Duality in a supersymmetric gauge theory from a perturbative viewpoint

Thomas A. Ryttov[1] and Robert Shrock[2]

^{1}CP^3-Origins, University of Southern Denmark, Campusvej 55, Odense, Denmark
^{2}C. N. Yang Institute for Theoretical Physics and Department of Physics and Astronomy,
Stony Brook University, Stony Brook, New York 11794, USA

We study duality in $\mathcal{N} = 1$ supersymmetric QCD in the non-Abelian Coulomb phase, order-by-order in scheme-independent series expansions. Using exact results, we show how the dimensions of various fundamental and composite chiral superfields, and the quantities a, c, a/c, and b at superconformal fixed points of the renormalization group emerge in scheme-independent series expansions in the electric and magnetic theories. We further demonstrate that truncations of these series expansions to modest order yield very accurate approximations to these quantities and suggest possible implications for non-supersymmetric theories.

I. INTRODUCTION

Transformations that allow one to deal with a strongly coupled quantum field theory as a weakly coupled field theory in a different form have proved to be very powerful throughout the history of physics. An important example is provided by the lattice formulation of quantum chromodynamics (QCD). Although the property of asymptotic freedom made possible perturbative calculations at large Euclidean energy/momentum scales μ in the deep ultraviolet (UV), the growth of the running gauge coupling $g(\mu)$ in the infrared (IR) prevented reliable perturbative calculations at low energies. This difficulty was surmounted by Wilson's formulation of the theory on a (Euclidean) lattice [1], in which the plaquette term in the action is multiplied by the coefficient $\beta_P = 2N_c/g_0^2$, where g_0 is the bare gauge coupling. Thus, the strong coupling limit $g_0 \to \infty$ is equivalent to $\beta_P \to 0$, allowing analytic strong-coupling Taylor series expansions in powers of β_P. From such an expansion, the area-law behavior of the Wilson loop and hence confinement in QCD were proved for strong g_0 [1]. In a different way, a duality transformation links two different regimes of a theory. For example, in the two-dimensional Ising model, a duality transformation maps the high-temperature regime to the low-temperature regime and led to the calculation of the critical temperature in this model. Another realization of duality occurs in the generalization of electromagnetic theory to include Dirac monopoles.

Here we will consider a theory for which duality relations have been very useful, namely an asymptotically free, vectorial, gauge theory (in $d = 4$ spacetime dimensions, at zero temperature) with $\mathcal{N} = 1$ supersymmetry, having a gauge group $\mathrm{SU}(N_c)$ and N_f massless chiral superfields Q^i and $\tilde{Q}_i$, $i = 1, \ldots, N_f$, transforming in the fundamental and conjugate fundamental representations of $\mathrm{SU}(N_c)$, respectively. The decomposition of the chiral superfields Q_i and $\tilde{Q}_i$ in terms of component fields is indicated in the standard manner as $Q_i = \phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i$ and $\tilde{Q}_i = \tilde{\phi}_i + \sqrt{2}\theta\tilde{\psi}_i + \theta\theta\tilde{F}_i$, where θ is a Grassmann variable, and ϕ_i, ψ_i, and F_i are the scalar, fermion, and auxiliary component fields. This theory is invariant under a global symmetry group $G_{gb} = \mathrm{SU}(N_f) \otimes \mathrm{SU}(N_f) \otimes \mathrm{U}(1)_B \otimes \mathrm{U}(1)_R$ symmetry, with the representations indicated in Table I. We call this theory supersymmetric quantum chromodynamics (SQCD). We denote

TABLE I. The matter content of the original (electric) and dual (magnetic) SQCD theories.

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q	N_c	N_f	1	1	$\frac{N_f - N_c}{N_f}$
$\tilde{Q}$	$\bar{N}_c$	1	$\bar{N}_f$	-1	$\frac{N_f - N_c}{N_f}$

	$\mathrm{SU}(N_f - N_c)$	$\mathrm{SU}(N_f)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
q	N_c	$\bar{N}_f$	1	$\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\bar{N}_c$	1	N_f	$-\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
ϕ	1	N_f	$\bar{N}_f$	0	$\frac{2(N_f - N_c)}{N_f}$

$\alpha = g^2/(4\pi)$. This theory has the appeal that many of its properties are well understood at a nonperturbative level [2–5]. The property of asymptotic freedom requires that N_f be less than an upper (u) bound, N_u, which, in the present theory, is $3N_c$. In this range, the theory is weakly interacting in the deep UV, so one can calculate its properties perturbatively. One can then investigate how it evolves ("flows") from the deep UV to the IR limit as $\mu \to 0$. For N_f slightly less than $3N_c$, the theory evolves to an infrared fixed point (IRFP) of the renormalization group (RG) at α_{IR}, at which point it is scale-invariant and is inferred [6] to be (super)conformally invariant. If N_f is in the interval I: $(3/2)N_c < N_f < 3N_c$, the theory flows to an IRFP in a (deconfined, chirally symmetric) non-Abelian Coulomb phase (NACP) [3]. Henceforth, we restrict our consideration to the NACP in this theory.

