Minimal $SU(3) \times SU(3)$ symmetry breaking patterns

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We study the vacua of an $SU(3) \times SU(3)$-symmetric model with a bifundamental scalar. Structures of this type appear in various gauge theories such as the renormalizable coloron model, which is an extension of QCD, or the trinification extension of the electroweak group. In other contexts, such as chiral or family symmetry, $SU(3) \times SU(3)$ is a global symmetry. As opposed to more general $SU(N) \times SU(N)$-symmetric models, the $N = 3$ case is special due to the presence of a trilinear scalar term in the potential. We find that the most general tree-level potential has only three types of minima: one that preserves the diagonal $SU(3)$ subgroup, one that is $SU(2) \times SU(2) \times U(1)$ symmetric, and a trivial one where the full symmetry remains unbroken. The phase diagram is complicated, with some regions where there is a unique minimum, and other regions where two minima coexist.

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I. INTRODUCTION

Several quantum field theories of interest for physics beyond the Standard Model have an $SU(3) \times SU(3)$ symmetry, which is spontaneously broken. The embedding of the QCD gauge group, $SU(3)_c$, into an $SU(3)_1 \times SU(3)_2$ gauge symmetry has been considered in various contexts, including dynamical symmetry breaking [1], rare Z decays [2], the study of heavy color-octet spin-1 particles such as the axigluon [3,4] or the coloron [5,6], composite Higgs models based on the top-seesaw mechanism [7], and so on. This requires the spontaneous breaking of the product group into its diagonal subgroup. A simple structure that achieves that breaking consists of a single scalar field that transforms in the bifundamental representation, with a potential that includes a trilinear interaction, as discussed in the renormalizable coloron model (ReCoM) [8–10]. A model of this type has been recently proposed as a solution to the strong CP problem [11].

The spontaneous breaking of an $SU(3) \times SU(3)$ symmetry down to its diagonal $SU(3)$ group is also encountered in certain tumbling theories [12], latticized extra dimensions [13], or the chiral symmetry of QCD with three light quark flavors [14].

Another example of a symmetry breaking pattern is given by the so-called trinification [15–17], which is an embedding of the $SU(2)_W \times U(1)_Y$ electroweak group into an $SU(3)_L \times SU(3)_R$ gauge group. In that case the symmetry breaking may be achieved in two steps, with the first one, $SU(3)_L \times SU(3)_R \to SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, being again due to the vacuum expectation value (VEV) of a bifundamental scalar. The same symmetry breaking pattern has been studied, in the case of global symmetries, as a possible origin for the mass hierarchy between the fermions of the third generation and those of the first two generations [18–20].

Here we study the scalar potential of the most general renormalizable potential for a scalar field that is an $SU(3) \times SU(3)$ bifundamental. Besides a mass term and two quartic terms, the potential includes a cubic term, or more precisely a trilinear interaction given by the determinant of the bifundamental, which in $SU(N) \times SU(N)$-symmetric models is specific only to the case $N = 3$ (the determinant term is also present for $N = 2$ or 4, but with a different mass dimension). The parameter space spanned by the coefficients of these four terms leads to a nontrivial vacuum structure that has not been fully explored thus far.

Given the applications mentioned above, we are particularly interested in identifying the regions of parameter space where the potential has global or local minima that are either $SU(3)$ symmetric or $SU(2) \times SU(2) \times U(1)$ symmetric.[1] Also, we would like to know if there exist vacua (either global or just local minima) with other symmetry properties. In the absence of the cubic term in

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[1]A comparison of the $SU(3)$-symmetric and $SU(2) \times SU(2) \times U(1)$-symmetric minima has been performed in [18], without differentiating between regions where a unique vacuum exists and regions where there is a local minimum in addition to the global minimum.

the potential, it has been known for a long time that there are no other nontrivial vacua [21]. In the presence of the cubic term, though, it is not immediately clear if other vacua exist. For example, in Ref. [15] it is speculated that the potential for the bifundamental scalar may have a minimum that preserves an $SU(2) \times U(1)$ group, and another minimum that breaks $SU(3) \times SU(3)$ down to $U(1) \times U(1)$. We will prove that such patterns of symmetry breaking are not possible, even when the vacuum resides in a local minimum instead of a global minimum.

Another question (partially addressed in [9,18]) is about the asymptotic behavior of the potential: what ranges of parameters make the potential bounded from below? We find a condition involving the two quartic couplings which is necessary and sufficient for that.

In Sec. II we present the renormalizable potential and the parameter space. In Secs. III–V we identify all possible local minima, and we compute the physical scalar masses in the $SU(3)$-symmetric and $SU(2) \times SU(2) \times U(1)$-symmetric vacua. The conditions for having a potential bounded from below are derived in Sec. VI. We analyze the phase diagram of this theory, including all global minima, in Sec. VII. Section VIII includes our conclusions.

