Fermion interparticle potentials in 5D and a dimensional restriction prescription to 4D

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This work sets out to compute and discuss effects of spin, velocity, and dimensionality on interparticle potentials systematically derived from gauge field-theoretic models. We investigate the interaction of fermionic particles by the exchange of a vector field in a parity-preserving description in five-dimensional (5D) space-time. A particular dimensional reduction prescription is adopted—reduction by dimensional restriction—and special effects, like a pseudospin dependence, show up in four dimensions (4D). What we refer to as pseudospin shall be duly explained. The main idea we try to convey is that the calculation of the potentials in five dimensions and the consequent reduction to four dimensions exhibits new effects that are not present if the potential is calculated in four dimensions after the action has been reduced.

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I. INTRODUCTION

Field-theoretic models *à la* Kaluza-Klein have had a remarkable revival after the line of papers quoted in Refs. [1–10], from which the activity known as Kaluza-Klein supergravities was boosted. An important question in connection with higher-dimensional models consists of computing quantum-mechanical effects. Two routes may be followed in connection with radiative corrections. In path *i*, one may carry out dimensional reduction by adopting some specific scheme, and then, once the reduction is carried out to some lower-dimensional space-time, quantum corrections are computed. Route *ii* proceeds in the reversed order: one computes the quantum effects directly in the higher-dimensional setup of the model and compares them, afterward, to the quantum corrections computed in the lower-dimensional version of the model with the towers of massive fields included. Procedures *i* and *ii* may not coincide. Actually, Álvarez and Faedo [11] carefully discussed this issue, and they found conditions in which the two routes yield quantum-mechanically equivalent results.

Back to 1983 and 1984, we point out a series of papers by Appelquist and Chodos [12,13], in which the authors consider the five-dimensional Kaluza-Klein model and compute the one-loop effective potential for the extra component of the metric in five dimensions, attaining, therefore, the gravitational analog of the Casimir effect. Appelquist *et al.* [14] also inspected how quantum effects may induce instabilities in the dimensional reduction process. Ever since, the issue of quantum corrections in higher dimensions and their residual effects in lower dimensions has become a very relevant activity in connection with models based on extra dimensions.

The main motivation of the present contribution lies in the problem of comparing results that follow if we adopt either of the routes, *i* or *ii*, namely, quantum effects computed prior to or after the dimensional reduction. We endeavor to tackle this question by considering a semiclassical aspect attainable from quantum field-theoretic models: interparticle interaction potentials derived from the mediation of some intermediating particle. Our paper sets out to work out a spin- and velocity-dependent interparticle potential between massive charged spin-$1/2$ particles in a five-dimensional formulation of parity-preserving electrodynamics. (The usual Dirac mass term explicitly breaks parity symmetry in five dimensions. We keep parity here as a good symmetry and double the fermion representation, as shall be clarified later on).

Even if no loop correction is computed, the tree-level one-scalar or one-photon exchange involves a quantum-mechanical object—the causal propagator—so we get a semiclassical potential in five dimensions to be suitably reduced to four dimensions. This shall eventually trigger some new effect in four dimensions, inherent to the fact that quantization has already been introduced in five dimensions. The idea of a pseudospin, which will show up in four dimensions as a result of imposing parity conservation in five dimensions, is a consequence of considering the fundamental interaction taking place in five dimensions, and the way to connect physics in four and five dimensions will be based on a procedure that we refer to as dimensional reduction by dimensional restriction. This shall be duly presented and discussed in Sec. IV. Had we first reduced the

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five-dimensional model and then calculated the interparticle potential, pseudospin interactions would not appear.

The main point of our investigation is indeed to claim that an interparticle potential in four space-time dimensions may exhibit extra spin effects that appear whenever we adopt the viewpoint that the quantum effects should be accounted for in five dimensions (where we consider that the fundamental physics takes place), rather than introducing quantum effects only after the dimensional reduction has been performed. In this scenario, deviations between theoretical results and experimental measurements could, in some cases, be originated from quantum-mechanical effects of physics that is processed in extra dimensions.

Even though truly fundamental physics in five dimensions should be associated to the five-dimensional anti-de Sitter (in connection with the gauge/gravity correspondence) or de Sitter spaces (in connection with the accelerated expansion of the Universe), we understand that we are dealing with physical effects that are far from being sensitive to possible effects of the cosmological constant. We are bound to the scales of the Standard Model. Actually, we are considering electromagnetic effects, and the length scales involved in the physics we investigate are very far above the curvature of five-dimensional Anti-de-Sitter space-time or five-dimensional de-Sitter space-time. This is our justification to consider that the fundamental physics underneath our present investigation is consistent with $(1+4)$ Minkowski space-time.

Our paper is organized according to the following outline. In Sec. II, we review the methodology for computing the spin- and velocity-dependent interparticle potentials. In Sec. III, we discuss the parity symmetry in 5D space-time for a massive Dirac spinor field and work out the potential for the Maxwell electrodynamics. In Sec. IV, we propose a prescription for restricting the interaction from five to four dimensions. Next, in Sec. V, we also obtain the potential for the Proca electrodynamics and show its asymptotic limits and restriction to four dimensions. Finally, in Sec. VI, we display our concluding comments. We shall adopt the natural units $\hbar = c = 1$.

II. METHODOLOGY AND USEFUL RESULTS

We consider an elastic scattering at tree level of two particles with initial and final states given by $(E_{1,i}, \mathbf{p}_{1,i}; E_{2,i}, \mathbf{p}_{2,i})$ and $(E_{1,f}, \mathbf{p}_{1,f}; E_{2,f}, \mathbf{p}_{2,f})$, respectively. It is convenient to work in the c.m. reference frame with parametrization in terms of the two independent momenta: the transfer momentum $\mathbf{q} = \mathbf{p}_{1,f} - \mathbf{p}_{1,i} = -(\mathbf{p}_{2,f} - \mathbf{p}_{2,i})$ and the average momentum $\mathbf{p} = (\mathbf{p}_{1,i} + \mathbf{p}_{1,f})/2 = (\mathbf{p}_{2,i} + \mathbf{p}_{2,f})/2$. In this case, we have $q^0 = 0$ and $\mathbf{q}\cdot\mathbf{p} = 0$ to simplify the amplitude.

In the first Born approximation [15], the interparticle potential in 4D space-time is obtained through Fourier integral of the nonrelativistic amplitude,

$$V = -\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \mathcal{M}_{\mathrm{NR}}, \tag{1}$$

where $\mathcal{M}_{\mathrm{NR}}$ is related to the Feynman amplitude, $\mathcal{M}$, by means of

$$\mathcal{M}_{\mathrm{NR}} = \frac{1}{\sqrt{2E_{1,i}}} \frac{1}{\sqrt{2E_{1,f}}} \frac{1}{\sqrt{2E_{2,i}}} \frac{1}{\sqrt{2E_{2,f}}} \mathcal{M}. \tag{2}$$

We assume the metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

To render this methodology more instructive and useful to the next sections, let us consider a particular case. For our purposes, it is convenient to work out the well-known electromagnetic interaction of fermions in four dimensions, described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi. \tag{3}$$

First of all, we need to exhibit the positive-energy solutions of the free Dirac equation,

$$[\gamma^\mu p_\mu - m]\psi(p) = 0. \tag{4}$$

Using the decomposition in terms of two-component spinors, $\psi = (\xi, \chi)^t$, and taking the gamma matrices in the Dirac representation, it is possible to eliminate χ and show that

$$\psi(p) = \frac{1}{\sqrt{E+m}} \begin{pmatrix} (E+m)\xi \\ \boldsymbol{\sigma}\cdot\mathbf{p}\xi \end{pmatrix}, \tag{5}$$

where we have normalized the spinor such that $\bar{\psi}(p)\psi(p) = 2m\xi^\dagger\xi$.

The basic spinor, ξ, may assume the two values

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

which refers to spin-up and -down configurations, respectively.

Here, we would like to fix some notations. Let us consider the possibility of a spin flip. Thus, we shall use ξ_i and ξ_f to indicate the initial and final spin states of the fermion. For this reason, it is also convenient to define the contractions

$$\delta = \xi_f^\dagger \xi_i, \qquad \langle \mathbf{S} \rangle = \xi_f^\dagger \frac{\boldsymbol{\sigma}}{2} \xi_i. \tag{7}$$

The previous expression is interpreted as the expectation value of the spin operator.

Now, we apply the Feynman rules for this scattering in the adopted c.m. frame,

$$
\begin{aligned}
i\mathcal{M} &= \bar{\psi}_1(\mathbf{p}+\mathbf{q}/2)\{ie_1\gamma^\mu\}\psi_1(\mathbf{p}-\mathbf{q}/2)\langle A_\mu A_\nu\rangle \\
&\quad\times\bar{\psi}_2(-\mathbf{p}-\mathbf{q}/2)\{ie_2\gamma^\nu\}\psi_2(-\mathbf{p}+\mathbf{q}/2) \\
&= -e_1e_2J^\mu_{(1)}\langle A_\mu A_\nu\rangle J^\nu_{(2)},
\end{aligned} \tag{8}
$$

where we are using the current $J^\mu = \bar{\psi}\gamma^\mu\psi$ and $\langle A_\mu A_\nu\rangle$ is the propagator in momentum space,

$$
\langle A_\mu A_\nu\rangle = -\frac{i}{q^2}\left[\eta_{\mu\nu} + (\alpha-1)\frac{q_\mu q_\nu}{q^2}\right], \tag{9}
$$

which is obtained after including the gauge-fixing term, $\frac{-1}{2\alpha}(\partial_\mu A^\mu)^2$, to the Lagrangian in Eq. (3).

