Black hole mass formula in the membrane paradigm

José P. S. Lemos[*]

Centro de Astrofísica e Gravitação—CENTRA, Departamento de Física, Instituto Superior Técnico—IST,
Universidade de Lisboa—UL, Avenida Rovisco Pais 1, Lisboa 1049-001, Portugal,
and Gravitational Physics Group, Faculty of Physics, University of Vienna,
Boltzmanngasse 5, A1090 Wien, Austria

Oleg B. Zaslavskii[†]

Department of Physics and Technology, Kharkov V. N. Karazin National University,
4 Svoboda Square, Kharkov 61022, Ukraine,
and Institute of Mathematics and Mechanics, Kazan Federal University,
18 Kremlyovskaya street, Kazan 420008, Russia

The membrane paradigm approach adopts a timelike surface, stretched out off the null event horizon, to study several important black hole properties. We use this powerful tool to give a direct derivation of the black hole mass formula in the static and stationary cases without and with electric field. Since here the membrane is a self-gravitating material system, we go beyond the usual applicability on test particles and test fields of the paradigm.

I. INTRODUCTION

The event horizon of a black hole acts like a membrane with well-defined matter fluid properties. This was understood first by Hawking and Hartle from matter fields entering the black hole and changing its area in a prescribed way [1] and Hanni and Ruffini [2] in an analysis of the lines of forces of an electrically charged particle around a black hole. It was fully recognized as a membrane with electric resistivity by Znajek [3] and Damour [4], who also showed that the membrane obeys a Navier-Stokes type equation with surface viscosity [5]. The membrane, being a substitute for the event horizon, is a lightlike hypersurface in these works. For practical reasons, it can be useful to stretch slightly the event horizon, into a stretched horizon, and turn the lightlike membrane into a timelike membrane. More specifically, the rationale is to replace the event horizon by a surface in its outside vicinity as if the horizon were stretched, with its interior being essentially a vacuum spacetime. This stretched horizon acts then like a 2-dimensional membrane evolving in time, and by imposing correct boundary conditions on this membrane, one finds the desired results. The stretched horizon is thus a timelike boundary, which makes this setting prone to using a $3 + 1$ formalism, with three spatial dimensions and one time dimension. In physical terms, some processes can be better understood and more intuitive in this timelike membrane than the lightlike boundary of the event horizon.

This idea of studying the event horizon as a stretched horizon, through a membrane in a $3 + 1$ spacetime formalism, was devised by Thorne and collaborators [6–9] and is called the membrane paradigm [10]. In this $3 + 1$ formalism, a black hole resembles a star, since now it is realized as a 3-dimensional space, with an interior, a surface membrane with appropriate boundary conditions, and an exterior, evolving in time. Following the approach, one performs all the calculations in the timelike membrane, located infinitesimally close to the true black hole horizon, imposes correct boundary conditions on it, and then finally takes the limit to the horizon, a lightlike surface, to find the correct desired results. The $3 + 1$ membrane paradigm is able to recover all the properties previously found, see Ref. [10], notably the interpretation of the membrane as a matter fluid with dissipative properties. The membrane paradigm approach has been developed and applied in several directions. We mention a few. The black hole complementarity idea was developed using the membrane paradigm [11], and a relation to the fuzzball model has been put forward [12], showing that the approach has found echo not only in astrophysics but also in fundamental physics. An action for the membrane was devised and applied to distinct black hole settings [13], and the black hole entropy formula was found through an Euclidean action membrane approach [14]. The quasinormal spectrum of black holes was explored using the membrane paradigm in Ref. [15], the physics of jets through the membrane paradigm was studied in Ref. [16], and the paradigm was applied to a setting with cosmological horizons in Ref. [17]. The

[*]joselemos@ist.utl.pt
[†]zaslav@ukr.net

Einstein field equations and the Navier-Stokes equations connection was developed in Ref. [18], and it was then realized that the fluid/gravity duality is in fact a parent of the membrane paradigm [19–22]. In this duality, fluids and gravity are similar, like in the membrane paradigm where the horizon, a pure gravitational entity, behaves as a membrane composed of matter with fluid behavior, and Einstein equations can be put in a Navier-Stokes form. The membrane paradigm was applied to other theories that contain black holes, namely, to $f(R)$ gravities [23], to the Gauss-Bonnet theory [24], to the Lovelock theory [25], and to Chern-Simons gravity [26].

Now, one interesting and important feature of black holes is that they possess a simple mass formula. For the Kerr-Newman family of black holes, this formula is the Smarr formula [27], and it expresses the mass of a black hole as a bilinear form in terms of the products of the surface gravity and the area, the electric potential and the electric charge, and the angular velocity and the angular momentum of the black hole. The formula was generalized to include black holes with surrounding matter [28] (see also Refs. [29,30]), and in an explicit form for extremal black holes [31], and in addition it was shown by Bardeen *et al.* that the Smarr mass formula is connected with the differential formula for the same quantities as was displayed in the first law of black hole mechanics [28]. The Smarr formula has also been deduced for a number of different types of black holes; see, e.g., Ref. [32]. The black hole mass formula as found in Ref. [28] is calculated from first principles using the Komar mass [33], a mass definition suited for vacuum spacetimes with a timelike Killing vector.

Since the membrane paradigm has shown to be a powerful tool for considering properties in the vicinity of a black hole, it is of interest to use the membrane paradigm to find from direct principles the black hole mass formula. Moreover, in using the membrane paradigm in this situation, we are paying attention to the properties of the membrane itself that are connected to its self-gravitation and thus going beyond the use of the paradigm for test fields in the vicinity of the horizon of a given black hole background. Now, to have a mass formula, one has to start with a mass definition, and for pure vacuum, one uses the Komar mass [33]. In the presence of nonvacuum spacetimes with matter field content, a more suitable mass definition is the Tolman mass [34] (see also Ref. [35]), which further requires that the spacetime is static or stationary. Using the Tolman mass definition, formulas for black holes were deduced in the quasiblack hole approach [36,37]. To calculate the black hole mass formula through the membrane paradigm, as is our intention, the Tolman mass is also the convenient definition. Given an interior and an exterior spacetime, the membrane, a matter field, reveals itself through the junction conditions [38–40]. The membrane is thus a cut between the inner and outer

spacetimes that provides the mass in a consistent manner. A particularly important case is when the membrane is at an infinitesimal distance from the horizon. In such a configuration, one should get the black hole mass formula.

The aim of the present paper is to give a direct derivation of the black hole mass formula through the membrane paradigm. When the membrane is far from its own gravitational radius, one gets a mass formula for the membrane spacetime in general. When the membrane comes close to its own gravitational radius, i.e., in the horizon limit, and upon using appropriate boundary conditions which signal the presence of a horizon, then the membrane formalism yields the black hole mass formula. For the inner region, one must note that in the horizon limit all compatible interior spacetimes, i.e., interior spacetimes with appropriate boundary and regularity conditions, give the same mass formula, so one can choose the simplest inner spacetime, i.e., Minkowski spacetime. The outer spacetime we consider is quite general. We only impose that it is static or stationary and the spatial sections are topological spheres. In this approach, it is the Einstein field equation at the membrane thin layer that conspires to give the black hole mass formula. In deriving the black hole mass formula, we consider cases in which there is matter outside the membrane and the black holes are distorted by outer force perturbations. We will use results given in Refs. [41–43].

The paper is organized as follows. In Sec. II, we derive the black hole mass formula from the Tolman mass for the simplest case, a static membrane with no electric field. We do not impose other symmetries, only staticity. In Sec. III, we derive the black hole mass formula from the Tolman mass for an electric shell. Again, we do not impose other symmetries, only staticity. In the horizon limit, we show that the electric potential is constant at the horizon. In Sec. IV, we derive the black hole mass formula from the Tolman mass for a rotating membrane. We only impose stationarity. In Sec. V, we conclude.