At this superconformal IRFP, it was conjectured in [3] that the original theory is dual to the IR limit of another $\mathcal{N} = 1$ supersymmetric theory with a gauge group SU($\tilde{N}_c$), where $\tilde{N}_c = N_f - N_c$, with matter content consisting of N_f chiral superfields q_i and $\tilde{q}^i$, $i = 1, ..., N_f$, in the fundamental and conjugate fundamental representations of SU($\tilde{N}_c$), respectively, together with a set of N_f^2 gauge-singlet "meson" chiral superfields ϕ_j^i, $1 \le i, j \le N_f$. The original and dual theories were called "electric" and "magnetic" in [3,4]. The dual theory also allows for a unique superpotential $W = \lambda \phi q \tilde{q}$, where λ is the superpotential coupling. Evidence for this conjectured equivalence includes the fact that the dual theory is also invariant under the same global G_{gb} symmetry, and satisfies 't Hooft anomaly matching [3–5].

The dual theory is asymptotically free for $N_f < 3\tilde{N}_c$, i.e., $N_f > (3/2)N_c$ and for N_f in the interval $\tilde{I}$: $(3/2)\tilde{N}_c < N_f < 3\tilde{N}_c$, it flows to a superconformal IRFP in the space of gauge and λ couplings, at which the physics is equivalent to that in the original theory. The weak-coupling region in the original theory corresponds to strong coupling in the dual theory, and vice versa. This duality is well supported by nonperturbative arguments, but one gains further insight by seeing how the duality relations emerge perturbatively. However, a conventional perturbative calculation, as a series expansion in powers of the gauge coupling, encounters the difficulty that although $\alpha_{IR} \to 0$ as $N_f \nearrow 3N_c$ at the upper end of the NACP, this theory becomes strongly coupled, and hence not amenable to this type of perturbative approach, as $N_f \searrow (3/2)N_c$ at the lower end of the NACP. A similar comment applies to conventional perturbative expansions in the magnetic theory, which has a weak gauge coupling as $N_f \nearrow 3\tilde{N}_c$, but is strongly coupled as $N_f \searrow (3/2)\tilde{N}_c$. Even in the regions of the electric and magnetic theories where they are weakly coupled, perturbative expansions in powers of gauge couplings are scheme-dependent.

Here we surmount this difficulty and present, for the first time, a scheme-independent perturbative understanding of the duality in the non-Abelian Coulomb phase of SQCD. We utilize the fact that $\alpha_{IR} \to 0$ as $N_f \nearrow 3N_c$, so that one can alternatively express physical quantities at an IRFP in the NACP as series in powers of the manifestly scheme-independent quantity [7–9]

$$\Delta_f = 3N_c - N_f. \qquad (1.1)$$

The expansion parameter in the dual theory is

$$\tilde{\Delta}_f = 3\tilde{N}_c - N_f = 2N_f - 3N_c. \qquad (1.2)$$

We will study dimensions of various (gauge-invariant) chiral superfield operators and of certain quantities a, c, a/c, and b characterizing RG flows, in both the original and dual theories, as expansions in both Δ_f and $\tilde{\Delta}_f$, and will show how various relations emerge order-by-order in these expansions. We address and answer an important question, namely how accurate are finite truncations of these series expansions as approximations to the exact results for the various scaling dimensions. Clearly, the series expansions in powers of Δ_f are most accurate at the upper end of the NACP, while the series expansions in powers of $\tilde{\Delta}_f$ are most accurate at the lower end of the NACP. From our analysis, we will show that a combination of finite-order expansions in these two dual expansion parameters Δ_f and $\tilde{\Delta}_f$ yields quite accurate approximations to physical quantities throughout the entire NACP. Our work thus demonstrates how perturbative calculations in the well-chosen scheme-independent expansion parameters Δ_f and $\tilde{\Delta}_f$ can provide insight into results based on abstract nonperturbative methods. Our present results extend our previous studies of anomalous dimensions in $\mathcal{N} = 1$ supersymmetric gauge theories [10,11] in showing how well a combination of perturbative expansions in Δ_f and $\tilde{\Delta}_f$, reproduce exactly known results for the anomalous dimensions of fundamental and composite chiral superfields, as well as quantities such as a and c. For this purpose, we make use of the powerful electric-magnetic duality in this theory. Just as exact results in SQCD [2–5] have provided new insights into QCD with many fermions (e.g., the NACP), so also, our quantitative demonstration of the accuracy of finite truncations of these scheme-independent series suggest that corresponding finite-order scheme-independent series [9] in the NACP of nonsupersymmetric theories might be similarly accurate.

