

Analytical attractor and the divergence of the slow-roll expansion in relativistic hydrodynamics

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We find the general analytical solution of the viscous relativistic hydrodynamic equations (in the absence of bulk viscosity and chemical potential) for a Bjorken expanding fluid with an ideal gas equation of state and a constant shear viscosity relaxation time. We analytically determine the hydrodynamic attractor of this fluid and discuss its properties. We show for the first time that the slow-roll expansion, a commonly used approach to characterize the attractor, diverges. This is shown to hold also in a conformal plasma. The gradient expansion is found to converge in an example where causality and stability are violated.

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I. INTRODUCTION

Relativistic hydrodynamics has played a key role in our understanding of the novel properties of the quark-gluon plasma (QGP) formed in ultrarelativistic heavy ion collisions (for a review, see [1]). The basic picture is that the hot and dense matter formed in these collisions behaves as a relativistic fluid in which dissipative effects are surprisingly small in comparison to other fluids in nature [2]. However, recent experimental observations [3–9] have suggested that the strongly interacting matter produced in small collision systems (such as proton-nucleus and even proton-proton collisions) also displays the same liquidlike properties found in large nucleus-nucleus collisions. This finding was accompanied by a number of theoretical studies on the emergence of hydrodynamic behavior from microscopic models (see, for instance, [10–21]), which have contributed to assess the domain of applicability of relativistic hydrodynamics as an effective theory for rapidly expanding systems.

Excluding the contribution from other conserved quantities (such as baryon number), the equations of motion of relativistic hydrodynamics stem from the conservation laws of energy and momentum, $\nabla_\mu T^{\mu\nu} = 0$, with $T^{\mu\nu}$ being the energy-momentum tensor of the fluid. Quite generally, one may write $T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}$ with $T^{\mu\nu}_{\text{ideal}}$ being the energy-momentum tensor of an ideal fluid constructed using the local energy density ε and flow velocity u_μ (i.e., the standard hydrodynamic fields) and $\Pi^{\mu\nu}$ being a dissipative contribution whose explicit form can only be found with additional assumptions. In the Landau frame [22] (used throughout this paper), in the absence of bulk viscous effects $\Pi^{\mu\nu} = \pi^{\mu\nu}$, with $\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$ being the shear stress tensor constructed using the tensor projector

$\Delta^{\mu\nu}_{\alpha\beta} = (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta)/2 - \Delta_{\alpha\beta} \Delta^{\mu\nu}/3$ defined by the projection operator transverse to the flow $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ ($g_{\mu\nu}$ is the spacetime metric). In the gradient expansion approach [23], the dissipative fluxes, such as $\pi^{\mu\nu}$, are organized as a formal expansion in powers of the spacetime gradients of the hydrodynamic fields taking into account all the possible structures compatible with the symmetries, whose conformal limit was originally worked out in the Landau frame to second order in gradients in [24,25] while Ref. [26] extended the expansion to third order.

However, in the relativistic regime this approach faces considerable challenges since the equations of motion obtained from this formalism generally display acausal behavior and instabilities (at least at the linear level) [27,28] already at first order in the gradient expansion, i.e., relativistic Navier-Stokes theory, which are not resolved by the inclusion of second order derivatives of the hydrodynamic fields [29] unless some type of resummation involving the hydrodynamic fields is employed [30]. With the recent evidence that the gradient series has zero radius of convergence in the relativistic regime both at strong coupling [31,32] and in kinetic theory models [33,34], it is unlikely that any of these problems are resolved perturbatively by going to even higher orders in the expansion. This motivates the search for a meaningful definition of viscous relativistic hydrodynamics that does not resort to an expansion in gradients of the hydrodynamic fields.

At least in the linear regime, causality and stability can be obtained by extending the set of dynamical variables to include not only the hydrodynamic fields but also the dissipative fluxes, as in Israel-Stewart (IS) theory [35]. Other approaches include, for instance, divergence type theories [36]. In IS $\pi^{\mu\nu}$ is defined dynamically via

additional equations of motion, which were originally determined by requiring that the second law of thermodynamics is satisfied. In this approach quantities such as $\pi^{\mu\nu}/(\varepsilon + P)$ (with $P = P(\varepsilon)$ being the equilibrium pressure) are assumed to be small, though it is important to remark that this assumption does not necessarily imply an expansion in gradients.

An interesting property displayed by IS theory and other more modern approaches such as [37] is that the first-order relaxation-type equations obeyed by $\pi^{\mu\nu}$ imply that these quantities must also be specified on the spacelike hypersurface that defines the initial value problem. Since the conservation laws couple the hydrodynamic fields to the dissipative fluxes, the solution for the hydrodynamic fields $\{\varepsilon, u_\mu\}$ is expected to be sensitive to the choices made for the initial conditions of $\pi^{\mu\nu}$. In equilibrium this dependence is of course erased but one may ask whether there is some type of non-equilibrium regime in which such a dependence is minimal. In this regard, one may define a nonequilibrium hydrodynamic attractor by the condition that for a large set of initial conditions,¹ the system's dynamics collapses at large times onto a single encompassing behavior, before true thermal equilibrium is reached.

This feature was observed [12] in a numerical simulation of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with large number of colors undergoing Bjorken flow [40] and its meaning was clarified in [41] via a study of conformal IS theory also assuming Bjorken symmetry. Since then, such dynamical attractor behavior has been investigated in other works [20,21,30,42–45]. The presence of an attractor solution restores the large degree of universality usually associated with hydrodynamic behavior without relying on the gradient expansion.

In Bjorken expanding systems the symmetries are so powerfully constraining that it is possible to investigate the large order limit of the gradient expansion in a systematic manner [31] (the same holds for fluids embedded in an expanding Universe [32]), which is not feasible in less symmetric situations. This allowed the authors of Ref. [41] to show that the hydrodynamic attractor corresponds to a resummation of the gradient series, establishing an interesting link between hydrodynamics and the mathematics of resurgence theory, later pursued by other works [32,42,46,47].

