$

which is compatible with the commutation relations

$$
\begin{aligned}
&[\hat{\mathbf{B}}, \mathbf{Q}_{\rm I}] = -\hat{\mathbf{Q}}_{\rm I} - (\mathbf{K}^{\frac{1}{2}} + \mathbf{K}^{-\frac{1}{2}})\bar{\mathbf{Q}}_{\bar{\rm I}},\\
&[\hat{\mathbf{B}}, \bar{\mathbf{Q}}_{\rm I}] = \hat{\bar{\mathbf{Q}}}_{\rm I} + (\mathbf{K}^{\frac{1}{2}} + \mathbf{K}^{-\frac{1}{2}})\mathbf{Q}_{\bar{\rm I}}. \qquad (2.17)
\end{aligned}
$$

Here hatted supercharges denote the ones at level 1 of the Yangian. We assume that we can use the evaluation representation and identify e.g. $\hat{\mathbf{Q}}_{\rm I} \sim \hat{u}\mathbf{Q}_{\rm I}$ with $\hat{u} = (x^+ + x^- + 1/x^+ + 1/x^-)/2$. One may check that the coproduct

$$
\begin{aligned}
\Delta(\hat{\mathbf{B}}) &= \hat{\mathbf{B}} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\mathbf{B}}\\
&\quad + \frac{i}{g}\sum_{\rm I=L,R}(\mathbf{K}^{-\frac{1}{4}}\mathbf{Q}_{\rm I} \otimes \mathbf{K}^{-\frac{1}{4}}\bar{\mathbf{Q}}_{\rm I} + \mathbf{K}^{\frac{1}{4}}\bar{\mathbf{Q}}_{\rm I} \otimes \mathbf{K}^{\frac{1}{4}}\mathbf{Q}_{\rm I}) \qquad (2.18)
\end{aligned}
$$

gives a symmetry of the R-matrix, both in the LL and the LR representations.

In order to determine the coproduct for the boost we follow the strategy used in Ref. [30] in the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$: we constrain an appropriate ansatz for the boost coproduct by imposing the commutation relations (2.2), while using the above coproducts for all other generators in the algebra. The coproduct that we find *in the fundamental representation*

$$
\Delta(\mathbf{J}) = \Delta'(\mathbf{J}) + \mathcal{T} \qquad (2.19)
$$

is analogous to the one of Ref. [30]. In particular, the contribution $\Delta'(\mathbf{J})$ remains the same, since it is found by imposing commutation relations of the bosonic q-Poincaré subalgebra. One has

$$
\begin{aligned}
\Delta'(\mathbf{J}) &= \left(1 - \frac{s_{12}}{h_1}\right)\mathbf{J} \otimes \mathbf{1} + \left(1 + \frac{s_{12}}{h_2}\right)\mathbf{1} \otimes \mathbf{J},\\
s_{12} &= \frac{g}{2}\frac{\sin p_1 + \sin p_2 - \sin(p_1 + p_2)}{w_1^{-1} - w_2^{-1}}, \qquad (2.20)
\end{aligned}
$$

where

$$
w_p = \frac{2h_p}{g \sin p} = 2\frac{1 + x_p^- x_p^+}{x_p^- + x_p^+}. \qquad (2.21)
$$

The tail $\mathcal{T}$ is obtained by imposing commutation relations between $\mathbf{J}$ and the supercharges, and we find

$$
\begin{aligned}
&\mathcal{T} = \mathcal{T}_{\mathbf{H}\hat{\mathbf{B}}} + \mathcal{T}_{\mathbf{Mb}} + \mathcal{T}_{\mathrm{L}} + \mathcal{T}_{\mathrm{R}} + \mathcal{T}_{1},\\
&\mathcal{T}_{\mathbf{H}\hat{\mathbf{B}}} = \frac{1}{2}\frac{1}{w_1 - w_2}\left(1 - \tan\frac{p}{2}\otimes\tan\frac{p}{2}\right)(\mathbf{H}\otimes\hat{\mathbf{B}} + \hat{\mathbf{B}}\otimes\mathbf{H}),\\
&\mathcal{T}_{\mathbf{Mb}} = \frac{1}{8}\frac{w_1 + w_2}{w_1 - w_2}(\mathbf{M}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{M}),\\
&\mathcal{T}_{\mathrm{J}} = \frac{1}{2}\frac{w_1 + w_2}{w_1 - w_2}(\mathbf{K}^{-\frac{1}{4}}\mathbf{Q}_{\mathrm{J}}\otimes\mathbf{K}^{-\frac{1}{4}}\bar{\mathbf{Q}}_{\mathrm{J}} - \mathbf{K}^{\frac{1}{4}}\bar{\mathbf{Q}}_{\mathrm{J}}\otimes\mathbf{K}^{\frac{1}{4}}\mathbf{Q}_{\mathrm{J}}).
\end{aligned}
\tag{2.22}
$$

Notice the strong analogies with the $\mathrm{AdS}_5/\mathrm{CFT}_4$ result of Ref. [30] when looking at the bilinear piece in supercharges and the contribution with the secret symmetry $\hat{\mathbf{B}}$. In the fundamental representation the terms $\mathcal{T}_{\mathbf{H}\hat{\mathbf{B}}} + \mathcal{T}_{\mathbf{Mb}}$ mix, but we can distinguish them by studying the coproduct both in the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{L}}$ and in the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{R}}$ fundamental representations. One may check that the above coproduct is a homomorphism for the commutation relations with $\mathbf{J}$ in both such representations.

Commutation relations do not fix $\mathcal{T}_1$, the contribution to the tail which is proportional to the identity operator. At the same time, the freedom of choosing $\mathcal{T}_1$ may be used to make sure that $\Delta(\mathbf{J})$ is a symmetry of the R-matrix. For example, in the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{R}}$ fundamental representation we can check that[5]

$$
\begin{aligned}
&\Delta^{op}_{\mathrm{RL}}(\mathbf{J})R_{\mathrm{LR}} - R_{\mathrm{LR}}\Delta_{\mathrm{LR}}(\mathbf{J})\\
&\quad = i[(h_1 - s_{12})\partial_{p_1} + (h_2 + s_{12})\partial_{p_2}]R_{\mathrm{LR}}\\
&\qquad + \mathcal{T}^{op}_{\mathrm{RL}}R_{\mathrm{LR}} - R_{\mathrm{LR}}\mathcal{T}_{\mathrm{LR}}\\
&\quad = (f_{\mathrm{LR}} + \mathcal{T}^{op}_{1,\mathrm{RL}} - \mathcal{T}_{1,\mathrm{LR}})R_{\mathrm{LR}}.
\end{aligned}
\tag{2.23}
$$

Notice the appearance of both $\mathcal{T}_{1,\mathrm{LR}}$ and $\mathcal{T}_{1,\mathrm{RL}}$, because of the opposite coproduct. A similar equation with just the labels $\mathrm{L}\leftrightarrow\mathrm{R}$ swapped is obtained when considering the representation $\varrho_{\mathrm{R}}\otimes\varrho_{\mathrm{L}}$. If we impose LR symmetry[6] $R_{\mathrm{LR}} = R_{\mathrm{RL}}$ as in Refs. [65,68], we find $f_{\mathrm{LR}} = f_{\mathrm{RL}}$ and we may impose also $\mathcal{T}_{1,\mathrm{LR}} = \mathcal{T}_{1,\mathrm{RL}}$. The crucial point here is that f_{LR} is a *scalar* factor. We omit its explicit expression, which is not illuminating nor important for the discussion. Then boost invariance follows by taking $\mathcal{T}_{1,\mathrm{LR}} = f_{\mathrm{LR}}/2 + \mathcal{T}^{\mathrm{symm}}_{\mathrm{LR}}$, where $\mathcal{T}^{\mathrm{symm}}_{\mathrm{LR}}$ is a contribution symmetric under "op" which drops out from the equations. The computation proceeds similarly for the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{L}}$ representation, where one finds a corresponding scalar factor f_{LL}.

Obviously, from this point of view the solutions depend on the normalization of the R-matrix. In fact, if in the above example we had normalized the R-matrix with a different scalar factor $R'_{\mathrm{LR}} = e^{\Phi_{12}}R_{\mathrm{LR}}$, then boost invariance would translate to $(f_{\mathrm{LR}} + \mathcal{T}^{op}_{1,\mathrm{LR}} - \mathcal{T}_{1,\mathrm{LR}} + \mathbb{D}\Phi_{12}) = 0$, where $\mathbb{D}\equiv i(h_1 - s_{12})\partial_{p_1} + i(h_2 + s_{12})\partial_{p_2}$. In other words, the solution for $\mathcal{T}_1$ would be further shifted by $\frac{1}{2}\mathbb{D}\Phi_{12}$. This consideration should be taken into account when constructing the physical S-matrices for the $\mathrm{AdS}_3\times S^3\times T^4$ case[7] that include the dressing factors of Refs. [93,94].