II. BLACK HOLE MASS FORMULA IN THE MEMBRANE PARADIGM: STATIC MEMBRANE WITH NO ELECTRIC FIELD

A. Preliminaries

1. Gravitational field

Let us consider a 4-dimensional spacetime with coordinates x^μ and interval ds given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where $g_{\mu\nu}$ is the metric representing the gravitational field and with μ, ν being spacetime indices, i.e., $\mu, \nu = 0, 1, 2, 3$, 0 being a time index and 1, 2, and 3 being spatial indices. Consider further that the spacetime is static, and write the metric in a $3 + 1$ split as

$$ds^2 = -N^2 dt^2 + g_{ik} dx^i dx^k, \qquad (2)$$

where N is a function of the spatial coordinates and i, $k = 1, 2, 3$ are the spatial indices.

Let us further consider that the 4-dimensional spacetime contains an infinitesimally thin membrane. The membrane is not necessarily a 2-sphere; it can be some surface with the topology of a 2-sphere. Inside the membrane, there is vacuum, and we assume spacetime there is flat. The assumption that spacetime inside the membrane is flat simplifies the calculations; other assumptions can be provided, giving quantitatively different results. However, when the membrane is at its own horizon radius, it does not matter what we have assumed for the spacetime inside; any compatible inner spacetime in this limit yields the same results. This is the basis of the membrane paradigm, namely, one excises the spacetime interior to the horizon, so it can be any, and works out the properties of the horizon seen as a membrane. So, assuming an inner flat spacetime is an assumption without loss of generality. Outside the membrane, spacetime is static and consistent with the existence of a membrane. Let us now state precisely the complete gravitational field, i.e., the complete spacetime.

Inside the membrane, the vacuum flat spacetime metric can be written in Gaussian coordinates (t, l, x^2, x^3), from Eq. (2), as

$$ds^2 = -N_0^2 dt^2 + dl^2 + \gamma_{ab} dx^a dx^b, \qquad (3)$$

where N_0 is some constant, the metric coefficients γ_{ab} do not depend on t, and $a, b = 2, 3$.

Outside, the membrane generates a generic static metric. In Gaussian coordinates (t, l, x^2, x^3), the metric of Eq. (2) takes the form

$$ds^2 = -N^2 dt^2 + dl^2 + \gamma_{ab} dx^a dx^b, \qquad (4)$$

where the metric coefficients N and γ_{ab} do not depend on t and $a, b = 2, 3$. Note that the determinants of the metrics have the relations $\sqrt{-g} = N\sqrt{g_3} = N\sqrt{\gamma}$, where g is the determinant of the spacetime metric [see Eq. (1)], g_3 is the determinant of the spatial 3-metric [see Eq. (2)], and γ is the determinant of the spatial 2-metric [see Eqs. (3) and (4)].

The membrane is located at some $l = l_0$ by assumption, and let the value of N be constant on the membrane, $N = N_0$. This assumption is not essential but simplifies some formulas. The matching conditions from the inside to the outside require, first, that $N_- = N_+ = N_0$ on both sides of the membrane, where $-$ corresponds to the inner side and $+$ corresponds to the outer side at l_0, and, second, that $\gamma_{ab}^- = \gamma_{ab}^+ \equiv \gamma_{0ab}$ also on both sides of the membrane. The matching conditions also require that the membrane has some stress-energy tensor.

We assume that *a priori* there is no horizon. When we take the limit to the membrane gravitational radius, i.e., to its own horizon, we obtain a stretched horizon where the properties of a black hole can be worked out.

2. Energy-momentum tensor

The spacetime has some energy-momentum tensor $T_{\mu\nu}$. We can divide the energy-momentum tensor as the sum of the inner energy-momentum tensor $T_{\text{in}\,\mu\nu}$, the membrane energy-momentum tensor $T_{\text{membrane}\,\mu\nu}$, and the outer energy-momentum tensor $T_{\text{out}\,\mu\nu}$,

$$T_{\mu\nu} = T_{\text{in}\,\mu\nu} + T_{\text{membrane}\,\mu\nu} + T_{\text{out}\,\mu\nu}. \qquad (5)$$

Since inside it is Minkowski, the energy-momentum tensor inside is zero,

$$T_{\text{in}\,\mu\nu} = 0. \qquad (6)$$

Since the membrane is infinitesimal, situated at l_0, we can write the membrane energy-momentum tensor as

$$T_{\text{membrane}\,\mu\nu} = S_{\mu\nu}\delta(l - l_0), \qquad (7)$$

where $\delta(l - l_0)$ is the Dirac delta function and $S_{\mu\nu}$ is a surface energy-momemtum tensor defined at the membrane. The outside energy-momentum tensor, with support on the outer region, can be divided if we wish into matter and other fields, so we can write

$$T_{\text{out}\,\mu\nu} = T_{\text{out matter}\,\mu\nu} + T_{\text{other fields}\,\mu\nu}. \qquad (8)$$

The term $T_{\text{out other fields}\,\mu\nu}$ can contain an electromagnetic field, for instance. In writing Eq. (8), we have assumed that there are no interaction cross terms between gravity and matter or between gravity and other fields. This means we assume minimal coupling throughout.

B. Tolman mass formula for a static membrane with no electric field

There are several mass formulas for spacetimes, notably the Komar mass formula [33] and the Tolman mass formula [34], both for static and stationary spacetimes. The two formulas are related [35], as our calculation will also show. The Tolman mass formula applies neatly for spacetimes that contain matter. Since our spacetime possesses a membrane, and so matter, we use the Tolman mass formula. We will first write a mass formula valid for a membrane at any radius l_0 larger than its own gravitational radius. Then, afterward, we will apply this formula to the case when the membrane is at its own gravitational radius, i.e., at the horizon. We will then find the black hole mass formula.

The Tolman mass for a given static spacetime with a stress-energy tensor $T_{\mu\nu}$ is defined as

$$M = \int_{\Sigma} (-T^0_0 + T^k_k)\sqrt{-g}d^3x, \qquad (9)$$

with g being the determinant of the metric $g_{\mu\nu}$ in Eq. (1), $d^3x = dx^1dx^2dx^3$, and the integral is performed over a 3-space Σ with $t = $ constant. In the metric Eq. (2), the mass formula is $M = \int_{\Sigma}(-T^0_0 + T^k_k)N\sqrt{g_3}d^3x$, with g_3 being the determinant of the metric g_{ik}. In addition, since $g_3 = \gamma$ where γ is the determinant of the 2-metric γ_{ab}, we have

$$M = \int_{\Sigma} (-T^0_0 + T^k_k)N\sqrt{\gamma}d^3x. \qquad (10)$$

We will resort to these formulas, mainly to Eqs. (9) and (10), for finding the Tolman mass.

Now, there are clearly tree distinct regions, as we have made explicit in the splitting of the energy-momentum tensor, Eq. (5). So, here, one should also split the mass formula into three parts, namely, the inner mass $M_{\rm in}$, the membrane mass $M_{\rm membrane}$, and the outside mass $M_{\rm out}$. Thus, we write

$$M = M_{\rm in} + M_{\rm membrane} + M_{\rm out} \qquad (11)$$

and calculate the contribution of each part to the total spacetime mass M.

Inside is vacuum, so from Eqs. (6) and (10), one has

$$M_{\rm in} = 0. \qquad (12)$$

The membrane term $M_{\rm membrane}$ is quite interesting. From Eq. (10), we write the contribution to the membrane mass as

$$M_{\rm membrane} = \int_{\rm membrane} (-T_{\rm membrane\,0}{}^0 + T_{\rm membrane\,k}{}^k)N\sqrt{\gamma}d^3x. \qquad (13)$$