The numerically obtained attractor solutions found so far indicate that it is possible to find universal hydrodynamic behavior far-from-equilibrium, regardless of the details of the initial state of the system. However, even though one may now associate hydrodynamic behavior with such

non-equilibrium attractors, it is not straightforward to clearly state its domain of validity or even how to clearly define attractors (besides by explicit numerical inspection involving a large number of initial conditions).

Given the simplicity of the hydrodynamic equations in Bjorken flow, in this case it is possible to use different ways to identify the non-equilibrium attractor, as discussed in [41]. One method involves defining the boundary condition of the fields at very early times. Another possibility is the resummation of the divergent gradient series. The last method is the analog of the slow-roll expansion used in cosmology [48] whose zeroth order term already generally gives a decent approximation for the attractor in the Bjorken case. Further progress in identifying the virtues and issues with these approaches could be achieved by having an analytical example where the fluid's evolution towards the attractor can be investigated in a simpler way.

In this paper we show that the equations of viscous relativistic hydrodynamics (neglecting effects from bulk viscosity and chemical potential) can be solved analytically when the shear relaxation time is constant, the equation of state of state is that of an ideal gas $\varepsilon = 3P$, and the system undergoes Bjorken expansion. Differently than other studies, here we analytically determine the hydrodynamic attractor of this system in closed form and discuss its properties. We perform the first study of the large order behavior of the slow-roll expansion and compare it to the analytical attractor. We find that the slow-roll expansion in hydrodynamics diverges. This is also the case for conformal fluids. We investigate the role played by the values of the transport coefficients on the convergence of the gradient expansion and show that the series can actually converge if the transport coefficients do not fulfill the standard conditions for causality and stability determined from well-known linearized analyses [27] (see also [49]). We also discuss the generalized gradient expansion series first presented in [33] and we apply it here to find solutions of IS theory. In contrast to the other series discussed, this one appears to converge and offers a very good description of the analytical solution already at second order.

This paper is organized as follows. In the next section we define the viscous hydrodynamic equations we use and obtain their full analytical solution, under the conditions mentioned above. We analytically determine in Sec. III the nonequilibrium attractor and discuss several of its properties. Our conclusions and outlook are presented in Sec. IV.

Definitions: Throughout this work we use natural units $\hbar = c = k_B = 1$ and Milne coordinates where $x^\mu = (\tau, x, y, \zeta)$, with proper-time and spacetime rapidity defined in terms of standard Minkowski coordinates as follows: $\tau = \sqrt{t^2 - z^2}$ and $\zeta = \tanh^{-1}(z/t)$. In these coordinates, the metric is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - dx^2 - dy^2 - \tau^2 d\zeta^2$, and the corresponding nonzero Christoffel symbols are $\Gamma_{\zeta\zeta}^\tau = \Gamma_{\tau\tau}^\zeta = 1/\tau$ and $\Gamma_{\tau\zeta}^\tau = \tau$.

¹Initial conditions for the dissipative fluxes are not, in fact, completely arbitrary. For instance, one may require the weak energy condition, $T_{\mu\nu}t^\mu t^\nu \geq 0$ where t^μ is an arbitrary time-like 4-vector [38], to be satisfied. For a discussion on related topics, see [39].

II. ANALYTICAL SOLUTION OF VISCOUS RELATIVISTIC HYDRODYNAMICS IN BJORKEN FLOW

The set of viscous relativistic hydrodynamic equations we use is given by

$$D\varepsilon + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0 \quad (1)$$

$$(\varepsilon + P)Du^\mu - \Delta^\mu_\lambda \nabla^\lambda P + \Delta^\mu_\lambda \nabla_\mu \pi^{\alpha\lambda} = 0 \quad (2)$$

$$\begin{aligned} \tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi} \theta \pi^{\mu\nu} + \tau_{\pi\pi} \Delta^{\mu\nu}_{\alpha\beta} \pi^{\alpha\lambda} \sigma^\beta_\lambda - 2\tau_R \Delta^{\mu\nu}_{\alpha\beta} \pi^\alpha_\lambda \omega^{\beta\lambda} \\ + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \end{aligned} \quad (3)$$

where $D = u^\mu \nabla_\mu$ is the comoving covariant derivative, $\theta = \nabla_\mu u^\mu$ is the local expansion rate, $\sigma_{\mu\nu} = \Delta^\alpha_{\mu\nu} \nabla_\alpha u_\beta$ is the shear tensor, $\omega_{\mu\nu} = (\Delta^\lambda_\mu \nabla_\lambda u_\nu - \Delta^\lambda_\nu \nabla_\lambda u_\mu)/2$ is the vorticity tensor, η is the shear viscosity, and τ_R is the shear relaxation time. We neglect all effects from bulk viscous pressure and assume an ideal gas equation of state, $\varepsilon = 3P$, at zero chemical potential. The equations above may be derived using the Boltzmann equation in the 14-moment approximation or in the relaxation time approximation (RTA), as shown in Refs. [37,50,51]. In the 14-moment approximation and for a massless gas, one can show that $\delta_{\pi\pi} = 4/3\tau_R$, $\tau_{\pi\pi} = 10/21\tau_R$ and $\eta = (\varepsilon + P)\tau_R/5$ [37,50,51]. For now, we assume a more general expression for $\tau_{\pi\pi}$, where $\tau_{\pi\pi} = \lambda\tau_R$. In this paper we further assume that τ_R is constant, an assumption that plays a crucial role in determining the analytical solution derived below.