With the current conservation, $q^\mu J_\mu = 0$, and $q^0 = 0$ in Eq. (8), the nonrelativistic amplitude, Eq. (2), can be written as

$$
\mathcal{M}_{\rm NR} = -\frac{e_1e_2}{\mathbf{q}^2}\frac{J^\mu_{(1)}J_{(2)\mu}}{(2E_1)(2E_2)}. \tag{10}
$$

One important step of this computation is to declare which approximation we are dealing with. Throughout this work, we consider corrections up to $\mathcal{O}(|\mathbf{p}^2|/m^2)$ in the amplitude, without counting the factor $1/\mathbf{q}^2$ in the previous equation.

Now, we study the currents in order to obtain an approximation to this amplitude. For the particle -1, we take the spinor solution, Eq. (5), and consider $E_{1,f} = E_{1,i} = E_1 \approx m_1 + \frac{1}{2m_1}(\mathbf{p}^2 + \frac{\mathbf{q}^2}{4})$ such that

$$
J^0_{(1)} \approx 2m_1\delta_1 + \frac{1}{m_1}[\mathbf{p}^2\delta_1 + i(\mathbf{q}\times\mathbf{p})\cdot\langle\mathbf{S}_1\rangle], \tag{11}
$$

$$
J^i_{(1)} \approx 2\mathbf{p}_i\delta_1 - 2i\epsilon_{ijk}\mathbf{q}_j\langle\mathbf{S}_{1,k}\rangle. \tag{12}
$$

The current $J^\mu_{(2)}$ is obtained by taking the prescription in the $J^\mu_{(1)}$, $\mathbf{q}\to-\mathbf{q}$, $\mathbf{p}\to-\mathbf{p}$, and changing the label $1\to 2$.

From these considerations, one could check that

$$
\begin{aligned}
\frac{J^\mu_{(1)}J_{(2)\mu}}{(2E_1)(2E_2)} &\approx \delta_1\delta_2\left[\left(1+\frac{\mathbf{p}^2}{m_1m_2}\right) - \frac{1}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\mathbf{q}^2\right] \\
&\quad + i\mathbf{q}\cdot\left\{\mathbf{p}\times\left[\delta_1\langle\mathbf{S}_2\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)+1\leftrightarrow 2\right]\right\} \\
&\quad - \frac{1}{m_1m_2}[\mathbf{q}^2\langle\mathbf{S}_1\rangle\cdot\langle\mathbf{S}_2\rangle - (\mathbf{q}\cdot\langle\mathbf{S}_1\rangle)(\mathbf{q}\cdot\langle\mathbf{S}_2\rangle)].
\end{aligned} \tag{13}
$$

Once we establish the nonrelativistic amplitude, Eq. (10) with Eq. (13), we use the prescription described in Eq. (1); i.e., we carry out the Fourier integral. For this calculation, we only need the massless limit of Eqs. (A1)–(A3) of the Appendix. Then, the interparticle potential is given by

$$
\begin{aligned}
V^{\rm Maxwell} &= e_1e_2\left\{\frac{\delta_1\delta_2}{4\pi r}\left[1+\frac{\mathbf{p}^2}{m_1m_2}\right] - \frac{\delta_1\delta_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\delta^3(\mathbf{r})\right. \\
&\quad - \frac{2}{3}\frac{\langle\mathbf{S}_1\rangle\cdot\langle\mathbf{S}_2\rangle}{m_1m_2}\delta^3(\mathbf{r}) + \frac{\mathbf{Q}_{ij}}{4\pi r^3}\frac{\langle\mathbf{S}_{1,i}\rangle\langle\mathbf{S}_{2,j}\rangle}{m_1m_2} \\
&\quad \left. - \frac{\mathbf{L}}{4\pi r^3}\cdot\left[\delta_1\langle\mathbf{S}_2\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)+1\leftrightarrow 2\right]\right\},
\end{aligned} \tag{14}
$$

where we defined the angular momentum, $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and quadrupole (or dipole-dipole) tensor $\mathbf{Q}_{ij} = \delta_{ij} - 3\frac{\mathbf{x}_i\mathbf{x}_j}{r^2}$.

The first contribution is the usual Coulomb interaction ($\sim 1/4\pi r$), which is the dominant term at large distances. Next, we have a velocity-dependent term, here parametrized in terms of the average momentum $\mathbf{p}$. It also has a spin-orbit coupling $\mathbf{L}\cdot\mathbf{S}$, quadrupole interaction, and contact terms, i.e., ones with Dirac delta $\delta^3(\mathbf{r})$. Because of our approximations, we do not have a higher-multipole contribution than a quadrupole. This result coincides with the one obtained in Refs. [16,17]. We shall see in the following sections that the calculation of the interparticle potential in 5D space-time follows procedures similar to the ones presented in this particular case.

III. MAXWELL ELECTRODYNAMICS IN FIVE DIMENSIONS

The properties of the interparticle interaction potentials in arbitrary dimensions have been already established in the literature for many situations; see Refs. [18–20]. However, there is a lack of attention to the study related to spin contributions for space-times with extra dimensions. In this section, we pursue an investigation of the spin as well as velocity-dependent interactions in space-time with one extra dimension. Initially, we concentrate our efforts in the Maxwell electrodynamics in 5D space-time. Keeping in mind that electromagnetism is also parity invariant in five dimensions, we start by studying how to implement the parity transformation on massive Dirac fermions.

Let us initiate by fixing other conventions. In 5D Minkowski space-time, we adopt the metric $\eta_{\hat{\mu}\hat{\nu}} = {\rm diag}(+,-,-,-,-)$, where $\hat{\mu}$, $\hat{\nu} = (0,i,4)$ with $i = (1,2,3)$. One possible choice to satisfy the Clifford algebra, $\{\gamma^{\hat{\mu}},\gamma^{\hat{\nu}}\} = 2\eta^{\hat{\mu}\hat{\nu}}$, is to take $\gamma^{\hat{\mu}} = (\gamma^\mu, \gamma^4 \equiv i\gamma_5)$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and γ^μ satisfies the Clifford algebra in four dimensions. Another possibility is $\gamma^{\hat{\mu}} = (i\gamma_5\gamma^\mu, \gamma^4 \equiv i\gamma_5)$. We consider the first one, which will be more convenient to the evaluations in the nonrelativistic limit.

The Lagrangian for a massive Dirac spinor field in five dimensions is given by

$$
\mathcal{L} = \bar{\psi}i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\psi - m\bar{\psi}\psi. \tag{15}
$$

We define the parity transformation in 5D space-time as $x_0' = x_0$, $\mathbf{x}' = -\mathbf{x}$ and $\mathbf{x}_4' = \mathbf{x}_4$. Thus, we maintain the usual transformation in four dimensions, and the extra dimension, $\mathbf{x}_4$, stays unaltered in order to have a discrete transformation. Let us propose the following parity transformation for the spinor field:

$$\psi'(x') = P\psi(x). \tag{16}$$

Now, we would like to find an explicit form for the matrix P. We start by imposing the invariance of the massless term in Eq. (15). Using Eq. (16) and $\overline{\psi'} = \psi'^{\dagger}\gamma^0 = \bar{\psi}\gamma^0 P^{\dagger}\gamma^0$, one could obtain the relations

$$P^{\dagger} = P^{-1}, \qquad \gamma^0\gamma^i P = -P\gamma^0\gamma^i, \qquad \gamma^0\gamma^4 P = P\gamma^0\gamma^4, \tag{17}$$

which provide us

$$P = i\gamma^1\gamma^2\gamma^3. \tag{18}$$

The factor i is just for future convenience. However, we have not finished yet; we need to consider the transformation of the mass term in Eq. (15). From the above result, we find that

$$m\overline{\psi'}\psi' = m\bar{\psi}\gamma^0 P^{\dagger}\gamma^0 P\psi = -m\bar{\psi}\psi, \tag{19}$$

so the mass term breaks the parity symmetry in five dimensions.

One way to circumvent this problem is to double the spinor field representation. A similar proposal was taken in Ref. [21], in the context of three-dimensional (3D) Minkowski space-time, also to conciliate the parity symmetry with massive fermions. Another possibility is to modify the mass term, as done in Ref. [22], but we will not follow this path here. Therefore, we define a doubled spinor field:

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \tag{20}$$

We also represent the gamma matrices as

$$\Gamma^{\hat{\mu}} = \begin{pmatrix} \gamma^{\hat{\mu}} & 0 \\ 0 & -\gamma^{\hat{\mu}} \end{pmatrix}; \tag{21}$$

then, the Dirac conjugate of Ψ takes the form $\bar{\Psi} = \Psi^{\dagger}\Gamma^0 = (\bar{\psi}, -\bar{\chi})$, with $\bar{\psi} = \psi^{\dagger}\gamma^0$ and $\bar{\chi} = \chi^{\dagger}\gamma^0$.

The Dirac Lagrangian for the doubled spinor field is given by

$$\begin{aligned} \mathcal{L} &= \bar{\Psi} i\Gamma^{\hat{\mu}}\partial_{\hat{\mu}}\Psi - m\bar{\Psi}\Psi \\ &= \bar{\psi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\psi + \bar{\chi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\chi - m\bar{\psi}\psi + m\bar{\chi}\chi. \end{aligned} \tag{22}$$

After these considerations, if we implement the parity transformation on Ψ as

$$\Psi' = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{23}$$

then it is possible to show that the Lagrangian of Eq. (22) is parity invariant, since the transformation exchanges

$$\bar{\psi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\psi \longleftrightarrow \bar{\chi} i\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\chi, \qquad -m\bar{\psi}\psi \longleftrightarrow m\bar{\chi}\chi. \tag{24}$$

It is worth mentioning that, similar to the 3D case [21] (with τ_3−QED), we could introduce other symmetries in the doubled field formalism. These possibilities shall be discussed in more details in the concluding comments.

