PHYSICAL REVIEW D **97,** 061502(R) (2018)

Rapid Communications

Linearized stability of extreme black holes

Lior M. Burko[1] and Gaurav Khanna[2]

[1]School of Science and Technology, Georgia Gwinnett College, Lawrenceville, Georgia 30043, USA

[2]Department of Physics, University of Massachusetts Dartmouth, Dartmouth, Massachusetts 02747, USA

(Received 28 September 2017; published 28 March 2018)

Extreme black holes have been argued to be unstable, in the sense that under linearized gravitational perturbations of the extreme Kerr spacetime the Weyl scalar ψ_4 blows up along their event horizons at very late advanced times. We show numerically, by solving the Teukolsky equation in $2+1$D, that all algebraically independent curvature scalar polynomials approach limits that exist when advanced time along the event horizon approaches infinity. Therefore, the horizons of extreme black holes are stable against linearized gravitational perturbations. We argue that the divergence of ψ_4 is a consequence of the choice of a fixed tetrad, and that in a suitable dynamical tetrad all Weyl scalars, including ψ_4, approach their background extreme Kerr values. We make similar conclusions also for the case of scalar field perturbations of extreme Kerr.

DOI: 10.1103/PhysRevD.97.061502

Black hole (BH) stability has been an important question in the understanding of their physical reality. Rigorous analyses have proved linear stability for Schwarzschild BHs for regular initial data [1]. For the rotating Kerr BH, linear stability has been proved rigorously only for massless scalar field perturbations for the nonextremal case [2], although mode stability has been demonstrated also for gravitational perturbations [3].

An interesting class of BHs is that of extreme ones: BHs which have vanishing surface gravity. In classical general relativity extreme BHs (maximally charged or maximally spinning BHs) behave differently, both physically and mathematically, from nonextremal ones, in a way that draws much attention to them and to nearly extreme BHs [4]. Extreme BHs also play an important role in supersymmetric and string theories, where it is easier to describe them quantum mechanically because of their vanishing surface gravity and consequently vanishing temperature for Hawking radiation [5].

Recently, it was argued that extreme BHs are unstable: fields (massless scalar fields or gravitational perturbations) or their transverse derivatives grow unboundedly along their event horizons (EHs). Specifically, Aretakis argued that extreme Reissner-Nordström BHs are linearly unstable under scalar field perturbations [6]: certain transverse derivatives of the time evolution of regular initial data grow unboundedly with advanced time.

Lucietti and Reall expanded Aretakis's result also for linearized vacuum gravitational perturbations of extreme Kerr BHs (EK) [7] and showed that for axisymmetric perturbations certain second transverse derivatives of the Weyl scalar ψ_4 and certain sixth transverse derivatives of the Weyl scalar ψ_0 blow up in the Hartle-Hawking (HH) tetrad along the EH with advanced time. The HH tetrad is a null tetrad in which the Kinnersley null-tetrad basis vectors $\mathbf{l}$, $\mathbf{n}$ are rescaled with the horizon function, so that they are regular on the EH, and specifically, for any finite value of advanced time the Weyl scalars on the EH are finite. For nonaxisymmetric gravitational perturbations Casals *et al.* showed that the HH Weyl scalar ψ_4 itself blows up along the EH and that each additional transverse derivative increases the blowup rate [8,9], and they concluded that spacetime curvature diverged. (Note that it was not claimed in [8] that curvature scalar invariants blow up. See also [9].) Lucietti and Reall [7] also suggested that when full nonlinearity is considered, spacetime would evolve such that either a null singularity would evolve instead of an EH, or spacetime would evolve to a nonextreme BH. (See also [10].) The suggestion that EK are linearly unstable and that spacetime may evolve a null singularity instead of a regular EH for EK is highly troubling in view of the importance of extreme BHs in both general relativity and string theory.

Our numerical experiment is to set a perturbation in the so-called Beetle-Burko scalar ξ [11], which in our case measures the deviation of curvature invariants from their background values. Horizon instability would imply that the perturbation ξ would not tend to a limit along the EH. The advantage of our approach is that we make an invariant statement on which all observers would agree. In practice, we solve the Teukolsky equation [12] for the Weyl scalars ψ_4 and ψ_0 (from which we construct ξ) in the HH tetrad for EK, using compactified hyperbolical coordinates similar to those used, say, in Ref. [13].