We impose Bjorken symmetry and, thus, in our coordinate system $u_\mu = (1, 0, 0, 0)$. This implies that only the first and the third equations above contain nontrivial information. The symmetries further constrain the expansion rate of the fluid, $\theta = 1/\tau$, and its shear tensor, $\sigma_{\mu\nu}$, which becomes diagonal

$$\sigma^\mu_\nu = \text{diag}\left(0, -\frac{1}{3\tau}, -\frac{1}{3\tau}, \frac{2}{3\tau}\right). \quad (4)$$

Furthermore, the shear stress tensor will also become diagonal and can be described with only one independent degree of freedom,

$$\pi^\mu_\nu = \text{diag}(0, -\pi/2, -\pi/2, \pi). \quad (5)$$

Therefore, even though homogeneous, the system is not static and, at sufficiently early times, the gradients θ and $\sigma^{\mu\nu}$ can become large enough to drive the system far away from local thermodynamic equilibrium. On the other hand, the gradients of any scalar, such as the chemical potential and temperature, are always zero, prohibiting the existence of any heat flow or diffusion. Finally, in this geometry the vorticity tensor is also always zero

$$\omega^{\mu\nu} = 0. \quad (6)$$

The evolution of the fluid is then described by the following set of coupled differential equations,

$$\frac{d\varepsilon}{d\tau} + \frac{(\varepsilon + P)}{\tau} - \frac{\pi}{\tau} = 0, \quad (7)$$

$$\tau_R \frac{d\pi}{d\tau} + \pi + \left(\frac{4}{3} + \lambda\right) \tau_R \frac{\pi}{\tau} = \frac{4\eta}{3\tau}. \quad (8)$$

It is convenient to rewrite these equations in terms of the dimensionless field, $\chi \equiv \pi/(\varepsilon + P)$, and define the dimensionless proper time variable $\hat{\tau} = \tau/\tau_R$, which is the inverse Knudsen, $K_N = 1/\hat{\tau}$, in Bjorken flow [33]. With these changes of variables, the equations simplify to

$$\frac{1}{\varepsilon \hat{\tau}^{4/3}} \frac{d(\varepsilon \hat{\tau}^{4/3})}{d\hat{\tau}} = \frac{4\chi}{3\hat{\tau}}, \quad (9)$$

and

$$\frac{d\chi}{d\hat{\tau}} + \lambda \frac{\chi}{\hat{\tau}} + \frac{4}{3\hat{\tau}} \chi^2 + \chi - \frac{3a}{4\hat{\tau}} = 0, \quad (10)$$

with

$$a = \frac{16}{9(\tau_R T)} \frac{\eta}{s}. \quad (11)$$

Causality and stability around equilibrium at the linearized level hold when $\eta/(s\tau_R T) \leq 1/2$ [49], i.e., $a \leq 8/9$. Even though $a = 16/45$ and $\lambda = 10/21$ in the 14-moment approximation, we kept a and λ above as arbitrary *constants* since the general analytical solution of these equations can be found for any $a \geq 0$ and $\lambda \in \mathbb{R}$, as we show below.²

Equation (10) is a Riccati equation that can be solved independently of (9), a direct consequence of the constant τ_R assumption.³ First order nonlinear ODEs of Riccati type can always be written as second order *linear* ODE's and, as

²We note that in conformal fluids $\tau_R \sim 1/T$ and η/s is constant, which still leads to a constant a , as defined in (11). However, there is no known analytical solution to (3) that smoothly connects to the Navier-Stokes solution at large times. Therefore, in this paper we only investigate the case of a conformal fluid numerically, see Sec. III D.

³Equations similar to (9) and (10) may be derived using the slightly more general equation of state, $\varepsilon = c_s^2 P$, where $0 < c_s^2 \leq 1/3$ is a constant. In this case, the equation of motion for χ would still be of Riccati type and the general procedure used to solve such equation, described in this section, would also lead to an analytical solution as long as effects from bulk viscosity are still neglected. For simplicity, in this paper we focus on the case where $c_s^2 = 1/3$, as remarked before.

a matter of fact, this can be done in the present case using a new variable $y(\hat{\tau})$ defined via

$$\frac{1}{y} \frac{dy}{d\hat{\tau}} = \frac{4\chi}{3\hat{\tau}}. \quad (12)$$

Inserting this into (10), provides

$$\frac{d^2 y}{d\hat{\tau}^2} + \left(1 + \frac{1+\lambda}{\hat{\tau}}\right) \frac{dy}{d\hat{\tau}} - \frac{a}{\hat{\tau}^2} y = 0. \quad (13)$$

This linear ODE can be solved and the general solution is

$$y(\hat{\tau}) = A\hat{\tau}^{-(\lambda+1)/2} \exp(-\hat{\tau}/2) \times \left[M_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau}) + \alpha W_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau}) \right], \quad (14)$$

where A and α are constants and $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are Whittaker functions.⁴ Using (14), one can find the following analytical solution for the energy density

$$\begin{aligned} \varepsilon(\hat{\tau}) &= \varepsilon_0 \left(\frac{\hat{\tau}_0}{\hat{\tau}} \right)^{4/3} \frac{y(\hat{\tau})}{y(\hat{\tau}_0)} \\ &= \varepsilon_0 \left(\frac{\hat{\tau}_0}{\hat{\tau}} \right)^{\frac{4}{3} + \frac{\lambda+1}{2}} \exp\left(-\frac{\hat{\tau} - \hat{\tau}_0}{2}\right) \\ &\quad \times \left[\frac{M_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau}) + \alpha W_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau})}{M_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau}_0) + \alpha W_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4a}}{2}}(\hat{\tau}_0)} \right] \end{aligned} \quad (15)$$

and the normalized shear stress tensor component

$$\chi(\hat{\tau}) = \frac{\pi}{\varepsilon + P} = \frac{3(\sqrt{4a + \lambda^2} - \lambda) M_{\frac{1}{2} - \frac{\lambda}{2}, \frac{1}{2}\sqrt{4a + \lambda^2}}(\hat{\tau}) - 6\alpha W_{\frac{1}{2} - \frac{\lambda}{2}, \frac{1}{2}\sqrt{4a + \lambda^2}}(\hat{\tau})}{8 \left(M_{-\frac{1+\lambda}{2}, \frac{1}{2}\sqrt{4a + \lambda^2}}(\hat{\tau}) + \alpha W_{-\frac{1+\lambda}{2}, \frac{1}{2}\sqrt{4a + \lambda^2}}(\hat{\tau}) \right)}. \quad (16)$$

One can see that the value of A in (14) does not enter in either ε or π . Thus, the constants that define the initial-value problem at $\hat{\tau} = \hat{\tau}_0 > 0$ are ε_0 and α , since the latter can be written in terms of $\chi(\hat{\tau}_0)$. One important constraint for this solution is that α must be such that $y(\hat{\tau})$ remains non-negative for all $\hat{\tau} \geq \hat{\tau}_0$ to make sure that the energy density is positive-definite and there are no zeros in the denominators of the expressions above. Equations (15) and (16) define the general analytical solution of the viscous hydrodynamic equations in Bjorken flow with a constant relaxation time. As such, they can be easily implemented in studies of different hydrodynamic schemes and their comparison to exact solutions in kinetic theory, such as [53].