Since the membrane is infinitesimally thin, there is the delta contribution given in Eq. (7). Moreover, due to the 2-dimensional character of the membrane, there are no radial stresses, i.e., $S^l_l = 0$. Performing then the integral over l across the membrane in Eq. (13), we get

$$M_{\rm membrane} = \int_{\rm membrane} (-S^0_0 + S^a_a)NdS, \qquad (14)$$

where $dS = \sqrt{\gamma(l_0)}d^2x$, and $d^2x = dx^2dx^3$. Given an outside spacetime and an inside one, one has from the junction conditions [38–40] the relation $8\pi S^\nu_\mu = [[K^\nu_\mu]] - \delta^\nu_\mu[[K]]$, where K^ν_μ is the extrinsic curvature tensor, and the symbol $[[...]]$ means $[[...]] = (...)_+ - (...)_-$ with $+$ being the value of the relevant quantity at the membrane from the outside and $-$ being the value of the same relevant quantity at the membrane from the inside. In our case, we

find $8\pi(-S^0_0 + S^a_a) = -2[[K^0_0]]$. Put n^μ as the unit vector normal to the membrane. Since $K_{\mu\nu} = -\nabla_\nu n_\mu$, where ∇_μ denotes covariant derivative, and $n_\mu = \frac{\partial l}{\partial x^\mu}$, we can calculate K^0_0. Indeed, $K^0_0 = -\nabla_0 n^0 = -\frac{1}{N}\frac{\partial N}{\partial l}$. As a result, we obtain $8\pi(-S^0_0 + S^a_a) = \frac{2}{N}[(\frac{\partial N}{\partial l})_+ - (\frac{\partial N}{\partial l})_-]$, but since inside is Minkowski and $N = N_0$, the expression becomes $8\pi(-S^0_0 + S^a_a) = \frac{2}{N}(\frac{\partial N}{\partial l})_+$. Thus, the membrane mass is

$$M_{\rm membrane} = \int_{\rm membrane} \sigma dS, \qquad (15)$$

where

$$\sigma = \frac{1}{4\pi}\left(\frac{\partial N}{\partial l}\right)_+ \qquad (16)$$

is the surface mass density of the membrane.

The contribution from the outside can be written as

$$M_{\rm out} = \int_{\rm out} (-T_{\rm out\,0}{}^0 + T_{\rm out\,k}{}^k)N\sqrt{\gamma}d^3x, \qquad (17)$$

where one can further split into matter fields and other fields, as we have done in Eq. (8).

Thus, putting together Eqs. (12), (15), (16), and (17), into Eq. (13), we obtain the mass formula from the Tolman definition for a spacetime with a membrane. As an example, if the outside spacetime is vacuum Schwarzschild, then $M_{\rm in} = 0$, $M_{\rm membrane} = M$, and $M_{\rm out} = 0$, so the Tolman formula gives $M = M_{\rm membrane} = M$. It is interesting to note that Tolman formula puts all the mass, i.e., membrane rest mass plus gravitational mass, in the membrane itself.

C. Black hole mass formula in the membrane paradigm with no electric field

1. Introduction

The Tolman formula is thus valid for a spacetime with a membrane and without horizons. We now want to test the Tolman formula in the strongest gravitational field possible, i.e., in the black hole limit. In this limit, we should take $N_0 \to 0$, where $N_0 \to 0$ is the value of N at the membrane.

Before we proceed, a remark is in order. In the above treatment, we assumed that the spacetime metric inside the membrane is the Minkowski metric. However, in the black hole limit, our results do not change if for the interior region we choose some other metric compatible with the exterior. Indeed, the continuity of the metric implies the boundary condition $N_- = N_+$, where for simplicity we have chosen $N_- = N_+ = N_0 = $ constant, on the boundary. The black hole limit under discussion means $N_0 \to 0$. Making a nonrestrictive assumption that $\frac{\partial N}{\partial l} \geq 0$, which holds for all reasonable spacetimes under consideration, one has that everywhere inside $N \to 0$ uniformly as well.

Therefore, $\left(\frac{\partial N}{\partial l}\right)_- \to 0$, and we again obtain Eq. (15) together with Eq. (16).

2. Black hole mass formula in the membrane paradigm with no electric field

In the black hole limit, write $l_h \equiv l_{0 \text{ at the horizon limit}}$, and note that indeed $N \to 0$. However, the possible problematic part for the mass, the membrane part given by Eq. (15) with (16), does not depend on N; rather, it depends on $\frac{\partial N}{\partial l}$. So, we have to calculate $\frac{\partial N}{\partial l}$ from the outside at the horizon limit. For that, we observe that for a regular horizon the metric potential N just outside the horizon must obey [41]

$$N = \kappa(l - l_h) + a(x^a)(l - l_h)^3 + \cdots, \qquad (18)$$

where κ is the surface gravity, a constant, and $a(x^a)$ is a function of the angular coordinates x^a. Thus,

$$\kappa = \lim_{N \to 0} \left(\frac{\partial N}{\partial l}\right)_+. \qquad (19)$$

Comparing Eqs. (16) and (19), we find also that $\lim_{N \to 0} \sigma = \frac{1}{4\pi}\kappa$. So, using that κ is a constant, we can perform the integral in Eq. (15) to obtain

$$M_{\text{membrane at the horizon}} = \frac{1}{4\pi}\kappa A, \qquad (20)$$

where $A \equiv A_h$ is the horizon area.

Inserting Eqs. (12), (17), and (20) into Eq. (11) yields the black hole mass formula

$$M = \frac{1}{4\pi}\kappa A + M_{\text{out}}. \qquad (21)$$

This black hole mass formula obtained through the membrane paradigm approach is the same as that obtained by other methods [28] (see also Refs. [29,30]). When $M_{\text{out}} = 0$, it is the Smarr formula for a static, i.e., Schwarzschild, black hole in general relativity [27].

Price and Thorne [8] in their detailed paper on the membrane paradigm, do not arrive at our Eq. (21) for the mass formula of a black hole. The key point in our derivation is the use of the Tolman mass definition which includes under its integral, not only the T_0^0 component at the membrane, but also the membrane T_k^k stress components. These latter contribute decisively to the integral at the horizon and thus to the black hole mass.

3. Interpretation of the surface gravity

The surface gravity of a body is defined as the acceleration of a test particle at the body's surface. If the body is a black hole, this acceleration at the black hole's surface, the horizon, is infinite, but the normalized surface gravity, or simply the black hole's surface gravity κ, defined as the proper acceleration times the redshift factor at the horizon, is finite. The black hole's surface gravity κ also defines the Hawking temperature.

But now note that, as a byproduct of the formalism we have developed within the membrane paradigm, we have obtained one more interpretation for the surface gravity. Indeed, comparing Eqs. (16) and (19) and using also Eq. (20), we can write

$$\lim_{N \to 0} \sigma = \frac{1}{4\pi}\kappa = \frac{M_{\text{membrane at the horizon}}}{A}, \qquad (22)$$

i.e., the quantity κ is nothing other than 4π times the surface energy density on the membrane in the horizon limit. This surface density is a constant in the limit under discussion. Such an interpretation of the surface gravity as a surface energy density, given in Eq. (22), could not be given in terms of a true black hole since a true black hole does not have any membrane on the horizon at all and the mass formula is obtained from a different approach. Indeed, in the black hole approach, the mass is defined at infinity and thus cannot be interpreted as localized at the horizon.

Our interpretation relies on self-gravitating effects since the expression for the membrane's surface stress-energy tensor implies the validity of the Einstein equations [38–40]. Indeed, Eq. (22) is a reflection of the Einstein equation, with $\frac{1}{4\pi}\kappa$ being related to the gravitational, geometrical, part of the equation and $\frac{M_{\text{membrane at the horizon}}}{A}$ being related to the matter, energy-momentum tensor, part of the equation. Equation (22) with the interpretation of the surface gravity κ as a surface energy density σ is new. Indeed, such an interpretation can be put forward only after one finds the black hole mass formula, Eq. (21), through the membrane paradigm, as we did.

III. BLACK HOLE MASS FORMULA IN THE MEMBRANE PARADIGM: STATIC MEMBRANE WITH ELECTRIC FIELD

A. Preliminaries

1. Gravitational and electric fields

We assume the same type of membrane configuration and the same type of gravitational field as in the previous section, so that Eqs. (1)–(4) still hold. But now, we consider further that there is in addition an electric field, and so one should consider the Einstein-Maxwell equations for finding a mass formula for the spacetime with an electric membrane and a mass formula in the black hole limit.