II. $SU(3) \times SU(3)$ WITH A SCALAR BIFUNDAMENTAL

Consider an $SU(3)_1 \times SU(3)_2$ symmetry with a scalar Σ transforming in the $(3, \bar{3})$ representation. Thus, Σ is a 3×3 matrix with complex entries. The renormalizable potential of Σ is given by

$$V(\Sigma) = -m_\Sigma^2 \mathrm{Tr}(\Sigma\Sigma^\dagger) - (\mu_\Sigma \det\Sigma + \mathrm{H.c.}) + \frac{\lambda}{2}[\mathrm{Tr}(\Sigma\Sigma^\dagger)]^2 + \frac{\kappa}{2}\mathrm{Tr}(\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger). \tag{2.1}$$

The dimensionless couplings λ and κ are real numbers. The mass-squared parameter, m_Σ^2, may be positive or negative. The phase rotation freedom of Σ allows us without loss of generality to choose the coefficient of the trilinear term (a mass parameter) to be real and satisfy

$$\mu_\Sigma \geq 0. \tag{2.2}$$

The potential $V(\Sigma)$ has an accidental Z_3 symmetry. If $\mu_\Sigma = 0$, then the Z_3 symmetry is enhanced to a global $U(1)_\Sigma$ symmetry, with Σ carrying nonzero global charge. We also note that when both $\mu_\Sigma = 0$ and $\kappa = 0$ the potential has an enhanced $SO(18)$ symmetry.

Even though the scalar Σ has 18 degrees of freedom, upon an $SU(3)_1 \times SU(3)_2$ transformation the most general form of its VEV is a diagonal 3×3 matrix. Furthermore, the diagonal $SU(3)_1 \times SU(3)_2$ transformations, associated with the T^3 and T^8 generators, can be used to get rid of two phases. Thus, the most general VEV of Σ has four real parameters:

$$\langle\Sigma\rangle = \mathrm{diag}(s_1, s_2, s_3)e^{i\alpha/3}, \quad \text{with} \quad s_i \geq 0, i = 1, 2, 3, \quad \text{and} \quad -\pi < \alpha \leq \pi. \tag{2.3}$$

The $1/3$ in the complex phase of the VEV is due to the Z_3 symmetry. We seek the values of s_i and α that correspond to local minima of the potential.

To identify the extrema of the $V(\Sigma)$ potential, we need to find s_i, $i = 1, 2, 3$ and α that satisfy the extremization (or more precisely stationarity) conditions, which are given by

$$\frac{1}{2}\frac{\partial V}{\partial s_1} = (\lambda + \kappa)s_1^3 + \lambda s_1(s_2^2 + s_3^2) - \mu_\Sigma s_2 s_3 \cos\alpha - m_\Sigma^2 s_1 = 0; \tag{2.4}$$

two analogous equations for $\partial V/\partial s_2$ and $\partial V/\partial s_3$ (the $i = 1, 2, 3$ indices are cyclical); and finally

$$\frac{\partial V}{\partial \alpha} = 2\mu_\Sigma s_1 s_2 s_3 \sin\alpha = 0. \tag{2.5}$$

This set of cubic equations in s_i appears difficult to solve analytically; however, the first three equations can be replaced by a set of quadratic and linear equations as follows:

$$\begin{aligned}
\frac{\partial V}{\partial s_1} - \frac{\partial V}{\partial s_2} &= 2(s_1 - s_2)[(\lambda + \kappa)(s_1^2 + s_2^2 + s_1 s_2) + \lambda(s_3^2 - s_1 s_2) + \mu_\Sigma s_3 \cos\alpha - m_\Sigma^2] = 0 \\
\frac{\partial V}{\partial s_2} - \frac{\partial V}{\partial s_3} &= 2(s_2 - s_3)[(\lambda + \kappa)(s_2^2 + s_3^2 + s_2 s_3) + \lambda(s_1^2 - s_2 s_3) + \mu_\Sigma s_1 \cos\alpha - m_\Sigma^2] = 0 \\
\frac{\partial V}{\partial s_2} + \frac{\partial V}{\partial s_3} &= 2(s_2 + s_3)[(\lambda + \kappa)(s_2^2 + s_3^2 - s_2 s_3) + \lambda(s_1^2 + s_2 s_3) - \mu_\Sigma s_1 \cos\alpha - m_\Sigma^2] = 0.
\end{aligned} \tag{2.6}$$

To find the solutions to the set of Eqs. (2.5) and (2.6) we will consider a few separate cases.