III. ANALYTICAL NONEQUILIBRIUM ATTRACTOR

In this section we investigate the solution of the hydrodynamic equations and the corresponding nonequilibrium

attractor. No significant change is observed when λ is taken into account and, thus, we set $\lambda = 0$ in the following (this approximation was also used in [41]). For convenience, we repeat the equation for χ in this case below

$$\hat{\tau} \frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0. \quad (17)$$

The general analytical solution of this equation can also be written in terms of Bessel functions

$$\begin{aligned} \chi(\hat{\tau}) &= \frac{3\sqrt{a}}{4} \\ &\quad \times \left[\frac{\alpha(K_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) + K_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}}{2})) + I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) - I_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}}{2})}{\alpha(K_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}_0}{2}) - K_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}_0}{2})) + I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}_0}{2}) + I_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}_0}{2})} \right] \end{aligned} \quad (18)$$

and the corresponding expression for the energy density is

$$\varepsilon(\hat{\tau}) = \varepsilon_0 e^{-\frac{1}{2}(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}} \right)^{\frac{5}{6}} \left[\frac{\alpha(K_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) - K_{\frac{1}{2}+\sqrt{a}}(\frac{\hat{\tau}}{2})) + I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) + I_{\frac{1}{2}+\sqrt{a}}(\frac{\hat{\tau}}{2})}{\alpha(K_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}_0}{2}) - K_{\frac{1}{2}+\sqrt{a}}(\frac{\hat{\tau}_0}{2})) + I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}_0}{2}) + I_{\frac{1}{2}+\sqrt{a}}(\frac{\hat{\tau}_0}{2})} \right]. \quad (19)$$

We note that $\alpha \leq 0$, which guarantees that the denominator of the above equations is always nonzero. Also, the general solution (18) cannot be simply decomposed in terms of an

attractor plus transient corrections. Both contributions are present in the numerator and the denominator of the analytical solution.

In all the previous studies on nonequilibrium attractors in Bjorken flow the attractor *per se* was only found numerically using basically three different approaches [41]:

⁴We refer the reader to Ref. [52] for more details concerning the analytical structure of these functions.

- (i) Explicit construction by solving the corresponding differential equation fixing a specific boundary condition at very early times.
- (ii) Resummation of the gradient series.
- (iii) Slow-roll expansion.

We will use the analytical solution found here to illustrate how these approaches fare at identifying the analytical attractor and its properties.

A. The attractor solution

From the analytical solution derived in the previous section, it is straightforward to see that the solution (18) completely loses the information about the initial conditions (encoded in α) at late times. This happens because the Bessel functions display the following asymptotic form for sufficiently large values of its argument, $K_\nu(x) \sim e^{-x}/\sqrt{x}$ and $I_\nu(x) \sim e^x/\sqrt{x}$. Therefore, the terms containing $K_\nu(\hat{\tau}/2)$ become significantly smaller compared to the terms containing $I_\nu(\hat{\tau}/2)$ as time increases. At a sufficiently long time, the solution (18) can be approximated as

$$\chi(\hat{\tau}) \rightarrow \chi_{\text{att}}(\hat{\tau}) = \frac{3\sqrt{a}}{4} \left[\frac{I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) - I_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}}{2})}{I_{\sqrt{a}-\frac{1}{2}}(\frac{\hat{\tau}}{2}) + I_{\sqrt{a}+\frac{1}{2}}(\frac{\hat{\tau}}{2})} \right]. \quad (20)$$

This expression corresponds to the exact solution for $\alpha = 0$ and it represents the nonequilibrium attractor solution of the hydrodynamic equations investigated here. The typical attractor behavior is illustrated in Fig. 1 for the RTA case where $a = 16/45$. One can see that (20) is the only solution of the differential equation that smoothly connects to the positive χ branch at early times, i.e., $\lim_{\hat{\tau} \rightarrow 0} \chi(\hat{\tau})|_{a=16/45} = 1/\sqrt{5}$.

When discussing the attractor solution in Bjorken flow, a common procedure consists in analyzing the behavior of χ

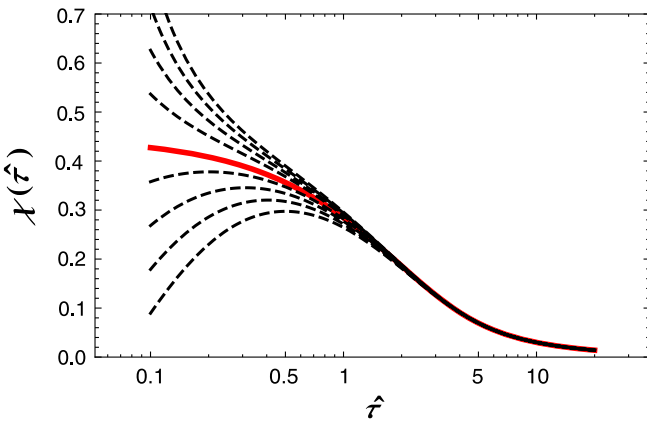


FIG. 1. Analytical nonequilibrium attractor (20) in solid red for the RTA value $a = 16/45$ (and $\lambda = 0$). The dashed curves correspond to the solution in Eq. (18) for different initial conditions (parametrized by α), which collapse onto the attractor before local equilibrium is reached (where χ vanishes).

at $\hat{\tau} \rightarrow 0$. In our case this gives two limiting values: $3\sqrt{a}/4$ for $\alpha = 0$ and $-3\sqrt{a}/4$ if $\alpha \neq 0$. Therefore, the attractor is the only solution that goes to $3\sqrt{a}/4$ at $\hat{\tau} = 0$. Indeed, this limiting behavior of the attractor in Bjorken flow has been used in previous works as a way to define it [41]. In this case, one may find the attractor numerically by identifying it as the solution that obeys this boundary condition.