Since the gravitational part has been treated in the last section, we now analyze the electromagnetic field which has its own special features. The electromagnetic field is characterized by a generic antisymmetric Maxwell tensor $F_{\mu\nu}$. The Maxwell equations for $F_{\mu\nu}$ are

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu, \tag{23}$$

i.e.,

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} F^{\mu\nu})}{\partial x^\nu} = 4\pi j^\mu. \tag{24}$$

where ∇_ν denotes covariant derivative and j^μ is the generic electric 4-current. There is another set of Maxwell equations, namely,

$$\frac{\partial}{\partial x^\alpha} (\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu}) = 0, \tag{25}$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. This last set of equations, Eq. (25), in turn, permits the electromagnetic Maxwell field $F_{\mu\nu}$ to be written in terms of a 4-potential A_μ as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{26}$$

The 4-potential A_μ can be split as $A_\mu = (\varphi, A_i)$, for some electric potential φ and a 3-potential A_i. For an electric ansatz, as we will assume, $A_i = 0$, and so

$$A_\mu = \varphi \delta_\mu^0, \tag{27}$$

where δ_μ^ν is the Kronecker delta. Putting Eq. (27) into Eq. (26) one finds that the only nonzero components of $F_{\mu\nu}$ are F_{0i}, i.e.,

$$F_{0i} = -\frac{\partial \varphi}{\partial x^i}. \tag{28}$$

This confirms that there is pure electric field, since the components F_{0i} are related to it. All this formulation is generic for an electric ansatz, and since the problem is static, the quantities are time independent.

We now specify that the interior spacetime is a flat spacetime with no electric field, that there is an electric membrane, and that there is some electric field outside compatible with the existence of the membrane. For the inside, we have a zero Maxwell tensor, and for the outside, we have a Maxwell tensor that we write as $F_{\mu\nu}^{\text{out}}$, having in mind that F_{0i}^{out} are the only nonzero components from Eq. (28). We can then write an expression for the Maxwell tensor field $F_{\mu\nu}$, valid throughout the whole spacetime, such that it represents zero Maxwell tensor inside the membrane, and $F_{\mu\nu}^{\text{out}}$ at and outside it. Thus, in the spirit of the membrane paradigm, we write for the Maxwell tensor the following expression,

$$F^{\mu\nu} = F_{\text{out}}^{\mu\nu} \theta(l - l_0), \tag{29}$$

where $\theta(l - l_0)$ is the step function. Equation (29) has implicitly in it that there are three regions, the inner, the

membrane, and the outer regions. It has also implicitly in it, due to the θ function, that inside we have $F_{\mu\nu}^{\text{in}} = 0$. Then, since Eq. (29) is a product, and the Maxwell equation, Eq. (24), involves a derivative for $F^{\mu\nu}$, the current j^μ is a sum of two currents, namely,

$$j_{\text{membrane}}^\mu + j_{\text{out}}^\mu = j^\mu. \tag{30}$$

Let us analyze each current at a time.

For the membrane current, j_{membrane}^μ, from the Maxwell equations, Eq. (24), and the specific form of the Maxwell field given in Eq. (29), we have that

$$j_{\text{membrane}}^\mu = \frac{1}{4\pi} F_{\text{out}}^{\mu\nu} \delta(l - l_0) \frac{\partial l}{\partial x^\nu}. \tag{31}$$

This term is very important and can yield a singular term when one lowers the membrane to its own gravitational radius or horizon. Instead of j_{membrane}, this current could be called j_{sing} since it can yield a singular term at the horizon. Recalling the electric anzatz, $A_i = 0$, and that the only components of $F_{\mu\nu}$ that do not vanish are F_{0i}, see Eq. (28), we find that the only nonzero component in Eq. (31) is

$$j_{\text{membrane}}^0 = \frac{1}{4\pi} F_{\text{out}}^{0l} \delta(l - l_0). \tag{32}$$

Since F^{0l} corresponds to a radial electric field E^l, say, we see that at the membrane there is an electric current related to the radial electric field. From this equation, we can find an expression for the total electric charge Q at the membrane, $l = l_0$. This will take a little detour. For some generic 4-current j^μ, the electric charge is defined as $Q = \int_\Sigma j^\mu d\Sigma_\mu$, where $d\Sigma_\mu \equiv \frac{1}{3!} \varepsilon_{\mu\alpha\beta\gamma} [\frac{\partial(x^\alpha, x^\beta, x^\gamma)}{\partial(a,b,c)}] da\, db\, dc$ is the 3-dimensional volume element of a hypersurface Σ parametrized by (a, b, c) through $x^\alpha = x^\alpha(a, b, c)$, $[\frac{\partial(x^\alpha, x^\beta, x^\gamma)}{\partial(a,b,c)}]$ is the 3×3 Jacobian determinant, and $\varepsilon_{\mu\alpha\beta\gamma}$ is the Levi-Civita tensor totally antisymmetric tensor which can be written as $\varepsilon_{\mu\alpha\beta\gamma} = \sqrt{-g} \epsilon_{\mu\alpha\beta\gamma}$ with $\epsilon_{\mu\alpha\beta\gamma}$ being the alternating symbol or Levi-Civita tensor density (being equal to 1 if μ, ν, α, and β is an even permutation of 0, 1, 2, 3, equal to -1 if it is an odd permutation of 0, 1, 2, 3, and equal to zero otherwise). Here, since the only nonzero component of j^μ is j^0, see Eq. (32), we have $Q = \int_\Sigma j^0 d\Sigma_0$. Parametrizing Σ by $a = l$, $b = x^2$, and $c = x^3$, we have $Q = \int_\Sigma j^0 \sqrt{-g} dl dx^2 dx^3$, or using Eq. (32), we get $Q = \frac{1}{4\pi} \int_\Sigma F_{\text{out}}^{0l} \delta(l - l_0) N \sqrt{\gamma} dl d^2 x$, where $d^2 x = dx^2 dx^3$. Integrating through the delta function, it becomes $Q = \frac{1}{4\pi} \int_{\text{membrane}} F_{\text{out}}^{0l}(l_0) N(l_0) \sqrt{\gamma(l_0)} d^2 x$. Defining $dS \equiv \sqrt{\gamma(l_0)} d^2 x$ as the invariant area element on the 2-dimensional membrane, we have $Q = \frac{1}{4\pi} \int_{\text{membrane}} F_{\text{out}}^{0l}(l_0) N(l_0) dS$. It is interesting to note that this result can also be found without the use of the δ function. Indeed, from $Q = \int_\Sigma j^\mu d\Sigma_\mu = \frac{1}{4\pi} \int_\Sigma \nabla_\nu F^{\mu\nu} d\Sigma_\mu$,

where Eq. (23) has been used, one can invoke Stokes' theorem which for this case reads $2 \int_\Sigma \nabla_\nu F^{\mu\nu} d\Sigma_\mu = \int_S F^{\mu\nu} dS_{\mu\nu}$, yielding $Q = \frac{1}{8\pi} \int_S F^{\mu\nu} dS_{\mu\nu}$, where $dS_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} [\frac{\partial(x^\alpha, x^\beta)}{\partial(e,f)}] de df$ is the 2-dimensional area element of the boundary of Σ, namely, a 2-surface S parametrized by $x^\alpha = x^\alpha(e,f)$, with $[\frac{\partial(x^\alpha, x^\beta)}{\partial(e,f)}]$ being the 2×2 Jacobian determinant. Then, one finds again $Q = \frac{1}{4\pi} \int_{\text{membrane}} F_{\text{out}}^{0l}(l_0) N(l_0) dS$. Thus, we can write for the total charge

$$Q = \int_{\text{membrane}} \sigma_e dS, \qquad (33)$$

where σ_e is the membrane's electric charge surface density, given by

$$\sigma_e = \frac{1}{4\pi} F_{\text{out}}^{0l} N, \qquad (34)$$

and where $F_{\text{out}}^{0l} N$ is evaluated at the membrane l_0 from the outside.

