Perfect relativistic magnetohydrodynamics around black holes in horizon penetrating coordinates

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Plasma accreting processes on black holes represent a central problem for relativistic astrophysics. In this context, here we specifically revisit the classical Ruffini-Wilson work developed for analytically modeling via geodesic equations the accretion of perfect magnetized plasma on a rotating Kerr black hole. Introducing the horizon penetrating coordinates found by Doran 25 years later, we revisit the entire approach studying Maxwell invariants, electric and magnetic fields, volumetric charge density and electromagnetic total energy. We finally discuss the physical implications of this analysis.

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I. INTRODUCTION

Observations confirm that many astrophysically collapsing objects are associated to multiple stars orbiting together, with the binary situation representing the most common system [1]. Because of the presence of magnetic fields in stellar atmospheres of about $10^{13}G$ in neutron stars for instance, it is expected that electromagnetism would play a crucial role in accretion processes [2]. The study of accretion onto stationary and moving stars initiated in the 1930s with many seminal papers by Bondi, Lyttelton and Hoyle [3,4]. The subject was refreshed in the 1960s realizing that active galactic nuclei (AGNs) are powered by accretion of magnetized plasma onto supermassive black holes [5,6]. Moreover, owing to the recent developments in observing the gamma-ray bursts (GRBs), it has been shown in the fireball model, depending on the duration of the burst, the inner engine is possibly associated with the accretion to a black hole or a neutron star merger [7,8]. In fireshell models instead [9,10], long and short GRBs have been divided into two subclasses, depending on whether or not a black hole (BH) is formed in the merger or in the hypercritical accretion process exceeding the Neutron Star critical mass [11]. Therefore, finding the mechanism for explaining the engine for astronomical objects that releases the electromagnetic radiation in highly energetic wavelengths, like AGNs and GRBs in search of possible ways to extract their energy represents one of the most important objectives in high energy astrophysics. In 1975, Ruffini and Wilson (RW) [2] and Damour [12] proposed to describe the accretion of a magnetized plasma onto a Kerr black hole by using Carter [13] geodesics solutions. The model is based on the assumption that an infinite conductivity condition for a magnetized plasma accreting onto a Kerr black hole holds i.e. $F_{\mu\nu}U^{\nu} = 0$ ($F_{\mu\nu}$ is the electromagnetic tensor and U^{ν} is matter four-velocity). As a result, torque will be exerted on the falling plasma which permits the extraction of rotational energy from the black hole and moreover electric charge is induced on the totally collapsed object. Many mechanisms have been proposed to extract energy from totally collapsed systems. In the 1970s, Damour and Ruffini [14] proposed a mechanism to extract energy from an already formed Kerr-Newman BH through the vacuum polarization process around the BH, leading to the notions of Dyadosphere [15–17] and later of Dyadotorus [18]. Blandford and Znajek [19] in the same years proposed on the other hand a different mechanism by considering the electron-positron pair production in the vicinity of a rotating black hole floating in a strong magnetic field. Using the force-free condition $F_{\mu\nu}J^{\nu} = 0$ [20] with J^{μ} being the electromagnetic four current density they incorporated their model into a possible

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scenario for AGNs and the possibility of extraction of the black hole's energy. In this article we revisit the RW work solution by exploiting a useful property of this approach, i.e. the possibility to find a frame adapted to the infalling plasma adopting the new form of Kerr metric in the coordinates system regular on the event horizon found by Doran [21] in 2000 and widely used especially in numerical relativity. At the time the RW article was published, these new very useful coordinates were unknown so it is important to revisit an old solution by using modern tools. The article is organized as follows. After this introduction, in Sec. II we present the classical RW analysis and we discuss the associated Maxwell invariants characterizing physically and mathematically the solution. In Sec. III we adopt the aforementioned coordinates in a selected RW solution case, recasting it in a new form which allows the detailed study of electric and magnetic fields, volumetric charge density, currents and of the electromagnetic energy for the associated normal observer. Finally in Sec. IV physical implications of this study are discussed.