B. Resummation of the gradient series

The formal gradient expansion solution is represented as the late time series $\chi(\hat{\tau}) = (3a/4) \sum_{n=0}^{\infty} c_n/\hat{\tau}^n$, where the corresponding coefficients of the series are given by

$$c_{n+1} = nc_n - a \sum_{m=0}^n c_{n-m}c_m, \quad (21)$$

with $c_0 = 0$ and $c_1 = 1$. It is interesting to notice that when causality and stability are fulfilled, i.e., for $a \leq 8/9$, the gradient series diverges since for large n the first term in (21) dominates leading to factorial growth.

Setting $a = 1$ is particularly interesting since in this case one can show that all $c_{n \geq 1} = 1$, which leads to

$$\chi(\hat{\tau}) = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{\hat{\tau}^n}. \quad (22)$$

In contrast to the other examples in Bjorken flow, this series has a nonzero radius of convergence, i.e., for any $\hat{\tau} > 1$ (which overlaps with the expected domain of the late time series) this can be summed up to give

$$\chi(\hat{\tau}) \rightarrow \frac{3}{4(\hat{\tau} - 1)}. \quad (23)$$

However, we remark that this nonzero radius of convergence was possible only when a was taken in the acausal region. Convergent series can also be obtained for other values of a , e.g., $a = 4$ and 9 . However, all these values are in the acausal regime.

Now, to see that resumming the gradient series does lead to the attractor, we note that the full analytical solution in the case $a = 1$ is

$$\chi(\hat{\tau})|_{a=1} = \frac{3}{4} \left[\frac{\frac{1}{\hat{\tau}} + \alpha(1 + \frac{1}{\hat{\tau}})e^{-\hat{\tau}}}{1 - \frac{1}{\hat{\tau}} + \frac{\alpha}{\hat{\tau}}e^{-\hat{\tau}}} \right] \quad (24)$$

and correspondingly the attractor solution (20) is simply

$$\chi(\hat{\tau})|_{a=1, \text{att}} = \frac{3}{4(\hat{\tau} - 1)}, \quad (25)$$

which matches the result obtained from the gradient series. Even though there is a pole in $\hat{\tau} = 1$, we note that the expression above is meaningful for larger times.

The current example with $a = 1$ shows in a very clear manner that the attractor can be defined via a resummation of the gradient series. Furthermore, it is straightforward to find a late time trans-series representation for the general analytical solution in (24). The first terms are

$$\chi(\hat{\tau}) = \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{\hat{\tau}^{n+1}} - \frac{3}{4} \alpha e^{-\hat{\tau}} \sum_{n=0}^{\infty} \frac{1}{\hat{\tau}^n} \left(1 + \frac{1}{\hat{\tau}} + \frac{n+1}{\hat{\tau}^2} \right) + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}}) \quad (26)$$

$$= \frac{3}{4} \frac{1}{(\hat{\tau}-1)} - \frac{3}{4} \alpha e^{-\hat{\tau}} \frac{\hat{\tau}^2}{(\hat{\tau}-1)^2} + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}}), \quad (27)$$

where one can see that α , the parameter that defines the initial condition, plays the role of the trans-series expansion parameter [46]. In this case, the contribution from each term in the trans-series can be easily determined since their corresponding power series representation converge (no Borel transforms are needed). For the causal configuration where $a \leq 8/9$, this is not the case and one must resort to resurgence theory to resum the series. In this case, one can compare the result from the resummation directly to the analytical expression for the attractor, which may lead to further insight into the application of resurgence ideas in hydrodynamics. However, this is beyond the scope of the present paper and we leave this interesting task for a future study.

C. Divergence of the slow-roll expansion

Reference [41] suggested another way to characterize the attractor based on the analog of the slow-roll expansion used in cosmology [48]. This can be done systematically [43] by including a small parameter ϵ (not to be confused with the energy density ϵ) in the differential equation

$$\epsilon \hat{\tau} \frac{d\chi}{d\hat{\tau}} + \frac{4}{3} \chi^2 + \hat{\tau} \chi - \frac{3a}{4} = 0 \quad (28)$$

where now $\chi = \chi(\hat{\tau}; \epsilon)$ is also a function of ϵ . The next step is to look for a series solution in powers of ϵ for χ

$$\chi(\hat{\tau}; \epsilon) = \sum_{n=0}^{\infty} \chi_n(\hat{\tau}) \epsilon^n. \quad (29)$$

Clearly, the full answer is only obtained in the limit $\epsilon \rightarrow 1$. In practice, ϵ is taken to 1 already after including only a few terms, given the apparent convergence of this procedure found in previous works. The zeroth order term gives two solutions and the one that recovers the Navier-Stokes (NS) limit at late times, $\chi_{\text{NS}} \sim 3a/(4\hat{\tau})$, is

$$\chi_0(\hat{\tau}) = \frac{3}{8} \left(\sqrt{\hat{\tau}^2 + 4a} - \hat{\tau} \right). \quad (30)$$

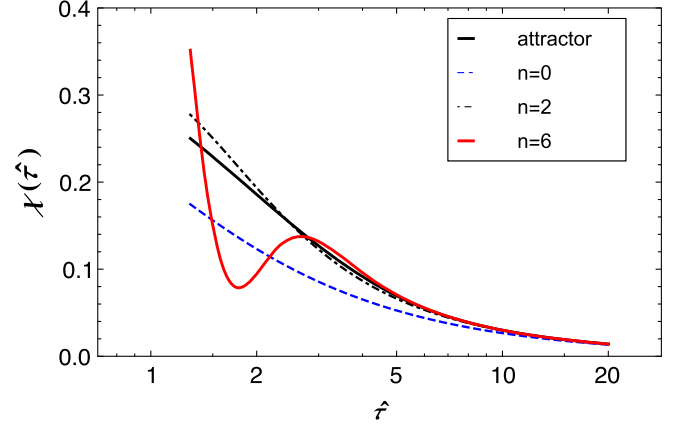


FIG. 2. Comparison between the analytical attractor in (25) for $a = 16/45$ and the result from the slow-roll expansion, computed at different orders.