A solution to the extremization conditions represents a local minimum if and only if the second-derivative matrix has only positive eigenvalues. Denoting that matrix by $\partial^2 V/(\partial s_i \partial s_j)$ with $i, j = 1, \ldots, 4$, where $s_4 \equiv \alpha\mu_\Sigma$, we find

$$\frac{1}{2}\frac{\partial^2 V}{\partial s_i \partial s_j} = \begin{pmatrix} (2\lambda+3\kappa)s_1^2+\Delta & 2\lambda s_1 s_2 - \mu_\Sigma s_3\cos\alpha & 2\lambda s_1 s_3 - \mu_\Sigma s_2 \cos\alpha & s_2 s_3 \sin\alpha \\ 2\lambda s_1 s_2 - \mu_\Sigma s_3\cos\alpha & (2\lambda+3\kappa)s_2^2+\Delta & 2\lambda s_2 s_3 - \mu_\Sigma s_1\cos\alpha & s_3 s_1 \sin\alpha \\ 2\lambda s_1 s_3 - \mu_\Sigma s_2\cos\alpha & 2\lambda s_2 s_3 - \mu_\Sigma s_1 \cos\alpha & (2\lambda+3\kappa)s_3^2+\Delta & s_1 s_2\sin\alpha \\ s_2 s_3 \sin\alpha & s_3 s_1\sin\alpha & s_1 s_2 \sin\alpha & s_1 s_2 s_3\cos\alpha/\mu_\Sigma \end{pmatrix}, \tag{2.7}$$

where we defined

$$\Delta \equiv \lambda(s_1^2+s_2^2+s_3^2) - m_\Sigma^2. \tag{2.8}$$

Let us first apply these minimization conditions to the extrema located at the trivial solution to Eq. (2.6), $s_1 = s_2 = s_3 = 0$, for any α. Three of the eigenvalues of $\partial^2 V/(2\partial s_i \partial s_j)$ are equal to $-m_\Sigma^2$, while the fourth one is zero (representing a flat direction along α). Thus, there is a minimum with $V(\Sigma) = 0$ at $s_1 = s_2 = s_3 = 0$ provided $m_\Sigma^2 < 0$.

III. $SU(3)$-SYMMETRIC VACUUM

We now search for minima that have $s_1 = s_2 = s_3 > 0$, so that the VEV preserves an $SU(3)$ symmetry, which is the diagonal subgroup of the $SU(3)_1 \times SU(3)_2$ symmetry. The three Eq. (2.6) are then replaced by a single quadratic equation:

$$(3\lambda+\kappa)s_1^2 = \mu_\Sigma s_1 \cos\alpha + m_\Sigma^2. \tag{3.1}$$

The extremization condition (2.5) becomes $\sin\alpha = 0$. The phase α is further constrained by requiring stability of the potential. The second-derivative matrix shown in Eq. (2.7) has an eigenvalue equal to the 44 entry, namely $s_1^3\cos\alpha/\mu_\Sigma$. Imposing that this be positive implies $\alpha = 0$.

For the range of parameters where

$$\mu_\Sigma^2 > -4(3\lambda+\kappa)m_\Sigma^2, \tag{3.2}$$

there are two solutions to the extremization conditions:

$$s_1 = s_2 = s_3 = \frac{1}{2(3\lambda+\kappa)}\left(\pm\sqrt{4(3\lambda+\kappa)m_\Sigma^2+\mu_\Sigma^2}+\mu_\Sigma\right). \tag{3.3}$$

Given that $s_i > 0$, the above solution with positive sign is valid only when

$$3\lambda+\kappa > 0, \tag{3.4}$$

while the solution with negative sign requires $m_\Sigma^2 < 0$.

We need to determine the regions of parameter space where these extrema satisfy the minimization conditions along the s_i directions with $i = 1, 2, 3$. The 3×3 upper-left block of the second-derivative matrix shown in Eq. (2.7) may be written as follows:

$$\begin{aligned} \mathcal{M}^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &\quad \times \frac{1}{2}\frac{\partial^2 V}{\partial s_i \partial s_j} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}_{11}^2 & \mathcal{M}_{12}^2 & 0 \\ \mathcal{M}_{12}^2 & \mathcal{M}_{22}^2 & 0 \\ 0 & 0 & \mathcal{M}_3^2 \end{pmatrix}, \end{aligned} \tag{3.5}$$

where the elements of the 2×2 upper-left block of $\mathcal{M}^2$ are given by

$$\begin{aligned} \mathcal{M}_{11}^2 &= \frac{1}{3\lambda+\kappa}[2(\lambda+\kappa)m_\Sigma^2 + (5\lambda+3\kappa)\mu_\Sigma s_1], \\ \mathcal{M}_{22}^2 &= 2\frac{2\lambda+\kappa}{3\lambda+\kappa}(m_\Sigma^2+\mu_\Sigma s_1), \\ \mathcal{M}_{12}^2 &= \frac{\sqrt{2}}{3\lambda+\kappa}[2\lambda m_\Sigma^2 - (\lambda+\kappa)\mu_\Sigma s_1], \end{aligned} \tag{3.6}$$

and the 33 entry of $\mathcal{M}^2$ is

$$\mathcal{M}_3^2 = \frac{2}{3\lambda+\kappa}[\kappa m_\Sigma^2 + (3\lambda+2\kappa)\mu_\Sigma s_1]. \tag{3.7}$$