It is natural to expect that there should be a *universal* form of $\mathcal{T}_1$, which should be valid independently of the representation that we consider. However, this does not mean that the above solutions $\mathcal{T}_{1,\mathrm{LL}}, \mathcal{T}_{1,\mathrm{LR}}$ found in the fundamental representation should coincide. In fact, $\mathcal{T}_1$ may receive contributions both from $\mathbf{H}$ and $\mathbf{M}$, which could be quite complicated; see e.g. the suggestion (5.9) towards a universal form of the other terms in the coproduct tail coming from the cobracket. Since their actions differ on L and R and their contributions mix, expressions in terms of $x^{\pm}_p$ could look quite different on $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{L}}$ and $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{R}}$. A possibility would be to inspect and compare the solutions for $\mathcal{T}_1$ in the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{L}}$ and in the $\varrho_{\mathrm{L}}\otimes\varrho_{\mathrm{R}}$ fundamental representations, when normalizing the R-matrix with the physical dressing factors, to see if the results suggest a universal form that evaluates as desired on both cases. We plan to return to this issue in the future.

A. Antipode

In this section we wish to determine the antipode of the boost $\mathbf{J}$. For all other generators $\mathbf{q}$, the antipode is implemented[8] by $S(\mathbf{q}(p)) = \mathcal{C}\mathbf{q}^{st}(\bar{p})\mathcal{C}^{-1}$, where $\mathcal{C}$ is the charge conjugation matrix, st denotes supertransposition and $\bar{p}$ is the analytic continuation of the momentum to the crossed region. In the representation $\varrho_{\mathrm{L}}\oplus\varrho_{\mathrm{R}} = \mathrm{span}\{\phi^{\mathrm{L}},\psi^{\mathrm{L}},\phi^{\mathrm{R}},\psi^{\mathrm{R}}\}$ we may choose

$$
\mathcal{C} = \sigma_1\otimes\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix},
\tag{2.24}
$$

which shows that charge conjugation is swapping the L and R representations. When crossing, we send $x^{\pm}\to 1/x^{\pm}$, with the caveat that we are more careful when dealing with the analytic continuation $\gamma_p\to -i(x^+_p)^{-1}(x^+_p/x^-_p)^{1/2}\gamma_p$.

[5]Here the subscripts LR and RL are used to denote the relevant representations.

[6]Imposing at the same time LR symmetry and (braiding and physical) unitarity singles out a particular class of normalization for the LR and RL blocks of the R-matrix; see Refs. [65,68]. A different normalization of the R-matrix results just in a shift of f_{LR} or f_{RL} as explained later.

[7]There is currently no proposal for the physical dressing factors that should solve the crossing equations of Refs. [65,67] in the $\mathrm{AdS}_3\times S^3\times S^3\times S^1$ case.

[8]In Eq. (B.11) of Ref. [68] the antipode is implemented differently on supercharges; see also Eq. (B.13). In that paper, one only looks at one two-dimensional representation (i.e. either L or R), and the swapping of L and R is therefore implemented on the labels of the supercharges rather than on the representations. Here we prefer to write the antipode formula in a more standard way. We still agree with Eq. (B.10) of Ref. [68].

Essentially, under crossing the coefficients a_p, b_p entering the definitions of the supercharges (2.4) transform as $a_p \to ib_p$ and $b_p \to ia_p$. With these prescriptions one finds that the antipode acts as $S(\mathbf{q}) = -\mathbf{q}$ on all supercharges $\mathbf{Q}_{\mathrm{I}}, \bar{\mathbf{Q}}_{\mathrm{I}}$, as well as generators $\mathbf{M}$, $\mathbf{H}$, $\mathbf{b}$ and $\hat{\mathbf{B}}$.

In order to find out how the antipode acts on $\mathbf{J}$ we follow the strategy of Ref. [30] and impose[9]

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(\mathbf{J}) = 0. \tag{2.25}$$

Let us separate the various contributions arising from the different terms that appear in the boost coproduct. The contribution related to $\Delta'(\mathbf{J})$ obviously does not differ from the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case [30]

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta'(\mathbf{J}) = \left(1 + \frac{\ell_p}{h_p}\right)(S(\mathbf{J}) + \mathbf{J}), \qquad \ell_p = \frac{g}{2} w_p^2 \left(\frac{dw_p}{dp}\right)^{-1} (\cos p - 1). \tag{2.26}$$

The tail of the boost coproduct contains factors of $(w_1 - w_2)^{-1}$ which potentially generate divergences when acting with the multiplication μ. As in $\mathrm{AdS}_5/\mathrm{CFT}_4$ we therefore need to carefully check that the divergences cancel in order to get a meaningful result. It is interesting to note that the piece of the tail $\mathcal{T}_{\mathbf{Mb}}$—which has no counterpart in $\mathrm{AdS}_5/\mathrm{CFT}_4$—is an essential ingredient in the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$, since without it the divergences would not cancel. When applying the multiplication μ we identify the two spaces appearing in the tensor product—where we have placed representations with the same mass m—and we take the limit $p_2 \to p_1$. We find

$$\begin{aligned}
\mu \circ (S \otimes \mathrm{id}) \circ \mathcal{T}_{\mathbf{H}\hat{\mathbf{B}}} &= \lim_{p_2 \to p_1} \frac{1}{w_1 - w_2} \left(-h_1 \hat{b}_1 \left(1 + \tan^2 \frac{p_1}{2}\right)\right)(\sigma_3 \oplus \sigma_3), \\
\mu \circ (S \otimes \mathrm{id}) \circ \mathcal{T}_{\mathbf{M}\hat{\mathbf{b}}} &= \lim_{p_2 \to p_1} \frac{1}{w_1 - w_2} \left(-\frac{m w_1}{2}\right)(\sigma_3 \oplus \sigma_3), \\
\mu \circ (S \otimes \mathrm{id}) \circ (\mathcal{T}_{\mathrm{L}} + \mathcal{T}_{\mathrm{R}}) &= \lim_{p_2 \to p_1} \frac{-w_1}{w_1 - w_2} (\mathbf{Q}_{\mathrm{L}} \bar{\mathbf{Q}}_{\mathrm{L}} - \bar{\mathbf{Q}}_{\mathrm{L}} \mathbf{Q}_{\mathrm{L}} + \mathbf{Q}_{\mathrm{R}} \bar{\mathbf{Q}}_{\mathrm{R}} - \bar{\mathbf{Q}}_{\mathrm{R}} \mathbf{Q}_{\mathrm{R}}) + \text{finite} \\
&= \lim_{p_2 \to p_1} \frac{1}{w_1 - w_2} m w_1 (\sigma_3 \oplus \sigma_3) + \text{finite}.
\end{aligned} \tag{2.27}$$

Here we wrote the secret symmetry as $\hat{\mathbf{B}} = \hat{b}_p(\sigma_3 \oplus \sigma_3)$. Since $w_p = \frac{2}{m} h_p \hat{b}_p (1 + \tan^2 \frac{p}{2})$, we find that all divergent terms cancel each other.

The piece of the tail containing the supercharges produces an additional finite contribution arising from the multiplication of factors of $\mathbf{K}$, which generate a factor of $(p_1 - p_2)$ canceling the pole. If we regularize $p_2 = p_1 + \epsilon$ and then take the limit $\epsilon \to 0$ we find that the finite contribution produced by $\mathcal{T}_{\mathrm{L}} + \mathcal{T}_{\mathrm{R}}$ is

$$\begin{aligned}
&\lim_{\epsilon \to 0} \frac{i\epsilon}{4} \left(1 - \frac{w(p_1 + \epsilon)}{w(p_1)}\right)^{-1} (\{\mathbf{Q}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{L}}\} + \{\mathbf{Q}_{\mathrm{R}}, \bar{\mathbf{Q}}_{\mathrm{R}}\}) = d_p \mathbf{1}, \\
&d_p \equiv -\frac{i}{4} w_p \left(\frac{dw_p}{dp}\right)^{-1} h_p.
\end{aligned} \tag{2.28}$$

Now that we have identified all the terms in the Eq. (2.25) we can solve it to determine the antipode of $\mathbf{J}$

$$S(\mathbf{J}) = -\mathbf{J} - \left(1 + \frac{\ell_p}{h_p}\right)^{-1} (c_p + d_p)\mathbf{1}. \tag{2.29}$$

The expression agrees with the one of $\mathrm{AdS}_5/\mathrm{CFT}_4$, except for a relative factor of 2 in the definition of d_p. We have included also a possible finite contribution c_p arising from the central part $\mathcal{T}_1$ of the tail of the boost coproduct.