Now, we are ready to start the steps for computing the interparticle potential in five dimensions. We shall follow the prescription described in Sec. II. First, we need to obtain the free positive-energy solution of the Dirac equation,

$$(\Gamma^{\hat{\mu}} p_{\hat{\mu}} - m)\Psi(p) = 0, \tag{25}$$

which is equivalent to

$$(\gamma^{\hat{\mu}} p_{\hat{\mu}} - m)\psi(p) = 0, \qquad (\gamma^{\hat{\mu}} p_{\hat{\mu}} + m)\chi(p) = 0. \tag{26}$$

Then, we consider the decomposition

$$\psi = \begin{pmatrix} \xi \\ \varphi \end{pmatrix}, \qquad \chi = \begin{pmatrix} \lambda \\ \zeta \end{pmatrix}, \tag{27}$$

where ξ, φ, λ, and ζ are two-component spinors. Using Eqs. (26), one can eliminate φ and λ, and the spinors reduce to

$$\psi(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\xi \\ (\boldsymbol{\sigma}\cdot\mathbf{p} - i\mathbf{p}_4)\xi \end{pmatrix}, \tag{28}$$

$$\chi(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} (\boldsymbol{\sigma}\cdot\mathbf{p} + i\mathbf{p}_4)\zeta \\ (E+m)\zeta \end{pmatrix}. \tag{29}$$

The two spinors above were normalized such that the doubled spinor field, Eq. (20), satisfies $\bar{\Psi}(p)\Psi(p) = 2m(\xi^{\dagger}\xi + \zeta^{\dagger}\zeta)$. Furthermore, they differ by a minus sign in the extra-dimensional term, i.e., in the $\mathbf{p}_4$ term. This sign is essential to maintain the parity symmetry in five dimensions and will play an important role in the spin interactions present in our 5D scenario. We expected to get more interactions in the doubled field formalism, and the parity-breaking case is recovered by taking $\zeta = 0$.

Since we are dealing with ξ and ζ, we introduce a label in the contractions given in Eq. (7), so we define

$$\delta_{\xi} = \xi_f^{\dagger}\xi_i, \qquad \delta_{\zeta} = \zeta_f^{\dagger}\zeta_i, \qquad \langle\mathbf{S}\rangle_{\xi} = \xi_f^{\dagger}\frac{\boldsymbol{\sigma}}{2}\xi_i, \\ \langle\mathbf{S}\rangle_{\zeta} = \zeta_f^{\dagger}\frac{\boldsymbol{\sigma}}{2}\zeta_i. \tag{30}$$

Having established the spinors solutions, we turn to the calculation of the doubled field vector current,

$$J^{\hat{\mu}} = \bar{\Psi}\Gamma^{\hat{\mu}}\Psi = \bar{\psi}\gamma^{\hat{\mu}}\psi + \bar{\chi}\gamma^{\hat{\mu}}\chi. \tag{31}$$

Inserting Eqs. (28) and (29) in Eq. (31) and using the adopted c.m. frame, one could show that the components of the current of the particle 1 assume the form

$$J^0_{(1)} = 2m_1(\delta_{\xi,1} + \delta_{\zeta,1}) + \frac{1}{m_1}\{(\delta_{\xi,1} + \delta_{\zeta,1})(\mathbf{p}^2 + \mathbf{p}_4^2) + i(\mathbf{q}\times\mathbf{p})\cdot[\langle\mathbf{S}_1\rangle_\xi + \langle\mathbf{S}_1\rangle_\zeta] + i\mathbf{q}_4\mathbf{p}\cdot[\langle\mathbf{S}_1\rangle_\xi - \langle\mathbf{S}_1\rangle_\zeta] - i\mathbf{p}_4\mathbf{q}\cdot[\langle\mathbf{S}_1\rangle_\xi - \langle\mathbf{S}_1\rangle_\zeta]\}, \tag{32}$$

$$J^i_{(1)} = 2(\delta_{\xi,1} + \delta_{\zeta,1})\mathbf{p}_i - 2i\epsilon_{ijk}\mathbf{q}_j[\langle\mathbf{S}_{1,k}\rangle_\xi + \langle\mathbf{S}_{1,k}\rangle_\zeta] + 2i\mathbf{q}_4[\langle\mathbf{S}_{1,i}\rangle_\xi - \langle\mathbf{S}_{1,i}\rangle_\zeta], \tag{33}$$

$$J^4_{(1)} = 2(\delta_{\xi,1} + \delta_{\zeta,1})\mathbf{p}_4 - 2i\mathbf{q}\cdot[\langle\mathbf{S}_1\rangle_\xi - \langle\mathbf{S}_1\rangle_\zeta]. \tag{34}$$

Before going to the amplitude, it is interesting to look carefully at these equations and their parity transformations. According to Eq. (23) and the spinor solution, Eqs. (28) and (29), one can check that the parity transformations of the components are $\xi' = \zeta$ and $\zeta' = -\xi$. That is the reason we put the factor i in $P = i\gamma^1\gamma^2\gamma^3$, i.e., to get a real transformation. Since $\mathbf{q}' = -\mathbf{q}$, $\mathbf{q}_4' = \mathbf{q}_4$, $\mathbf{p}' = -\mathbf{p}$, and $\mathbf{p}_4' = \mathbf{p}_4$, we note some specific linear combinations of the spins in order to keep the parity property of the vector in five dimensions. For example, in the second term of Eq. (34), we have $-2i\mathbf{q}'\cdot[\langle\mathbf{S}_1\rangle'_\xi - \langle\mathbf{S}_1\rangle'_\zeta] = -2i\mathbf{q}\cdot[\langle\mathbf{S}_1\rangle_\xi - \langle\mathbf{S}_1\rangle_\zeta]$, which is consistent with $J'^4_{(1)} = J^4_{(1)}$. A similar argument holds for the other terms. Therefore, it is suggestive to define

$$\langle\mathbf{S}^\pm\rangle = \langle\mathbf{S}\rangle_\xi \pm \langle\mathbf{S}\rangle_\zeta, \tag{35}$$

which can be understood as the bilinears

$$\langle\mathbf{S}^\pm\rangle = (\xi_f^\dagger, \zeta_f^\dagger) \quad \mathbf{S}^\pm\begin{pmatrix}\xi_i\\ \zeta_i\end{pmatrix}, \tag{36}$$

where

$$\mathbf{S}^\pm = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \pm\boldsymbol{\sigma}\end{pmatrix}. \tag{37}$$

The $\langle\mathbf{S}^+\rangle$ can be interpreted as an expectation value of the spin, because the operator $\mathbf{S}^+$ satisfies the $SU(2)$ algebra, $[\mathbf{S}_i^+, \mathbf{S}_j^+] = i\epsilon_{ijk}\mathbf{S}_k^+$, and its expectation value is even under parity, $\langle\mathbf{S}^+\rangle' = \langle\mathbf{S}^+\rangle$, as true spin should be. On the other hand, the operator $\mathbf{S}^-$ and its expectation value do not satisfy these properties. Once $\mathbf{S}^-$ is given by the combination of two spin $\boldsymbol{\sigma}/2$ and under parity satisfies $\langle\mathbf{S}^-\rangle' = -\langle\mathbf{S}^-\rangle$, we shall call it pseudospin. We highlight that the pseudospin we introduce here is not the same as the pseudospin that appears in other contexts; for example, in condensed matter systems [23] and nuclear physics [24,25].

It is also convenient to define $\Delta = \delta_\xi + \delta_\zeta$, which is parity invariant, $\Delta' = \Delta$.

After these definitions, we can recast the components of the vector current, Eqs. (32)–(34), as follows:

$$J^0_{(1)} = 2m_1\Delta_1 + \frac{1}{m_1}[\Delta_1(\mathbf{p}^2 + \mathbf{p}_4^2) + i(\mathbf{q}\times\mathbf{p})\cdot\langle\mathbf{S}_1^+\rangle + i\mathbf{q}_4(\mathbf{p}\cdot\langle\mathbf{S}_1^-\rangle) - i\mathbf{p}_4(\mathbf{q}\cdot\langle\mathbf{S}_1^-\rangle)], \tag{38}$$

$$J^i_{(1)} = 2\Delta_1\mathbf{p}_i - 2i\epsilon_{ijk}\mathbf{q}_j\langle\mathbf{S}_{1,k}^+\rangle + 2i\mathbf{q}_4\langle\mathbf{S}_{1,i}^-\rangle, \tag{39}$$

$$J^4_{(1)} = 2\Delta_1\mathbf{p}_4 - 2i\mathbf{q}\cdot\langle\mathbf{S}_1^-\rangle. \tag{40}$$

They exhibit all the contributions of the components in four dimensions, see Eqs. (11) and (12), and new terms associated with the extra dimension.

We can now proceed to evaluating the amplitude in the context of Maxwell electrodynamics in five dimensions. In a way similar to what was done in Sec. II, one can show that

$$\mathcal{M}^{5D}_{\mathrm{NR}} = -\frac{g_1 g_2}{\mathbf{q}^2 + \mathbf{q}_4^2}\frac{J^{\hat{\mu}}_{(1)}J_{(2)\hat{\mu}}}{(2E_1)(2E_2)}, \tag{41}$$

where $g_{1(2)}$ denotes the coupling constant in five dimensions.