The eigenvalues of $\mathcal{M}^2$ are the squared masses of the radial modes. $SU(3)$ invariance implies that two eigenvalues are equal, $\mathcal{M}_2^2 = \mathcal{M}_3^2$, because they are the squared masses of different components (associated with the T^3 and T^8 generators) of an $SU(3)$-octet scalar. The third eigenvalue represents the squared mass of an $SU(3)$-singlet scalar, and is given by

$$\mathcal{M}_1^2 = 2m_\Sigma^2 + \mu_\Sigma s_1. \tag{3.8}$$

The minimization condition $\mathcal{M}_1^2 > 0$ is equivalent to

$$\frac{1}{3\lambda+\kappa}\left(\sqrt{4(3\lambda+\kappa)m_\Sigma^2+\mu_\Sigma^2} \pm \mu_\Sigma\right) > 0, \tag{3.9}$$

where the $+$ or $-$ sign corresponds to the sign chosen for the extremum (3.3). This condition can never be satisfied by the negative solution (since $m_\Sigma^2 < 0$ in that case), which thus is at most a saddle point.

The minimization condition (3.9) is automatically satisfied by the positive solution [given the constraint (3.4) in that case], so only $\mathcal{M}_3^2 > 0$ remains to be imposed:

$$(3\lambda+2\kappa)\left(\sqrt{4(3\lambda+\kappa)m_\Sigma^2+\mu_\Sigma^2}+\mu_\Sigma\right) > -2\kappa(3\lambda+\kappa)\frac{m_\Sigma^2}{\mu_\Sigma}. \tag{3.10}$$

For $m_\Sigma^2 > 0$ we find that the positive solution from (3.3) represents a local minimum if and only if either $\kappa \geq 0$ or else

$$\kappa < 0 \quad \text{and} \quad (3\lambda+2\kappa)\mu_\Sigma^2 > \kappa^2 m_\Sigma^2. \tag{3.11}$$

For $m_\Sigma^2 \leq 0$ the positive solution is a local minimum when

$$-3\lambda<\kappa \text{ and } \lambda<0,$$

or

$$-\frac{3}{2}\lambda<\kappa \text{ and } \lambda>0,$$

or

$$-2\lambda<\kappa<-\frac{3}{2}\lambda<0 \text{ and } (3\lambda+2\kappa)\mu_\Sigma^2>\kappa^2 m_\Sigma^2. \tag{3.12}$$

To derive the above conditions we used the constraints (3.2) and (3.4). The value of the potential at the $SU(3)$-symmetric vacuum is given by

$$V_{\text{min}}^{(3)} = -\frac{3}{2(3\lambda+\kappa)}\left[m_\Sigma^4+\frac{m_\Sigma^2\mu_\Sigma^2}{3\lambda+\kappa} + \frac{\mu_\Sigma^4+\mu_\Sigma(4(3\lambda+\kappa)m_\Sigma^2+\mu_\Sigma^2)^{3/2}}{6(3\lambda+\kappa)^2}\right]. \tag{3.13}$$

We will discuss the conditions for a global minimum in Sec. VII.

Among the 18 degrees of freedom in Σ, there are 8 massless Nambu-Goldstone bosons (NGBs). If $SU(3)\times SU(3)$ is a gauge symmetry, the 8 NGBs become the longitudinal degrees of freedom for a heavy spin-1 field transforming as an octet under the unbroken $SU(3)$. The remaining 10 degrees of freedom are massive and can be decomposed into $8+1+1'$ under the unbroken $SU(3)$ vacuum symmetry [8]. The heavy octet scalar has a squared mass given in Eq. (3.7), and one of the singlet scalars has a squared mass given in Eq. (3.8), with s_1 being the positive solution shown in Eq. (3.3). The mass of the remaining heavy singlet scalar is

$$M_{\mathbf{1}'} = \sqrt{3\mu_\Sigma s_1}. \tag{3.14}$$

In the $\mu_\Sigma \to 0$ limit, this state becomes the NGB associated with the global $U(1)_\Sigma$ symmetry mentioned after Eq. (2.2).

IV. $SU(2)\times SU(2)\times U(1)$-SYMMETRIC VACUUM

We now seek minima with two of the s_i vanishing, so that the VEV preserves an $SU(2)\times SU(2)\times U(1)$ symmetry. It is sufficient to set $s_1 > 0$ and $s_2 = s_3 = 0$, as this is equivalent up to $SU(3)_1\times SU(3)_2$ transformations to the cases $s_1 = s_2 = 0$ or $s_1 = s_3 = 0$. Another transformation, along the diagonal generators, can be used in this case to eliminate the phase α from the VEV (2.3). The extremization conditions (2.6) take a simple form,

$$(\lambda+\kappa)s_1^2 = m_\Sigma^2. \tag{4.1}$$

For $(\lambda+\kappa)m_\Sigma^2 > 0$ the extremum is at

$$s_1 = \frac{|m_\Sigma|}{\sqrt{|\lambda+\kappa|}}. \tag{4.2}$$

Using the same rotation on the second-derivative matrix $\partial^2 V/(2\partial s_i \partial s_j)$ as in Eq. (3.5), we find the eigenvalues

$$\mathcal{M}_1^2 = 2m_\Sigma^2,$$
$$\mathcal{M}_2^2 = -\frac{\kappa}{\lambda+\kappa}m_\Sigma^2 - \mu_\Sigma s_1,$$
$$\mathcal{M}_3^2 = -\frac{\kappa}{\lambda+\kappa}m_\Sigma^2 + \mu_\Sigma s_1. \tag{4.3}$$