For the outside current, j_{out}^μ, we have from the Maxwell equations, Eq. (24), and the form of the Maxwell field, Eq. (29), the following equation $\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} F_{\text{out}}^{\mu\nu})}{\partial x^\nu} = 4\pi j_{\text{out}}^\mu$. If there are only electric fields this equation yields

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} F_{\text{out}}^{0k})}{\partial x^k} = 4\pi j_{\text{out}}^0. \qquad (35)$$

2. Electromagnetic energy-momentum tensor

The formulas for the energy-momentum tensor given in Eqs. (5)–(8) are still valid when one has a static spacetime with an electric field. However, it is important to isolate now the electric part of the energy-momentum tensor, $T_\mu^{\text{em}\nu}$. By definition,

$$T^{\text{em}\,\nu}_{\;\;\mu} = \frac{1}{4\pi} \left(F_\alpha^\mu F_\nu^\alpha - \frac{1}{4} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta} \right), \qquad (36)$$

where $F_{\mu\nu}$ is the Maxwell tensor. In our case, since F_{0i} are the only components that do not vanish, the electromagnetic energy-momentum tensor has the following nontrivial components:

$$T^{\text{em}\,0}_{\;\;0} = -\frac{1}{8\pi N^2} F_{0i} F_{0j} g^{ij}, \qquad (37)$$

$$T^{\text{em}\,i}_{\;\;j} = \frac{1}{8\pi N^2} (F_0^k F_{0k} \delta_j^i - 2 F_0^i F_{0j}). \qquad (38)$$

Since we are after a mass formula, and we use the Tolman mass definition given in Eq. (10), we still need to develop in the mass formula the electromagnetic part of the energy-momentum tensor.

B. Tolman mass formula for an electric membrane

To calculate the contribution to the mass from the electric charge of the membrane, we use the results above. From Eqs. (37) and (38), we have that the combination $-T^{\text{em}\,0}_{\;\;0} + T^{\text{em}\,k}_{\;\;k}$ that enters into the Tolman formula, Eq. (10), is $-T^{\text{em}\,0}_{\;\;0} + T^{\text{em}\,k}_{\;\;k} = -\frac{1}{4\pi} F_{0k} F^{0k}$. It follows from Eq. (10) that the contribution of the electromagnetic field to the electromagnetic mass M_{em} is

$$M_{\text{em}} = \frac{1}{4\pi} \int_\Sigma F_{0k} F^{0k} N \sqrt{\gamma} d^3 x. \qquad (39)$$

From Eq. (28), i.e., $F_{0k} = -\frac{\partial\varphi}{\partial x^k}$, we have $M_{\text{em}} = -\frac{1}{4\pi} \int_\Sigma \varphi_{,k} F^{0k} N \sqrt{\gamma} d^3 x$, where the integration is over the whole 3-space Σ, and to simplify the notation, we use here $_{,k} \equiv \frac{\partial}{\partial x^k}$. Performing the integral by parts gives $-\int_\Sigma \varphi_{,k} F^{0k} N \sqrt{\gamma} d^3 x = -\int_\Sigma (\varphi F^{0k} N \sqrt{\gamma})_{,k} d^3 x + \int_\Sigma \varphi (F^{0k} N \sqrt{\gamma})_{,k} d^3 x$. From Gauss' theorem, the first term can be swapped to a surface integral at infinity, $-\int_\Sigma (\varphi F^{0k} N \sqrt{\gamma})_{,k} d^3 x = -\int_{S_\infty} (\varphi F^{0k} N) n_k dS$, where n_k is the normal to the 2-surface at infinity, S_∞. Now, this surface term vanishes because at infinity one has $N \to 1$, $F^{0k} \sim \frac{1}{l^2}$, $dS \sim l^2$, $\varphi \sim \frac{1}{l}$. Thus, M_{em} reduces to $M_{\text{em}} = \frac{1}{4\pi} \int_\Sigma \varphi (F^{0k} N \sqrt{\gamma})_{,k} d^3 x$. From Eq. (29), we can then write $M_{\text{em}} = \frac{1}{4\pi} \int_\Sigma \varphi (F_{\text{out}}^{0k} \theta(l - l_0) N \sqrt{\gamma})_{,k} d^3 x$. So, splitting $\theta(l - l_0)$, we have $M_{\text{em}} = \frac{1}{4\pi} \int_\Sigma \varphi_{\text{out}} F_{\text{out}}^{0l} \delta(l - l_0) N \sqrt{\gamma} d^3 x + \int_{\Sigma_{\text{out}}} \varphi_{\text{out}} (F_{\text{out}}^{0k} N \sqrt{\gamma})_{,k} d^3 x$, where φ_{out} is the electric potential in the outer region and Σ_{out} is the outer 3-space. The first term is related to j_{membrane}^0 and after taking care of the δ function can be written as $\frac{1}{4\pi} \int_{\text{membrane}} \varphi_{\text{out}} F_{\text{out}}^{0l} N dS$ with $dS = \sqrt{\gamma} d^2 x$, and the second term is related to j_{out}^0. Thus, we can divide M_{em} as

$$M_{\text{em}} = M_{\text{membrane em}} + M_{\text{out em}}, \qquad (40)$$

with

$$M_{\text{membrane em}} = \int_{\text{membrane}} \varphi_{\text{out}} \sigma_e dS, \qquad (41)$$

where $\sigma_e = \frac{1}{4\pi} F_{\text{out}}^{0l} N$ as in Eq. (34), and

$$M_{\text{out em}} = \int_{\Sigma_{\text{out}}} \varphi_{\text{out}} j_{\text{out}}^0 N \sqrt{\gamma} d^3 x. \qquad (42)$$

This is the contribution of the electromagnetic mass to the Tolman mass.

The full mass M is then given by the sum of the membrane gravitational matter contribution, Eqs. (15) and (16); plus the membrane electrical contribution, Eq. (41); plus the outer contribution involving all matter

and nonelectric fields, Eq. (17); plus the outer electric contribution, Eq. (42).

C. Black hole mass formula in the membrane paradigm with electric field

1. Constancy of the electric potential at the horizon

When we want to treat the black hole limit, i.e., take $N \to 0$, we have to be careful with the physical quantities so that they do not blow up at the horizon. So, to avoid problems and confusion caused by pure coordinate effects when the original frame becomes degenerate in the horizon or near-horizon limit, it is convenient to use normalized components in an orthonormal basis. Thus, we must use normalized variables, the hat variables, such that $X_{\hat{0}} = \frac{X_0}{N}$. For the electric components $F_{\text{out}\,0i}$, the only components of $F_{\text{out}\,\mu\nu}$ that exist in this problem, we have from Eq. (28) that $F_{\text{out}\,\hat{0}i} = \frac{F_{\text{out}\,0i}}{N} = -\frac{1}{N}\frac{\partial\varphi_{\text{out}}}{\partial x^i}$, and these quantities must be finite at the horizon. This is equivalent to the requirement of finiteness of

$$F_{\text{out}\,\hat{0}l} = \frac{F_{\text{out}\,0l}}{N} = -\frac{1}{N}\frac{\partial\varphi_{\text{out}}}{\partial l}, \qquad (43)$$

$$F_{\text{out}\,\hat{0}a} = \frac{F_{\text{out}\,0a}}{N} = -\frac{1}{N}\frac{\partial\varphi_{\text{out}}}{\partial x^a}. \qquad (44)$$

But for a regular horizon, the metric potential N just outside the horizon must obey Eq. (18). Therefore, from Eqs. (18) and (43), we obtain that in this limit

$$\varphi_{\text{out}} - \Phi = b(x^a)(l - l_{\text{h}})^2 + O((l - l_{\text{h}})^3), \qquad (45)$$

for some function $b(x^a)$, where $\Phi \equiv \varphi_{\text{h}} = \varphi_{\text{membrane at the horizon}} = \varphi_{\text{out at the horizon}}$ is constant, having the meaning of the potential at the membrane when this is at the horizon. By continuity, the electric potential remains constant inside, in the vacuum region, with value Φ. Thus, we have the important result that at the horizon the electric potential is a constant,

$$\Phi = \varphi_{\text{out at the horizon}} = \text{constant}. \qquad (46)$$

In addition, Eq. (45) means that the normal component of $F_{\hat{0}l}$ has a jump; indeed, $(F_{\text{out}\hat{0}l})_+$ is $(F_{\text{out}\hat{0}l})_+ = -\frac{1}{N}(\frac{\partial\varphi_{\text{out}}}{\partial l})_+ = -\frac{2b}{\kappa}$, and $(F_{\text{in}\hat{0}l})_- = -\frac{1}{N}(\frac{\partial\varphi_{\text{in}}}{\partial l})_- = 0$, where again the subscripts $+$ and $-$ mean the evaluation at the horizon from the outside and the inside, respectively. So, $(F_{\text{out}\hat{0}l})_+ - (F_{\text{in}\hat{0}l})_- = -\frac{2b}{\kappa}$; i.e., there is a jump in the normal electric component. Thus, comparing with the formulas above, see Eqs. (34) and (43), i.e., $F_{\text{ou}\hat{0}l} = -4\pi\sigma_{\text{e}}$, we have $b = 2\pi\sigma_{\text{e}}\kappa$ at the horizon. On the other hand, the tangential components of the electric field go as $F_{\text{out}\,0a} \sim (l - l_{\text{h}})^2 \to 0$, and so $F_{\text{out}\,\hat{0}a} \sim (l - l_{\text{h}}) \to 0$. Thus, $(F_{\text{out}\,\hat{0}a})_+ = 0$, and $(F_{\text{in}\,\hat{0}a})_- = 0$. We

conclude that there is no jump in the tangential electric components.