After some manipulations, we find that

$$\begin{aligned}\frac{J^{\hat{\mu}}_{(1)}J_{(2)\hat{\mu}}}{(2E_1)(2E_2)} \approx{}& \Delta_1\Delta_2\left[\left(1 + \frac{\mathbf{p}^2}{m_1m_2} + \frac{\mathbf{p}_4^2}{m_1m_2}\right) + -\frac{1}{8}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)(\mathbf{q}^2 + \mathbf{q}_4^2)\right] + i\mathbf{q}\cdot\left\{\mathbf{p}\times\left[\Delta_1\langle\mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1m_2}\right)\right]\right.\\ &\left.- \mathbf{p}_4\left[\Delta_1\langle\mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1m_2}\right)\right] + 1\leftrightarrow 2\right\} + i\mathbf{q}_4\mathbf{p}\left[\Delta_1\langle\mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1m_2}\right) + 1\leftrightarrow 2\right]\\ &+ \frac{\mathbf{q}_4\mathbf{q}}{m_1m_2}\cdot[(\langle\mathbf{S}_1^+\rangle\times\langle\mathbf{S}_2^-\rangle) + (\langle\mathbf{S}_2^+\rangle\times\langle\mathbf{S}_1^-\rangle)] - \frac{1}{m_1m_2}[\mathbf{q}^2\langle\mathbf{S}_1^+\rangle\cdot\langle\mathbf{S}_2^+\rangle + \mathbf{q}_4^2\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle]\\ &+ \frac{1}{m_1m_2}[(\mathbf{q}\cdot\langle\mathbf{S}_1^+\rangle)(\mathbf{q}\cdot\langle\mathbf{S}_2^+\rangle) - (\mathbf{q}\cdot\langle\mathbf{S}_1^-\rangle)(\mathbf{q}\cdot\langle\mathbf{S}_2^-\rangle)].\end{aligned} \tag{42}$$

Finally, we only need to compute the Fourier integral,

$$V_{5D} = -\int \frac{d^4\mathbf{q}}{(2\pi)^4} e^{i\mathbf{q}\cdot\mathbf{R}} \mathcal{M}_{\mathrm{NR}}^{5D}. \tag{43}$$

As explained in the Appendix, we shall use $\mathbf{R}$ to denote the 4D Euclidean vector, so we avoid confusion with $\mathbf{r}$, used for the 3D case. Therefore, using the massless Fourier integrals in four dimensions, given by Eqs. (A7)–(A9), we obtain

$$\begin{aligned} V_{5D}^{\mathrm{Maxwell}} = g_1 g_2 \Bigg\{ & \frac{\Delta_1\Delta_2}{4\pi^2 R^2}\left(1+\frac{\mathbf{p}^2}{m_1 m_2}+\frac{\mathbf{p}_4^2}{m_1 m_2}\right) - \frac{\Delta_1\Delta_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\delta^4(\mathbf{R}) - \frac{1}{2m_1 m_2}[\langle \mathbf{S}_1^+\rangle\cdot\langle \mathbf{S}_2^+\rangle + \langle \mathbf{S}_1^-\rangle\cdot\langle \mathbf{S}_2^-\rangle]\delta^4(\mathbf{R}) \\ & - \frac{(\mathbf{r}\times\mathbf{p})}{2\pi^2 R^4}\cdot\left[\Delta_1\langle \mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1 m_2}\right) + 1\leftrightarrow 2\right] - \frac{(\mathbf{x}_4\mathbf{p}-\mathbf{p}_4\mathbf{r})}{2\pi^2 R^4}\cdot\left[\Delta_1\langle \mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1 m_2}\right) + 1\leftrightarrow 2\right] \\ & - \frac{2}{\pi^2 m_1 m_2}\frac{\mathbf{x}_4\mathbf{r}}{R^6}\cdot[(\langle \mathbf{S}_1^+\rangle\times\langle \mathbf{S}_2^-\rangle) + (\langle \mathbf{S}_2^+\rangle\times\langle \mathbf{S}_1^-\rangle)] - \frac{1}{\pi^2 R^4}\frac{1}{m_1 m_2}\Bigg[\left(1-\frac{2r^2}{R^2}\right)(\langle \mathbf{S}_1^+\rangle\cdot\langle \mathbf{S}_2^+\rangle - \langle \mathbf{S}_1^-\rangle\cdot\langle \mathbf{S}_2^-\rangle) \\ & + \frac{2}{R^2}(\langle \mathbf{S}_1^+\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^+\rangle\cdot\mathbf{r}) - \frac{2}{R^2}(\langle \mathbf{S}_1^-\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^-\rangle\cdot\mathbf{r})\Bigg]\Bigg\}. \end{aligned} \tag{44}$$

The dominant contribution at large distances is given by the first term ($\sim 1/4\pi^2 R^2$). Similar to the 3D case, see Eq. (14), we obtain a spin-orbit coupling, $\boldsymbol{L}\cdot\boldsymbol{S}^+$, where $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ is the 3D angular momentum, and an extra component that couples with pseudospin, namely, the term related to $(\mathbf{x}_4\mathbf{p}-\mathbf{p}_4\mathbf{r})\cdot\boldsymbol{S}^-$. As anticipated at the beginning, the doubled spinor formalism provides new interactions in five dimensions compared the usual (nondoubled) formalism. For example, we observe a nontrivial coupling between spin and pseudospin of the type $\langle \mathbf{S}^+\rangle\times\langle \mathbf{S}^-\rangle$ with a $\boldsymbol{x}_4\boldsymbol{r}/R^6$ power-law decay. We can check that this contribution only exists in the parity-invariant case. For instance, let us examine its (pseudo) spin dependence,

$$\begin{aligned} &(\langle \mathbf{S}_1^+\rangle\times\langle \mathbf{S}_2^-\rangle) + (\langle \mathbf{S}_2^+\rangle\times\langle \mathbf{S}_1^-\rangle) \\ &\quad = 2(\langle \mathbf{S}_1\rangle_\zeta\times\langle \mathbf{S}_2\rangle_\xi) + 2(\langle \mathbf{S}_2\rangle_\zeta\times\langle \mathbf{S}_1\rangle_\xi), \end{aligned} \tag{45}$$

so, in the parity-breaking case, we take $\zeta = 0$, which implies $\langle \mathbf{S}\rangle_\zeta = 0$ and leads to a trivial contribution. A similar argument holds for the last term of the potential, the quadrupolelike interaction, since

$$\begin{aligned} &(\langle \mathbf{S}_1^+\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^+\rangle\cdot\mathbf{r}) - (\langle \mathbf{S}_1^-\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^-\rangle\cdot\mathbf{r}) \\ &\quad = 2(\langle \mathbf{S}_1\rangle_\zeta\cdot\mathbf{r})(\langle \mathbf{S}_2\rangle_\xi\cdot\mathbf{r}) + 2(\langle \mathbf{S}_1\rangle_\xi\cdot\mathbf{r})(\langle \mathbf{S}_2\rangle_\zeta\cdot\mathbf{r}), \end{aligned} \tag{46}$$

which also vanishes when $\langle \mathbf{S}\rangle_\zeta = 0$.

In the next section, we shall develop a prescription to extract a 4D potential from a 5D result. As we will see, this prescription enables us to bring some pseudospin contributions to four dimensions.

IV. RESTRICTION TO FOUR DIMENSIONS

To go over into a four-dimensional scenario, one may consider many different procedures, which are all based on at least one ansatz. For example, a usual case assumes compactified extra dimensions that lead to the Kaluza-Klein expansion modes. Also, we could impose a trivial reduction [1], in which only field configurations that do not depend on the extra dimensions are considered and only the so-called zero modes are accounted for. Another possibility is to carry out the dimensional reduction by spontaneous compactification [2]. In this case, we look for the solutions of the equations of motion that factorize into a four-dimensional space-time and an internal space. On the other hand, one could also consider warped geometries—the so-called brane worlds scenarios—in which the extra dimensions are noncompact [26]. We also highlight a dimensional reduction prescription that does not assume compactified extra dimensions nor take dynamical solutions. That is the case of the Legendre reduction [27], normally adopted for the construction of off-shell supersymmetric models.

After we have presented and discussed our results for the interparticle potentials directly in five dimensions, our purpose in the present section is to carry out the dimensional reduction to four dimensions. Nevertheless, instead of reducing the action and rederiving the potentials in four dimensions from the reduced action, we pursue the attainment of the four-dimensional potentials by directly reducing the expressions calculated in five dimensions in the previous section. So, we adopt the viewpoint already alluded to in the Introduction of our paper: the quantum-mechanical calculation in our case is a semiclassical derivation of the potential, and it is performed in the higher dimension to be, after that, reduced to the lower dimension. We proceed along the lines of the works quoted in Refs. [11–14], aiming at a four-dimensional result that already brings the semiclassical imprints from the five-dimensional physics. And what we actually conclude is that this procedure differs from the scheme of first reducing the action to then derive the potential from the reduced action. We converge to the claims of the papers by Álvarez and Faedo [11], who state that the two paths (reduction of the action followed by the inclusion of quantum effects or, alternatively, quantum effects worked out in the higher dimension to then reduce the quantum-corrected quantities to lower dimensions) may not be equivalent. We shall refer to the procedure we follow here as reduction by dimensional restriction.