The minimization condition $\mathcal{M}_1^2 > 0$ is satisfied provided $m_\Sigma^2 > 0$, which implies $\kappa > -\lambda$. As m_Σ is real and its sign is irrelevant, we choose $m_\Sigma > 0$. Given that $\mathcal{M}_3^2 > \mathcal{M}_2^2$, it remains to impose $\mathcal{M}_2^2 > 0$, so that

$$-\lambda < \kappa < 0 \quad \text{and} \quad \mu_\Sigma < -\frac{\kappa m_\Sigma}{\sqrt{\lambda+\kappa}}. \tag{4.4}$$

Thus, an $SU(2)\times SU(2)\times U(1)$-symmetric local minimum exists at

$$s_1 = \frac{m_\Sigma}{\sqrt{\lambda+\kappa}}, \qquad s_2 = s_3 = 0. \tag{4.5}$$

The value of the potential at this minimum is

$$V_{\text{min}}^{(2,2,1)} = -\frac{m_\Sigma^4}{2(\lambda+\kappa)} < 0. \tag{4.6}$$

The degrees of freedom in the Σ field are grouped into 9 massless NGBs and 9 massive scalars. The latter can be decomposed into two real scalars transforming as $(2,2)_0$ under the unbroken $SU(2)\times SU(2)\times U(1)$ vacuum symmetry, and a real singlet scalar. The squared masses of these heavy scalars are given by the eigenvalues of the second-derivative matrix, shown in Eq. (4.3). More precisely, the mass of the real singlet scalar is

$$M_{(1,1)_0} = \sqrt{2} m_\Sigma, \tag{4.7}$$

while the two $SU(2)\times SU(2)$ bifundamentals have nondegenerate masses:

$$M_{(2,2)_0} = \sqrt{|\kappa| s_1^2 \pm \mu_\Sigma s_1}. \tag{4.8}$$

V. ABSENCE OF LESS SYMMETRIC VACUA

Let us now seek extrema with two of the s_i equal but nonzero, so that the remaining symmetry of the Σ VEV is the diagonal $SU(2)\times U(1)$ subgroup of $SU(3)_1\times SU(3)_2$. It is sufficient to consider the case

$$s_2 = s_3 > 0 \quad \text{and} \quad s_2 \neq s_1 \geq 0, \tag{5.1}$$

because $SU(3)_1\times SU(3)_2$ transformations can connect this extremum to the ones with permutations of the $i = 1, 2, 3$ indices ($s_3 \neq s_1 = s_2 > 0$ or $s_2 \neq s_1 = s_3 > 0$). The extremization conditions Eqs. (2.6) and (2.5) are in this case given by

$$\begin{aligned}
&s_1 \sin\alpha = 0,\\
&(\lambda+\kappa)(s_1^2+s_2^2+s_1 s_2) + \lambda s_2(s_2 - s_1) + \mu_\Sigma s_2 \cos\alpha = m_\Sigma^2,\\
&(\lambda+\kappa)s_2^2 + \lambda(s_1^2+s_2^2) - \mu_\Sigma s_1 \cos\alpha = m_\Sigma^2.
\end{aligned} \tag{5.2}$$

The solution $s_1 = 0$ to the first equation implies $\cos\alpha = 0$, due to the last two equations above. At this extremum, the second-derivative matrix [see Eq. (2.7)] is block diagonal, with one of the 2×2 blocks having the determinant equal to $-s_2^4 < 0$. Thus, at least one of the eigenvalues is negative so that the extremum at $s_1 = 0$ is only a saddle point.

The other solution to the first Eq. (5.2), $\sin\alpha = 0$, leads to more complications. One of the eigenvalues of the second-derivative matrix is given by its 44 entry, and it is positive only for $\cos\alpha = 1$. Imposing this condition as well as the positivity condition (5.1), we find that the extremization conditions (5.2) have a solution,

$$\begin{aligned}
s_1 &= -\frac{\mu_\Sigma}{\kappa} > 0,\\
s_2 = s_3 &= \left(\frac{\kappa^2 m_\Sigma^2 - (\lambda+\kappa)\mu_\Sigma^2}{\kappa^2(2\lambda+\kappa)}\right)^{1/2} > 0,\\
\alpha &= 0,
\end{aligned} \tag{5.3}$$

only for

$$\kappa < 0 \quad \text{and} \quad \frac{1}{2\lambda+\kappa}[(\lambda+\kappa)\mu_\Sigma^2 - \kappa^2 m_\Sigma^2] < 0. \tag{5.4}$$