The major technological innovation in this study is boundary conditions (BC) that allow us to track the evolution of the fields on the EH accurately: the fields are actually "evolved" on the boundary (which is the EH in our computational setup) as opposed to computed using the

BC in conjunction with data from the "bulk." The more common approach is to evolve the fields in the bulk, i.e. compute the source term in the bulk and update the values of the field via time stepping, and then use this evolved data in the bulk along with the imposed BC (or a simple extrapolation) to compute the fields on the boundary. This approach has the advantage that it is computationally cheaper and fairly simple to implement. However, it inherently relies on a high degree of smoothness in the solution, thus resulting in some inaccuracy in cases wherein a sharp physical feature is present. Given that is precisely what is expected here, we took the alternate approach of evolving the fields everywhere, including at the boundary itself. To do this, the source term is computed at the boundary and the field values are updated at every time step. Now, computing the source term involves computing derivatives at the boundary, and that is done using a high-order, one-sided, finite-difference stencil. This approach generated results that were consistent with several of the test cases that we used to validate our computational framework. Detailed results from these tests appear below.

The numerical scheme that we used is presented in detail in Ref. [14] along with several stability, convergence and other tests. We summarize the approach as follows: (i) the Teukolsky equation, written in hyperboloidal coordinates (based on the ingoing Kerr coordinate system) is first cast into a $(2+1)$D form by separating out the axisymmetric φ dependence; (ii) the resulting equation is rewritten in first-order hyperbolic form; and (iii) a time-explicit, two-step Richtmeyer-Lax-Wendroff, second-order finite-difference evolution scheme is implemented. We also developed a new fifth-order Weighted Essentially Non-Oscillatory (WENO) finite-difference scheme [15] with third-order Shu-Osher explicit time stepping [16]. This method was used to cross-check the results obtained with the second-order code, and to obtain results that were inaccurate with the second-order code. The initial data for the evolved fields is specified as a "truncated" (in order for it to be compactly supported) Gaussian pulse placed in the strong field, with or without support on the EH: in the code's compactified hyperboloidal coordinates (ρ, τ) [13], the Gaussian pulse is centered at $\rho = 1.0M$ or $5.0M$, respectively, and is of width $0.1M$ with a truncation window of $4.0M$ width.

We next find numerically the behavior of the Weyl scalars ψ_0 and ψ_4 and their ∂_ρ gradients along the EH as functions of advanced time v ("Eddington coordinate"). The gradient $\partial_\rho \propto \partial_r$, r being the ingoing Kerr radial coordinate. We note that ρ is regular on the EH, so that these gradients are effectively gradients with respect to a Kruskal-like coordinate. We further note that as $v \to \infty$ the ∂_ρ gradients of ψ_4, ψ_0 and also of a scalar field ϕ become transverse (i.e., ∂_ρ becomes proportional to ∂_u, u being retarded time), with the relative error at finite late advanced times decaying like v^{-2}. For simplicity of discussion, we refer to ∂_ρ as a transverse gradient hereafter.

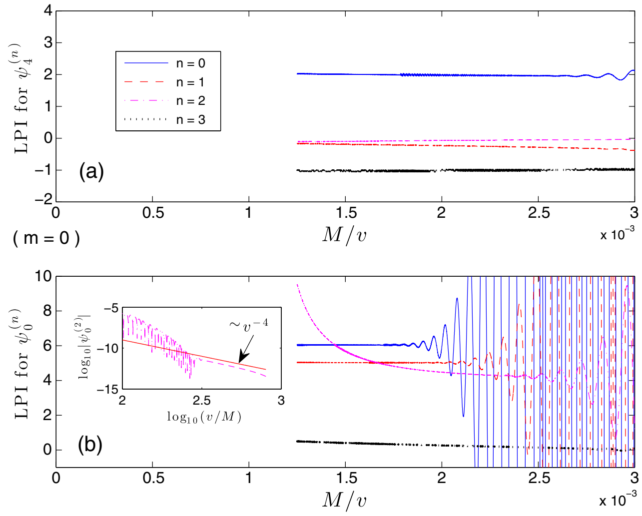

FIG. 1. The LPIs for the real parts of (a) ψ_4 and (b) ψ_0 and for their first three ∂_ρ derivatives along the EH as functions of advanced time, v, for axisymmetric $(m = 0)$ perturbations of EK. The inset in (b) shows $\partial_\rho^2 \psi_0$ as a function of v. The imaginary parts of the fields behave qualitatively similarly at late times. Initial data have no support on the EH. Data here and in the figures below are extracted on the surface $\theta = \pi/4$.