By inspecting the canonical mass dimension of the fields and coupling constants in five and four dimensions, we have $[\psi_{(5)}] = 2$, $[A_{(5)\hat{\mu}}] = 3/2$, $[g_{(5)}] = -1/2$, and $[\psi_{(4)}] = 3/2$, $[A_{(4)\mu}] = 1$, $[g_{(4)} \equiv e] = 0$, respectively. If L denotes a length in the extra dimension, then the factor $\sqrt{L}$ restores the correct mass dimension in four dimensions, such that $g_{(4)} = g_{(5)}/\sqrt{L}$, $\psi_{(4)} = \psi_{(5)}\sqrt{L}$, and $A_{(4)\hat{\mu}} = A_{(5)\hat{\mu}}\sqrt{L}$. For our purposes, we only need the relation between the coupling constants, which is independent of the dimensional reduction scheme.

Now, we propose the following procedure. First, we define the average in the extra dimension of the potential in five dimensions,

$$\langle V_{5D}\rangle_L = \frac{1}{L}\int_{-L/2}^{L/2} d\mathbf{x}_4 V_{5D}, \tag{47}$$

and then we extend this to a noncompact case, by taking the limit $L \to \infty$,

$$V_{\text{res}} = \lim_{L\to\infty} \langle V_{5D}\rangle_L. \tag{48}$$

We refer to the potential V_{res}, defined in the previous equation, as a restricted potential or a restriction of the 5D potential to 4D space-time.

In principle, one could think that this prescription is only a statistical procedure, since in Eq. (47) we have an average in the box $-L/2 < \mathbf{x}_4 < L/2$ with equal probability $1/L$ and after we take the limit of a noncompact box. However, we shall also give a physical meaning to this. If we substitute Eq. (43) in Eq. (47), we have that Eq. (48) can be recast as

$$V_{\text{res}} = \lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} d\mathbf{x}_4 \left\{ -\int \frac{d\mathbf{q}_4}{2\pi}\int \frac{d^3\mathbf{q}}{(2\pi)^3} \times e^{i\mathbf{q}_4\mathbf{x}_4} e^{i\mathbf{q}\cdot\mathbf{r}} \mathcal{M}_{\text{NR}}^{5D}[g_{(5)}] \right\}. \tag{49}$$

In what follows, we shall use the relation $g_{(5)} = \sqrt{L} g_{(4)}$ between the fermionic coupling constants in five and four dimensions. Before doing that, let us recall that the nonrelativistic tree-level amplitude is proportional to the square of $g_{(5)}$ [see, for example, Eq. (41)], so the factor $1/L$ cancels against a factor coming from the coupling constants in $\mathcal{M}_{\text{NR}}^{5D}[g_{(5)}] = L\mathcal{M}_{\text{NR}}^{5D}[g_{(4)}]$. Next, we take the limit $L \to \infty$ and interchange the integrals over $\mathbf{x}_4$ and spatial momentum; this yields

$$\begin{aligned} V_{\text{res}} &= -\int \frac{d^3\mathbf{q}}{(2\pi)^3}\int \frac{d\mathbf{q}_4}{2\pi}\left(\int d\mathbf{x}_4 e^{i\mathbf{q}_4\mathbf{x}_4}\right) e^{i\mathbf{q}\cdot\mathbf{r}}\mathcal{M}_{\text{NR}}^{5D}[g_{(4)}] \\ &= -\int \frac{d^3\mathbf{q}}{(2\pi)^3}\int d\mathbf{q}_4 \delta(\mathbf{q}_4) e^{i\mathbf{q}\cdot\mathbf{r}}\mathcal{M}_{\text{NR}}^{5D}[g_{(4)}] \end{aligned} \tag{50}$$

or, equivalently,

$$V_{\text{res}} = -\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} [\mathcal{M}_{\text{NR}}^{5D}|_{\mathbf{q}_4=0}], \tag{51}$$

where it is implicit $g_{(5)} \to g_{(4)}$ in the above amplitude. Equation (50) highlights that our reduction prescription naturally leads to $\mathbf{q}_4 = 0$, in view of the Dirac delta function, which comes out upon integration over $\mathbf{x}_4$.

Once we take $\mathbf{q}_4 = 0$ in Eq. (51), we could read the prescription as a restriction of the interaction to a subspace of the 5D space-time, without loss of the properties of the particles in five dimensions, namely, Δ, (pseudo)spin $\langle \mathbf{S}^{\pm}\rangle$, and momentum $\mathbf{p}$, $\mathbf{p}_4$. For this reason, we shall avoid the expression dimensional reduction. This prescription is just a restriction to the scattering amplitude, in which the transfer momentum of the extra dimension, $\mathbf{q}_4$, could be considered negligible compared to $\mathbf{q}$ in the process. Remember that we are considering an elastic scattering, so we also have $q^0 = 0$. Here, we highlight that we assumed $\mathbf{r} \neq 0$, so the restricted potential will not contemplate contact terms, i.e., ones with $\delta^3(\mathbf{r})$. To go over into Eq. (51), we interchanged integrals and assumed nonsingular functions.

In our procedure, we draw attention to the fact that, by taking $\mathbf{q}_4 = 0$, we are not setting the fifth component of the individual momenta to zero; in other words, we do not disregard the dependence of the fields on the extra space coordinate, $\mathbf{x}_4$. What is zero here is the fifth component of the momentum transfer: the interaction of the matter currents with the intermediate boson does not transfer momentum along the fourth spatial component $(\mathbf{q}_4)$ of the momentum transfer. On the other hand, on the basis of our assumption given by Eq. (48) to carry out the dimensional reduction, the average taken over the extra dimension goes from minus to plus infinity; this means that we are neither dropping out the $\mathbf{x}_4$ dependence nor assuming $\mathbf{x}_4$ to be compact. We are rather adopting the viewpoint of a noncompact extra dimension, along the lines of the idea proposed by Randall and Sundrum in the works [26]. This is the true reason why we do not consider the influence of the Kaluza-Klein tower of massive states. Perhaps, we should also stress that, by considering the plane wave solutions given in Eqs. (28) and (29), we are already anticipating that noncompact dimensions will be present in our approach, which also confirms that Kaluza-Klein massive states are not considered here.

Though it is not our case in the present work, we would like to point out that Kaluza-Klein massive states, which appear as a consequence of the compactness of the extra dimension, are of a very high mass, and at the energy compatible with the calculation of (low-energy) interparticle potentials, they may be fairly well disregarded. Actually, they decouple. The momenta transfer in this sort of considerations is very low to excite the massive Kaluza-Klein states associated to the compact extra dimensions. If these states were present, they would contribute as virtual particles running inside the momentum-space loop integrals that appear in the radiative corrections.

Another point that we wish to stress is that, once the extra coordinate $\mathbf{x}_4$ is noncompact [which becomes explicit when we take the limit in Eq. (48)], through the uncertainty principle, by fixing $\mathbf{q}_4 = 0$, which is our basic assumption, we completely lose the localization on $\mathbf{x}_4$; this supports our prescription of taking the limit $L \to \infty$ of Eq. (47). All possible values of $\mathbf{x}_4$ are allowed (complete uncertainty on $\mathbf{x}_4$) once $\mathbf{q}_4 = 0$; this supports our prescription of taking the average on $\mathbf{x}_4$.

The procedure of taking the integral in the extra dimension is not exclusive to this work. In Ref. [28], the authors applied this integration in the fields and currents—they called it a concatenation—and this prescription was used in the off-shell electrodynamics (see, for example, Ref. [29]). In our case, a different point of view is adopted. First, we carry out the interparticle potential in five dimensions, taking into account all the contribution of the fields and currents in five dimensions, and then we integrate the potential.

Now, let us apply the prescription to the $V_{5D}^{\mathrm{Maxwell}}$; i.e., we use Eq. (48) and integrate Eq. (44), or, equivalently, we take the 3D Fourier integral of the amplitude, Eqs. (41) and (42), with $\mathbf{q}_4 = 0$, which leads to the following result (for $\mathbf{r} \neq 0$):

$$\begin{aligned} V_{\mathrm{res}}^{\mathrm{Maxwell}} = \frac{e_1 e_2}{4\pi r}\Bigg\{ & \Delta_1\Delta_2\left(1+\frac{\mathbf{p}^2+\mathbf{p}_4^2}{m_1 m_2}\right) + \\ & -\frac{\mathbf{L}}{r^2}\cdot\left[\Delta_1\langle \mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1 m_2}\right)+1\leftrightarrow 2\right] \\ & +\frac{\mathbf{Q}_{ij}}{r^2}\frac{1}{m_1 m_2}[\langle \mathbf{S}_{1,i}^+\rangle\langle \mathbf{S}_{2,j}^+\rangle - \langle \mathbf{S}_{1,i}^-\rangle\langle \mathbf{S}_{2,j}^-\rangle] \\ & +\frac{\mathbf{p}_4\mathbf{r}}{r^2}\cdot\left[\Delta_1\langle \mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1 m_2}\right)+1\leftrightarrow 2\right]\Bigg\}. \end{aligned} \tag{52}$$

By comparing $V_{\mathrm{res}}^{\mathrm{Maxwell}}$ with V^{Maxwell}, calculated directly in four dimensions, Eq. (14), we note some similarities after using the following dictionary: $\delta \leftrightarrow \Delta$ and $\langle \mathbf{S}\rangle \leftrightarrow \langle \mathbf{S}^+\rangle$. We do not obtain modifications for the Coulomb term, and due our approximations, we also do not have interactions that couple $\mathbf{S}^+$ with $\mathbf{S}^-$. The $\mathbf{p}_4$ contribution appears coupled to the pseudospin in a way similar to spin-orbit coupling, $\mathbf{L}\cdot\langle \mathbf{S}^+\rangle$. We highlight a new contribution to the quadrupole term, namely, a pseudospin interaction, proportional to $\mathbf{Q}_{ij}\langle \mathbf{S}_{1,i}^-\rangle\langle \mathbf{S}_{2,j}^-\rangle/r^2$. This new contribution is related to the interaction intermediate by the extra component of the $A^{\hat{\mu}}$. Even if $\boldsymbol{q}_4 = 0$, we have some contributions from the current, Eq. (40), which exhibits the coupling $\mathbf{q}\cdot\langle \mathbf{S}^-\rangle$.