II. RUFFINI-WILSON-DAMOUR MODEL FOR ACCRETING PLASMA INTO THE KERR BLACK HOLE

The Kerr space-time is stationary and axisymmetric and in Boyer-Linquist (BL) coordinates reads [22]

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left[r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2, \quad (1)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Here M and a are the total mass and specific angular momentum respectively characterizing the space-time. In this black hole solution, the (outer) event horizon is located at $r_+ = M + \sqrt{M^2 - a^2}$ and Boyer-Lindquist coordinates are singular there. As anticipated, here we consider the model for a Kerr black hole accreting plasma with infinite conductivity. We start from Maxwell equations set on Kerr background where both the effects of electromagnetic and matter energy-momentum tensors of the accreting plasma are assumed to be negligible upon the Kerr background. The electromagnetic Maxwell tensor in terms of the vector potential A_μ is [23]

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (2)$$

so that Maxwell equations read

$$F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu \quad (3)$$

and

$$F_{[\mu\nu,\alpha]} = 0. \quad (4)$$

The motion of charged particles around the black hole satisfies [12]

$$mU_{\mu;\nu}U^\nu = qF_{\mu\nu}U^\nu \quad (5)$$

in which q and m are charge and mass of falling particle, respectively, while $F_{\mu\nu}$ is electromagnetic tensor. In general charged particles will be accelerated by the electromagnetic force and will not be geodesic. What RW and Damour proposed [2,12] was to adopt a self-consistent method for analyzing the magnetohydrodynamics (MHD) problem starting from the zeroth order assumption to have geodesic motion for the charged fluid. If one assumes in fact the approximation that $F_{\mu\nu}U^\nu = 0$ (perfect plasma condition [24]) for the matter fluid MHD equations in union with the request that pressure terms and quantity $F_{\mu\alpha}J^\alpha$ (magnetic force term) being negligible [12], the fluid motion in this zeroth order approximation will be given by the geodesic equations in Kerr space-time,

$$U_{\mu;\nu}U^\nu = 0, \quad (6)$$

analytically solved by Carter [13]. In the case of the RW solution, one considers $U_\phi = 0$ (i.e. zero axial component of the angular momentum [22]) and $U_t = -1$ at infinity so that U_θ is a constant of motion on the particle's trajectory. The geodesics four-velocity vector field of interest for the RW solution is [2,12]

$$U^t = \frac{\Sigma(r^2 + a^2) + 2Mra^2\sin^2\theta}{\Sigma\Delta} \quad (7)$$

$$U^r = -\frac{[-\Delta U_\theta{}^2 + 2Mr(r^2 + a^2)]^{\frac{1}{2}}}{\Sigma} \quad (8)$$

$$U^\theta = \frac{U_\theta}{\Sigma} \quad (9)$$

$$U^\phi = \frac{2MRa}{\Sigma\Delta}, \quad (10)$$

with U_θ constant. Let us consider now the electromagnetic field associated to the accreting plasma. Since we consider overall neutral stationary and axisymmetric configurations, the electromagnetic field expressed in terms of the vector potential requires A_ϕ only [12]. Consequently the only nonvanishing components of the electromagnetic tensor result in $F_{\phi r} = A_{\phi,r}$, $F_{\phi\theta} = A_{\phi,\theta}$ (derived from the vector potential) in union with another term $F_{r\theta}$, reexpressible as we will see briefly, in terms of the vector potential again. By using now the infinite conductivity condition

$$F_{\mu\nu}U^\nu = 0 \quad (11)$$

(which does not necessarily imply the force-free one i.e. $F_{\mu\nu}J^\nu \neq 0$ unless $J^\nu = \rho U^\nu$ with ρ being the charge density, i.e. a charged dust [25]), inserting into Maxwell equations and performing relatively simple manipulations [2,12] one gets

$$A_\phi = A(\theta_\infty) = A_\phi(\theta, r) \tag{12}$$

in which

$$\theta_\infty = \theta - U_\theta \zeta(r) \tag{13}$$

with

$$\zeta(r) = \int_r^\infty \frac{dr'}{\sqrt{-(r'^2 - 2Mr' + a^2)U_\theta^2 + 2Mr'(r'^2 + a^2)}}. \tag{14}$$