Similarly to the discussion in Ref. [30], we remark that although we have solved Eq. (2.25), the equation where the antipode acts on the second space $\mu \circ (\mathrm{id} \otimes S) \circ \Delta(\mathbf{J}) = 0$ should hold as well. Following calculations similar to the above ones, in that case one would find $S(\mathbf{J}) = -\mathbf{J} - (1 + \ell_p/h_p)^{-1}(c'_p - d_p)\mathbf{1}$, where c'_p is the contribution from $\mathcal{T}_1$ which is possibly different from the previous c_p. Notice the change of sign in front of d_p. We conclude that we may have a consistent antipode on $\mathbf{J}$ only if the contribution of $\mathcal{T}_1$ is such that the two results agree. An analogous question was encountered in Ref. [30], and originally left unanswered. It has subsequently become clear that it is always possible to reverse engineer the tail of the boost coproduct to incorporate the contribution from a dressing phase which is a solution of the crossing equation.[10] The same argument applies in this context, which confirms that the boost, although not capable of

[9] This follows from one of the axioms of Hopf algebras $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mathbf{1} \circ \epsilon$ after setting $\epsilon(\mathbf{J}) = 0$.

[10] Cf. Ref. [30] (revision to appear).

constraining the dressing factor, is nevertheless a genuine symmetry of the complete S-matrix.[11]

It would be interesting to see whether it is possible to find such a $\mathcal{T}_1$, which at the same time makes sure that the boost coproduct is a symmetry of the R-matrix normalized with the physical dressing factors of Refs. [93,94].

III. SEMICLASSICAL LIMIT

We achieve the semiclassical limit by rescaling the generators $\mathbf{J} \to g\mathbf{J}$ and $\mathbf{P} \to \mathbf{P}/g$ and then taking $g \to \infty$. This corresponds to the BMN limit of Ref. [95], although from our point of view this is really a contraction of the algebra and not just of the representation. We obtain

$$\begin{aligned}
\{\mathbf{Q}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{L}}\} &= \frac{1}{2}(\mathbf{H} + \mathbf{M}), \\
\{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} &= -\frac{1}{2}\mathbf{P}, \\
\{\mathbf{Q}_{\mathrm{R}}, \bar{\mathbf{Q}}_{\mathrm{R}}\} &= \frac{1}{2}(\mathbf{H} - \mathbf{M}), \\
\{\bar{\mathbf{Q}}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{R}}\} &= -\frac{1}{2}\mathbf{P}, \\
[\mathbf{J}, \mathbf{H}] &= i\mathbf{P}, \\
[\mathbf{J}, \mathbf{Q}_{\mathrm{I}}] &= -\frac{i}{2}\bar{\mathbf{Q}}_{\bar{\mathrm{I}}}, \\
[\mathbf{J}, \mathbf{P}] &= i\mathbf{H}, \\
[\mathbf{J}, \bar{\mathbf{Q}}_{\mathrm{I}}] &= -\frac{i}{2}\mathbf{Q}_{\bar{\mathrm{I}}},
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
[\hat{\mathbf{B}}, \mathbf{Q}_{\mathrm{I}}] &= -\hat{\mathbf{Q}}_{\mathrm{I}} - 2\bar{\mathbf{Q}}_{\bar{\mathrm{I}}}, \\
[\hat{\mathbf{B}}, \bar{\mathbf{Q}}_{\mathrm{I}}] &= \hat{\bar{\mathbf{Q}}}_{\mathrm{I}} + 2\mathbf{Q}_{\bar{\mathrm{I}}}. \\
[\mathbf{b}, \mathbf{Q}_{\mathrm{L}}] &= -2\mathbf{Q}_{\mathrm{L}}, \\
[\mathbf{b}, \bar{\mathbf{Q}}_{\mathrm{L}}] &= +2\bar{\mathbf{Q}}_{\mathrm{L}}, \\
[\mathbf{b}, \mathbf{Q}_{\mathrm{R}}] &= +2\mathbf{Q}_{\mathrm{R}}, \\
[\mathbf{b}, \bar{\mathbf{Q}}_{\mathrm{R}}] &= -2\bar{\mathbf{Q}}_{\mathrm{R}},
\end{aligned} \tag{3.2}$$

which shows that the deformed algebra turns into a standard classical superalgebra. It contains in particular the Poincaré algebra in two dimensions (spanned by $\mathbf{P}$, $\mathbf{H}$, $\mathbf{J}$) as a subalgebra. There is a clear interpretation of the above limit at the level of the worldsheet. In fact, in the strict semiclassical limit only the quadratic part of the Hamiltonian in the light-cone gauge survives (see e.g. Ref. [2]), and the boost invariance on the worldsheet, which was broken by the gauge in the full Hamiltonian, is restored. One may therefore derive the corresponding Noether charge $\mathbf{J} = \int d\sigma(\sigma\mathcal{H} + \tau\mathcal{P})$, where $\mathbf{H} = \int d\sigma\mathcal{H}$, $\mathbf{P} = -\int d\sigma\mathcal{P}$, and σ, τ parametrize the worldsheet. The canonical quantization of the usual Poisson brackets will then reproduce the above commutation relations involving the boost. We refer to Ref. [30] for the explicit calculations in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case. Our findings concerning the deformed boost invariance at finite g suggest that the symmetry associated to $\mathbf{J}$ should be implemented nonlocally on the worldsheet, as indicated by the form of the coproduct.

The centrally extended $\mathfrak{su}(1|1)_{\mathrm{L}} \oplus \mathfrak{su}(1|1)_{\mathrm{R}}$ superalgebra in the semiclassical limit can be obtained as a contraction of $\mathfrak{sl}(1|2)$. The superalgebra $\mathfrak{sl}(1|2)$ is generated by 3×3 matrices M_{ij} with zeros everywhere except 1 at entry ij that are supertraceless $\mathrm{Str}(A) = A_{11} - A_{22} - A_{33} = 0$.

$$\begin{tikzcd}
U_q(\mathfrak{sl}(1|2)) \arrow[r, dashed, "?"'] \arrow[d, "q \to 1"'] & q\text{-Poincaré superalgebra} \arrow[d, "g \to \infty"] \\
\mathfrak{sl}(1|2) \arrow[r, "\varepsilon \to 0"'] & \text{Poincaré superalgebra}
\end{tikzcd}$$

A Serre-Chevalley basis for $\mathfrak{sl}(1|2)$ with both simple roots fermionic may be given by

$$\begin{aligned}
&\mathbf{e}_1 = M_{21}, \qquad \mathbf{f}_1 = M_{12}, \qquad \mathbf{h}_1 = M_{11} + M_{22}, \\
&\mathbf{e}_2 = -M_{13}, \qquad \mathbf{f}_2 = M_{31}, \qquad \mathbf{h}_2 = -M_{11} - M_{33},
\end{aligned} \tag{3.3}$$

so that

$$\begin{aligned}
&[\mathbf{h}_i, \mathbf{h}_j] = 0, \qquad [\mathbf{h}_i, \mathbf{e}_j] = a_{ij}\mathbf{e}_j, \\
&[\mathbf{h}_i, \mathbf{f}_j] = -a_{ij}\mathbf{f}_j, \qquad \{\mathbf{e}_i, \mathbf{f}_j\} = \delta_{ij}\mathbf{h}_i,
\end{aligned} \tag{3.4}$$

with a symmetric Cartan matrix $a_{ij} = (\sigma_1)_{ij}$. The two remaining generators may be found by taking $\mathbf{e}_{12} = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{f}_{12} = -\{\mathbf{f}_2, \mathbf{f}_1\}$. If we identify the above generators with

$$\begin{aligned}
\mathbf{Q}_{\mathrm{L}} &= \sqrt{\frac{\varepsilon}{2}}(\mathbf{f}_1 + i\mathbf{e}_2), & \mathbf{Q}_{\mathrm{R}} &= \sqrt{\frac{\varepsilon}{2}}(i\mathbf{e}_1 + \mathbf{f}_2), \\
\bar{\mathbf{Q}}_{\mathrm{L}} &= -\sqrt{\frac{\varepsilon}{2}}(\mathbf{e}_1 + i\mathbf{f}_2), & \bar{\mathbf{Q}}_{\mathrm{R}} &= -\sqrt{\frac{\varepsilon}{2}}(i\mathbf{f}_1 + \mathbf{e}_2), \\
\mathbf{H} &= i\varepsilon(-\mathbf{e}_{12} + \mathbf{f}_{12}), & \mathbf{P} &= -i\varepsilon(\mathbf{h}_1 + \mathbf{h}_2), \\
\mathbf{J} &= -\frac{i}{2}(\mathbf{e}_{12} + \mathbf{f}_{12}), & \mathbf{M} &= -\varepsilon(\mathbf{h}_1 - \mathbf{h}_2)
\end{aligned} \tag{3.5}$$

and then take $\varepsilon \to 0$ we indeed reproduce the (anti)commutation relations of the q-Poincaré superalgebra in the

[11] Access to a universal formulation of the boost coproduct would of course allow a first-principle derivation of the dressing phase; however this is not yet available, and a subject for future study.

semiclassical limit. Notice that we have been careful to identify $\mathbf{P}$ with a Cartan generator.