To see if the extremum (5.3) may be a minimum, we use the mass-squared matrix $\mathcal{M}^2$ of Eq. (3.5), which in this case has the following nonzero elements:

$$\begin{aligned}
\mathcal{M}_{11}^2 &= \frac{1}{2\lambda+\kappa}\left[-\kappa m_\Sigma^2 + (\lambda+\kappa)(4\lambda+3\kappa)\frac{\mu_\Sigma^2}{\kappa^2}\right],\\
\mathcal{M}_{22}^2 &= 2m_\Sigma^2 - 2(\lambda+\kappa)\frac{\mu_\Sigma^2}{\kappa^2},\\
\mathcal{M}_{12}^2 &= -\sqrt{2}(2\lambda+\kappa)\frac{\mu_\Sigma}{\kappa} s_2,\\
\mathcal{M}_3^2 &= \frac{2\kappa}{2\lambda+\kappa}\left[m_\Sigma^2 - (3\lambda+2\kappa)\frac{\mu_\Sigma^2}{\kappa^2}\right].
\end{aligned} \tag{5.5}$$

The determinant of $\mathcal{M}^2$ is given by $-(2\lambda+\kappa)s_2^2(\mathcal{M}_3^2)^2$, so a necessary minimization condition is

$$2\lambda + \kappa < 0, \tag{5.6}$$

which in conjunction with (5.4) implies $\lambda+\kappa < 0$ and $m_\Sigma^2 < 0$. Another necessary minimization condition is that the trace of the upper 2×2 block of $\mathcal{M}^2$ (or equivalently the sum of the first and second eigenvalues) is positive, $\mathcal{M}_{11}^2 + \mathcal{M}_{22}^2 > 0$, which leads to

$$(\lambda+\kappa)\mu_\Sigma^2 > -\kappa(4\lambda+\kappa)m_\Sigma^2. \tag{5.7}$$

The remaining minimization condition is $\mathcal{M}_3^2 > 0$, implying

$$(3\lambda+2\kappa)\mu_\Sigma^2 < \kappa^2 m_\Sigma^2, \tag{5.8}$$

which is incompatible with (5.7). Thus, the solution (5.3) is only a saddle point.

Let us finally seek solutions to the extremization conditions (2.6) and (2.5) where $s_i \neq s_j$ for all $i \neq j$, with $i, j = 1, 2, 3$. Note that when $s_2 \neq s_3$ the last two equations in (2.6) are equivalent to

$$\begin{aligned}
\kappa s_2 s_3 &= -\mu_\Sigma s_1 \cos\alpha,\\
(\lambda+\kappa)(s_2^2+s_3^2) &= m_\Sigma^2 - \lambda s_1^2.
\end{aligned} \tag{5.9}$$

From Eq. (2.4) it follows that

$$s_1^2 = \frac{1}{2\lambda+\kappa}\left(m_\Sigma^2 - (\lambda+\kappa)\frac{\mu_\Sigma^2}{\kappa^2}\cos^2\alpha\right). \tag{5.10}$$

As at most one s_i vanishes, we can take s_1, $s_2 > 0$, so Eq. (2.5) becomes $\mu_\Sigma s_3 \sin\alpha = 0$. The solution with $s_3 = \cos\alpha = 0$ is not allowed because Eqs. (5.9) and (5.10) imply $s_1 = s_2$. The solution with $s_3 > 0$ and $\sin\alpha = 0$ is less obvious, but it also leads to $s_1 = s_2$. Thus, there is no extremum when all three s_i's are different.

VI. ASYMPTOTIC BEHAVIOR

A necessary condition for the existence of a global minimum is that there are no runaway directions at large field values. In other words, $V(\Sigma)$ must have a lower limit as $s_i \to \infty$.

At large field values, where the μ_Σ and m_Σ terms can be neglected, the potential (2.1) has the following asymptotic form:

$$V_\infty = \frac{\lambda}{2}(s_1^2 + s_2^2 + s_3^2)^2 + \frac{\kappa}{2}(s_1^4 + s_2^4 + s_3^4). \qquad (6.1)$$

Hence, in the case where $s_1 = s_2 = s_3 \to \infty$, the condition that $V(\Sigma)$ is bounded from below is $3\lambda + \kappa > 0$ (this was also derived in [9]). A separate necessary condition for $V(\Sigma)$ to be bounded from below is obtained in the case where $s_i \to \infty$ for a single value of i: $\lambda + \kappa > 0$. These two conditions (which agree with those stated in [18] using a different notation) can be combined as follows:

$$\kappa > \max\{-\lambda, -3\lambda\}, \qquad (6.2)$$

which is a necessary condition to have $V(\Sigma)$ bounded from below.