This allows one to finally compute by differentiation $F_{\phi,\theta} = A_{\phi,\theta}$ and subsequently

$$A_{\phi,r} = -\frac{U^\theta}{U^r} A_{\phi,\theta} \tag{15}$$

as well as

$$F_{r\theta} = -\frac{U^\phi}{U^r} A_{\phi,\theta}. \tag{16}$$

Inserting into Maxwell equations, one finally obtains the corresponding four current density J^μ. Taking the non-geodesic and nonrotation-free timelike vector $n_\mu = -1/\sqrt{-g^{tt}}\delta^0{}_\mu$ representing the normal to the $t =$ constant slices one can write the magnetosphere's charge outside the black hole:

$$Q_{\mathrm{mag}} = \int J^\mu n_\mu \sqrt{h} d^3x \equiv \int J^t \sqrt{-g} d^3x, \tag{17}$$

where

$$h = \frac{\Sigma}{\Delta}[(r^2 + a^2)\Sigma + 2Mra^2 \sin^2\theta] \sin^2\theta \tag{18}$$

is the determinant of the metric $h_{\mu\nu}$ induced on the $t =$ const slice. By using the Gauss theorem [2,26], this volumetric integral can be reexpressed as the two surface integrals involving the Maxwell tensor $F_{\mu\nu}$, i.e. Q_∞ (an integral evaluated at infinite radial distance and vanishing for the overall charge neutrality hypothesis) and Q_{hole} (an integral on the outer event horizon's surface). Ruffini and Wilson then have arrived to the result that the total charge induced on the black hole surface is opposite to the total volumetric charge of the outer space, i.e. $Q_{\mathrm{hole}} = -Q_{\mathrm{mag}}$. Algebraic manipulations moreover show that the electromagnetic invariants of the solution are

$$\begin{aligned}
\mathcal{F} &\equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = (\mathbf{B}^2 - \mathbf{E}^2) \\
&= \frac{2MrA_{\phi,\theta}^2}{\Sigma \sin^2\theta[2Mr(r^2 + a^2) - \Delta U_\theta^2]} \\
&\equiv \frac{2MrA_{\phi,r}^2}{\Sigma \sin^2\theta U_\theta^2} \geq 0 \\
\mathcal{G} &\equiv \frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B} = 0,
\end{aligned} \tag{19}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields as measured by a generic observer. As expected, there must exist a frame in which he/she measures a magnetic field only, while the electric one vanishes as well as the volumetric charge density as shown in the following. As a by-product of this calculation, the conditions $\mathcal{F} \geq 0$ and $\mathcal{G} = 0$ show that no vacuum polarization process *á la Schwinger* is possible [14,18,27,28]. The fact that one of the Maxwell invariants in this solution is zero, while in the Kerr-Newman (KN) case both are nonvanishing (see [18] for instance), means that the RW process does not form such a type of charged black hole. This result is not unexpected because KN black holes are solutions of electrovacuum Einstein-Maxwell equations while the RW solution, although in a perturbative scheme, describes a nonelectrovacuum situation not included in the uniqueness theorem for black holes [22]. In order to avoid the complexity of writing Eq. (13) in terms of elliptic integrals, we shall use for our analysis now the condition $U_\theta = 0$. This simplifies remarkably any calculation because from Eq. (13) $\theta_\infty = \theta$, so that A_ϕ will be a function of θ only, i.e. $A_\phi = F(\theta)$. Following RW, we assume $F(\theta) = A_0|\cos\theta|$ (A_0 is a constant). Moreover, the first and second distributional derivatives in terms of θ of this solution are given by

$$\begin{aligned}
\frac{d}{d\theta} F(\theta) &= -A_0 \sin\theta \frac{\cos\theta}{|\cos\theta|} \equiv -A_0 \sin\theta \frac{|\cos\theta|}{\cos\theta}, \\
\frac{d^2}{d\theta^2} F(\theta) &= 2A_0 \sin^2\theta \delta(\cos\theta) - A_0 \frac{\cos^2\theta}{|\cos\theta|} \\
&\equiv 2A_0 \sin^2\theta \delta(\cos\theta) - A_0|\cos\theta|.
\end{aligned} \tag{20}$$