2. Black hole mass formula in the membrane paradigm with electric field

Now, to find the black hole mass formula, we start by using Eq. (41). Note that Eq. (41) is independent of N. Take then the limit $N \to 0$, and take into account the constancy of the potential in the horizon limit, Eq. (46). Use Eq. (33) in Eq. (41) to obtain

$$M_{\text{membrane at the horizon em}} = \Phi Q, \qquad (47)$$

where $M_{\text{membrane at the horizon em}}$ is the contribution from the electromagnetic mass of the membrane to the black hole mass when this is at the gravitational radius.

In the outer region, if charged, there is also a contribution from the regular part of the current, namely, $M_{\text{out em}} = \frac{1}{4\pi}\int_{\Sigma_{\text{out}}}\varphi_{\text{out}}j^0_{\text{out}}N\sqrt{\gamma}d^3x$, see Eq. (42), and where Σ_{out} is the 3-space outside the horizon. Further, the outer current can be written as $j^\mu_{\text{out}} = \rho_{\text{out em}}u^\mu$, and $\rho_{\text{out em}}u^\mu$ is an invariant charge density, with u^μ being the 4-velocity of the source. In our case, the only component of the current is $j^0_{\text{out}} = \frac{\rho_{\text{out em}}}{N}$. Then,

$$M_{\text{em out}} = \frac{1}{4\pi}\int_{\Sigma_{\text{out}}}\varphi_{\text{out}}\rho_{\text{out em}}\sqrt{\gamma}d^3x, \qquad (48)$$

valid outside the horizon.

Thus, the total mass M of a black hole is the sum of the mass formula for a nonelectric black hole given in Eq. (21), plus the electric mass coming from the membrane at the horizon given in Eq. (47), plus the outer mass involving the sum of Eq. (17) of all the matter and other nonelectric fields outside the black hole with the outer electric mass given by Eq. (48). Summing all mass contributions one finds the black hole mass formula with an electric field, namely,

$$M = \frac{1}{4\pi}\kappa A + \Phi Q + M_{\text{out}}. \qquad (49)$$

Equation (49) is the mass formula for electric black holes, now derived in the membrane paradigm. This black hole mass formula obtained through the membrane paradigm approach is the same as that obtained by other methods [28] (see also Refs. [29,30]). When $M_{\text{out}} = 0$, Eq. (49) is the Smarr formula for a static and electric, i.e., Reissner-Nordström, black hole in general relativity [27].

3. Extremal electric case and the interpretation for the surface gravity

The extremal case corresponds to $\kappa = 0$. Thus,

$$M = \Phi Q + M_{\text{out}}. \qquad (50)$$

For $M_{\mathrm{out}} = 0$ and for Reissner-Nordström, one finds $\Phi = 1$ and $M = Q$, which is the well-known relation for electrical extremal black holes. In principle, this derivation for the extremal case through the limits suffices. However, if one prefers, a more consistent way of derivation in the extremal case requires taking into account another asymptotic form of the lapse function for small N. By definition, the extremal case implies that for a black hole $N \sim \exp(-\alpha l)$, where α is some positive constant and $l \to \infty$. Then, the above equations show that the membrane stresses remain finite but their contribution to the surface mass vanishes because of the form of the lapse function N. As a result, indeed, $M_{\mathrm{membrane\ at\ the\ horizon}} = 0$.

Since in the extremal case $\kappa = 0$, our interpretation for κ implies that the horizon now has no surface energy density; all the energy density it has comes from the electric field.

IV. BLACK HOLE MASS FORMULA IN THE MEMBRANE PARADIGM: ROTATING MEMBRANE

A. Preliminaries

1. Gravitational and rotational fields

A mass formula for a spacetime with a rotational axially symmetric background can also be derived using the membrane paradigm. The considerations follow the same lines as before. We restrict ourselves to the electrically uncharged case.

A general rotating axially symmetric metric can be written in (t, r, θ, ϕ) coordinates as

$$ds^2 = -N^2 dt^2 + g_{ll}dl^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}(d\phi - \omega dt)^2, \quad (51)$$

where the metric functions N, g_{ll}, $g_{\theta\theta}$, $g_{\phi\phi}$, and ω are functions of l and θ in general. We assume there is a rotating shell that separates the vacuum, Minkowski, inside spacetime from the outside spacetime, and thus from Eq. (51), we can specify the inside and outside metrics.

The metric for the vacuum, Minkowski, inner spacetime is written, putting $g_{ll} = 1$, as

$$ds^2 = -N_0^2 dt^2 + dl^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}(d\phi - \omega_0 dt)^2, \quad (52)$$

where N_0 and ω_0 are conveniently chosen constants. The metric for the spacetime outside the membrane is, putting also $g_{ll} = 1$, written as

$$ds^2 = -N^2 dt^2 + dl^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}(d\phi - \omega dt)^2, \quad (53)$$

where N, $g_{\theta\theta}$, $g_{\phi\phi}$, and ω are functions of l and θ in general.

2. Energy-momentum tensor

As in the static case, we assume that the spacetime has some energy-momentum tensor $T_{\mu\nu}$ and divide the energy-momentum tensor as the sum of inner energy-momentum tensor $T_{\mathrm{in}\mu\nu}$, the membrane energy-momentum tensor $T_{\mathrm{membrane}\,\mu\nu}$, and the outer energy-momentum tensor $T_{\mathrm{out}\,\mu\nu}$,

$$T_{\mu\nu} = T_{\mathrm{in}\,\mu\nu} + T_{\mathrm{membrane}\,\mu\nu} + T_{\mathrm{out}\,\mu\nu}. \quad (54)$$

Inside is Minkowski, and so inside,

$$T_{\mathrm{in}\,\mu\nu} = 0. \quad (55)$$

The membrane is infinitesimal, situated at l_0, so we put

$$T_{\mathrm{membrane}\,\mu\nu} = S_{\mu\nu}\delta(l - l_0), \quad (56)$$

where $\delta(l - l_0)$ is the Dirac delta function and $S_{\mu\nu}$ is a surface energy-momentum tensor defined at the membrane. The outside energy-momentum tensor is divided into matter and other fields, so

$$T_{\mathrm{out}\,\mu\nu} = T_{\mathrm{out\ matter}\,\mu\nu} + T_{\mathrm{out\ other\ fields}\,\mu\nu}. \quad (57)$$

The term $T_{\mathrm{out\ other\ fields}\,\mu\nu}$ can contain an electromagnetic field, for instance.