II. BACKGROUND AND CALCULATIONAL METHODS

In this section we review some background and methods relevant for our calculations. The series expansion of β in powers of α is

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\alpha}{4\pi}\right)^\ell, \tag{2.1}$$

where b_ℓ is the ℓ-loop coefficient. The first two coefficients in the expansion (2.1) are scheme-independent, while higher-order coefficients are scheme-dependent. For this theory, the first two coefficients are [12]

$$b_1 = 3N_c - N_f \tag{2.2}$$

and [13]

$$b_2 = 2[3N_c^2 - (2N_c - N_c^{-1})N_f]. \tag{2.3}$$

For N_f slightly less than $3N_c$, the two-loop beta function has a (scheme-independent) IR zero at

$$\alpha_{\mathrm{IR},2\ell} = \frac{2\pi(3N_c - N_f)}{(2N_c - N_c^{-1})N_f - 3N_c^2}. \tag{2.4}$$

If N_f is only slightly less than $3N_c$, then this IRFP occurs at weak coupling. Indeed, in the limit $N_c \to \infty, N_f \to \infty$ with the ratio $r = N_f/N_c$ fixed, the IRFP in the scaled coupling $\xi = \alpha N_c$ (which is finite in this limit) occurs at

$$\xi_{\mathrm{IR},2\ell} = \frac{2\pi(3-r)}{2r-3} \tag{2.5}$$

and becomes arbitrarily weak as $r \nearrow 3$.

One can express a (gauge-invariant) physical quantity $\mathcal{P}$ at this IRFP as a power series in the coupling,

$$\mathcal{P} = \sum_{n=0}^{\infty} c_{\mathcal{P},n} \alpha_{\mathrm{IR}}^n, \tag{2.6}$$

where the $c_{\mathcal{P},n}$ are n-loop coefficients, and it is understood implicitly here that α_{IR} is the value of α at the IRFP, calculated to the n-loop order, to match the n-loop order of the expansion of $\mathcal{P}$. One important example of a physical quantity is the (full) scaling dimension $D_{\mathcal{O}}$ of a physical operator $\mathcal{O}$ at an IRFP, which, in general, differs from its free-field value, due to interactions. We write

$$D_{\mathcal{O}} = D_{\mathcal{O},\mathrm{free}} - \gamma_{\mathcal{O}}, \tag{2.7}$$

where $D_{\mathcal{O},\mathrm{free}}$ is the free-field scaling dimension of $\mathcal{O}$ and $\gamma_{\mathcal{O}}$ is the anomalous dimension at the IRFP. In general (not assuming that the theory is at an IRFP) the anomalous dimension can be expanded as a conventional series in powers of the coupling as

$$\gamma_{\mathcal{O}} = \sum_{\ell=1}^{\infty} c_{\mathcal{O},\ell} \alpha^\ell. \tag{2.8}$$

At an IRFP, one can calculate this to a given n-loop order by computing the n-loop value of α and substituting it in Eq. (2.8). This was done for $\mathcal{O} = \bar{\psi}\psi = \sum_{i=1}^{N_f} \bar{\psi}_i\psi_i$ in [10], [14,15], and effects of scheme dependence were studied in [16]. Connections between the $\overline{\mathrm{DR}}$ [17] and NSVZ [2] schemes have been discussed in [18].

As a physical quantity, $\mathcal{P}$ cannot depend on the scheme used in the conventional calculation as the power series (2.9). However, this property is not maintained beyond the lowest orders in Eq. (2.9). It is therefore of great value to devise a complementary approach in which one expresses a physical quantity evaluated at the IRFP $\alpha = \alpha_{\mathrm{IR}}$ as an expansion in powers of a variable such that, at every order in this expansion, the result is scheme-independent. The key property that makes this possible is the fact that both α_{IR} and Δ_f vanish as N_f approaches $3N_c$ from below. Because of this, one can recast the conventional series (2.9) as a power series in the manifestly scheme-independent variable Δ_f. By the same logic, one can recast a series for $\mathcal{P}$ in the dual, magnetic theory as a series in powers of $\tilde{\Delta}_f$. This successfully accomplishes three goals: (i) achieving series for the quantity $\mathcal{P}$ at the IRFP in the non-Abelian Coulomb phase in both the electric and magnetic theories that are manifestly scheme-independent, (ii) having a series in the magnetic theory involving a small expansion parameter $\tilde{\Delta}_f$ that can be used accurately in the region near the lower end of the NACP where the expansion parameter Δ_f in the electric theory is getting large, and hence (iii) having accurate perturbative expansions starting from both ends and extending throughout the entirety of the non-Abelian Coulomb phase.

For a (gauge-invariant) quantity $\mathcal{P}$ at the superconformal IRFP, we write the expansions of $\mathcal{P}$ in powers of Δ_f and $\tilde{\Delta}_f$ as

$$\mathcal{P} = \sum_{n=0}^{\infty} p_n \Delta_f^n = \sum_{n=0}^{\infty} \tilde{p}_n \tilde{\Delta}_f^n. \tag{2.9}$$

In the first sum in Eq. (2.9), one takes N_c as fixed and computes $\mathcal{P}$ as a function of the variable Δ_f, or equivalently, N_f. In the second sum, for the dual theory, one takes $\tilde{N}_c = N_f - N_c$ as fixed; while varying N_f, one can keep $\tilde{N}_c$ fixed by formally varying N_c oppositely to N_f. The nth-order coefficients have a definite relation to terms in conventional perturbative expansions.