What we expect to find on physical grounds in general is that due to the perfect plasma condition, in the plasma comoving frame no electric field would be present and consequently no volumetric nor surface charge densities (sources of the electric field) would be observed. On the other hand, a purely magnetic field would be there, whose source would be a spatial current density only. If the magnetic field would be discontinuous (so undefined on a sheet for instance), then Dirac delta functions are expected to occur in the spatial current density accounting for this (a situation known in the literature as current sheets and introduced to avoid magnetic monopoles [20]). The presence of Dirac deltas and other distributions must not alarm the reader. These appear in current densities, which require always a spatial integration in order to obtain finite quantities as charges or currents, making the presence of distributions almost fine for many practical purposes then. However, it is important to stress at this stage that the use of mathematical distributions in classical electrodynamics, mechanics, fluid dynamics or other physical context is a very useful tool for describing with a reasonable level of accuracy a system bypassing the necessity to account for the more involved physics occurring in the regions where these mathematical entities manifest their presence. We are ready now to reexpress this specific RW solution in horizon penetrating coordinates and show quantitatively that our expectations are correct.

III. HORIZON PENETRATING COORDINATES ANALYSIS

Let us transform the Kerr metric from Boyer-Lindquist (BL) coordinates (t, r, θ, ϕ) to Doran Painlevé-Gullstrand-like (hereafter DPG) horizon penetrating coordinates (T, R, Θ, Φ) [21,29] via the transformation

$$T = t - \int^r f(r)dr, \qquad R = r, \qquad \Theta = \theta,$$
$$\Phi = \phi - \int^r \frac{a}{r^2 + a^2} f(r)dr, \tag{21}$$

where

$$f(r) = -\frac{\sqrt{(2Mr)(r^2 + a^2)}}{\Delta} \tag{22}$$

so that

$$dT = dt - f(r)dr, \qquad dR = dr, \qquad d\Theta = d\theta,$$
$$d\Phi = d\phi - \frac{a}{r^2 + a^2} f(r)dr. \tag{23}$$

Substituting Eq. (23) into the BL Kerr spacetime element line gives

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dT^2 + 2\sqrt{\frac{2Mr}{r^2 + a^2}}dTdr \\ &- \frac{2a(2Mr)}{\Sigma}\sin^2\theta dTd\Phi \\ &+ \sin^2\theta\left[r^2 + a^2 + \frac{a^2(2Mr)}{\Sigma}\sin^2\theta\right]d\Phi^2 \\ &- 2a\sin^2\theta\sqrt{\frac{2Mr}{r^2 + a^2}}drd\Phi \\ &+ \frac{\Sigma}{r^2 + a^2}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{24}$$

where we have decided to denote here and in the following R with r and Θ with θ due to the coincidence of these coordinates in BL and DPG coordinates. The transformed RW vector potential results in

$$A_\mu = \left[0, -a\sqrt{\frac{2Mr}{r^2 + a^2}}\frac{F(\theta)}{\Delta}, 0, F(\theta)\right], \tag{25}$$

where we have as anticipated $F(\Theta) \equiv F(\theta) = A_0|\cos\theta|$. Regarding BL geodesics, the transformed four-velocity in the DPG observer (covariant vector only is here shown due to its compact form) results in

$$\begin{aligned} U_\mu = \Bigg[&-1, -\frac{\sqrt{-\Delta U_\theta^2 + 2Mr(r^2 + a^2)}}{\Delta} \\ &+ \frac{\sqrt{2Mr(r^2 + a^2)}}{\Delta}, U_\theta, 0\Bigg]. \end{aligned} \tag{26}$$

The $U_\theta = 0$ case shows that

$$U_\mu = [-1, 0, 0, 0] \tag{27}$$

with

$$U^\mu = \left[1, -\frac{\sqrt{(2Mr)(r^2 + a^2)}}{\Sigma}, 0, 0\right]. \tag{28}$$

These equations show that U^μ here is the $T = \text{const}$ normal geodesic observer with T being the local proper time of observers in free fall along trajectories of constant θ and Φ [21]. Transforming the RW solution with $U_\theta = 0$ from BL to DPG coordinates, the electromagnetic field tensor has the only nonvanishing component:

$$F_{\Phi\theta} = A_{\Phi,\theta} \equiv \frac{dF(\theta)}{d\theta} = -A_0 \sin\theta\frac{|\cos\theta|}{\cos\theta}. \tag{29}$$