B. Tolman mass and angular momentum for a rotating membrane

We start with the angular momentum as it will be useful for the expression for the mass. The Tolman spatial vector momentum is defined as $J_i = \int T_i^0 \sqrt{-g}d^3x$, which in our case from the metric Eq. (53) can be put in the form $J_i = \int T_i^0 N\sqrt{g_3}d^3x$, where we have used $-g = N^2 g_3$, g_3 being the determinant of the spatial 3-metric. The only momentum that matters here is the angular momentum J_ϕ. Moreover, $g_3 = g_{\theta\theta}g_{\phi\phi}$ from the metric Eq. (53). So, defining $\gamma = g_{\theta\theta}g_{\phi\phi}$, we write the total angular momentum J_ϕ as given by

$$J_\phi = -\int T_\phi^0 N\sqrt{\gamma}d^3x. \quad (58)$$

Now, the total value of the angular momentum (58) can be split into three contributions, namely, the inner vacuum, the membrane, and the outer region, such that,

$$J_\phi = J_{\phi\,\mathrm{in}} + J_{\phi\,\mathrm{membrane}} + J_{\phi\,\mathrm{out}}. \quad (59)$$

For the inside, since there is no matter inside, see Eq. (55), we have from Eq. (58)

$$J_{\phi\,\mathrm{in}} = 0. \quad (60)$$

For the membrane, its angular momentum can be calculated through the expression

$$J_{\phi\,\text{membrane}} = -\int_{\text{membrane}} T^0_\phi N\sqrt{\gamma}d^3x. \qquad (61)$$

Now, from Eq. (56), we have $T_{\text{membrane}\phi}{}^0 = S^0_\phi\delta(l-l_0)$. Since $8\pi S^\nu_\mu = [[K^\nu_\mu]] - \delta^\nu_\mu[[K]]$, we have here $8\pi S^0_\phi = [[K^0_\phi]] - \delta^0_\phi[[K]]$, i.e., $8\pi S^0_\phi = [[K^0_\phi]]$, and since the inside is flat, we have $[[K^0_\phi]] = K_{+\phi}{}^0$. We suppress the $+$ index in the following as it is obvious that all quantities are calculated for the outside. For the outside, we have $K^0_\phi = K_{\phi 0}g^{00} + K_{\phi\phi}g^{0\phi}$. Since $K_{\mu\nu} = -\nabla_\nu n_\mu$ and $n_\mu = \frac{\partial l}{\partial x^\mu}$, we can calculate $K_{\phi 0}$ and $K_{\phi\phi}$. We find $K_{\phi 0} = +\Gamma^l_{\phi 0}n_l = -\frac{1}{2}g^{ll}\frac{\partial g_{\phi 0}}{\partial l} = \frac{1}{2}(g_{\phi\phi}\frac{\partial\omega}{\partial l} + \omega\frac{\partial g_{\phi\phi}}{\partial l})$, where we have used $g^{ll} = 1$, $n_l = 1$, and $g_{\phi 0} = -\omega g_{\phi\phi}$. Also, $K_{\phi\phi} = +\Gamma^l_{\phi\phi}n_l = -\frac{1}{2}g^{ll}\frac{\partial g_{\phi\phi}}{\partial l} = -\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial l}$. In addition, $g^{00} = -\frac{1}{N^2}$, and $g^{0\phi} = -\frac{\omega}{N^2}$. Then, $K^0_\phi = -\frac{1}{2}\frac{g_{\phi\phi}}{N^2}\frac{\partial\omega}{\partial l}$, and so $8\pi S^0_\phi = -\frac{1}{2}\frac{g_{\phi\phi}}{N^2}\frac{\partial\omega}{\partial l}$. All quantities are evaluated at the membrane $l = l_0$ from the outside. Integrating through the delta function, we obtain

$$J_{\phi\,\text{membrane}} = \int_{\text{membrane}} j\,dS, \qquad (62)$$

where

$$j \equiv \frac{1}{16\pi}\frac{g_{\phi\phi}}{N}\frac{\partial\omega}{\partial l} \qquad (63)$$

is an angular momentum surface density, the quantities are evaluated at the membrane from the outside, and dS is the 2-dimensional surface spanned by $t = $ constant and $l = $ constant. For the outside, when there is matter and other fields outside the membrane, one has

$$J_{\text{out}} = -\int_{\text{out}} T^0_\phi N\sqrt{\gamma}d^3x. \qquad (64)$$

Now, we evaluate the mass in the stationary case, resorting to the Tolman mass formula. The Tolman mass for a given stationary spacetime with a stress-energy tensor $T_{\mu\nu}$ is defined as in the static case, see Eq. (9), i.e.,

$$M = \int_\Sigma (-T^0_0 + T^k_k)\sqrt{-g}d^3x, \qquad (65)$$

with g being the determinant of the metric $g_{\mu\nu}$ in, e.g., Eq. (53), $d^3x = dx^1dx^2dx^3$, and the integral is performed over a 3-space Σ with $t = $ constant. Using, from the metric Eq. (53), that $g = -N\gamma$ with $\gamma = g_{\theta\theta}g_{\phi\phi}$, we have

$$M = \int_\Sigma (-T^0_0 + T^k_k)N\sqrt{\gamma}d^3x. \qquad (66)$$

Again, we divide the mass into three pieces,

$$M = M_{\text{in}} + M_{\text{membrane}} + M_{\text{out}}. \qquad (67)$$

For the inside vacuum, we have

$$M_{\text{in}} = 0. \qquad (68)$$

For the membrane, we apply the junction conditions. Since $8\pi S^\nu_\mu = [[K^\nu_\mu]] - \delta^\nu_\mu[[K]]$, we have here $8\pi(-S^0_0 + S^a_a) = -2[[K^0_0]]$. Put n^μ as the unit vector normal to the boundary surface, the membrane. Also, $K_{\mu\nu} = -\nabla_\nu n_\mu$, where at the boundary surface $N = $ constant and the normal unit vector is $n_\mu = \frac{\partial l}{\partial x^\mu}$. Further calculations then give

$$M_{\text{membrane}} = \int_{\text{membrane}} (\sigma + 2\omega j)dS, \qquad (69)$$

where σ has the expression given in Eq. (16), ω is the membrane angular velocity with $\omega = \omega_0$ defined in Eq. (52), and j is given in Eq. (63).

For the outside, we have

$$M_{\text{out}} = \int_{\text{out}} (-T_{\text{out}\,0}{}^0 + T_{\text{out}\,k}{}^k)N\sqrt{\gamma}d^3x, \qquad (70)$$

where one can further split into matter fields and other fields.

Thus, putting together Eqs. (68), (69), and (70) into Eqs. (67), we obtain the mass formula from the Tolman definition for a spacetime with a membrane.

C. Black hole mass formula in the membrane paradigm with rotation

1. Constancy of the angular velocity at the horizon

Now, let $l_h \equiv l_{0\,\text{at the horizon limit}}$. The membrane is approaching the horizon. In the nonextremal case, we have that N behaves as given in Eq. (18); see Ref. [42]. It still involves κ, the surface gravity at the horizon obeying $\kappa = $ constant different from zero. For the metric coefficient ω, we assume the validity of the expansion near the horizon [42,43] $\omega = \Omega + a_2N^2 + \cdots$, for some constant a_2, and so from Eq. (18), we get

$$\omega = \Omega + O((l - l_h)^2), \qquad (71)$$

where $\Omega \equiv \omega_h$ is the horizon value of ω obeying

$$\Omega = \text{constant}. \qquad (72)$$

2. Black hole mass formula in the membrane paradigm with rotation

To obtain the black hole mass formula in the membrane paradigm with rotation, let us focus on the membrane term above. Using Eq. (69) and with the help of Eqs. (16) and (63), we can put the σ term, now in the guise of κ, and the ω term, now as Ω, outside the integral, since κ and Ω are constants at the horizon. For the κ term we get $\frac{\kappa A}{4\pi}$ after

integration, with A being the horizon area. For the Ω term we get $2\Omega J$ after integration, where J is the angular momentum of the membrane at the horizon, i.e., the angular momentum of the black hole. So, Eq. (69) at the horizon together with Eq. (70) and Eq. (68) in Eq. (67), lead to

$$M = \frac{\kappa A}{4\pi} + 2\Omega J + M_{\text{out}}. \tag{73}$$

This is precisely the black hole mass formula for rotating black holes derived from the membrane paradigm. It was derived in Ref. [28] (see also Refs. [29,30]) from black hole theory using Komar's mass and angular momentum definitions, which are appropriate in a vacuum spacetime with one timelike and one spacelike Killing vector. In the vacuum case, $M_{\text{out}} = 0$, Eq. (73) is the Smarr formula for a Kerr black hole [27].