At an IRFP, $\gamma_{\mathcal{O}}$ can also be expanded in a scheme-independent series as

$$\gamma_{\mathcal{O}} = \sum_{n=1}^{\infty} \kappa_{\mathcal{O},n} \Delta_f^n. \tag{2.10}$$

The coefficient $\kappa_{\mathcal{O},n}$ depends on the terms in the beta function of the theory in Eq. (2.1) up to loop order $\ell = n + 1$ and on the terms in the conventional expansion of

$\gamma_{\mathcal{O}}$ in powers of α in Eq. (2.8) up to loop order $\ell = n$, inclusive, but does not receive contributions from higher-loop terms. This is a very powerful observation which can be used in two different ways. First, it allows for a calculation of any physical quantity in a manifestly scheme-independent way, order-by-order, even if one only knows the gauge coupling beta function and $D_{\mathcal{O}}$ to some finite loop order. Second, if, on the other hand, one knows an exact expression for a quantity $\mathcal{P}$ at the IRFP, then one can calculate its expansions in powers of Δ_f and $\tilde{\Delta}_f$ in Eq. (2.9) exactly, to all orders, without doing any explicit loop computations.

In the analysis in the present work, we will use the second method, since the exact expressions for the relevant operator dimensions are known in this $\mathcal{N} = 1$ supersymmetric gauge theory. To make our discussion self-contained, we will also review the explicit method of calculating the coefficients $\kappa_{\mathcal{O},n}$ from knowledge of the coefficients b_n and $c_{\mathcal{P},n}$ in conventional series in powers of α. (For further details of this method, we refer the reader to our previous papers in [9].) For definiteness, we consider the case of the gauge-invariant quadratic chiral superfield operator product $\tilde{Q}Q$ (where a sum over flavor indices is understood) and focus on the anomalous dimension $\gamma_{\tilde{\psi}\psi,\text{IR}}$ of the associated fermionic bilinear $\tilde{\psi}\psi$ at the IRFP, where, as defined above, ψ_i and $\tilde{\psi}_i$ are the fermion field components in the chiral superfields Q_i and $\tilde{Q}_i$. This anomalous dimension is the analogue, in this supersymmetric theory, of the quantity $\gamma_{\tilde{\psi}\psi,\text{IR}}$ in nonsupersymmetric gauge theories that we have calculated scheme-independent Δ_f expansions for in previous work [9]. The conventional expansion for $\gamma_{\tilde{\psi}\psi,\text{IR}}$ in the electric theory has the form (2.9) with $c_{\tilde{\psi}\psi,n} \equiv c_n$. For the scheme-independent expansion in the electric theory we write

$$\gamma_{\tilde{\psi}\psi,\text{IR}} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j. \qquad (2.11)$$

To calculate κ_j, one begins by writing $a_{\text{IR}} = \alpha_{\text{IR}}/(4\pi)$ as a series expansion in Δ_f:

$$a_{\text{IR}} = \sum_{j=1}^{\infty} a_j \Delta_f^j. \qquad (2.12)$$

For an IR zero of β, we have

$$\sum_{\ell=1}^{\infty} b_\ell a^\ell = 0. \qquad (2.13)$$

Next, one expands the coefficients b_ℓ in Taylor series around $N_f = N_u$, i.e., $\Delta_f = 0$. Since each b_ℓ is a finite polynomial in N_f, its Taylor series around $\Delta_f = 0$ truncates at a maximal degree denoted $r_{\max}(\ell)$. One then substitutes the resulting expansions for b_ℓ and the Δ_f expansion for a_{IR}, Eq. (2.12), in Eq. (2.13), yielding

$$\sum_{\ell=1}^{\infty} b_\ell a^\ell = 0 = \sum_{\ell=1}^{\infty} \left[\left(\sum_{r=0}^{r_{\max}(\ell)} b_\ell^{(r)} \Delta_f^r \right) \left(\sum_{j=1}^{\infty} a_j \Delta_f^j \right)^\ell \right]$$

$$= \sum_{n=1}^{\infty} k_n \Delta_f^n. \qquad (2.14)$$

Since the sum in the last line of Eq. (2.14) is zero for all Δ_f, each coefficient k_n must be zero. This yields a set of linear equations that one can solve for the a_n in terms of the Taylor series coefficients of b_ℓ in the expansion about $\Delta_f = 0$. Next, one inserts these expressions for the a_n in the Δ_f expansion for a_{IR}. Then, one carries out similar Taylor series expansions of the c_n around $\Delta_f = 0$ and substitutes these, together with the expansion (2.12) of the a_{IR}, in the series $\gamma_{\tilde{\psi}\psi,\text{IR}} = \sum_{\ell=0}^{\infty} c_\ell a_{\text{IR}}^\ell$. This finally yields the coefficients κ_j in Eq. (2.11). This, then, is the procedure to obtain the coefficients in the scheme-independent series expansion of $\gamma_{\tilde{\psi}\psi,\text{IR}}$ in powers of Δ_f. The same type of procedure is used to obtain the coefficients in the scheme-independent series expansions of other physical quantities at the IRFP.