In consequence of the modulus in the vector potential, the Maxwell tensor is undefined on the equator while the Maxwell invariants, shown in Fig. 1, are

$$\begin{aligned} \mathcal{F} &\equiv \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\mathbf{B}^2 - \mathbf{E}^2) \\ &= \frac{A_0^2}{(r^2 + a^2)\Sigma} \geq 0 \\ \mathcal{G} &\equiv \frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} = 0, \end{aligned} \tag{30}$$

where regularity on the equator of the first invariant is an artifact of the identity $(|\cos\theta|/\cos\theta)^2 = 1$.

In the orthonormal locally Lorentzian frame naturally associated to the DPG normal observer [21],

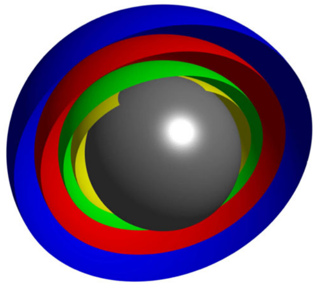

FIG. 1. Maxwell invariant $\mathcal{F}$ in DPG coordinates for a Kerr black hole, with parameters $A_0 = 0.1M$, $a = M = 1$ and $U_\theta = 0$. Colored shells represent selected isosurfaces: $\mathcal{F} = 3\times 10^{-3}$ (yellow), $\mathcal{F} = 2\times 10^{-3}$ (green), $\mathcal{F} = 1\times 10^{-3}$ (red), $\mathcal{F} = 5\times 10^{-4}$ (blue). The outer event horizon is represented by the gray solid sphere. Up to a $1/(8\pi)$ multiplicative constant factor, this plot gives also the energy density measured by the DPG normal observer discussed in the text.

$$
\begin{aligned}
e^{(0)}{}_\mu &= [-1, 0, 0, 0] \\
e^{(1)}{}_\mu &= \left[\sqrt{\frac{2Mr}{\Sigma}}, \sqrt{\frac{\Sigma}{r^2+a^2}}, 0, -\sqrt{\frac{2Mr}{\Sigma}}a\sin^2\theta\right] \\
e^{(2)}{}_\mu &= [0, 0, \sqrt{\Sigma}, 0] \\
e^{(3)}{}_\mu &= [0, 0, 0, \sqrt{r^2+a^2}\sin\theta],
\end{aligned} \tag{31}
$$

the tetradic Maxwell tensor $F_{(a)(b)} = e_{(a)}{}^\mu e_{(b)}{}^\mu F_{\mu\nu}$ has the only nonvanishing component,

$$
F_{(3)(4)} = -F_{(4)(3)} \equiv B_{(1)} \equiv B_{\hat{r}} = -\frac{A_0}{\sqrt{(r^2+a^2)\Sigma}}\frac{|\cos\theta|}{\cos\theta}. \tag{32}
$$

In such a frame the electric field is zero everywhere, as expected from the initial infinite conductivity assumption, while the magnetic field is nonzero. We point out that the modulus function in the vector potential produces a Maxwell tensor discontinuous on the equator so distributions are expected to occur in the four-current density J^μ in order to account for this fact, i.e.,

$$
J^T = 0, \tag{33}
$$

$$
J^r = \frac{\pi M r a A_0[\delta(\cos\theta)\Sigma\sin^2\theta - (a^2+r^2)|\cos\theta|]}{\sqrt{2Mr(r^2+a^2)}\Sigma^3}, \tag{34}
$$

$$
J^\theta = \frac{\pi M^2 r a A_0[a^2(a^2-r^2)\cos^2\theta - 3a^2r^2 - 5r^4]}{4\sqrt{2}(Mr(r^2+a^2))^{\frac{3}{2}}\Sigma^3} \times \sin\theta\frac{|\cos\theta|}{\cos\theta}, \tag{35}
$$