One may be tempted to construct $U_q(\mathfrak{sl}(1|2))$ and try to recover the q-Poincaré superalgebra under study as a contraction of $U_q(\mathfrak{sl}(1|2))$; in other words the idea would be that of closing the following diagram. In Ref. [30] it was shown that in the case of AdS_5/CFT_4—in that case $\mathfrak{sl}(1|2)$ is replaced by the $\mathfrak{d}(2,1;\alpha)$ superalgebra—the naive limits fail to achieve the desired contraction and to close the diagram corresponding to the one above. Here we are faced with the same mechanism. The problem lies in the fact that in the q-deformed case the (exponentials of the) Cartan elements will appear as

$$\varepsilon\frac{q^{\mathbf{h}_1\pm\mathbf{h}_2}-q^{-(\mathbf{h}_1\pm\mathbf{h}_2)}}{q-q^{-1}}, \tag{3.6}$$

where the explicit ε comes from the normalization of the generators. When considering the combination $\mathbf{h}_1+\mathbf{h}_2$ it appears natural to take $q=e^{w\varepsilon/2}$, so that factors of $e^{i\mathbf{P}}$ will naturally appear after taking the $\varepsilon\to 0$ limit. However, this would at the same time leave unwanted factors of $e^{\mathbf{M}}$ coming from $\mathbf{h}_1-\mathbf{h}_2$, which would prevent us from matching with the desired superalgebra.

We should note, however, that the current situation is much simpler than the AdS_5/CFT_4 case. There, in fact, the unwanted factors are exponentials of the Cartans of the $\mathfrak{su}(2)$ subalgebra, meaning that it is not obvious how to implement the semiclassical limit only at the level of these generators without spoiling other commutation relations. Here, instead, $\mathbf{M}$ is a central element of the superalgebra (after taking $\varepsilon\to 0$); in other words it appears only on the right-hand side of anticommutation relations. Therefore, it would be enough to define a new generator $\mathbf{M}'\equiv\frac{1}{w}(e^{\frac{w}{2}\mathbf{M}}-e^{-\frac{w}{2}\mathbf{M}})$ to mimic the wanted (anti)commutation relations, where $\mathbf{M}$ is replaced by $\mathbf{M}'$. Although this trick seems to work at the level of commutation relations, we do not expect that it will go through when including also the coproducts.

IV. UNIVERSAL r-MATRIX

In this section we wish to construct a universal classical r-matrix for AdS_3/CFT_2. Besides its intrinsic importance, it will also be a necessary tool for the next section, where we will use it to compute the cobracket of the various generators, in particular the boost.

A. Universal r-matrix and the CYBE

We want the r-matrix to agree with the semiclassical expansion of the quantum R-matrix given in Eqs. (2.11) and (2.12), i.e. in the $g\to\infty$ limit we should have

$$R=1+g^{-1}(r+r_0)+\mathcal{O}(g^{-2}),\qquad r_0=\phi_0\mathbf{1}\otimes\mathbf{1}. \tag{4.1}$$

The part proportional to the identity, r_0, is sensitive to the normalization and we will not consider it. To take the semiclassical limit in the fundamental representation we rewrite[12]

$$x^{\pm}=x\left(\sqrt{1-\frac{m^2x^2}{g^2(x^2-1)^2}}\pm\frac{imx}{g(1-x^2)}\right), \tag{4.2}$$

and send $g\to\infty$. After rewriting the semiclassical expansion of the quantum R-matrix in terms of the semiclassical spectral parameter u [related to x as $u=x+1/x, x=\frac{1}{2}(u+\sqrt{u^2-4})$] we find that it can be written as

$$r=\frac{-i}{u_1-u_2}\Bigg[2\sum_{\mathrm{I=L,R}}(\mathbf{Q}_\mathrm{I}\otimes\bar{\mathbf{Q}}_\mathrm{I}-\bar{\mathbf{Q}}_\mathrm{I}\otimes\mathbf{Q}_\mathrm{I})\\+\frac{u_2}{u_1}\mathbf{H}\otimes\mathbf{B}_0+\frac{u_1}{u_2}\mathbf{B}_0\otimes\mathbf{H}+\frac{1}{2}(\mathbf{M}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{M})\Bigg]. \tag{4.3}$$

All the generators appearing above are assumed to be written in the semiclassical limit. Moreover, $\mathbf{B}_0$ corresponds to the level 0 of the secret symmetry, so that $\mathbf{B}_0\sim u^{-1}\hat{\mathbf{B}}$. Crucially, the above expression matches with the semiclassical expansion of R both in the $\varrho_\mathrm{L}\otimes\varrho_\mathrm{L}$ and in the $\varrho_\mathrm{L}\otimes\varrho_\mathrm{R}$ representations.[13]

We will interpret the above result as the r-matrix in the *evaluation representation*. If we assume that it comes from a universal expression after identifying the charges at each level n as $\mathbf{q}_n=u^n\mathbf{q}$, it is easy to reverse engineer a candidate form for the *universal r-matrix*

$$\begin{aligned}
r&=-i\left(2r_\mathrm{L}+2r_\mathrm{R}+r_{\mathbf{HB}}+\frac{1}{2}r_{\mathbf{Mb}}\right),\\
r_\mathrm{I}&=\sum_{n=0}^{\infty}(\mathbf{Q}_{\mathrm{I},-1-n}\otimes\bar{\mathbf{Q}}_{\mathrm{I},n}-\bar{\mathbf{Q}}_{\mathrm{I},-1-n}\otimes\mathbf{Q}_{\mathrm{I},n}),\\
r_{\mathbf{HB}}&=\sum_{n=-1}^{\infty}\mathbf{B}_{-1-n}\otimes\mathbf{H}_n+\sum_{n=1}^{\infty}\mathbf{H}_{-1-n}\otimes\mathbf{B}_n,\\
r_{\mathbf{Mb}}&=\sum_{n=0}^{\infty}(\mathbf{M}_{-1-n}\otimes\mathbf{b}_n+\mathbf{b}_{-1-n}\otimes\mathbf{M}_n).
\end{aligned} \tag{4.4}$$

Although the above proposal for a universal expression matches with the known results, it is important to further test it by checking whether it satisfies the classical Yang-Baxter equation (CYBE) without specifying any

[12] The semiclassical limit for massless representations should be taken with some care. See the end of this section for a discussion on this.

[13] In fact, the terms $\mathbf{B}_0\otimes\mathbf{H}$ and $\mathbf{b}\otimes\mathbf{M}$ mix, but they can be distinguished by comparing the expansion of the R-matrix both in the $\varrho_\mathrm{L}\otimes\varrho_\mathrm{L}$ and in the $\varrho_\mathrm{L}\otimes\varrho_\mathrm{R}$ representations.

representation. To do that we will follow the strategy used in Ref. [16] to check the CYBE for the universal r-matrix of AdS_5/CFT_4. We start by noticing that the above r-matrix may be rewritten as

$$r = r^{\mathrm{can}} + \mathsf{r}, \qquad \mathsf{r} \equiv -i(\mathbf{B}_0 \otimes \mathbf{H}_{-1} - \mathbf{H}_{-1} \otimes \mathbf{B}_0), \quad (4.5)$$

where we interpret r^{can} as the canonical universal r-matrix of the loop algebra $\mathfrak{u}(1|1)_{\mathrm{L}} \oplus \mathfrak{u}(1|1)_{\mathrm{R}}$. This superalgebra is spanned by the supercharges $\mathbf{Q}_{\mathrm{I}}, \bar{\mathbf{Q}}_{\mathrm{I}}$, $\mathrm{I} = \mathrm{L}, \mathrm{R}$, the central elements $\mathbf{H}$, $\mathbf{M}$ and the inner automorphisms $\mathbf{B}_0$, $\mathbf{b}$, which are linear combinations of the inner automorphisms acting separately on the two copies of $\mathfrak{u}(1|1)$. The universal r-matrix of the loop algebra is built according to the generic construction as Refs. [96,97]