Figure 1 shows the local power indices (LPIs) [17] for the axisymmetric $(m = 0)$ case. For the field $\zeta(v)$ we define the LPI q as $q \coloneqq -v\zeta_{,v}\zeta^{-1}$. We denote by $\psi_i^{(n)}$ the nth transverse derivative of ψ_i. We find for $m = 0$ that for $n = 0, 1, 2, 3$ the corresponding q values are $2, 0, 0, -1$ for ψ_4 and 6,5,4,1 for ψ_0. The instability in the field ψ_4 is manifest in its third derivative, in accordance with the conclusions of [7]: $q < 0$ implies unbounded growth with advanced time along the EH. (We comment that the results here have initial data that are unsupported on the EH. For initial data that are supported on the EH we find results in agreement with [7].)

In Fig. 2 we show the fields ψ_4 and ψ_0 for the nonaxisymmetric case $(m = 2)$. Our results for ψ_4 are in agreement with the results of [8,18] for $n = 0, 1, 2, 3$; that is, the late time behavior is found to be $\psi_4^{(n)}(v \gg M) \sim v^{3/2+n}$ and $\psi_0^{(n)}(v \gg M) \sim v^{-5/2+n}$.

The gravitational case is Ricci flat, and therefore all scalars made with R or $R_{\mu\nu}$ or their derivatives vanish identically. Curvature therefore depends only on the Weyl tensor. A general spacetime in 4D has 14 algebraically independent scalars that determine the curvature [19]. In vacuum there are only four nonvanishing such scalars, because any curvature invariant can be expressed as a function of a set of the six fundamental (real) invariant eigenvalues of the Weyl tensor. Since the traces of both the Weyl tensor and its dual vanish, there are four independent scalars left [11]. These scalars may be taken to be the real and imaginary parts of the invariants I, J, where $I \coloneqq \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma}$ and $J \coloneqq \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\rho\sigma}_{\alpha\beta}\tilde{C}^{\alpha\beta\mu\nu}$, $\tilde{C}_{\mu\nu\rho\sigma}$ being

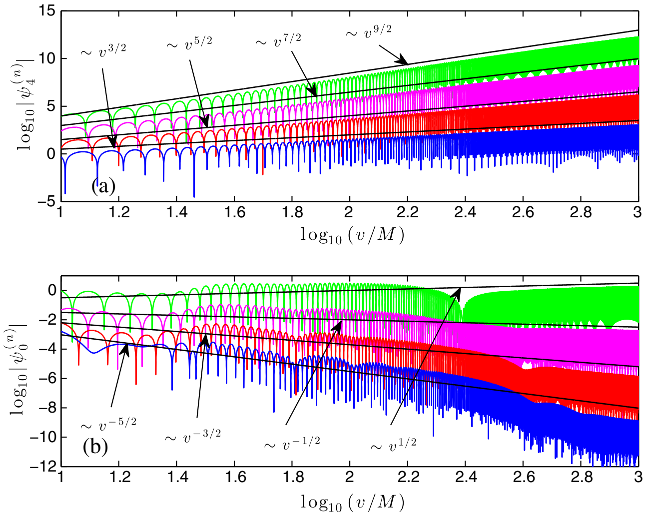

FIG. 2. The (real parts of the) fields (a) ψ_4 and (b) ψ_0 and their first three transverse derivatives along the EH as functions of advanced time for nonaxisymmetric ($m = 2$) perturbations of EK, for initial data that have support on the EH. In panel (a) we show four reference lines, corresponding to $v^{3/2+n}$, and in panel (b) we show the reference lines for $v^{-5/2+n}$. The imaginary parts behave qualitatively similarly at late times.