The other terms with $n \geq 1$ can be found from the recurrence relation

$$\chi_n(\hat{\tau}) = -\frac{1}{\sqrt{\hat{\tau}^2 + 4a}} \left(\hat{\tau} \frac{d\chi_{n-1}}{d\hat{\tau}} + \frac{4}{3} \sum_{m=1}^{n-1} \chi_{n-m} \chi_m \right). \quad (31)$$

Each term of the series can be determined analytically, which may be used to study the large order behavior of the slow-roll expansion in hydrodynamics. We show in Fig. 2 a comparison between the analytical attractor in (25) (solid black curve) and the result from the slow-roll expansion computed at different orders for $a = 16/45$. One can see that there is an improvement when going from 0 to 2nd order as the latter gives a good representation for the attractor for $\hat{\tau} \geq 3$. However, as we increase the order of the expansion, already at $n = 6$ the result oscillates significantly, which indicates that the slow-roll expansion does not converge. In fact, this is indeed the case as shown in Fig. 3 which shows for the first time the large order behavior of the slow-roll expansion in hydrodynamics. One can see that for different values of $\hat{\tau}$ the series appears to diverge. This behavior persisted for all values of a in the causal regime⁵ Therefore, both the gradient series and the slow-roll expansion diverge in hydrodynamics. However, this divergence does not mean that such series are not useful. As a matter of fact, when properly truncated divergent series provide extremely powerful approximations to the solutions of several problems [54].

To illustrate that this is the case here, we plot the relative difference between the attractor and the two different series representations. In Fig. 4 we plot

⁵The maximum number of terms we could investigate numerically goes to roughly 100. Going to larger times also did not change this behavior. We also checked values of a in the acausal regime. Again, no qualitative difference was found.

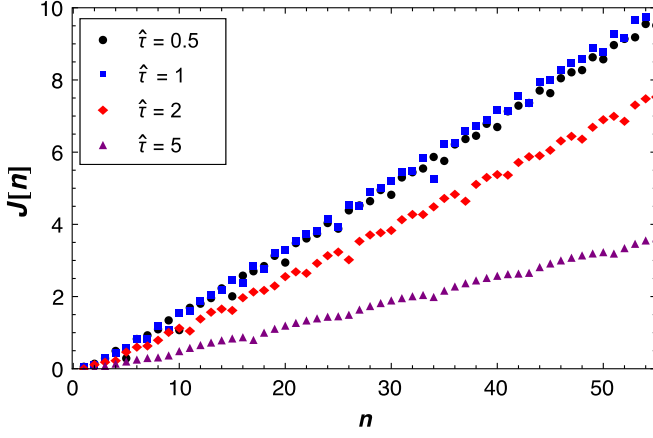


FIG. 3. Large order behavior of the slow-roll expansion in hydrodynamics. $J[n] = |\chi_n|^{1/n}$ as a function of n for $a = 16/45$ and different times $\hat{\tau} = 0.5, 1, 2, 5$.

$$R[n] = \frac{\int_{\hat{\tau}_0}^{\hat{\tau}_f} d\hat{\tau} |\chi_{\text{atr}}(\hat{\tau}) - \sum_{m=0}^n \chi_m(\hat{\tau})|}{\int_{\hat{\tau}_0}^{\hat{\tau}_f} d\hat{\tau} \chi_{\text{att}}(\hat{\tau})} \quad (32)$$

for the slow-roll expansion and also the corresponding expression for the gradient series ($\hat{\tau}_0 = 1$ and $\hat{\tau}_f = 50$). This quantity is defined in a way that maximizes the differences between these functions and the attractor. We see that $n = 2$ seems to be the optimal truncation for the gradient series while for the slow-roll expansion one finds $n = 3$. Altogether, the slow-roll expansion provides a much more accurate description of the attractor than the gradient series does at any order in the truncation (this is still the case when larger values of n are considered). However, for large values of n , the description already becomes very poor. Nevertheless, we note that the truncated slow-roll series is found to oscillate *around* the attractor while the gradient series completely misses the behavior of the attractor, leading to very different divergence patterns.

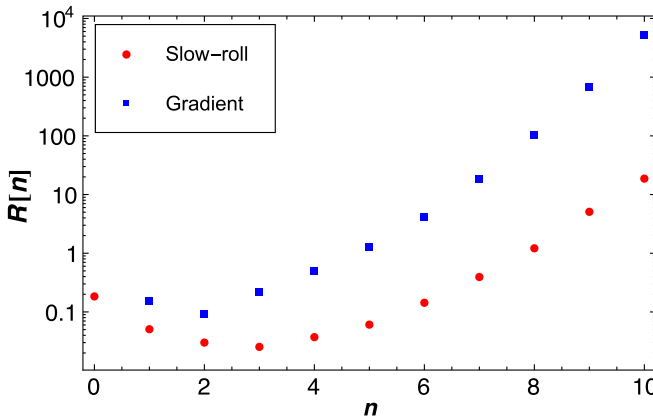


FIG. 4. Comparison between the relative difference between the attractor and the gradient and slow-roll expansions defined in (32) as a function of the truncation order n (with $a = 16/45$).

Therefore, the gradient expansion and the slow-roll series cannot be used to systematically approximate the hydrodynamic solution via the inclusion of higher order contributions. Nevertheless, the optimal truncation of these series can be extremely useful as they provide excellent approximations for the solution of the equations in the attractor regime.

D. Divergence of the slow-roll expansion in conformal hydrodynamics

To show that the divergence of the slow-roll expansion is not particular to the model studied here, in this section we determine the large order behavior of this series also in conformal hydrodynamics. In this case, the equations of motion are still given by (8) but now $\tau_R = c_R/T$, with c_R being a constant. We still assume $\lambda = 0$ for simplicity. We also note that in contrast with the previous case involving a constant relaxation time, in a conformal fluid $c_\eta = \eta/s$ is constant and, for instance, for a massless gas within the 14-moment approximation $c_R = 5\eta/s$ [37,55,56].