$$r^{\mathrm{can}} = -i \sum_{n=0}^{\infty} T^A_{-1-n} \otimes T^B_n g_{AB}, \quad (4.6)$$

where T^A_n are the generators at level n, and g_{AB} is the (inverse of) an invariant nondegenerate bilinear form.[14] To reproduce our r we take

$$g(\mathbf{Q}_{\mathrm{I}}, \bar{\mathbf{Q}}_{\mathrm{J}}) = -\frac{1}{2}\delta_{\mathrm{IJ}}, \quad g(\mathbf{H}, \mathbf{B}) = 1, \quad g(\mathbf{M}, \mathbf{b}) = 2, \quad (4.7)$$

and one may check that the above bilinear form is invariant and nondegenerate on $\mathfrak{u}(1|1)_{\mathrm{L}} \oplus \mathfrak{u}(1|1)_{\mathrm{R}}$. In what follows we will actually consider a *deformation* of the loop algebra of $\mathfrak{u}(1|1)_{\mathrm{L}} \oplus \mathfrak{u}(1|1)_{\mathrm{R}}$, as suggested by the strategy of Ref. [16]. We write the (anti)commutation relations as

$$\begin{aligned}
\{\mathbf{Q}_{\mathrm{L},m}, \bar{\mathbf{Q}}_{\mathrm{L},n}\} &= \frac{1}{2}(\mathbf{H}_{m+n} + \mathbf{M}_{m+n}),\\
\{\mathbf{Q}_{\mathrm{L},m}, \mathbf{Q}_{\mathrm{R},n}\} &= -\beta \mathbf{H}_{m+n-1},\\
\{\mathbf{Q}_{\mathrm{R},m}, \bar{\mathbf{Q}}_{\mathrm{R},n}\} &= \frac{1}{2}(\mathbf{H}_{m+n} - \mathbf{M}_{m+n}),\\
\{\bar{\mathbf{Q}}_{\mathrm{L},m}, \bar{\mathbf{Q}}_{\mathrm{R},n}\} &= -\beta \mathbf{H}_{m+n-1},
\end{aligned} \quad (4.8)$$

and

$$\begin{aligned}
[\mathbf{b}_m, \mathbf{Q}_{\mathrm{L},n}] &= -2\mathbf{Q}_{\mathrm{L},m+n},\\
[\mathbf{b}_m, \bar{\mathbf{Q}}_{\mathrm{L},n}] &= +2\bar{\mathbf{Q}}_{\mathrm{L},m+n},\\
[\mathbf{b}_m, \mathbf{Q}_{\mathrm{R},n}] &= +2\mathbf{Q}_{\mathrm{R},m+n},\\
[\mathbf{b}_m, \bar{\mathbf{Q}}_{\mathrm{R},n}] &= -2\bar{\mathbf{Q}}_{\mathrm{R},m+n},\\
[\mathbf{B}_m, \mathbf{Q}_{\mathrm{I},n}] &= -\mathbf{Q}_{\mathrm{I},m+n} - 2\beta \bar{\mathbf{Q}}_{\bar{\mathrm{I}},m+n-1},\\
[\mathbf{B}_m, \bar{\mathbf{Q}}_{\mathrm{I},n}] &= +\bar{\mathbf{Q}}_{\mathrm{I},m+n} + 2\beta \mathbf{Q}_{\bar{\mathrm{I}},m+n-1}.
\end{aligned} \quad (4.9)$$

The undeformed loop algebra is recovered at $\beta = 0$. When setting $\beta = 1$, instead, we reproduce the superalgebra that is of interest to us; in particular, the commutators involving the secret symmetry $\mathbf{B}$ reduce to the ones in Eq. (3.2). To match them we also need the identification $\mathbf{H}_{-1} \sim \frac{1}{2}\mathbf{P}$.

We will now prove that r satisfies the CYBE at $\beta = 1$; we will actually prove it for generic β. We use the fact that r^{can} satisfies the CYBE at $\beta = 0$; therefore there are only two types of additional contributions to compute:

(1) Those proportional to β (coming from deformed commutators) when computing

$$[r^{\mathrm{can}}_{12}, r^{\mathrm{can}}_{13}] + [r^{\mathrm{can}}_{13}, r^{\mathrm{can}}_{23}] + [r^{\mathrm{can}}_{12}, r^{\mathrm{can}}_{23}]. \quad (4.10)$$

(2) Those coming from the "mixed terms"

$$\begin{aligned}
&[\mathsf{r}_{12}, r^{\mathrm{can}}_{13}] + [\mathsf{r}_{13}, r^{\mathrm{can}}_{23}] + [\mathsf{r}_{12}, r^{\mathrm{can}}_{23}] + [r^{\mathrm{can}}_{12}, r^{\mathrm{can}}_{13}]\\
&\quad + [r^{\mathrm{can}}_{13}, \mathsf{r}_{23}] + [r^{\mathrm{can}}_{12}, \mathsf{r}_{23}].
\end{aligned} \quad (4.11)$$

Notice that terms of the form $[\mathsf{r}, \mathsf{r}]$ are automatically 0 since $\mathbf{B}_m$ and $\mathbf{H}_m$ commute. For contributions of type 1 we find

$$\begin{aligned}
[r^{\mathrm{can}}_{12}, r^{\mathrm{can}}_{13}]&: \ -4\beta \sum_{m,n=0}^{\infty} \mathcal{X}_{[-3-n-m,n,m]},\\
[r^{\mathrm{can}}_{13}, r^{\mathrm{can}}_{23}]&: \ -4\beta \sum_{m,n=0}^{\infty} \mathcal{X}_{[-1-n,-1-m,m+n-1]},\\
[r^{\mathrm{can}}_{12}, r^{\mathrm{can}}_{23}]&: \ +4\beta \sum_{m,n=0}^{\infty} \mathcal{X}_{[-1-n,n-m-2,m]},
\end{aligned} \quad (4.12)$$

where we defined

$$\begin{aligned}
\mathcal{X}_{[n_1,n_2,n_3]} \equiv (&\mathbf{H}_{n_1} \otimes \mathbf{Q}_{\mathrm{L},n_2} \otimes \mathbf{Q}_{\mathrm{R},n_3} + \mathbf{Q}_{\mathrm{L},n_1} \otimes \mathbf{Q}_{\mathrm{R},n_2} \otimes \mathbf{H}_{n_3}\\
&- \mathbf{Q}_{\mathrm{L},n_1} \otimes \mathbf{H}_{n_2} \otimes \mathbf{Q}_{\mathrm{R},n_3} + \mathrm{L} \leftrightarrow \mathrm{R}) + \mathbf{Q} \leftrightarrow \bar{\mathbf{Q}}.
\end{aligned} \quad (4.13)$$

To avoid long expressions, here we are not writing all the terms explicitly. For each term that we write explicitly there are three additional ones, obtained by first exchanging the labels $\mathrm{L} \leftrightarrow \mathrm{R}$, and then $\mathbf{Q} \leftrightarrow \bar{\mathbf{Q}}$ everywhere. Summing up the above results we obtain

[14] In our conventions $[\![T^A, T^B]\!] = f^{AB}_C T^C$, where $[\![,]\!]$ denotes a (anti)commutator. We also define the metric as $g^{AB} = g(T^A, T^B)$, so that g_{AB} is the inverse metric. The metric is symmetric in the block of bosonic generators, while it is antisymmetric in the block of fermionic generators. The Killing form of $\mathfrak{u}(1|1)_{\mathrm{L}} \oplus \mathfrak{u}(1|1)_{\mathrm{R}}$ is degenerate.