III. RESULTS FOR MESON AND BARYON OPERATORS

In the original (electric) theory, gauge-singlet composite chiral superfields include the meson-type operators $M^i{}_j = Q^i \tilde{Q}_j$, and baryon- and antibaryon-type operators $B^{i_1 \cdots i_{N_c}} = \epsilon_{i_1 \cdots i_{N_c}} Q^{i_1} \cdots Q^{i_{N_c}}$ and $\tilde{B}_{i_1 \cdots i_{N_c}} = \epsilon^{i_1 \cdots i_{N_c}} \tilde{Q}_{i_1} \cdots \tilde{Q}_{i_{N_c}}$, respectively. Similarly, in the dual (magnetic) theory, in addition to $\phi^i{}_j$, one has the dual baryon and antibaryon operators $b_{i_1 \cdots i_{\tilde{N}_c}} = \epsilon^{i_1 \cdots i_{\tilde{N}_c}} q_{i_1} \cdots q_{i_{\tilde{N}_c}}$ and $\tilde{b}^{i_1 \cdots i_{\tilde{N}_c}} = \epsilon_{i_1 \cdots i_{\tilde{N}_c}} \tilde{q}^{i_1} \cdots \tilde{q}^{i_{\tilde{N}_c}}$. Duality dictates that the meson operators are matched in the electric and magnetic theories, and similarly for the baryon operators and the antibaryon operators. Hence, the meson operators in the electric and magnetic theories must have the same dimensions, and similarly for the baryon and antibaryon operators.

In a superconformal theory, the dimension $D_{\mathcal{O}}$ of a gauge-invariant chiral superfield operator $\mathcal{O}$ is related to the R-charge, $R_{\mathcal{O}}$, of the operator via $D_{\mathcal{O}} = (3/2)R_{\mathcal{O}}$. This implies, in particular, that the scaling dimension of the (composite) electric and (fundamental) magnetic mesons is

$$D_M = D_\phi \equiv D = \frac{3(N_f - N_c)}{N_f} = \frac{3\tilde{N}_c}{N_f}. \qquad (3.1)$$

Note that $D_{M,\text{free}} = 2$ while $D_{\phi,\text{free}} = 1$. We want to understand how Eq. (3.1) emerges order-by-order in the scheme-independent expansions in the original and dual theories. To calculate the series expansion of D in powers of Δ_f, we substitute $N_f = 3N_c - \Delta_f$ in Eq. (3.1) and expand, obtaining
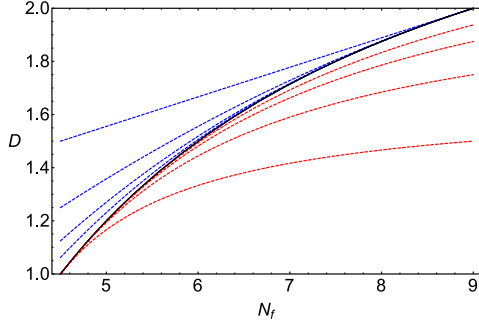
FIG. 1.   The scaling dimension D of the meson chiral superfield, as computed to $O(\Delta_f^s)$ in the original (electric) theory (blue dashed curves) and to $O(\tilde{\Delta}_f^s)$ in the dual (magnetic) theory (red dashed curves), together with the exact D (black solid curve), for the illustrative case $N_c = 3$. From top to bottom for the blue curves and from bottom to top for the red curves, these depict D_s with $s = 1, \ldots, 4$.

$$D = 2 - \sum_{n=1}^{\infty} \left(\frac{\Delta_f}{3N_c} \right)^n. \qquad (3.2)$$

Equivalently, in the dual theory, substituting $N_f = 3\tilde{N}_c - \tilde{\Delta}_f$ in Eq. (3.1) and expanding, we find

$$D = 1 + \sum_{n=1}^{\infty} \left(\frac{\tilde{\Delta}_f}{3\tilde{N}_c} \right)^n. \qquad (3.3)$$

These series have respective radii of convergence $|\Delta_f| = 3N_c$ and $|\tilde{\Delta}_f| = 3\tilde{N}_c$, and hence converge throughout the entire NACP, since the maximal values of Δ_f and $\tilde{\Delta}_f$ in the NACP are $(3/2)N_c$ and $(3/2)\tilde{N}_c$. The truncations of the series (3.2) and (3.3) to order $n = s$ are denoted D_s.