The equation for the energy density in the conformal fluid is still the same as (9) but the corresponding equation for normalized shear stress tensor component is

$$c_R \tau \frac{d\chi}{d\tau} + \frac{4c_R}{3} \chi^2 + \chi(\tau T) - \frac{4}{3} c_\eta = 0. \quad (33)$$

We now follow [41] and define the variable $w = \tau T$ (the reciprocal of the Knudsen number for this conformal fluid), with which one can eliminate T from the equation above and find a single equation that determines the state of the fluid

$$\frac{\bar{w}}{3} (\chi + 2) \chi' + \frac{4}{3} \chi^2 + \chi \bar{w} - \frac{4}{3} c_{\eta/R} = 0, \quad (34)$$

where $\bar{w} = w/c_R$, $c_{\eta/R} = c_\eta/c_R$ [43], and $\chi' = d\chi/d\bar{w}$. We follow the same procedure as before to obtain the slow-roll expansion $\chi(\bar{w}) = \sum_{n=0}^{\infty} \epsilon^n \chi_n(\bar{w})$. The zeroth order term that recovers the NS limit is

$$\chi_0(\bar{w}) = \frac{1}{8} \left(\sqrt{9\bar{w}^2 + 64c_{\eta/R}} - 3\bar{w} \right) \quad (35)$$

while the higher order terms are given by

$$\chi_n(\bar{w}) = -\frac{1}{\sqrt{9\bar{w}^2 + 64c_{\eta/R}}} \left[2\bar{w}\chi'_{n-1} + \bar{w}\chi_{n-1}\chi'_0 + \sum_{m=1}^{n-1} (4\chi_{n-m}\chi_m + \bar{w}\chi_{n-m-1}\chi'_m) \right]. \quad (36)$$

The terms can be computed analytically but now the expressions are considerably more complicated. This limits our ability to go to a very large order in this expansion, in

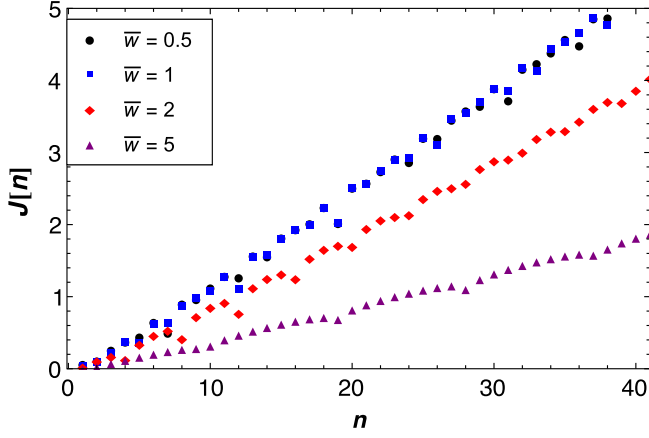


FIG. 5. Large order behavior of the slow-roll expansion in conformal hydrodynamics. $J[n] = |\chi_n|^{1/n}$ as a function of n with $c_{\eta/R} = 1/5$ and $\bar{w} = 0.5, 1, 2, 5$.

comparison to the constant τ_R case. Our results for this series are shown in Fig. 5 for $c_R = 5c_\eta$. Until the order we were able to compute, the series is found to diverge. We also checked that the same behavior holds when the values for c_η and c_R are taken from strongly coupled $\mathcal{N} = 4$ SYM theory [24]. This shows that the divergence of the slow-roll expansion is not an exclusive feature of the set of hydrodynamic equations obtained when the shear relaxation time is constant.

E. Generalized gradient expansion

In [33] a new type of expansion was proposed to provide a different resummation of the famous Chapman-Enskog series for the Boltzmann equation [23]. After just a few iterations, this new series appeared to converge very rapidly to the exact solution for the shear stress tensor computed using the Boltzmann equation in the relaxation time approximation.

In this approach, the coefficients of the gradient expansion [see (21)] are allowed to depend on time, i.e., we assume the following representation for the solution

$$\chi(\hat{\tau}) = \frac{3a}{4} \sum_{n=0}^{\infty} \frac{c_n(\hat{\tau})}{\hat{\tau}^n}. \quad (37)$$

This expansion is, in principle, more general, since it allows the expansion coefficients to have a time dependence that cannot be expanded in powers of $1/\hat{\tau}$. The time dependence of the generalized coefficients $c_n(\hat{\tau})$ cannot be determined *a priori*, but must be obtained by solving a simple set of coupled first order linear differential equations, which can be solved analytically. These equations are obtained by inserting (37) into (17) and collecting the terms with the same power in $1/\hat{\tau}$. This procedure rearranges the terms of the expansion in a specific way that it naturally captures non-perturbative exponentially small terms in Knudsen

number at late times $\sim e^{-\hat{\tau}}$. This is mathematically justified if the series (37) converges absolutely.

Another important point concerns the initial conditions. To solve the original equation for χ one needs to specify the initial condition $\chi_0 \equiv \chi(\hat{\tau}_0)$ defined at some initial time $\hat{\tau}_0$. On the other hand, since the coefficients c_n 's now obey first order differential equations, one also needs to specify their initial conditions at $\hat{\tau}_0$. It is natural to assume that the initial condition for the full solution is taken care of by the zeroth order term, i.e., $c_0(\hat{\tau}_0) = 4\chi_0/(3a)$, with $c_{n>0}(\hat{\tau}_0) = 0$ —this considerably simplifies the solutions of our hierarchy of equations order by order [33]. Also, it shows that (37) has the potential to capture *both* the late time asymptotics as well as the early time dynamics driven by the initial condition.

In our case, the equation at zeroth order and its solution are given by

$$\frac{dc_0}{d\hat{\tau}} + c_0 = 0 \Rightarrow c_0(\hat{\tau}) = \frac{4\chi_0}{3a} e^{-(\hat{\tau}-\hat{\tau}_0)} \quad (38)$$

and at first order one finds

$$\begin{aligned} \frac{dc_1}{d\hat{\tau}} + c_1 &= 1 - ac_0(\hat{\tau})^2 \Rightarrow c_1(\hat{\tau}) \\ &= (1 - e^{-(\hat{\tau}-\hat{\tau}_0)}) \left(1 - \frac{16\chi_0^2}{9a} e^{-(\hat{\tau}-\hat{\tau}_0)} \right). \end{aligned} \quad (39)$$

One can see that the zeroth order solution decays exponentially in time with a rate given by the relaxation time—this generates all the other nonperturbative terms in Knudsen number $\sim e^{-1/K_N}$. The differential equation that determines the higher order terms ($n \geq 1$) is

$$\frac{dc_{n+1}}{d\hat{\tau}} + c_{n+1} = nc_n - a \sum_{m=0}^n c_{n-m} c_m. \quad (40)$$

This equation can be easily solved iteratively to determine the coefficients at arbitrary order in an analytical manner. Clearly, at late times the solutions of these equations give coefficients that are *asymptotic* to those defined by (21). However, we emphasize that in contrast to the usual gradient expansion the current procedure leads to a late time expansion that also includes exponentially small terms characteristic of resurgent behavior.