We now prove that (6.2) is also a sufficient condition to have a bounded potential. For $\lambda \geq 0$, the condition becomes $\kappa > -\lambda$ so that

$$\begin{aligned} V_\infty &> \frac{\lambda}{2}(s_1^2 + s_2^2 + s_3^2)^2 - \frac{\lambda}{2}(s_1^4 + s_2^4 + s_3^4) \\ &= \lambda(s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2) \geq 0. \end{aligned} \qquad (6.3)$$

For $\lambda < 0$, condition (6.2) becomes $\kappa > -3\lambda > 0$, which implies

$$\begin{aligned} V_\infty &> -\frac{\kappa}{6}(s_1^2 + s_2^2 + s_3^2)^2 + \frac{\kappa}{2}(s_1^4 + s_2^4 + s_3^4) \\ &= \frac{\kappa}{12}[(2s_1^2 - s_2^2 - s_3^2)^2 + 3(s_2^2 - s_3^2)^2] \geq 0. \end{aligned} \qquad (6.4)$$

Therefore, (6.2) is the sufficient and necessary condition to have $V(\Sigma)$ bounded from below.

VII. GLOBAL MINIMUM

As established in Secs. II–V, the renormalizable potential for a single bifundamental scalar allows only three possible vacua:

$$\begin{aligned} &SU(3)\times SU(3)\ \text{vacuum:}\ s_1 = s_2 = s_3 = 0 \\ &SU(3)\ \text{vacuum:} \\ &\quad s_1 = s_2 = s_3 = \frac{1}{2(3\lambda+\kappa)}\left(\sqrt{4(3\lambda+\kappa)m_\Sigma^2 + \mu_\Sigma^2} + \mu_\Sigma\right), \\ &\quad \alpha = 0 \\ &SU(2)\times SU(2)\times U(1)\ \text{vacuum:} \\ &\quad s_1 = \frac{m_\Sigma}{\sqrt{\lambda+\kappa}}, \qquad s_2 = s_3 = 0. \end{aligned} \qquad (7.1)$$

Let us analyze which of these local minima represents a global minimum of the potential. To this end we need to impose first the condition that $V(\Sigma)$ is bounded from below, namely (6.2). In this case the regions of parameter space where the $SU(3)$-symmetric and $SU(2)\times SU(2)\times U(1)$-symmetric vacua exist, namely (3.13) and (4.4), are simpler.

Three regions of parameter space have a single vacuum:

$$\begin{aligned} &m_\Sigma > 0, \kappa < 0 \quad \text{and} \quad \frac{\mu_\Sigma}{m_\Sigma} < \frac{-\kappa}{\sqrt{3\lambda+2\kappa}} \\ &\qquad \Rightarrow SU(2)\times SU(2)\times U(1)\ \text{vacuum} \\ &m_\Sigma > 0 \quad \text{and} \quad \left\{\kappa \geq 0\ \text{or}\ \frac{\mu_\Sigma}{m_\Sigma} > \frac{-\kappa}{\sqrt{\lambda+\kappa}}\right\} \\ &\qquad \Rightarrow SU(3)\ \text{vacuum} \\ &m_\Sigma^2 < 0 \quad \text{and} \quad \frac{-\mu_\Sigma^2}{m_\Sigma^2(3\lambda+\kappa)} < 4 \\ &\qquad \Rightarrow SU(3)\times SU(3)\ \text{vacuum} \end{aligned} \qquad (7.2)$$

where again we choose $m_\Sigma > 0$ when $m_\Sigma^2 > 0$.

In the other regions there is competition between two vacua. Studying the sign of the potential at the $SU(3)$-symmetric minimum, $V_{\min}^{(3)}$ of Eq. (3.13), we find[2]

$$\begin{aligned} &m_\Sigma^2 < 0 \quad \text{and} \quad \frac{9}{2} < \frac{-\mu_\Sigma^2}{m_\Sigma^2(3\lambda+\kappa)} \\ &\qquad \Rightarrow \begin{cases} SU(3) & \text{global min.} \\ SU(3)\times SU(3) & \text{local min.} \end{cases} \\ &m_\Sigma^2 < 0 \quad \text{and} \quad 4 < \frac{-\mu_\Sigma^2}{m_\Sigma^2(3\lambda+\kappa)} < \frac{9}{2} \\ &\qquad \Rightarrow \begin{cases} SU(3)\times SU(3) & \text{global min.} \\ SU(3) & \text{local min.} \end{cases} \end{aligned} \qquad (7.3)$$

[2]This result agrees with the one derived in Appendix A of Ref. [9], namely the $SU(3)$ global minimum at $r_\Delta < 3/2$ in the notation used there. The competition between the $SU(2)\times SU(2)\times U(1)$-symmetric minimum and the $SU(3)$-symmetric minimum is not discussed in Ref. [9].