In Fig. 1 we plot the values of D calculated from (3.2) to $O(\Delta_f^s)$, i.e., D_s, in the electric theory and D_s calculated from (3.3) in the magnetic theory for $1 \leq s \leq 4$, together with the exact result, for the illustrative case, $N_c = 3$. The maximal order $s = 4$ is chosen to be the same as the highest order in our SI expansions in nonsupersymmetric theories, to facilitate accuracy comparisons.

Because the coefficients of Δ_f^n and $\tilde{\Delta}_f^n$ in the expansions (3.2) and (3.3) are positive, several monotonicity properties follow (for N_f in the NACP): (i) for fixed s, D_s is a monotonically increasing function of N_f; (ii) for fixed N_f, D_s, as calculated in the electric theory, decreases monotonically with s; and (iii) D_s, as calculated in the magnetic theory, increases monotonically with s. As is evident from Fig. 1, the respective fractional accuracies of the Δ_f and $\tilde{\Delta}_f$ series expansions in (3.2) and (3.3) are highest near the upper and lower ends of the NACP, respectively. Thus, by combining these two perturbative calculations, we achieve an excellent approximation to the exact expression (3.1) throughout all of the NACP, even with a modest value of the
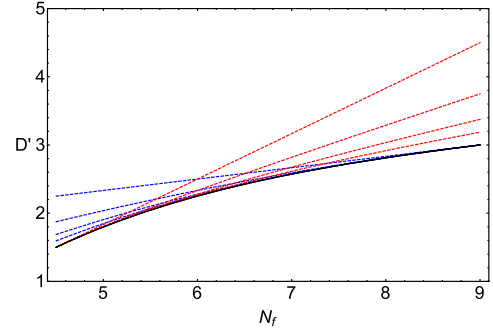

FIG. 2.   The scaling dimension D' of the baryon operators, as computed in the electric theory (blue dashed curves) and magnetic theory (red dashed curves), together with the exact result (black solid curve), for the illustrative value, $N_c = 3$. From top to bottom for both the blue and red curves, these depict D_s' with $s = 1, \ldots, 4$.

truncation order, s such as $s = 4$. This makes use of the full power of the duality, since it allows one to treat the strong-coupling regime in the original theory via a perturbative calculation in the weak-coupling regime of the dual theory, and vice versa.

We next consider the baryon-type operators. In both the electric and magnetic theories, their scaling dimensions have to agree, and are

$$D_B = D_b \equiv D' = \frac{3N_c(N_f - N_c)}{2N_f} = \frac{3N_c\tilde{N}_c}{2N_f}. \qquad (3.4)$$

As with the mesons, we want to understand how this expression for D' emerges order-by-order in perturbation theory, as calculated in both the original and dual theories. In the original (electric) theory, we find

$$D' = N_c - \frac{N_c}{2} \sum_{n=1}^{\infty} \left(\frac{\Delta_f}{3N_c} \right)^n, \qquad (3.5)$$

while in the dual (magnetic) theory we find

$$D' = \tilde{N}_c - \frac{\tilde{N}_c}{2} \sum_{n=1}^{\infty} \left(\frac{\tilde{\Delta}_f}{3\tilde{N}_c} \right)^n. \qquad (3.6)$$

As before, these series converge throughout the entire NACP. The truncations of these series to order $n = s$ are denoted D_s'.

In Fig. 2 we plot the values of D' calculated from (3.5) to $O(\Delta_f^s)$ in the electric theory and from (3.6) to $O(\tilde{\Delta}_f^s)$ in the magnetic theory, for $1 \leq s \leq 4$, together with the exact result. Because the coefficients of Δ_f^n and $\tilde{\Delta}_f^n$ in the expansions (3.5) and (3.6) are positive, several monotonicity properties follow for the calculations in both the electric and magnetic theories (for N_f in the NACP): (i) for fixed s, D_s' is a monotonically increasing function of N_f;

and (ii) for fixed N_f, D'_s is a monotonically decreasing function of s. The same comments about respective fractional accuracies of the Δ_f and $\tilde{\Delta}_f$ calculations made for Fig. 1 hold here, so that again, by combining these two perturbative calculations, we obtain an excellent approximation to the exact expression (3.4) throughout all of the NACP even with a modest truncation order, s, such as $s = 4$.