$$-4\beta\left(\sum_{m=0}^{\infty}\sum_{n=m+2}^{\infty}+\sum_{n=0}^{\infty}\sum_{m=n-1}^{\infty}-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\right)\mathcal{X}_{[-1-n,n-m-2,m]}=-4\beta\mathcal{X}_{[-1,-1,-1]}, \tag{4.14}$$

where we first relabeled the summed indices, and then used the identity

$$\left(\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}+\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\right)F_{mn}=0, \tag{4.15}$$

which is valid due to the cancellation of the domains for any collection of objects F_{mn} labeled by m and n. We will now show that $-4\beta\mathcal{X}_{[-1,-1,-1]}$ is exactly canceled by the contributions of type 2. We find

$$\begin{aligned}
[\mathsf{r}_{12},r^{\text{can}}_{13}]:&\ +\sum_{n=0}^{\infty}(2\mathbf{Q}_{\mathrm{L},-1-n}\otimes\mathbf{H}_{-1}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}+4\beta\mathbf{Q}_{\mathrm{L},-2-n}\otimes\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{R},n}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}},\\
[r^{\text{can}}_{12},\mathsf{r}_{13}]:&\ -\sum_{n=0}^{\infty}(2\mathbf{Q}_{\mathrm{L},-1-n}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}\otimes\mathbf{H}_{-1}+4\beta\mathbf{Q}_{\mathrm{L},-2-n}\otimes\mathbf{Q}_{\mathrm{R},n}\otimes\mathbf{H}_{-1}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}},\\
[\mathsf{r}_{13},r^{\text{can}}_{23}]:&\ +\sum_{n=0}^{\infty}(2\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{L},-1-n}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}+4\beta\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{L},-1-n}\otimes\mathbf{Q}_{\mathrm{R},n-1}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}},\\
[r^{\text{can}}_{13},\mathsf{r}_{23}]:&\ -\sum_{n=0}^{\infty}(2\mathbf{Q}_{\mathrm{L},-1-n}\otimes\mathbf{H}_{-1}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}+4\beta\mathbf{Q}_{\mathrm{L},-1-n}\otimes\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{R},n-1}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}},\\
[\mathsf{r}_{12},r^{\text{can}}_{23}]:&\ -\sum_{n=0}^{\infty}(2\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{L},-1-n}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}+4\beta\mathbf{H}_{-1}\otimes\mathbf{Q}_{\mathrm{L},-2-n}\otimes\mathbf{Q}_{\mathrm{R},n}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}},\\
[r^{\text{can}}_{12},\mathsf{r}_{23}]:&\ +\sum_{n=0}^{\infty}(2\mathbf{Q}_{\mathrm{L},-1-n}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n}\otimes\mathbf{H}_{-1}+4\beta\mathbf{Q}_{\mathrm{L},-1-n}\otimes\mathbf{Q}_{\mathrm{R},n-1}\otimes\mathbf{H}_{-1}+\mathrm{L}\leftrightarrow\mathrm{R})+\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}}.
\end{aligned} \tag{4.16}$$

It is easy to see that all β-independent terms cancel each other, while the β-dependent ones leave a finite result due to some shifts in the levels in some expressions. The result

$$4\beta\sum_{n=0}^{\infty}(\mathcal{X}_{[-1,-1-n,n-1]}-\mathcal{X}_{[-1,-2-n,n]})=4\beta\mathcal{X}_{[-1,-1,-1]} \tag{4.17}$$

exactly cancels the contributions of type 1, and the CYBE is checked for generic β. Notice that, in order for the calculation to work, it was crucial to have shifts of -1 in the levels in the β-dependent terms of the (anti)commutation relations, as well as the additional r.

B. Massless representations, semiclassical limit and the r-matrix

The parametrization (4.2) of the Zhukovski variables is not adequate in the massless limit $m\to 0$, since it would imply $x^+=x^-$ and $p=2\pi n$. A different parametrization is therefore needed in the massless case, and we can find it e.g. by sending $m\to 0$ only after redefining $x=1+\frac{m}{2\xi}$ (or $x=-1-\frac{m}{2\xi}$) in Eq. (4.2). We find

$$x^{\pm}=\pm\frac{i\xi}{g}+\sqrt{1-\frac{\xi^2}{g^2}},\quad\text{or}\quad x^{\pm}=\pm\frac{i\xi}{g}-\sqrt{1-\frac{\xi^2}{g^2}}, \tag{4.18}$$

where the first parametrization implies[15] $p>0$ while the second one implies $p<0$. Therefore, we need to distinguish between *worldsheet left- and right-movers*. In both cases the energy is $2g\sin(p/2)=2\xi$. The coefficients parametrizing the supercharges in Eq. (2.4) are just $a_p=-b_p=\sqrt{\xi}$ in the first parametrization, and $a_p=+b_p=\sqrt{\xi}$ in the second one. Let us emphasize that we have not taken the $g\to\infty$ limit yet. Notice that the secret symmetry in Eq. (2.16) vanishes in the massless limit, since $x^+=1/x^-$ when $m=0$. Furthermore, the spectral parameter $\hat{u}=(x^++x^-+1/x^++1/x^-)/2$ reduces to $\pm 2\sqrt{1-\xi^2/g^2}$, where the sign $\pm$ depends on which of the above parametrizations is chosen. Therefore, semiclassically $\hat{u}\to u=\pm 2$.

Let us make a comment on the g dependence. In the massless case we may parametrize $x^{\pm}=e^{\pm ip/2}$, so that there is no explicit g dependence. This is not a good

[15]Here we assume $-\pi<p<\pi$.

parametrization if we want to take a semiclassical limit $g \to \infty$, since for example the massless-massless R-matrix would not expand as $1 + \mathcal{O}(1/g)$. If instead we use the parametrization above in terms of ξ, we reintroduce the missing g dependence, and it makes sense to expand our results at large g. This is similar to what one does in the BMN limit [95], where one first rescales $p \to p/g$.

Let us now discuss the semiclassical limit of the R-matrix. First we consider the case of massless-massive scattering, where the mass of the second excitation is generic but not 0. To obtain the classical r-matrix in the fundamental representation we first consider the massless-massive R-matrix, where $x_1^\pm$ are parametrized in terms of Eq. (4.18) and $x_2^\pm$ in terms of Eq. (4.2). Then we send $g \to \infty$ and we obtain $R = 1 + r/g + \mathcal{O}(1/g^2)$. We have checked that what we obtain coincides with the r-matrix in the evaluation representation as written in Eq. (4.3).

Particular care is needed when taking the semiclassical limit in the case of massless-massless scattering. In fact, we must scatter a left- with a right-mover, i.e. we must use the first parametrization in Eq. (4.18) for one excitation and the second one for the other. From the operational point of view, this is done to avoid the appearance of infinities. Physically it is justified by the fact that we want the two massless excitations to travel in opposite directions, so that they have the chance to meet, since they both go at the speed of light. Then we extract the classical r-matrix from the semiclassical expansion of the massless-massless R-matrix, $R = 1 + r/g + \mathcal{O}(1/g^2)$. We get $r = -i\sqrt{\xi_1}\sqrt{\xi_2}M$, where $M = \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+$. Also this result matches with the classical r-matrix written in the evaluation representation in Eq. (4.3), and this can be seen quite simply. In fact, all terms in r containing $\mathbf{M}$ or $\mathbf{B}$ obviously vanish. The only contributions come from the supercharges, and

$$\mathbf{Q}_\mathrm{L} \otimes \bar{\mathbf{Q}}_\mathrm{L} - \bar{\mathbf{Q}}_\mathrm{L} \otimes \mathbf{Q}_\mathrm{L} + \mathbf{Q}_\mathrm{R} \otimes \bar{\mathbf{Q}}_\mathrm{R} - \bar{\mathbf{Q}}_\mathrm{R} \otimes \mathbf{Q}_\mathrm{R}$$
$$= (a_1\bar{a}_2 - \bar{b}_1 b_2)\sigma_- \otimes \sigma_+ - (\bar{a}_1 a_2 - b_1\bar{b}_2)\sigma_+ \otimes \sigma_-, \tag{4.19}$$

where we have used the parametrization coefficients as in Eq. (2.4). Now it is crucial that we are taking two massless excitations in opposite kinematical regimes, i.e. $a_1 = -b_1$ and $a_2 = +b_2$. Recalling that in our parametrization we have *real* coefficients ($\bar{a} = a, \bar{b} = b$), this means that the contributions add up instead of canceling $(a_1\bar{a}_2 - \bar{b}_1 b_2) = 2a_1 a_2 = (\bar{a}_1 a_2 - b_1\bar{b}_2)$. Using $u_1 = -u_2 = 2$ we obtain $r = -i\sqrt{\xi_1}\sqrt{\xi_2}M$ as wanted.

V. COBRACKET

Although the majority of the results—e.g. the R-matrix in Eqs. (2.11) and (2.12) and the coproducts in Eqs. (2.10) and (2.19)—are only given in the fundamental representation, the universal r-matrix proposed in the previous section allows us to go towards a universal formulation. In particular, if we consider the invariance of the R-matrix under a generic generator $\mathbf{q}$ as in Eq. (2.9) and we implement a semiclassical expansion, we find that $\delta(\mathbf{q}) \equiv \Delta_{(1)}(\mathbf{q}) - \Delta_{(1)}^{op}(\mathbf{q})$ may be obtained by computing the commutator $\delta(\mathbf{q}) = [\mathbf{q} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{q}, r]$. One gets this result after expanding the coproduct as $\Delta(\mathbf{q}) = \mathbf{q} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{q} + g^{-1}\Delta_{(1)}(\mathbf{q}) + \mathcal{O}(g^{-2})$. In other words $\delta(\mathbf{q})$, which we call the cobracket of $\mathbf{q}$, can be derived in universal form thanks to the knowledge of the universal r-matrix.