$$
J^\Phi = \frac{\pi A_0 \delta(\cos\theta)}{2\Sigma(r^2+a^2)}. \tag{36}
$$

It is interesting to plot the current density lines in spacetime. These are numerically obtained by solving the differential equations set $\frac{dx^\alpha}{d\lambda} = J^\alpha$, where λ parametrizes each curve although the presence of distributions makes the procedure nontrivial. We point out that the current density lines are A_0 independent because such a common multiplicative factor can be reabsorbed in the affine parameter λ. On the equatorial plane, i.e. $\theta = \pi/2$, the component J^θ is undefined, however in comparison to the remaining components of the four current density which contains diverging Dirac deltas activated on the same plane, it can be safely neglected there. Consequently, current density lines which start equatorially remain there confined. More in detail these current density lines are given by the differential equation

$$
\frac{dr}{d\Phi} = \lim_{\theta\to\frac{\pi}{2}}\frac{J^r}{J^\Phi} = a\frac{\sqrt{2M(r^2+a^2)}}{r^{\frac{3}{2}}}. \tag{37}
$$

Integration of this relation for different initial values of radius and angles on the equator leads to nonaxisymmetric spiraling configurations. For configurations starting off the equator instead, the current density lines have an axisymmetric structure. The absence of continuity and differentiability for the vector field J^μ on the equatorial plane implies that uniqueness for the solution is not guaranteed, so in principle two curves are allowed to meet on an equatorial point, closing in this way the electrical circuit [30]. This scenario is shown in Fig. 2 produced by using MATLAB®. In Boyer-Linquist coordinates this plot looks different especially close to the horizon because of the dragging of inertial frames which would make current density lines whirl around the black hole reaching the horizon asymptotically only as shown in Fig. 3.

Figure 2 would seem somewhat analogous to what happens for a classical Faraday conducting and rotating disk immersed into a uniform magnetic field. In the latter charge density occurs [31] together with spiraling current

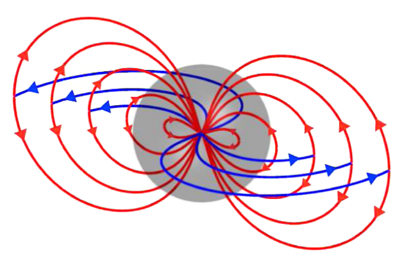

FIG. 2. Current density lines in DPG coordinates for a Kerr black hole with parameters $A_0 = 0.1M$, $a = M = 1$ and $U_\theta = 0$, where standard spherical coordinates have been used for visualization purposes only. Red lines represent off equatorial current density lines while blue ones denote the equatorial ones. The outer event horizon is represented by the grey transparent sphere. The current density lines are continued inside the horizon due to the use of horizon penetrating coordinates.

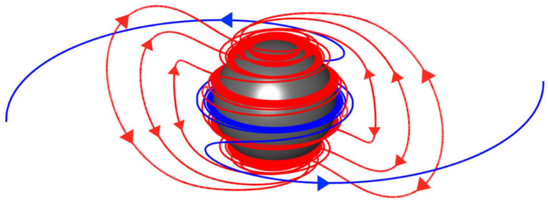

FIG. 3. Current density lines in BL coordinates for a Kerr black hole, with parameters $A_0 = 0.1M$, $a = M = 1$ and $U_\theta = 0$. Red lines represent off equatorial current density lines while the blue colored denote the equatorially confined ones. The outer event horizon is represented by the gray solid sphere.

density lines [32]. However, here we have a major difference in the fact that charge density $J^\mu U_\mu \equiv J_{(0)} = \rho$ measured by the normal DPG observer (28) vanishes in the whole spacetime. This is consistent with a zero electric field in this plasma comoving frame, as expected from basic Newtonian plasma physics in which charge density and electric field disappear in a locally comoving and corotating frame (in general difficult to be found) characterized by $\vec{\nabla} \times \vec{v} = 0$ [33]. This better explains the zero volumetric charge calculated by Ruffini and Wilson in their classical work [2]. As a consequence the DPG observer measures a purely spatial current density "paying the price" for sustaining the magnetic field. Of course projecting the J^μ in DPG coordinates obtained for $U_\theta = 0$ on different timelike congruences, for instance a geodesic four-velocity characterized by $U_\theta \neq 0$, leads to complicated volumetric charge densities localized in space-time instead. These results can be better understood in a geometrical language by starting from the infinite conductivity condition $F^{\mu\nu} U_\nu = 0$. Let us perform its divergence so that