For all dimensional reduction schemes, the main requirement should be that the dominant contribution to the potential at large distances in the reduced four space-time dimensions, namely, the monopole-monopole interaction, decays with r^{-1} and respects the well-known Coulomb's law [30,31]. In our prescription, however, we arrive at a Coulomb potential and obtain an extra contribution to the quadrupole term due the presence of the pseudospin. We highlight that this is an effect driven by our reduction prescription. It is worth it to decompose the (pseudo)spin contributions to the quadrupole interaction in terms of the expectation values $\langle \mathbf{S}\rangle_\xi$ and $\langle \mathbf{S}\rangle_\zeta$ [see definitions in Eq. (30)]. By using Eq. (35), we obtain

$$\begin{aligned} & \frac{\mathbf{Q}_{ij}}{4\pi r^3}\frac{\langle \mathbf{S}_{1,i}^+\rangle\langle \mathbf{S}_{2,j}^+\rangle - \langle \mathbf{S}_{1,i}^-\rangle\langle \mathbf{S}_{2,j}^-\rangle}{m_1 m_2} \\ & \quad = 2\frac{\mathbf{Q}_{ij}}{4\pi r^3}\frac{\langle \mathbf{S}_{1,i}\rangle_\xi\langle \mathbf{S}_{2,j}\rangle_\zeta + \langle \mathbf{S}_{1,i}\rangle_\zeta\langle \mathbf{S}_{2,j}\rangle_\xi}{m_1 m_2}. \end{aligned} \tag{53}$$

Hence, the quadrupole interaction appears only as couplings between $\langle \mathbf{S}\rangle_\xi$ and $\langle \mathbf{S}\rangle_\zeta$. For this reason, in the parity-breaking case ($\zeta \to 0$), this interaction vanishes.

V. PROCA ELECTRODYNAMICS IN FIVE DIMENSIONS

In the previous section, we discussed the interparticle potential for the Maxwell electrodynamics in five dimensions. We wish now to generalize our results for a massive (boson) particle exchanged in the scattering process, described by the Proca Lagrangian,

$$\mathcal{L}_{\mathrm{Proca}} = -\frac{1}{4}F_{\hat{\mu}\hat{\nu}}^2 + \frac{1}{2}m^2 A_{\hat{\mu}}^2. \tag{54}$$

The propagator is given by

$$\langle A_{\hat{\mu}} A_{\hat{\nu}}\rangle = -\frac{i}{q^2 - m^2}\left(\eta_{\hat{\mu}\hat{\nu}} - \frac{q_{\hat{\mu}} q_{\hat{\nu}}}{m^2}\right). \tag{55}$$

In a way similar to what was done in Sec. II, we arrive at $\mathcal{M} = i g_1 g_2 J_{(1)}^{\hat{\mu}}\langle A_{\hat{\mu}} A_{\hat{\nu}}\rangle J_{(2)}^{\hat{\nu}}$. After using the relation between $\mathcal{M}$ and $\mathcal{M}_{\mathrm{NR}}$, Eq. (2), we have

$$\mathcal{M}_{\mathrm{NR}}^{5D} = -\frac{g_1 g_2}{\mathbf{q}^2 + \mathbf{q}_4^2 + m^2}\frac{J_{(1)}^{\hat{\mu}} J_{(2)\hat{\mu}}}{(2E_1)(2E_2)}, \tag{56}$$

where, in the last step, we have used the current conservation and $q^0 = 0$.

Note that the current contraction, presented in the last expression, was carried out in Eq. (42). Thus, considering the prescription in Eq. (43) and using the Fourier integrals, Eqs. (A4)–(A6) in the Appendix, we obtain

$$\begin{aligned}
V_{5D}^{\mathrm{Proca}} = g_1 g_2 \Bigg\{ & \frac{mK_1}{4\pi^2 R}\Delta_1\Delta_2\left[1+\frac{\mathbf{p}^2}{m_1m_2}+\frac{\mathbf{p}_4^2}{m_1m_2}+\frac{m^2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\right] \\
& -\frac{\Delta_1\Delta_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\delta^4(\mathbf{R}) - \frac{\langle\mathbf{S}_1^+\rangle\cdot\langle\mathbf{S}_2^+\rangle+\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle}{2m_1m_2}\delta^4(\mathbf{R}) \\
& +\frac{m^3K_1}{4\pi^2 R}\frac{\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle}{m_1m_2} - \frac{1}{4\pi^2R^2}\left[\frac{4mK_1}{R}+2m^2K_0-\frac{r^2}{R}\left(\frac{8mK_1}{R^2}+\frac{4m^2K_0}{R}+m^3K_1\right)\right] \\
& \times\frac{\langle\mathbf{S}_1^+\rangle\cdot\langle\mathbf{S}_2^+\rangle-\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle}{m_1m_2} - \frac{1}{4\pi^2R^2}\left(\frac{2mK_1}{R}+m^2K_0\right)\left[(\mathbf{r}\times\mathbf{p})\cdot\left(\Delta_1\langle\mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)\right)\right. \\
& \left.+(\mathbf{x}_4\mathbf{p}-\mathbf{p}_4\mathbf{r})\cdot\left(\Delta_1\langle\mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)\right)+1\leftrightarrow 2\right] \\
& -\frac{1}{4\pi^2R^3}\left(\frac{8mK_1}{R^2}+\frac{4m^2K_0}{R}+m^3K_1\right)\frac{1}{m_1m_2}[\mathbf{x}_4\mathbf{r}\cdot[(\langle\mathbf{S}_1^+\rangle\times\langle\mathbf{S}_2^-\rangle) \\
& +(\langle\mathbf{S}_2^+\rangle\times\langle\mathbf{S}_1^-\rangle)]+(\langle\mathbf{S}_1^+\rangle\cdot\mathbf{r})(\langle\mathbf{S}_2^+\rangle\cdot\mathbf{r})-(\langle\mathbf{S}_1^-\rangle\cdot\mathbf{r})(\langle\mathbf{S}_2^-\rangle\cdot\mathbf{r})]\Bigg\}.
\end{aligned} \tag{57}$$

This potential exhibits all the velocity- and (pseudo) spin-dependence interactions of the Maxwell case, Eq. (44), which is recovered in the massless limit. In the Proca potential, the power-law decay depends on the modified Bessel function, $K_\nu(z)$, and the range of $z = mR$. In the sequel, we shall study its asymptotic behaviors as R goes to infinity and zero, respectively, and then present its form upon the reduction by dimensional restriction.

According to Refs. [32,33], the behavior of the $K_\nu(z)$ when $z \to \infty$ is given by

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}\left[1+O\left(\frac{1}{z}\right)\right], \tag{58}$$

which holds for $|\arg z| < 3\pi/2$. Since we are using real values $z = mR$, the previous condition is automatically satisfied. Applying this result in Eq. (57), we obtain the following asymptotic limit:

$$\begin{aligned}
V_{5D}^{\mathrm{Proca}}|_{R\to\infty} \sim \frac{g_1g_2}{4}\sqrt{\frac{m}{2\pi^3}}e^{-mR}\Bigg\{ & \frac{\Delta_1\Delta_2}{R^{3/2}}\left(1+\frac{\mathbf{p}^2}{m_1m_2}+\frac{\mathbf{p}_4^2}{m_1m_2}\right) \\
& +\frac{m^2}{8}\frac{\Delta_1\Delta_2}{R^{3/2}}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)+\frac{m^2}{m_1m_2}\frac{1}{R^{3/2}}\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle \\
& -m\frac{(\mathbf{r}\times\mathbf{p})}{R^{5/2}}\cdot\left[\Delta_1\langle\mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)+1\leftrightarrow 2\right] \\
& -m\frac{(\mathbf{x}_4\mathbf{p}-\mathbf{p}_4\mathbf{r})}{R^{5/2}}\cdot\left[\Delta_1\langle\mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2}+\frac{1}{m_1m_2}\right)+1\leftrightarrow 2\right] \\
& -\frac{m^2}{m_1m_2}\frac{\mathbf{x}_4\mathbf{r}}{R^{7/2}}\cdot[(\langle\mathbf{S}_1^+\rangle\times\langle\mathbf{S}_2^-\rangle)+(\langle\mathbf{S}_2^+\rangle\times\langle\mathbf{S}_1^-\rangle)] \\
& +\frac{m^2}{m_1m_2}\frac{r^2}{R^{7/2}}[\langle\mathbf{S}_1^+\rangle\cdot\langle\mathbf{S}_2^+\rangle-\langle\mathbf{S}_1^-\rangle\cdot\langle\mathbf{S}_2^-\rangle] \\
& -\frac{m^2}{m_1m_2}\frac{1}{R^{7/2}}[(\langle\mathbf{S}_1^+\rangle\cdot\mathbf{r})(\langle\mathbf{S}_2^+\rangle\cdot\mathbf{r})-(\langle\mathbf{S}_1^-\rangle\cdot\mathbf{r})(\langle\mathbf{S}_2^-\rangle\cdot\mathbf{r})]\Bigg\}.
\end{aligned} \tag{59}$$

Here, we notice a peculiar behavior. In addition to the common factor e^{-mR}, we also have fractional power-law decay in all terms. The dominant monopole-monopole contribution decays with $R^{3/2}$. The velocity- and (pseudo)spin-dependent terms are suppressed by the mass fermions and have higher power-law decay.