We present the results for the cobrackets of all the generators of the deformed loop algebra $\mathfrak{u}(1|1)_\mathrm{L} \oplus \mathfrak{u}(1|1)_\mathrm{R}$ of the previous section. In universal form they read

$$\delta(\mathbf{Q}_{\mathrm{L},m}) = i\sum_{n=0}^{m}[\mathbf{H}_{m-n-1} \otimes \mathbf{Q}_{\mathrm{L},n} - \mathbf{Q}_{\mathrm{L},m-n} \otimes \mathbf{H}_{n-1}] + i\sum_{n=0}^{m-1}[2\beta(\mathbf{H}_{m-n-2} \otimes \bar{\mathbf{Q}}_{\mathrm{R},n} - \bar{\mathbf{Q}}_{\mathrm{R},m-n-1} \otimes \mathbf{H}_{n-1})$$
$$+ \mathbf{M}_{m-n-1} \otimes \mathbf{Q}_{\mathrm{L},n} - \mathbf{Q}_{\mathrm{L},m-n-1} \otimes \mathbf{M}_n], \tag{5.1}$$

$$\delta(\bar{\mathbf{Q}}_{\mathrm{L},m}) = -i\sum_{n=0}^{m}[\mathbf{H}_{m-n-1} \otimes \bar{\mathbf{Q}}_{\mathrm{L},n} - \bar{\mathbf{Q}}_{\mathrm{L},m-n} \otimes \mathbf{H}_{n-1}] - i\sum_{n=0}^{m-1}[2\beta(\mathbf{H}_{m-n-2} \otimes \mathbf{Q}_{\mathrm{R},n} - \mathbf{Q}_{\mathrm{R},m-n-1} \otimes \mathbf{H}_{n-1})$$
$$+ \mathbf{M}_{m-n-1} \otimes \bar{\mathbf{Q}}_{\mathrm{L},n} - \bar{\mathbf{Q}}_{\mathrm{L},m-n-1} \otimes \mathbf{M}_n], \tag{5.2}$$

$$\delta(\mathbf{Q}_{\mathrm{R},m}) = i\sum_{n=0}^{m}[\mathbf{H}_{m-n-1} \otimes \mathbf{Q}_{\mathrm{R},n} - \mathbf{Q}_{\mathrm{R},m-n} \otimes \mathbf{H}_{n-1}] + i\sum_{n=0}^{m-1}[2\beta(\mathbf{H}_{m-n-2} \otimes \bar{\mathbf{Q}}_{\mathrm{L},n} - \bar{\mathbf{Q}}_{\mathrm{L},m-n-1} \otimes \mathbf{H}_{n-1})$$
$$- \mathbf{M}_{m-n-1} \otimes \mathbf{Q}_{\mathrm{R},n} + \mathbf{Q}_{\mathrm{R},m-n-1} \otimes \mathbf{M}_n], \tag{5.3}$$

$$\delta(\bar{\mathbf{Q}}_{\mathrm{R},m}) = -i\sum_{n=0}^{m}[\mathbf{H}_{m-n-1} \otimes \bar{\mathbf{Q}}_{\mathrm{R},n} - \bar{\mathbf{Q}}_{\mathrm{R},m-n} \otimes \mathbf{H}_{n-1}] - i\sum_{n=0}^{m-1}[2\beta(\mathbf{H}_{m-n-2} \otimes \mathbf{Q}_{\mathrm{L},n} - \mathbf{Q}_{\mathrm{L},m-n-1} \otimes \mathbf{H}_{n-1})$$
$$- \mathbf{M}_{m-n-1} \otimes \bar{\mathbf{Q}}_{\mathrm{R},n} + \bar{\mathbf{Q}}_{\mathrm{R},m-n-1} \otimes \mathbf{M}_n], \tag{5.4}$$

$$\delta(\mathbf{b}_m) = 4i\sum_{n=0}^{m-1}[\mathbf{Q}_{\mathrm{L},m-n-1}\otimes\bar{\mathbf{Q}}_{\mathrm{L},n} - \mathbf{Q}_{\mathrm{R},m-n-1}\otimes\bar{\mathbf{Q}}_{\mathrm{R},n} + \bar{\mathbf{Q}}_{\mathrm{L},m-n-1}\otimes\mathbf{Q}_{\mathrm{L},n} - \bar{\mathbf{Q}}_{\mathrm{R},m-n-1}\otimes\mathbf{Q}_{\mathrm{R},n}], \tag{5.5}$$

$$\delta(\mathbf{B}_m) = 2i\sum_{\mathrm{I=L,R}}\sum_{n=0}^{m-1}[\mathbf{Q}_{\mathrm{I},m-n-1}\otimes\bar{\mathbf{Q}}_{\mathrm{I},n} + \bar{\mathbf{Q}}_{\mathrm{I},m-n-1}\otimes\mathbf{Q}_{\mathrm{I},n}] + 4i\beta\sum_{\mathrm{I=L,R}}\sum_{n=0}^{m-2}[\mathbf{Q}_{\bar{\mathrm{I}},m-n-2}\otimes\mathbf{Q}_{\mathrm{I},n} + \bar{\mathbf{Q}}_{\bar{\mathrm{I}},m-n-2}\otimes\bar{\mathbf{Q}}_{\mathrm{I},n}]. \tag{5.6}$$

Notice that the cobrackets of the barred supercharges are obtained through complex conjugation of their nonbarred correspondences, together with the exchange $\mathbf{Q}_\mathrm{I}\leftrightarrow\bar{\mathbf{Q}}_\mathrm{I}$. Also, the signs of the terms involving the generator $\mathbf{M}$ keep us from easily writing the cobrackets of the supercharges in the more compact forms $\delta(\mathbf{Q}_{\mathrm{I},m})$ and $\delta(\bar{\mathbf{Q}}_{\mathrm{I},m})$. Since $\mathbf{M}_m$, $\mathbf{H}_m$ are central elements of the loop algebra of $\mathfrak{u}(1|1)_\mathrm{L}\oplus\mathfrak{u}(1|1)_\mathrm{R}$, their cobrackets are trivial.

A. Cobracket of the boost

We now wish to compute the cobracket of the boost $\delta(\mathbf{J}) = [\mathbf{J}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{J}, r]$. As in Ref. [30] we use

$$[\mathbf{J},\mathbf{B}_0] = -2i\mathbf{B}_{-1}, \tag{5.7}$$

which is motivated by the fundamental representation. Moreover, if we define the action of the boost on a generic generator $\mathbf{q}_0$ at level 0 as $\tilde{\mathbf{q}}_0\equiv[\mathbf{J},\mathbf{q}_0]$, we will assume that the boost acts on the level n $\mathbf{q}_n\sim u^n\mathbf{q}_0$ as

$$[\mathbf{J},\mathbf{q}_n] = \tilde{\mathbf{q}}_n + in\left(2\mathbf{q}_{n-1} - \frac{1}{2}\mathbf{q}_{n+1}\right), \tag{5.8}$$

where $\tilde{\mathbf{q}}_n\sim u^n\tilde{\mathbf{q}}_0$. This commutator is justified by the result in the evaluation representation. In universal form we find

$$\begin{aligned}
\delta(\mathbf{J}) &= -i\left(2\delta_\mathrm{L}(\mathbf{J}) + 2\delta_\mathrm{R}(\mathbf{J}) + \delta_{\mathbf{HB}}(\mathbf{J}) + \frac{1}{2}\delta_{\mathbf{Mb}}(\mathbf{J})\right),\\
\delta_\mathrm{I}(\mathbf{J}) &= i\sum_{m=0}^{\infty}[\mathbf{Q}_{\mathrm{I},-m}\otimes\bar{\mathbf{Q}}_{\mathrm{I},m} - \bar{\mathbf{Q}}_{\mathrm{I},-m}\otimes\mathbf{Q}_{\mathrm{I},m}] - \frac{i}{2}[\mathbf{Q}_{\mathrm{I},0}\otimes\bar{\mathbf{Q}}_{\mathrm{I},0} - \bar{\mathbf{Q}}_{\mathrm{I},0}\otimes\mathbf{Q}_{\mathrm{I},0}],\\
\delta_{\mathbf{HB}}(\mathbf{J}) &= i\left(\sum_{m=0}^{\infty}\mathbf{B}_{-m}\otimes\mathbf{H}_m + \sum_{m=1}^{\infty}\mathbf{H}_{-m}\otimes\mathbf{B}_m\right),\\
\delta_{\mathbf{Mb}}(\mathbf{J}) &= i\sum_{m=0}^{\infty}[\mathbf{M}_{-m}\otimes\mathbf{b}_m + \mathbf{b}_{-m}\otimes\mathbf{M}_m] - \frac{i}{2}[\mathbf{M}_0\otimes\mathbf{b}_0 + \mathbf{b}_0\otimes\mathbf{M}_0].
\end{aligned} \tag{5.9}$$