$$(F^{\mu\nu} U_\nu)_{;\mu} \equiv F^{\mu\nu}{}_{;\mu} U_\nu + F^{\mu\nu} U_{\nu;\mu} = 0 \tag{38}$$

which, by using Maxwell equations, becomes

$$-4\pi J^\nu U_\nu + F^{\mu\nu} U_{\nu;\mu} = 0. \tag{39}$$

A simple check shows that the geodesics of RW analysis in Boyer-Linquist coordinates describing the matter flow are rotation free, i.e. $\omega_{\alpha\beta} = u_{[\alpha;\beta]} = 0$. Regarding DPG coordinates, the normal observer U^μ forming—by definition—a congruence which is hypersurface orthogonal to spacelike hypersurfaces [34] (in our case $T =$ constant ones) has again zero rotation because of the Frobenius theorem [26]. These two apparently different situations expressed in different coordinates refer to the same observer. Incidentally we remind that the fluid comoving observer and the normal one are the ones naturally involved in discussing relativistic Ohm's law and the ideal MHD condition (see exercise 11.16 in [35] and also [36]). The consequence of vanishing rotation tensor in the cases discussed above is that $U_{\nu;\mu}$ is a symmetric tensor, whose contraction with the always antisymmetric Maxwell tensor gives zero, i.e. $F^{\mu\nu} U_{\nu;\mu} = 0$. Consequently, for this geodesic observer we must have in Eq. (39) that $J^\nu U_\nu \equiv \rho = 0$, i.e. he/she always measures zero charge density (and zero electric field because of $F^{\mu\nu} U_\nu \equiv E^\mu = 0$) everywhere although a purely spatial current density accounting for the existence of the magnetic field is measured. More general geodesics (or accelerated observers) characterized by a nonzero rotation (i.e. $U_{\nu;\mu}$ nonsymmetric) would generate on the other hand a nonzero charge density. This result is of course in the spirit of relativity where local charge density is relative to the observer [33], still leading to important physical consequences as it happens for instance in the case of the well-known astrophysical Goldreich-Julian charge density mechanism [37]. These results show moreover also that already in the $U_\theta = 0$ case we cannot write the current density as a charged dust one $J^\mu \neq \rho U^\mu$. We compute now the electromagnetic energy stored outside the event horizon through a $T = \text{const}$ hypersurface as measured by DPG normal observers, always in the $U_\theta = 0$ case. We remark that this is a quasilocal quantity and depends on different possible cuts of spacetime [38,39]. In formulas,

$$E_\Sigma(U) = \int_\Sigma T^{(\text{em})}_{\mu\nu} U^\mu d\Sigma^\nu, \tag{40}$$

where Σ is a bounded hypersurface containing a portion of spacetime, and U^μ is the four-velocity of the normal observer. Local energy density turns out to be then

$$\mathcal{E} = T^{(\text{em})}_{\mu\nu} U^\mu U^\nu \equiv \frac{A_0^2}{8\pi(r^2 + a^2)\Sigma}, \tag{41}$$

where $T^{(\text{em})}_{\mu\nu}$ is the electromagnetic energy-momentum tensor expressed in DPG coordinates. Let us assume that the boundary S of Σ^ν be the 2-surface $r = \mathcal{R} = \text{const}$, so that energy (40) turns out to be

$$\begin{aligned} E(\mathcal{N})_{(r_+,\mathcal{R})} &= 2\pi \int_{r_+}^{\mathcal{R}} \int_0^\pi \mathcal{E}(\mathcal{N}) \sqrt{h_\mathcal{N}} dr d\theta \\ &= \frac{A_0^2}{2a} \left[\arctan \frac{\mathcal{R}}{a} - \arctan \frac{r_+}{a} \right], \end{aligned} \tag{42}$$

where $h_\mathcal{N} = \Sigma^2 \sin^2\theta$ is the determinant of the induced metric. The apparently singular behavior of the total energy in the $a \to 0$ limit can be removed by setting $a = 0$ in (42) before integration so that

$$E(\mathcal{N})_{(r_+,\mathcal{R})} = \frac{A_0^2}{2} \left[\frac{1}{r_+} - \frac{1}{\mathcal{R}} \right]. \tag{43}$$