Let us now consider the situation in which $z \to 0$. The asymptotic limits are given by Refs. [34,35], respectively,

$$K_0(z) \sim -\log\left(\frac{z}{2}\right)[1 + O(z^2)] - \gamma[1 + O(z^2)], \qquad (60)$$

$$K_1(z) \sim \frac{1}{z}[1 + O(z^2)] + \frac{z}{2}\log\left(\frac{z}{2}\right)[1 + O(z^2)], \qquad (61)$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

By taking into account these limits in Eq. (57), one may obtain

$$\begin{aligned}
V^{\text{Proca}}_{5D}|_{R\to 0} \sim \frac{g_1 g_2}{4\pi^2 R^2}\Bigg\{ & \Delta_1\Delta_2\left[1 + \frac{\mathbf{p}^2}{m_1 m_2} + \frac{\mathbf{p}_4^2}{m_1 m_2} + \frac{m^2}{8}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\right] \\
& + \frac{m^2}{m_1 m_2}\langle \mathbf{S}_1^-\rangle \cdot \langle \mathbf{S}_2^-\rangle - \left[\frac{4}{R^2} - 2m^2\left(\log\left(\frac{mR}{2}\right) + \gamma\right)\right. \\
& \left. - \frac{r^2}{R^2}\left(\frac{8}{R^2} + m^2\left(1 - 4\gamma - 4\log\left(\frac{mR}{2}\right)\right)\right)\right] \frac{\langle \mathbf{S}_1^+\rangle \cdot \langle \mathbf{S}_2^+\rangle - \langle \mathbf{S}_1^-\rangle \cdot \langle \mathbf{S}_2^-\rangle}{m_1 m_2} \\
& - \left(\frac{2}{R^2} - m^2\left(\log\left(\frac{mR}{2}\right) + \gamma\right)\right)\left[(\mathbf{r}\times\mathbf{p})\cdot\left(\Delta_1\langle \mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1 m_2}\right)\right)\right. \\
& \left. + (\mathbf{x}_4\mathbf{p} - \mathbf{p}_4\mathbf{r})\cdot\left(\Delta_1\langle \mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1 m_2}\right)\right) + 1 \leftrightarrow 2\right] \\
& - \frac{1}{R^2}\left(\frac{8}{R^2} + m^2\left(1 - 4\gamma - 4\log\left(\frac{mR}{2}\right)\right)\right)\frac{1}{m_1 m_2}[\mathbf{x}_4\mathbf{r}\cdot[(\langle \mathbf{S}_1^+\rangle \times \langle \mathbf{S}_2^-\rangle) \\
& + (\langle \mathbf{S}_2^+\rangle \times \langle \mathbf{S}_1^-\rangle)] + (\langle \mathbf{S}_1^+\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^+\rangle\cdot\mathbf{r}) - (\langle \mathbf{S}_1^-\rangle\cdot\mathbf{r})(\langle \mathbf{S}_2^-\rangle\cdot\mathbf{r})]\Bigg\}. \qquad (62)
\end{aligned}$$

Next, we present the restriction to four dimensions of the Proca electrodynamics studied in five dimensions. Following the prescription discussed in Sec. IV, we take the limit $\mathbf{q}_4 \to 0$ in the nonrelativistic amplitude, Eq. (56) with Eq. (42), and work out the Fourier integrals in three dimensions for $r \neq 0$ [see Eqs. (A1)–(A3) in Appendix], which leads to

$$\begin{aligned}
V^{\text{Proca}}_{\text{res}} = e_1 e_2 \frac{e^{-mr}}{4\pi r}\Bigg\{ & \Delta_1\Delta_2\left[\left(1 + \frac{\mathbf{p}^2 + \mathbf{p}_4^2}{m_1 m_2}\right)\right] + \frac{m^2}{m_1 m_2}\langle \mathbf{S}_1^+\rangle \cdot \langle \mathbf{S}_2^+\rangle \\
& - \mathbf{L}\cdot\left[\Delta_1\langle \mathbf{S}_2^+\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1 m_2}\right) + 1 \leftrightarrow 2\right]\frac{(1+mr)}{r^2} \\
& + \frac{1}{m_1 m_2}\frac{\mathbf{Q}_{ij}^{(m)}}{r^2}[\langle \mathbf{S}_{1,i}^+\rangle\langle \mathbf{S}_{2,j}^+\rangle - \langle \mathbf{S}_{1,i}^-\rangle\langle \mathbf{S}_{2,j}^-\rangle] \\
& + \mathbf{p}_4\mathbf{r}\cdot\left[\Delta_1\langle \mathbf{S}_2^-\rangle\left(\frac{1}{2m_2^2} + \frac{1}{m_1 m_2}\right) + 1 \leftrightarrow 2\right]\frac{(1+mr)}{r^2}\Bigg\}, \qquad (63)
\end{aligned}$$

where we defined

$$\mathbf{Q}_{ij}^{(m)} = (1 + mr)\delta_{ij} - (3 + 3mr + m^2 r^2)\frac{\mathbf{x}_i\mathbf{x}_j}{r^2}. \qquad (64)$$

It is worth it to compare this potential with the Maxwell case, Eq. (14). Again, the pseudospin contributions appear in a coupling with $\mathbf{p}_4$ and in the quadrupole term.

Finally, let us discuss an illustrative case in which we do not take $\mathbf{q}_4 = 0$ but we consider small contributions. For the sake of simplicity, we assume the monopole-monopole interaction, described by the following amplitude:

$$\mathcal{M}^{5D}_{\text{NR}} = -\frac{g_1 g_2}{\mathbf{q}^2 + \mathbf{q}_4^2 + m^2}\Delta_1\Delta_2. \qquad (65)$$

By plugging this amplitude into Eq. (43) and carrying out the integration $\int d^3\mathbf{q}$, one arrives at

$$V = g_1 g_2 \Delta_1\Delta_2 \int \frac{d\mathbf{q}_4}{(2\pi)} e^{i\mathbf{q}_4\mathbf{x}_4}\left[\frac{1}{4\pi r}e^{-mr\sqrt{1+\mathbf{q}_4^2/m^2}}\right]. \qquad (66)$$

Now, if one considers $\mathbf{q}_4^2 \ll m^2$, it is possible to approximate $e^{-mr\sqrt{1+\mathbf{q}_4^2/m^2}} \approx e^{-mr}e^{-r\mathbf{q}_4^2/2m}$, and the Fourier integral above reduces to a Gaussian one, which, for large distances (with $mr \gg m^2\mathbf{x}_4^2$), leads to the potential

$$V \approx g_1 g_2 \left[\frac{\Delta_1 \Delta_2}{4\pi r} e^{-mr}\right] \sqrt{\frac{m}{2\pi r}} \left(1 - \frac{m^2 \mathbf{x}_4^2}{2mr} + \cdots\right). \quad (67)$$

From this example, we conclude that, if we take even a small $\mathbf{q}_4$ contribution, the dominant term decays with $e^{-mr}/r^{3/2}$ rather than showing the usual Yukawa-like profile in four dimensions given by e^{-mr}/r. The term with $\mathbf{x}_4$ dependence falls off with $e^{-mr}/r^{5/2}$. In our prescription, after imposing the dimensional restriction, we arrive at the condition $\mathbf{q}_4 = 0$ and, consequently, obtain the expected Yukawa (or Coulomb in the massless case) dominant interaction in four dimensions. Moreover, we have shown that new contributions appear in the spin sector due the presence of pseudospin as an inheritance of 5D space-time.

VI. CONCLUDING COMMENTS

As it has been discussed over the past sections, our main endeavour in this paper was to compute photon-mediated and Proca-mediated parity-preserving interparticle potentials in five dimensions to get their four-dimensional description by adopting a particular scheme, which we refer to as reduction by dimensional restriction. The main feature of this procedure is the prescription that the mediating particle (in the cases we considered, massless and massive Abelian vector bosons) does not transfer momentum in the extra spatial dimension ($\mathbf{q}_4 = 0$). Our claim is that the physics of the interaction process exchanges momentum only along the $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ directions in momentum-transfer space ($\mathbf{q}_0 = 0$ for an elastic scattering). With this assumption, we correctly get the right space dependence of the potentials. Had we considered a nontrivial momentum transfer along $\mathbf{q}_4$, the dependence of the four-dimensional potentials would not be the right ones; they would fall off faster with distance (in the case of the monopole contribution, as an illustration, it would be r^{-2} rather than r^{-1}).

The behavior of the potentials with the particles spatial separation, velocities, and spins is also worked out in details in the tree-level approximation. And as a consequence of setting up the physics in five dimensions, there emerges, in four dimensions, an extra degree of freedom that we name here pseudospin; that is not the same as the pseudospin that appears in other contexts, as we have previously pointed out. Actually, the appearance of the pseudospin in our prescription seems to be a new feature, and we wish to go deeper into this point. The potential obtained in Eq. (52) and the decomposition of the quadrupole interaction in terms of two spins in four dimensions, Eq. (53), are our departure to better understand the role of the pseudospin in four-dimensional physical processes. This quadrupole-type contribution is a nontrivial consequence of the scheme we are referring to as dimensional restriction. We have, in particular, already initiated pursuing a study of the pseudospin in connection with the multipole structure of the fermionic current, with particular attention to a possible relationship between the pseudospin and the electron and muon electric dipole moments in models in which CP violation occurs. We intend to report on that elsewhere in a forthcoming work.