Corresponding to the global symmetry groups in G_{gb}, there are conserved currents. We will focus on the conserved current $J_\mu^{(B)}$ associated with the $U(1)_B$ baryon number symmetry. The two-point correlation function for this current (in flat space) is

$$\langle J_\mu^{(B)}(x) J_\nu^{(B)}(0) \rangle = \frac{b}{(4\pi)^2} (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) \frac{1}{x^4}. \quad (3.7)$$

Here b is a function of the couplings of the theory and changes from b_{UV} to b_{IR} along the RG flow. For SQCD, b_{UV} is given by its respective free-field values, $b_{UV} = 2N_f/N_c$ and $b_{UV} = 2N_f/\tilde{N}_c$ in the electric and magnetic theories, while $b_{IR} = 6$ at the IRFP in the non-Abelian Coulomb phase in both of these theories [19]. Hence, calculating $b_{UV} - b_{IR}$, we have, in the electric theory,

$$b_{UV} - b_{IR} = -\frac{2\Delta_f}{N_c}, \quad (3.8)$$

and, in the magnetic theory,

$$b_{UV} - b_{IR} = -\frac{2\tilde{\Delta}_f}{\tilde{N}_c}. \quad (3.9)$$

These results show that higher-order contributions to b_{IR} in powers of Δ_f, and equivalently in powers of $\tilde{\Delta}_f$, vanish. This again shows the value of the scheme-independent series expansion method, since the zero coefficients of the respective Δ_f^n and $\tilde{\Delta}_f^n$ terms with $n \geq 2$ in Eqs. (3.8) and (3.9) involve complicated cancellations when computed via conventional (scheme-dependent) series expansions in powers of couplings.

IV. ANALYSIS OF a AND c

The trace of the energy-momentum tensor in four spacetime dimensions, in the presence of a curved background metric $g_{\mu\nu}$, is [20]

$$T^\mu_{\ \mu} = \frac{1}{(4\pi)^2} (c W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - a E_4) \quad (4.1)$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and E_4 is the Euler density. In $d = 2$, it was proved that c decreases monotonically along an RG flow [21], but this monotonicity does not hold in $d = 4$. The quantity a encodes important information

about the flow of a quantum field theory between two RG fixed points and satisfies the inequality that $a_{UV} - a_{IR} > 0$ (called the a theorem) [19,22–26]. This is in accord with the Wilsonian notion of thinning of degrees of freedom along an RG flow [27]. For an asymptotically free theory, a_{UV} is given by the (massless) free-field content of the theory [20]: $a_{UV} = (1/48)(9N_v + N_\chi)$, where here N_v and N_χ denote the numbers of vector and chiral superfields, respectively [20]. In the original (electric) theory, $N_v = N_c^2 - 1$ and $N_\chi = 2N_c N_f$, while in the dual (magnetic) theory, $N_v = \tilde{N}_c^2 - 1$ and $N_\chi = 2\tilde{N}_c N_f + N_f^2$. The duality at the superconformal IRFP dictates that the value of a_{IR} must be identical in the electric and magnetic theories; it is [19] $a_{IR} = (3/16)[2N_c^2 - 1 - 3(N_c^4/N_f^2)]$. Note that this result for a_{IR} does not contain any factors $\sim e^{-\text{const.}/\alpha_{IR}}$ that are generic manifestations of instanton effects. Calculating a series expansion for a_{IR} in the electric theory, in powers of Δ_f, we obtain

$$a_{IR} = a_{UV} - \frac{1}{144} \sum_{n=2}^{\infty} \frac{(n+1)\Delta_f^n}{(3N_c)^{n-2}}. \quad (4.2)$$

Equation (4.2) shows how the a theorem is satisfied at each order in powers of Δ_f. In the magnetic theory, we find

$$a_{IR} = a_{UV} - \frac{1}{144}\left[21\tilde{\Delta}_f^2 + \sum_{n=3}^{\infty} \frac{(n-11)\tilde{\Delta}_f^n}{(3\tilde{N}_c)^{n-2}}\right]. \quad (4.3)$$

Here again, the a theorem is satisfied at each order in powers of $\tilde{\Delta}_f^n$; in this case, the result follows because the leading-order term in the square brackets, $21\tilde{\Delta}_f^2$, dominates over higher-order terms $\propto \tilde{\Delta}_f^n$ with $3 \leq n \leq 10$ with opposite sign.

In Fig. 3 we plot a_{IR} as computed to first through fourth order in Δ_f in the electric theory and in $\tilde{\Delta}_f$ in the magnetic theory, together with the exact result, for the illustrative case $N_c = 3$.

Although the coefficient c in Eq. (4.1) does not obey a monotonicity relation along RG flows or an inequality on the sign of $c_{UV} - c_{IR}$, it is also of interest to calculate scheme-independent series expansions for this difference. For the (asymptotically free) $\mathcal{N} = 1$ supersymmetric gauge theory considered here, $c_{UV} = (1/24)(3N_v + N_\chi)$, where, as above, N_v and N_χ denote the numbers of (massless) vector and chiral superfields [20]. The duality at the superconformal IRFP implies that the value of c_{IR} must be identical in the electric and magnetic theories; it is [19] $c_{IR} = (1/16)[7N_c^2 - 2 - (9N_c^4/N_f^2)]$. Calculating a series expansion for c_{IR} in the electric theory, in powers of Δ_f, we obtain