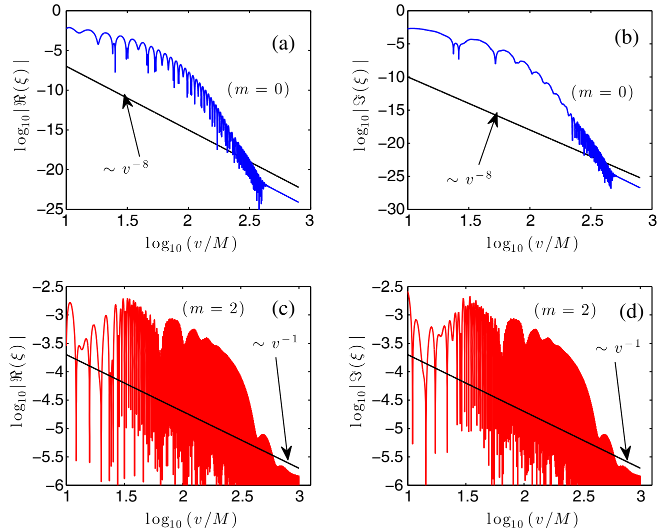

FIG. 3. The real and imaginary parts of ξ as functions of advanced time along the EH of EK. Panels (a) and (b) are for the real and imaginary parts, respectively, of the axisymmetric ($m = 0$) case, and panels (c) and (d) are for the real and imaginary parts, respectively, of the nonaxisymmetric ($m = 2$) case. The $m = 2$ results are obtained for initial data having support on the EH, and the $m = 0$ results are obtained for initial data that do not have support on the EH.

the self-dual of the Weyl tensor. Our spacetime is even more restricted, because the HH tetrad is a transverse frame ($\psi_1 = 0 = \psi_3$). Since the background is a known EK spacetime, specifically the Weyl scalar ψ_2 is known and is constant along the EH, only two algebraically independent curvature scalars remain. These scalars can be taken to be the real and imaginary parts of $\xi \coloneqq \psi_0\psi_4$ [11].

We show next that along the EH of EK both the real and the imaginary parts of ξ vanish at late advanced times, so that $I \to 3\psi_2^2$ and $J \to -\psi_2^3$. As ψ_2 is that of the background EK, i.e., finite along the EH, both I and J have limits that exist as $v \to \infty$. As I, J exhaust all the algebraically independent curvature invariants, all scalars made from polynomials in the Weyl tensor have limits that exist as $v \to \infty$. We therefore show that the EH of EK does not evolve an instability in ξ.

Specifically, Fig. 3 shows the real and imaginary parts of ξ for both the axisymmetric and nonaxisymmetric cases along the EH of EK as functions of advanced time. We find that in the axisymmetric case $\Re(\xi), \Im(\xi) \sim v^{-8}$ for $v \gg M$. In the nonaxisymmetric case $\Re(\xi), \Im(\xi) \sim v^{-1}$ for $v \gg M$, so that in either case spacetime curvature along the EH decays to that of the EK background at late advanced times. As argued above, this demonstrates that the EH of the EK spacetime is indeed stable against linearized gravitational perturbations [20].

One may ask why the blowing up of the Weyl scalar ψ_4 does not signify instability, as claimed by [7,8]. After all, ψ_4 is a scalar under coordinate transformations, and therefore all observers would presumably agree on its blowing up. The resolution of this conundrum is that the Weyl scalars are not invariant under transformations of the tetrad vectors. Indeed, under type-III rotations the null tetrad basis vectors $\mathbf{l} \to A^{-1}\mathbf{l}$, $\mathbf{n} \to A\mathbf{n}$, $\mathbf{m} \to e^{i\vartheta}\mathbf{m}$, and $\bar{\mathbf{m}} \to e^{i\vartheta}\bar{\mathbf{m}}$, where the two real parameters A, ϑ describe rescaling and rotation, correspondingly, of the tetrad vectors [21]. We can choose ϑ in a way that makes, say, $\Re(\psi_4) = 0$, or if we choose another value of ϑ we can make $\Im(\psi_4) = 0$. More importantly, we can choose the rescaling function $A = M/v$; i.e., as our null observer moves along the EH she continuously rescales her tetrad vector $\mathbf{l}$ linearly in advanced time, and her tetrad vector $\mathbf{n}$ inversely in advanced time. Correspondingly, $\psi_4 \to \psi_4' \sim v^{-2}\psi_4$, and $\psi_0 \to \psi_0' \sim v^2\psi_0$. Therefore, $\psi_4' \sim v^{-1/2}$ and $\psi_0' \sim v^{-1/2}$ as $v \gg M$. We refer to this dynamical HH tetrad as the symmetric tetrad. We conclude that the blowup of ψ_4 in the HH tetrad is a consequence of a problem with the tetrad: if one generalizes the tetrad to a dynamical HH tetrad ("the symmetric tetrad") in which the basis vectors are continuously rescaled as discussed above, both Weyl scalars ψ_4' and ψ_0' decay to zero. The Beetle-Burko scalar ξ is, however, invariant also under tetrad vector transformations, and therefore is unchanged by this rescaling. Both curvature invariants I, J approach their EK values at late advanced times along the EH. EK are stable because there exist observers for whom initially small ψ_4', ψ_0' remain small along the EH and decay to zero. Notice that a family of observers, separated by time translations, who fall into EK and make measurements in the symmetric tetrad are non-parallel-propagated observers. In this family of observers, asymptotically late daughters see no instability.