As expected, $\delta_{\mathbf{HB}}(\mathbf{J})$ is identical[16] to the case of AdS_5/CFT_4, and one may notice close similarities also in the contributions with the supercharges. After going to the evaluation representation we obtain

$$\begin{aligned}
\delta(\mathbf{J}) = \frac{u_1+u_2}{u_1-u_2}&\left(\sum_{\mathrm{I=L,R}}(\mathbf{Q}_\mathrm{I}\otimes\bar{\mathbf{Q}}_\mathrm{I} - \bar{\mathbf{Q}}_\mathrm{I}\otimes\mathbf{Q}_\mathrm{I})\right.\\
&\left.+\frac{1}{4}(\mathbf{M}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{M})\right) + \frac{1}{u_1-u_2}(\mathbf{H}\otimes\mathbf{B}_1\\
&+\mathbf{B}_1\otimes\mathbf{H}).
\end{aligned} \tag{5.10}$$

There are obvious analogies between the cobracket and the exact coproduct given in Eqs. (2.19), (2.20) and (2.22). As in the case of AdS_5/CFT_4, the result suggests that the semiclassical spectral parameter u_i is replaced at the quantum level by w_i. Certain terms in the exact coproduct $\Delta(\mathbf{J})$ are not captured by the cobracket, either because they start entering at orders higher than $1/g$ (e.g. the contribution with $\tan\frac{p}{2}\otimes\tan\frac{p}{2}$ in $\mathcal{T}_{\mathbf{H}\hat{\mathbf{B}}}$) or because they are symmetric under the action of "op" [e.g. the correction to the trivial coproduct in $\Delta'(\mathbf{J})$].

[16]In the above expression we have already used the identification $\mathbf{P}_n\sim 2\mathbf{H}_{n-1}$. We refer to Ref. [30] for the expression of $\delta_{\mathbf{HB}}(\mathbf{J})$ before this identification.

VI. CONCLUSIONS

In this paper we have shown that the q-Poincaré supersymmetry is not exclusive to the AdS_5/CFT_4 integrable problem, and that it can be realized also in AdS_3/CFT_2. This suggests that, similarly to what happened for the secret symmetry, also the invariance under the boost $\mathbf{J}$ should be viewed as one of the several common features shared by the AdS/CFT integrable models. It would be interesting to identify other manifestations of $\mathbf{J}$ in AdS/CFT. In particular, a background recently found to be integrable is $AdS_2 \times S^2 \times T^6$, with superisometry $\mathfrak{psu}(1,1|2)$. The holographic dual might either be a superconformal quantum mechanics, or a chiral CFT [98,99]. In Ref. [100] an exact S-matrix theory was built, realizing a centrally extended $\mathfrak{psu}(1|1)$ Lie superalgebra. The Yangian, bonus symmetry and Bethe ansatz have been studied in Refs. [100–103]. On the one hand, observing the boost symmetry also in the AdS_2 case, which appears to be amenable to a similar algebraic treatment as its higher-dimensional analogues, would confirm the universal nature of the symmetry we are finding. On the other hand, the AdS_2 integrable structure is in several ways more subtle, and therefore progress towards the complete solution of the model is harder to achieve, and it is decorated with open questions. Discovering the boost symmetry in that setup could represent a crucial step in overcoming some of these open problems, and we plan to return to this issue in future work.

Since the action of $\mathbf{J}$ includes taking a derivative with respect to the worldsheet momentum, the boost invariance is sensitive to the normalization of the S-matrix. Nevertheless, a different normalization of S would produce only a shift in the tail of $\Delta(\mathbf{J})$ proportional to the identity matrix. Since we can only reverse engineer the boost coproduct and we cannot fix it *a priori*, we cannot obtain constraints on the dressing phases of AdS_3/CFT_2. In a scenario where the boost coproduct were instead known in universal form, the dressing phases would need to satisfy certain differential equations, and one could further test the proposals of Refs. [93,94]. It would be therefore very interesting to find alternative ways to fix the tail of the boost coproduct, including its contribution proportional to the identity. The achievement of this goal would certainly require some additional inputs, and the specification of which AdS_3 background is studied. In fact the dressing phases of $AdS_3 \times S^3 \times S^3 \times S^1$ and of $AdS_3 \times S^3 \times T^4$ are expected to be different.

Let us mention that the AdS_3 backgrounds that we are considering can in general be supported by a mixture of Neveu-Schwarz-Neveu-Schwarz (NSNS) and Ramond-Ramond (RR) fluxes. It is known that in the generic case the off-shell algebra is essentially the same as the one in the pure RR case considered here, and that the representations will depend on an additional parameter $\hbar$ corresponding to the relative amount of the fluxes [67,71].[17] It would be interesting to extend the deformed boost invariance to the generic case of mixed fluxes: since the dispersion relation depends on $\{\mathbf{Q}_{\mathrm{L}}, \overline{\mathbf{Q}}_{\mathrm{L}}\} = \frac{1}{2}(\mathbf{H} + \mathbf{M} + \hbar\mathbf{P})$, that would correspond to a deformation of the q-Poincaré algebra considered here, and it would be nice to investigate it also in the pure NSNS limit.

Motivated by the desire of better understanding the boost symmetry, we have also proposed a universal expression for the classical r-matrix of the AdS_3/CFT_2 integrable system. Its structure resembles the one of the AdS_5/CFT_4 classical r-matrix of Beisert and Spill [16]. With this result we complete some information that was missing in AdS_3/CFT_2, and we contribute to putting AdS_3/CFT_2 in a status closer to the one of its higher-dimensional cousin.

The appearance of the boost invariance in both AdS_5/CFT_4 and AdS_3/CFT_2 gives us further confidence that $\mathbf{J}$ should not be just an accidental symmetry of the fundamental representations of the underlying symmetries, and that $\mathbf{J}$ may help to shed some light on the universal formulation of the corresponding quantum groups.

ACKNOWLEDGMENTS

We would like to thank Antonio Pittelli for discussions, and for sharing with us a copy of the manuscript "Yangian symmetry of string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ with mixed 3-form flux" [104] while in preparation. We thank Ben Hoare, Marius de Leeuw and Juan Miguel Nieto for discussions. We also thank Olof Ohlsson Sax and Antonio Pittelli for comments on the manuscript. The work of R. B. was supported by the ERC advanced Grant No. 341222. J. S. and A. T. thank the STFC under the Consolidated Grant Project No. ST/L000490/1.

Note added.—While writing this manuscript we received the interesting paper [104], where the universal classical r-matrix of AdS_3/CFT_2 is justified from the RTT formulation. In order to match our results with those of Ref. [104] we need to identify the generators as

$$\begin{aligned}
&\mathfrak{Q}_L^{(n)} = -\mathbf{Q}_{\mathrm{L},n}, \qquad \mathfrak{S}_L^{(n)} = \bar{\mathbf{Q}}_{\mathrm{L},n}, \\
&\mathfrak{H}_L^{(n)} = -\frac{1}{2}(\mathbf{H}_n + \mathbf{M}_n), \qquad \mathfrak{H}_R^{(n)} = -\frac{1}{2}(\mathbf{H}_n - \mathbf{M}_n), \\
&\mathfrak{Q}_R^{(n)} = -\mathbf{Q}_{\mathrm{R},n}, \qquad \mathfrak{S}_R^{(n)} = \bar{\mathbf{Q}}_{\mathrm{R},n}, \\
&\mathfrak{B}_L^{(n)} = -\frac{1}{2}\mathbf{b}_{n+1} - \mathbf{B}_{n+1}, \qquad \mathfrak{B}_R^{(n)} = \frac{1}{2}\mathbf{b}_{n+1} - \mathbf{B}_{n+1},
\end{aligned} \tag{6.1}$$

where the notation of each paper is used, and the identification of $\mathbf{b}_n$, $\mathbf{B}_n$ is to be understood up to central elements.

[17] It may be useful, especially when attempting to include the boost, to reformulate the construction and have already the commutation relations, rather than just the representations, deformed by this additional parameter. For example, instead of having $\{\mathbf{Q}_{\mathrm{L}}, \bar{\mathbf{Q}}_{\mathrm{L}}\} = \frac{1}{2}(\mathbf{H} + \mathbf{M})$ where $\mathbf{M}$ has eigenvalues $\hbar$ [67,71], one may prefer to write $m + \hbar p$ where $\mathbf{M}$ has eigenvalues m. The generator $\mathbf{M}$ would then remain central even when including a generator acting as the derivative with respect to momentum.

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