$$c_{IR} = c_{UV} + \frac{N_c}{48}\left(2\Delta_f - \sum_{n=2}^{\infty} \frac{(n+1)\Delta_f^n}{(3N_c)^{n-1}}\right). \quad (4.4)$$
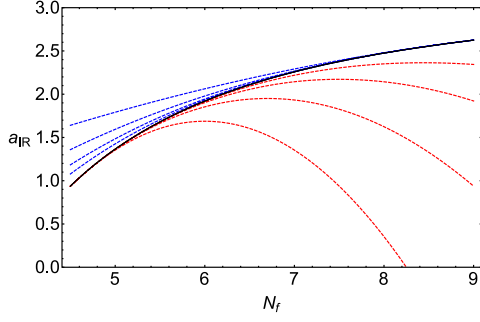
FIG. 3. The quantity a_{IR}, as computed in the electric theory (blue dashed curves) and magnetic theory (red dashed curves), together with the exact result (black solid curve), for the illustrative case $N_c = 3$. From top to bottom, the blue curves are for a_{IR} calculated to $O(\Delta_f^s)$ with $s = 1, \ldots, 4$, and from bottom to top, the red curves are for a_{IR} calculated to $O(\tilde{\Delta}_f^s)$ with $s = 1, \ldots, 4$.

FIG. 4. The central charge c_{IR}, as computed in the electric theory (blue dashed curves) and magnetic theory (red dashed curves), together with the exact result (black solid curve), for the illustrative case $N_c = 3$. For the electric theory, from top to bottom, the curves refer to the calculation of c_{IR} to $O(\Delta_f^s)$ with $s = 1, \ldots, 4$, while for the magnetic theory, from bottom to top, the curves refer to the calculation of c_{IR} to $O(\tilde{\Delta}_f^s)$ with $s = 1, \ldots, 4$.

In the magnetic theory, we find

$$c_{\mathrm{IR}} = c_{\mathrm{UV}} - \frac{\tilde{N}_c}{48}\left(4\tilde{\Delta}_f + \frac{5\tilde{\Delta}_f^2}{\tilde{N}_c} + \sum_{n=3}^{\infty} \frac{(n-11)\tilde{\Delta}_f^n}{(3\tilde{N}_c)^{n-1}}\right). \quad (4.5)$$

In Fig. 4 we plot c_{IR} as computed to first through fourth order in Δ_f in the electric theory and in $\tilde{\Delta}_f$ in the magnetic theory, together with the exact result (black solid curve), for the illustrative case $N_c = 3$. The curves in Fig. 4 show the same result as we demonstrated for operator dimensions and for a_{IR}, namely that by combining Δ_f and $\tilde{\Delta}_f$ series expansions, one can get very good approximations to exact nonperturbative expressions, even for modest truncation orders.

In a superconformal field theory, the ratio a/c is restricted, by the positivity of energy flux, to lie in the interval $1/2 \le a/c \le 3/2$ [28–30]. (This ratio a/c saturates its upper bound of $3/2$ if the superconformal theory contains only vector superfields, while it saturates its lower bound of $1/2$ if this theory contains only chiral superfields.) We have calculated Δ_f and $\tilde{\Delta}_f$ series expansions for $a_{\mathrm{IR}}/c_{\mathrm{IR}}$ and, as with a_{IR} and c_{IR}, we have found that that a combination of modest-length series in these two variables gives a very good approximation to the exact result.

V. CONCLUSIONS

In conclusion, we have shown, for the first time, how exact relations for dimensions of chiral superfields and for the quantities a, c, a/c, and b in the non-Abelian Co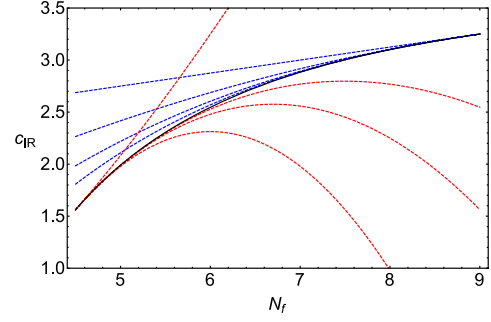ulomb phase of SQCD emerge order-by-order in scheme-independent perturbative series expansions, as calculated in the original (electric) theory in powers of Δ_f and in the dual magnetic theory in powers of $\tilde{\Delta}_f$. Our series expansions have a fundamental advantage over conventional series expansions of these quantities in powers of α_{IR}, namely that our expansions are scheme-independent, while conventional expansions in powers of α_{IR} are scheme-dependent. We have addressed and answered the question of how well finite-order truncations of these scheme-independent series expansions approximate exact expressions. We have demonstrated that a combination of truncated Δ_f and $\tilde{\Delta}_f$ series expansions of modest order yield quite accurate approximations to exact results for these quantities throughout the entire non-Abelian Coulomb phase. Although these results do not apply directly to nonsupersymmetric theories, they encourage one in the conjecture that corresponding finite-order series [9] in the NACP of a nonsupersymmetric gauge theory might be similarly accurate.

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