In order to investigate how this series describes the analytical attractor in (20) we set $\hat{\tau}_0 = 0$ and $\chi_0 = 3\sqrt{a}/4$, which gives $c_0(\hat{\tau}) = e^{-\hat{\tau}}/\sqrt{a}$ and $c_1(\hat{\tau}) = (1 - e^{-\hat{\tau}})^2$ for the first terms. We show in Fig. 6 a comparison between the analytical attractor and the result obtained from the new series, which approaches the analytical solution already at second order. While we have not been able to verify if this new expansion converges absolutely, in Fig. 7 we show that the relative absolute difference between the analytical attractor and the new expansion,

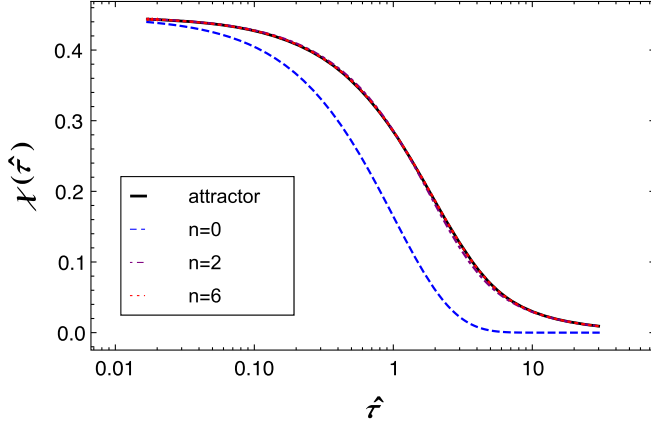


FIG. 6. Comparison between the analytical attractor in (25) for $a = 16/45$ and the result from the generalized gradient expansion, computed at different orders.

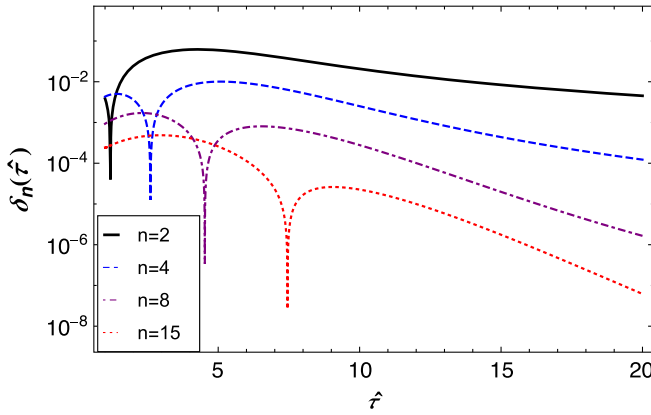


FIG. 7. Relative absolute difference between the analytical attractor and the generalized gradient series for $a = 16/45$ computed at different orders.

$$\delta_n(\hat{\tau}) = \frac{|\chi_{\text{att}}(\hat{\tau}) - \frac{3a}{4} \sum_{m=0}^n \frac{c_m(\hat{\tau})}{\hat{\tau}^m}|}{\chi_{\text{att}}(\hat{\tau})}, \quad (41)$$

decreases significantly when more terms are included in the expansion (our maximum number of terms here was 15). Even if this series is later shown to also be divergent, one can see that it provides an excellent approximation to the attractor already at low orders in comparison to previous approaches.

IV. CONCLUSIONS

In this paper we investigated the solutions of Israel-Stewart theory under Bjorken scaling, in the absence of bulk viscous pressure contributions, assuming an equation of state $\varepsilon = 3P$ and zero chemical potential, and for a constant relaxation time. Our goal was to investigate the emergent universal behavior of these solutions at late times where all the information about the initial conditions is lost.

We demonstrated that the equations of motion of Israel-Stewart theory under these conditions can be solved analytically. We determined an analytical expression for the hydrodynamic attractor for the first time and checked if it could be reproduced, even in an approximate form, by a series expansion. In particular, we considered two expansion methods that are commonly employed in this area: the gradient expansion and the slow-roll series.

When analyzing the gradient expansion, we confirmed that the series diverges for the values of transport coefficients that arise from the Boltzmann equation. Interestingly enough, we found that the series can converge depending on the values of $\eta/(sT\tau_R)$ and in these cases the series can even be explicitly resummed. However, we note that this was only possible for parameter choices that lead to acausal propagation in the fluid and, consequently, are unphysical.

More importantly, we demonstrated for the first time that the slow-roll expansion, which is widely employed to find approximate expressions for the hydrodynamic attractor, has zero radius of convergence. This was found by showing that the terms in the series display factorial growth, for all values of time. We note that this result also holds for a conformal fluid. Nevertheless, both series investigated have an optimal truncation that is actually able to provide a reasonable description of the attractor solution at late times. Therefore, these expansions can still be used to describe the universal hydrodynamic properties of a fluid, even though they cannot be systematically improved by the inclusion of higher order terms.

Finally, we showed an example of a series that appears to converge rapidly. This expansion was first proposed in [33], to find approximate solutions of the Boltzmann equation, where it also appeared to converge. Within this approach, the coefficients of the gradient expansion are assumed to display a non-trivial time dependence, which cannot be simply expanded in powers of Knudsen number. Such a time dependence is obtained directly from the equations of motion, by obtaining and solving the simple first order differential equations satisfied by each coefficient. If this series does in fact converge, it is currently the only option to systematically approximate the hydrodynamic attractor of a given system. Despite the apparent success of this method, we stress that it has only been developed so far in Bjorken flow and it remains a challenge to generalize it to more general flow patterns.

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