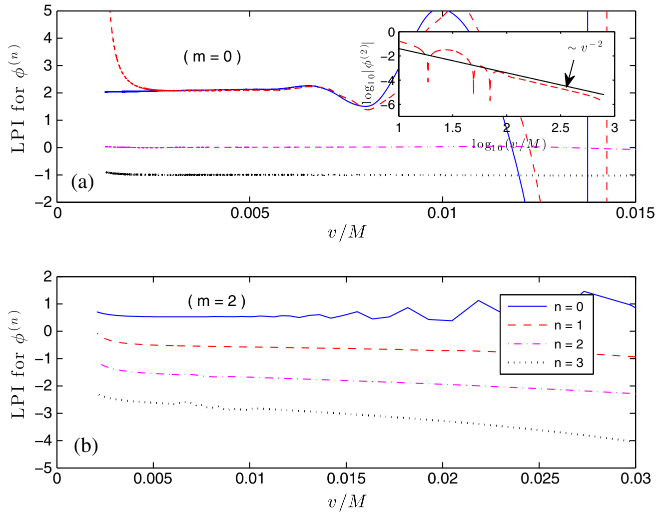

FIG. 4. The local power indices for a scalar field perturbation for the (a) axisymmetric ($m = 0$) and (b) nonaxisymmetric ($m = 2$) cases as functions of advanced time along the EH of EK. In panel (a) the inset shows $\partial_\rho^2\phi$ as a function of advanced time.

Our analysis does not show that any curvature scalar of higher order (i.e., a curvature scalar that includes gradients of the Weyl scalars) does not blow up along the EH with infinite advanced time. However, we cannot rule out the possibility that curvature scalars of high enough orders do. If that is the case, there might be a nonscalar polynomial singularity ("whimper singularity") evolving, which would be asymptotically delayed [22]. Whimper singularities have the feature that in a suitably rotated tetrad the singular behavior disappears. That is, they signify a problem with parallel transport, not a genuine singularity of spacetime, in the sense that there could be (null) observers who are not parallelly propagated, who experience no singular behavior.

Consider next a scalar field. Figure 4 shows the LPIs for the axisymemtric ($m = 0$) and nonaxisymmetric ($m = 2$) cases. In both cases we obtain asymptotic LPI values that agree with [8]. Specifically, in the axisymmetric case we find $q = 2, 2, 0, -1$ and in the nonaxisymmetric case $q = 1/2, -1/2, -3/2, -5/2$ for $n = 0, 1, 2, 3$, respectively. We find that the scalar field itself decays to zero with advanced time, but transverse gradients thereof blow up, consistent with previous results.

Consider for simplicity an EH null observer on the rotation axis of EK. The gradient of the scalar field, $\partial_\rho\phi$, blows up for $m = 2$ with advanced time. However, observers who use different coordinates disagree on what the gradient is. The only observer-independent way to consider the gradient is to consider a scalar under coordinate transformations. Specifically, $(\nabla_\alpha\phi\nabla^\alpha\phi)^{1/2} \sim (\partial_\rho\phi\partial_v\phi)^{1/2} \sim v^{-1/2} \to 0$ as $v \to \infty$. Consider next higher-order gradients, say, $\nabla_{\alpha_1,\ldots,\alpha_n}\phi$. Also in this case, the scalar $(\nabla_{\alpha_1,\ldots,\alpha_n}\phi\nabla^{\alpha_1,\ldots,\alpha_n}\phi)^{1/2} \sim v^{-1/2}$ vanishes at infinite advanced time.