In order to induce a net charge on the surface of the black hole, Ruffini and Wilson adopted a nonvanishing $U_\theta(\theta) = -U_\theta(\pi - \theta) = \text{const}$. Due to the absence in the literature of Doran-like coordinates for nonzero U_θ we will not extend here the previous analysis leaving it to the future studies. We can finally verify *a posteriori* that the initial assumption which led us to obtain our analytical solution result satisfied i.e. the role of the magnetic force term $f_\mu = F_{\mu\nu}J^\nu$ being negligible. From Eqs. (29) and (33), we obtain in fact

$$f_T = 0, \qquad f_r = 0, \qquad f_\theta = \frac{A_0^2 \sin\theta\cos\theta\pi\delta(\cos\theta)}{2|\cos\theta|\Sigma(r^2+a^2)}$$
$$f_\Phi = \frac{A_0^2\pi a\sqrt{M}\sin^2\theta[r^2(5r^2+3a^2)+a^2\cos^2\theta(r^2-a^2)]}{4\sqrt{2r}(r^2+a^2)^{\frac{3}{2}}\Sigma^3}, \tag{44}$$

which is decreasing with the distance from the black hole as well as its invariant norm. In order to have an "order of magnitude" estimate of these magnetic effects, we evaluate for the sake of simplicity the square of the norm of f^μ at a representative point P of the event horizon r_+ (where relativistic effects are expected to be more relevant), off the equatorial plane (where the previously discussed mathematical distributions are present). We select moreover an extreme black hole with $a = M$ so that the point P is characterized by $r_+ = M$ together with a generic angle Φ and for simplicity $\theta = \pi/4$. We write also $A_0 = \alpha M$ with α being a dimensionless number whose modulus is assumed in our analysis to be much smaller than 1. This choice satisfies the test field approximation in our perturbation problem as it can be easily seen by comparing from formula (43) the total electromagnetic energy outside the spherical black hole, i.e. $E(\mathcal{N})_{(r_+,+\infty)} = A_0^2/(2M) \equiv \alpha^2 M/2$ or the extreme rotating one in formula (42), i.e. $E(\mathcal{N})_{(r_+,+\infty)} = \pi A_0^2/(8M) \equiv \alpha^2\pi M/8$ with the total spacetime mass M measured at infinity as described by the well-known laws of black hole mechanics [40]. It is evident in fact that the value $\alpha = 0.1$ chosen for the plots is small enough for having physically sound results. Coming back to the magnetic force term estimate at a point of the black hole surface, after some simple algebra, we obtain $\sqrt{f_\mu f^\mu}|_P = \frac{2\pi}{27}\frac{\alpha^2}{M^3}$. Using Newtonian physics in a very first approximation which neglects rotation, the gravitational force density pointwise roughly behaves as $\rho M/r^2$ with $\rho \sim \dot{M}/r^2$ (here $\dot{M} = dM/dr$) estimated for the sake of simplicity by using Tolman-Oppenheimer-Volkoff equations [23]. The ratio of the magnitude of magnetic force term f^μ over the gravitational effects above, estimated on a point sufficiently close the horizon (i.e. $r \sim M$), goes as $\alpha^2/\dot{M}$. This shows that in our situation a test electromagnetic field or a nontrivial accretion process will make magnetic forces negligible leading—in practice—to a geodesic motion for plasma.

IV. CONCLUSIONS

In this work we revisited analytically the Ruffini-Wilson work developed for studying the accretion of perfect neutral plasma onto a Kerr black hole. After an analysis of the Maxwell invariants, selecting the $U_\theta = 0$ condition we introduced in the discussion the Doran-Painlevé-Gullstrand coordinates. The normal observer in these coordinates is equivalent to the comoving plasma one so measuring both zero electric field and volumetric charge density everywhere, in agreement with the infinite conductivity condition for plasma. The formulation here presented can possibly be used also to study the high-energy astrophysical phenomena due to the accreting process of highly magnetized neutral plasma into the rotating black holes. This aspect, as well as the more complex situations of nonvanishing U_θ or even U_ϕ, will be investigated in future studies.

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Correction: The email address was missing for the corresponding author at publication; the footnote has now been inserted.