It is worth it to mention that we have considered in this paper a particular way to introduce the fermion mass without breaking parity in five dimensions. We have doubled the fermion representation and defined parity in a particular way, by imposing that the fermions of the doublet are exchanged into one another upon the action of parity transformation. In connection with the doubling of the spinors that represent the fermion in five dimensions, we may introduce a number of different symmetries, as we highlight in the next paragraph. As a new possibility that opens up, it would be interesting to understand how these symmetries may affect four-dimensional physics, especially in association with the electron's and muon's electric and magnetic dipole moments. This shall be the object of our immediate interest.

In a way similar to what was done in three dimensions [21], one may introduce other global (or local) phase transformations for the doubled spinor field in five dimensions. These transformations are defined by using 8×8 matrices, namely,

$$\tau_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_4 = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\tau_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (68)$$

It is possible to show that the massless term $\bar{\Psi} i \Gamma^{\hat{\mu}} \partial_{\hat{\mu}} \Psi$ is invariant under these transformations. However, the mass term breaks the τ_3 and τ_4 symmetries. Only the τ_5 case is consistent with a mass term. Furthermore, if we consider a local τ_5 symmetry, we obtain the current $J_5^{\hat{\mu}} \equiv \bar{\Psi}\Gamma^{\hat{\mu}}\tau_5\Psi$ and an Abelian gauge field, $B^{\hat{\mu}}$, both pseudovectors in five dimensions. In this case, beyond the Maxwell-like term in the Lagrangian, we could also introduce a Chern-Simons term in five dimensions, without breaking the parity symmetry. This particular case could be more explored in connection with topological superconductors [36], for which a Chern-Simons term plays an important role.

Finally, we point out that applying the dimensional restriction prescription to go from a $(1+3)$- to a $(1+2)$-dimensional space-time may be of interest in the inspection of low-dimensional systems in condensed matter physics, such as graphene and the charge/spin Hall effect.

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APPENDIX: FOURIER INTEGRALS

Below, we present some useful Fourier integrals in three and four dimensions. Let us initiate with the well-known 3D massive case,

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+m^2} = \frac{e^{-mr}}{4\pi r}, \tag{A1}$$

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+m^2}\mathbf{q}_i = \frac{i\mathbf{x}_i}{4\pi r^3}(1+mr)e^{-mr}, \tag{A2}$$

$$\begin{aligned}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+m^2}\mathbf{q}_i\mathbf{q}_j &= \frac{\delta_{ij}}{3}\delta^3(\mathbf{r}) + \frac{e^{-mr}}{4\pi r^3}\bigg[(1+mr)\delta_{ij} \\ &\quad - (3+3mr+m^2r^2)\frac{\mathbf{x}_i\mathbf{x}_j}{r^2}\bigg],\end{aligned} \tag{A3}$$

where $r=\sqrt{\mathbf{r}^2}$ and $i,\ j=1,\ 2,\ 3$. From these equations, one can directly obtain the massless limit.

To avoid confusion, we shall use $\mathbf{R}$ to denote the 4D (Euclidean) vector and $\mathbf{x}_I$ for its components, with capital letter $I=1,\ 2,\ 3,\ 4$. The 4D massive Fourier integrals are given by

$$\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2+m^2} = \frac{m}{4\pi^2 R}K_1(mR), \tag{A4}$$

$$\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2+m^2}\mathbf{q}_I = \frac{i\mathbf{x}_I}{4\pi^2R^2}\left[\frac{2mK_1(mR)}{R}+m^2K_0(mR)\right], \tag{A5}$$

$$\begin{aligned}&\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2+m^2}\mathbf{q}_I\mathbf{q}_J \\ &= \frac{1}{4}\delta_{IJ}\delta^4(\mathbf{R}) + \frac{\delta_{IJ}}{4\pi^2R^2}\left[\frac{2mK_1(mR)}{R}+m^2K_0(mR)\right] \\ &\quad - \frac{\mathbf{x}_I\mathbf{x}_J}{4\pi^2R^3}\left[\frac{8mK_1(mR)}{R^2}+\frac{4m^2K_0(mR)}{R}+m^3K_1(mR)\right],\end{aligned} \tag{A6}$$

where $R=\sqrt{\mathbf{R}^2}$ and $K_\nu(z)$ is the modified Bessel function of the second kind with order ν.

By using the asymptotic limits, Eqs. (60) and (61), it is possible to work out the massless limits, which take the form

$$\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2} = \frac{1}{4\pi^2R^2}, \tag{A7}$$

$$\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2}\mathbf{q}_I = \frac{i}{2\pi^2}\frac{\mathbf{x}_I}{R^4}, \tag{A8}$$

$$\int \frac{d^4\mathbf{q}}{(2\pi)^4}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2}\mathbf{q}_I\mathbf{q}_J = \frac{1}{4}\delta_{IJ}\delta^4(\mathbf{R}) + \frac{1}{2\pi^2R^4}\left[\delta_{IJ}-4\frac{\mathbf{x}_I\mathbf{x}_J}{R^2}\right]. \tag{A9}$$

[1] J. Scherk and J. Schwarz, Phys. Lett. B **57**, 463 (1975).
[2] E. Cremmer and J. Scherk, Nucl. Phys. **B103**, 399 (1976); **B108**, 409 (1976).
[3] Z. Horvath, L. Palla, E. Cremmer, and J. Scherk, Nucl. Phys. **B127**, 57 (1977).
[4] L. Palla, in *Proceedings, 19th International Conference, ICHEP 1978, Tokyo, Japan* (1978), p. 629.
[5] Z. Horvath and L. Palla, Nucl. Phys. **B142**, 327 (1978).
[6] W. Nahm, Nucl. Phys. **B135**, 149 (1978).
[7] J. Scherk and J. H. Schwarz, Nucl. Phys. **B153**, 61 (1979); Phys. Lett. B **82**, 60 (1979); **84**, 83 (1979).
[8] P. G. O. Freund and M. A. Rubin, Phys. Lett. B **97**, 233 (1980).
[9] C. Orzalesi, Multidimensional Unified Theories, University of Parma, Preprint IFPR/TH/63, 1980; Fortsch. Phys. **29**, 413 (1981).
[10] C. Orzalesi and M. Pauri, Phys. Lett. B **107**, 186 (1981).
[11] E. Álvarez and A. F. Faedo, J. High Energy Phys. 05 (2006) 046; Phys. Rev. D **74**, 124029 (2006); J. Phys. A **40**, 6641 (2007).
[12] T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); Phys. Rev. D **28**, 772 (1983).
[13] A. Chodos, Comments Nucl. Part. Phys. **13**, 171 (1984); *Quantum Aspects of Kaluza-Klein Theories*, Yale University Preprint YTP/84-05, 1984.
[14] T. Appelquist, A. Chodos, and E. Myers, Phys. Lett. **127B**, 51 (1983).
[15] M. Maggiore, *A Modern Introduction to Quantum Field Theory* (Oxford University, New York, 2005).
[16] P. C. Malta, L. P. R. Ospedal, K. Veiga, and J. A. Helayël-Neto, Adv. High Energy Phys. **2016**, 2531436 (2016); Adv. High Energy Phys. **2017**, 9152437(E) (2017).
[17] G. P. de Brito, P. C. Malta, and L. P. R. Ospedal, Phys. Rev. D **95**, 016006 (2017).

[18] A. Accioly, J. Helayël-Neto, F. E. Barone, F. A. Barone, and P. Gaete, Phys. Rev. D **90**, 105029 (2014).
[19] A. Accioly, J. Helayël-Neto, G. Correia, G. Brito, J. de Almeida, and W. Herdy, Phys. Rev. D **93**, 105042 (2016).
[20] A. Accioly, G. Correia, G. P. De Brito, J. De Almeida, and W. Herdy, Mod. Phys. Lett. A **32**, 1750048 (2017).
[21] N. Dorey and N. E. Mavromatos, Nucl. Phys. **B386**, 614 (1992).
[22] J. Kociński, J. Phys. A **32**, 4257 (1999).
[23] Z. F. Ezawa, *Quantum Hall Effects* (World Scientific, Singapore, 2008).
[24] H. Liang, Phys. Scr. **91**, 083005 (2016).
[25] J. N. Ginocchio, Symmetry **8**, 16 (2016).
[26] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
[27] M. F. Sohnius, K. S. Stelle, and P. C. West, Nucl. Phys. **B173**, 127 (1980); Phys. Lett. **92B**, 123 (1980).
[28] R. Arshansky, L. P. Horwitz, and Y. Lavie, Found. Phys. **13**, 1167 (1983).
[29] M. C. Land and L. P. Horwitz, J. Phys. Conf. Ser. **437**, 012011 (2013).
[30] E. R. Williams, J. E. Faller, and H. A. Hill, Phys. Rev. Lett. **26**, 721 (1971).
[31] L. P. Fulcher, Phys. Rev. A **33**, 759 (1986).
[32] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series (U.S. Government Printing Office, Washington, D.C., 1972), Vol. 55.
[33] http://functions.wolfram.com/03.04.06.0010.01.
[34] http://functions.wolfram.com/03.04.06.0039.01.
[35] http://functions.wolfram.com/03.04.06.0040.01.
[36] X.-L. Qi, E. Witten, and S.-C. Zhang, Phys. Rev. B **87**, 134519 (2013).