We cannot calculate the perturbations of the Riemann tensor in the scalar field case, as we have a fixed Kerr background. However, we can use the (linearized) Einstein equations to find the Ricci tensor: we write the Einstein equations as $R_{\mu\nu} = 8\pi(T_{\mu\nu} - Tg_{\mu\nu}/2)$. We can then calculate the scalar field energy-momentum tensor from the scalar field perturbation ϕ, $T^{\mu\nu}[\phi] = (g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi$. The Ricci scalar $R \sim (\partial_\rho\phi\partial_v\phi) \sim v^{-1} \to 0$ as $v \to \infty$ for $m = 2$. The curvature scalar $R_{\mu\nu}R^{\mu\nu} \sim (\partial_\rho\phi\partial_v\phi)^2 \sim v^{-2}$ in the nonaxisymmetric case. We conjecture that all other curvature scalar polynomials made with R and $R_{\mu\nu}$ are also well behaved as $v \to \infty$ along the EH. We cannot, however, find a compete set of algebraically independent scalar polynomials as we did in the gravitational case.

We next examine scalars constructed from gradients of R and $R_{\mu\nu}$. Consider $\nabla^\mu R\nabla_\mu R$. Comparing with R^2, we now introduce one additional ∂_ρ and one additional ∂_v derivative. The effects of both tend to cancel each other, and this scalar behaves like v^{-2} in the nonaxisymmetric case. We also find that scalars such as $\nabla^\sigma R^{\mu\nu}\nabla_\sigma R_{\mu\nu} \sim v^{-2}$ and $R^{\mu\nu}\nabla_\mu R\nabla_\nu R \sim v^{-3}$. We did not find a scalar made with derivatives of the curvature that does not decay to zero. We propose that in this case, neither a scalar polynomial singularity nor a nonscalar polynomial one evolves.

We thank S. Aretakis, S. Gralla, A. Ori, and A. Strominger (and his research group) for discussions. G. K. acknowledges research support from the National Science Foundation (Grant No. PHY-1701284) and Air Force Research Laboratory (Grant No. 10-RI-CRADA-09). Many of the computations for this work were performed on systems provided by Microway Inc. and Nvidia.

[1] M. Dafermos, G. Holzegel, and I. Rodnianski, arXiv:1601.06467.
[2] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, arXiv:1402.7034.
[3] B. F. Whiting, J. Math. Phys. (N.Y.) **30**, 1301 (1989).
[4] S. E. Gralla, A. Zimmerman, and P. Zimmerman, Phys. Rev. D **94**, 084017 (2016).
[5] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).

[6] S. Aretakis, arXiv:1006.0283.
[7] J. Lucietti and H. S. Reall, Phys. Rev. D **86**, 104030 (2012).
[8] M. Casals, S. E. Gralla, and P. Zimmerman, Phys. Rev. D **94**, 064003 (2016).
[9] S. E. Gralla and P. Zimmerman, arXiv:1711.00855.
[10] K. Murata, H. S. Reall, and N. Tanahashi, Classical Quantum Gravity **30**, 235007 (2013).
[11] C. Beetle and L. M. Burko, Phys. Rev. Lett. **89**, 271101 (2002).
[12] S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[13] L. M. Burko, G. Khanna, and A. Zenginoğlu, Phys. Rev. D **93**, 041501(R) (2016).
[14] A. Zenginoğlu and G. Khanna, Phys. Rev. X **1**, 021017 (2011).
[15] G.-S. Jiang and C.-W. Shu, J. Comput. Phys. **126**, 202 (1996).
[16] S. Gottlieb, C.-W. Shu, and E. Tadmor, SIAM Rev. **43**, 89 (2001).
[17] L. M. Burko and A. Ori, Phys. Rev. D **56**, 7820 (1997).
[18] M. Richartz, C. A. R. Herdeiro, and E. Berti, Phys. Rev. D **96**, 044034 (2017).
[19] A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969).
[20] The conclusion that observer-independent quantities exhibit no instability was made independently and at the same time for the case of Ricci curvature in S. Hadar and H. S. Reall, J. High Energ. Phys. 12 (**2017**), 062. It was later shown in greater generality also in [9].
[21] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[22] R. A. Sussman, J. Math. Phys. (N.Y.) **29**, 945 (1988).