3. Extremal rotating case

For the extremal rotating case, one has $N = O((l - l_{\text{h}})^3)$ [42]. For the metric coefficient ω, we assume the validity of the expansion near the horizon [42,43] $\omega = \Omega + a_1 N + \ldots$, i.e., $\omega = \Omega + O((l - l_{\text{h}})^3) + \ldots$. This all leads to $\kappa = 0$, so that we can use Eq. (73) directly and write the mass formula for the extremal case as

$$M = 2\Omega J + M_{\text{out}}. \tag{74}$$

For $M_{\text{out}} = 0$ and for an extremal Kerr, one finds $\Omega = \frac{1}{2M}$ and $M^2 = J$, which is the well-known relation for rotating extremal black holes.

4. Black hole mass formula in the membrane paradigm with electric field and rotation

If the membrane has electric field and rotation, developing the calculations done previously, one finds the following mass formula:

$$M = \frac{\kappa A}{4\pi} + \Phi Q + 2\Omega J + M_{\text{out}}. \tag{75}$$

In the vacuum case, $M_{\text{out}} = 0$, it is the Smarr formula for a Kerr-Newman black hole [27].

V. CONCLUSIONS

In replacing the black hole event horizon by a self-gravitating material membrane located slightly above the horizon itself, i.e., using the membrane paradigm formalism, we have been able to derive the mass formula for static and rotating black holes without and with electric fields. A fundamental element in the derivation is the setting of the proper boundary conditions on the membrane when it approaches its own gravitational or horizon radius. We found that both the membrane paradigm and the standard black hole theory yield exactly the same result.

We have thus filled an important gap in the membrane paradigm investigations and showed that the membrane paradigm formalism is able to reproduce correctly one of the basics of black hole physics, namely, the mass formula. This has been achieved in a completely different perspective than in the original derivations since now there is no black hole and we have dealt with a timelike surface only in the form of a membrane.

[1] S. W. Hawking and J. B. Hartle, Energy and angular momentum flow into a black hole, Commun. Math. Phys. **27,** 283 (1972).

[2] R. S. Hanni and R. Ruffini, Lines of force of a point charge near a Schwarxschild black hole, Phys. Rev. D **8,** 3259 (1973).

[3] R. L. Znajek, The electric and magnetic conductivity of a Kerr hole, Mon. Not. R. Astron. Soc. **185,** 833 (1978).

[4] T. Damour, Black-hole eddy currents, Phys. Rev. D **18,** 3598 (1978).

[5] T. Damour, Surface effects in black hole physics, in *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North-Holland, Amsterdam, 1982), p. 587.

[6] D. A. Macdonald and K. S. Thorne, Black-hole electrodynamics: An absolute-space/universal-time formulation, Mon. Not. R. Astron. Soc. **198,** 345 (1982).

[7] W. H. Zurek and K. S. Thorne, Statistical Mechanical Origin of the Entropy of a Rotating, Charged Black Hole, Phys. Rev. Lett. **54,** 2171 (1985).

[8] R. H. Price and K. S. Thorne, Membrane viewpoint on black holes: Properties and evolution of the stretched horizon, Phys. Rev. D **33,** 915 (1986).

[9] W. M. Suen, R. H. Price, and I. H. Redmount, Membrane viewpoint on black holes: Gravitational perturbations of the horizon, Phys. Rev. D **37,** 2761 (1988).

[10] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, London, 1986).

[11] L. Susskind, L. Thorlacius, and J. Uglum, The stretched horizon and black hole complementarity, Phys. Rev. D **48,** 3743 (1993).

[12] S. D. Mathur, Membrane paradigm realized?, Gen. Relativ. Gravit. **42,** 2331 (2010).

[13] M. K. Parikh and F. Wilczek, An action for black hole membranes, Phys. Rev. D **58,** 064011 (1998).

[14] J. P. S. Lemos and O. B. Zaslavskii, Membrane paradigm and entropy of black holes in the Euclidean action approach, Phys. Rev. D **84,** 064017 (2011).

[15] A. O. Starinets, Quasinormal spectrum and the black hole membrane paradigm, Phys. Lett. B **670,** 442 (2009).

[16] R. F. Penna, Black hole jet power from impedance matching, Phys. Rev. D **92,** 084017 (2015).

[17] T. Wang, Cosmological horizons as new examples of membrane paradigm, Classical Quantum Gravity **32,** 195006 (2015).

[18] C. Eling, I. Fouxon, and Y. Oz, The incompressible Navier-Stokes equations from black hole membrane dynamics, Phys. Lett. B **680,** 496 (2009).

[19] V. E. Hubeny, The fluid/gravity correspondence: A new perspective on the membrane paradigm, Classical Quantum Gravity **28,** 114007 (2011).

[20] I. Bredberg and A. Strominger, Black holes as incompressible fluids on the sphere, J. High Energy Phys. 05 (2012) 043.

[21] N. Pinzani-Fokeeva and M. Taylor, Towards a general fluid/gravity correspondence, Phys. Rev. D **91,** 044001 (2015).

[22] W. Fischler and S. Kundu, Membrane paradigm, gravitational Θ-term and gauge/gravity duality, J. High Energy Phys. 04 (2016) 112.

[23] S. Chatterjee, M. K. Parikh, and S. Sarkar, The black hole membrane paradigm in $f(R)$ gravity, Classical Quantum Gravity **29,** 035014 (2012).

[24] T. Jacobson, A. Mohd, and S. Sarkar, Membrane paradigm for Einstein-Gauss-Bonnet gravity, Phys. Rev. D **95,** 064036 (2017).

[25] S. Kolekar and D. Kothawala, Membrane paradigm and horizon thermodynamics in Lanczos-Lovelock gravity, J. High Energy Phys. 02 (**2012**) 006.

[26] T.-Y. Zhao and T. Wang, Membrane paradigm of black holes in Chern-Simons modified gravity, J. Cosmol. Astropart. Phys. 06 (2016) 019.

[27] L. Smarr, Mass Formula for Black Holes, Phys. Rev. Lett. **30,** 71 (1973).

[28] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31,** 161 (1973).

[29] B. Carter, Black hole equilibrium states, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), p. 57.

[30] J. M. Bardeen, Rapidly rotating stars, disks, and black holes, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), p. 241.

[31] R. Meinel, On the black hole limit of rotating fluid bodies in equilibrium, Classical Quantum Gravity **23,** 1359 (2006).

[32] H.-S. Liu, H. L., and C. N. Pope, Generalized Smarr formula and the viscosity bound for Einstein-Maxwell-dilaton black holes, Phys. Rev. D **92,** 064014 (2015).

[33] A. Komar, Positive-definite energy density and global consequences for general relativity, Phys. Rev. **129,** 1873 (1963).

[34] R. Tolman, *Relativity, Thermodynamics and Cosmology*, (Dover, New York, 1987), Sec. 92.

[35] G. Abreu and M. Visser, Tolman Mass, Generalized Surface Gravity, and Entropy Bounds, Phys. Rev. Lett. **105,** 041302 (2010).

[36] J. P. S. Lemos and O. B. Zaslavskii, The mass formula for quasiblack holes, Phys. Rev. D **78,** 124013 (2008).

[37] J. P. S. Lemos and O. B. Zaslavskii, The angular momentum and mass formulas for rotating stationary quasiblack holes, Phys. Rev. D **79,** 044020 (2009).

[38] W. Israel, Singular hypersurfaces and thin shells in general relativity, Il Nuovo Cimento **44,** 1 (1966).

[39] A. H. Taub, Space-times with distribution valued curvature tensors, J. Math. Phys. **21,** 1423 (1980).

[40] C. W. Misner, K. S. Thorne, and J. A. W. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 21.13.

[41] A. J. M. Medved, D. Martin, and M. Visser, Dirty black holes: Symmetries at stationary non-static horizons, Phys. Rev. D **70,** 024009 (2004).

[42] C. Barcelo, S. Liberati, S. Sonego, and M. Visser, Fate of gravitational collapse in semiclassical gravity, Phys. Rev. D **77,** 044032 (2008).

[43] I. V. Tanatarov and O. B. Zaslavskii, Dirty rotating black holes: regularity conditions on stationary horizons, Phys. Rev. D **86,** 044019 (2012).

Correction: The previously published verso running head contained a spelling error and was replaced.