For the remaining region of parameter space,

$$m_\Sigma > 0, \kappa < 0 \ \text{ and } \ \frac{-\kappa}{\sqrt{3\lambda+2\kappa}} < \frac{\mu_\Sigma}{m_\Sigma} < \frac{-\kappa}{\sqrt{\lambda+\kappa}}, \qquad (7.4)$$

there is competition between the $SU(3)$ and $SU(2)\times SU(2)\times U(1)$ local minima. We need to compare the values of the potential at these minima, which are given in Eqs. (3.13) and (4.6). The $SU(3)$ minimum is deeper, $V^{(3)}_{\rm min} < V^{(2,2,1)}_{\rm min}$, if and only if[3]

$$\frac{\mu_\Sigma}{m_\Sigma} > \left(\frac{(4\lambda+2\kappa)^{3/2}}{\sqrt{\lambda+\kappa}} - 2(4\lambda+\kappa)\right)^{1/2} \equiv \xi(\lambda,\kappa). \qquad (7.5)$$

One can check that the function defined above, $\xi(\lambda,\kappa)$, is real and positive in this region of parameter space. As a result, we find the following possible vacua:

$$m_\Sigma > 0, \kappa < 0 \quad \text{and} \quad \xi(\lambda,\kappa) < \frac{\mu_\Sigma}{m_\Sigma} < \frac{-\kappa}{\sqrt{\lambda+\kappa}}$$
$$\Rightarrow \begin{cases} SU(3) & \text{global min.} \\ SU(2)\times SU(2)\times U(1) & \text{local min.} \end{cases}$$
$$m_\Sigma > 0, \kappa < 0 \quad \text{and} \quad \frac{-\kappa}{\sqrt{3\lambda+2\kappa}} < \frac{\mu_\Sigma}{m_\Sigma} < \xi(\lambda,\kappa)$$
$$\Rightarrow \begin{cases} SU(2)\times SU(2)\times U(1) & \text{global min.} \\ SU(3) & \text{local min.} \end{cases} \qquad (7.6)$$

The phase diagram of this model, based on Eqs. (7.2), (7.3) and (7.6), is shown in Fig. 1 in the $\lambda^{-1/2}\mu_\Sigma/m_\Sigma$ versus κ/λ plane, for $m_\Sigma > 0$ and $\lambda > 0$. Note that for $\lambda > 0$ the lower limit $\kappa/\lambda > -1$ is required in order to have the potential bounded from below, while there is no upper limit on κ/λ at tree level.

The region where the global minimum is $SU(2)\times SU(2)\times U(1)$-symmetric lies below the solid blue line in Fig. 1, which is given by the function $\xi(\lambda,\kappa)/\sqrt{\lambda}$ [see Eq. (7.5)]. In the region above or to the right of that line, the global minimum is $SU(3)$ symmetric.

A change of parameters that crosses the boundary between these two regions represents a first-order phase transition: both local minima exist for parameter points between the blue dashed line and the red dotted line of Fig. 1. In between these two minima there is a shallow saddle point, of coordinates given in (5.3), which is $SU(2)\times U(1)$ symmetric. In Fig. 2 we show the potential for a point ($\mu_\Sigma/m_\Sigma = 0.2$, $\kappa = -0.21$, $\lambda = 1$) from the $\xi(\lambda,\kappa)/\sqrt{\lambda}$ curve, where the $SU(3)$-symmetric vacuum and the $SU(2)\times SU(2)\times U(1)$-symmetric vacuum have the same depth and are global minima. The shallowness of the potential around both minima is related to the smallness of $|\mu_\Sigma/m_\Sigma|$ and $|\kappa/\lambda|$. The mass of the "angular mode" is parametrically smaller than the "radial mode."

[3]This inequality has also been obtained in [18], without proving that the Σ field configurations $\mathrm{diag}(u_s,u_s,u_s)$ and $\mathrm{diag}(u_t,0,0)$ are minima of the full tree-level potential, or that other minima do not exist. An argument about the absence of other minima is given in [19] based on a conjecture [22] (using a method [23] to identify extrema). The conditions for symmetry breaking in the case $m_\Sigma^2 < 0$ have not been discussed in [18,19].

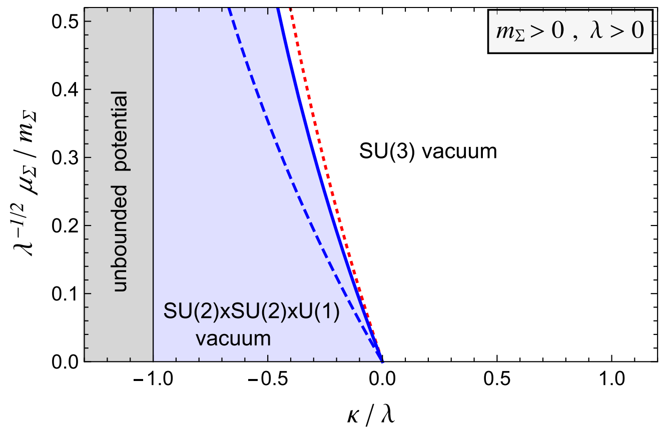

FIG. 1. Phase diagram of the $SU(3)\times SU(3)$ model with a scalar bifundamental, for $m_\Sigma^2 > 0$ and $\lambda > 0$, in the plane of $\mu_\Sigma/(m_\Sigma\sqrt{\lambda})$ versus the ratio of quartic couplings κ/λ. The global minimum is $SU(2)\times SU(2)\times U(1)$-symmetric in the blue shaded region, and $SU(3)$-symmetric in the unshaded region. Between the dashed blue line and the solid blue line there is also an $SU(3)$-symmetric local minimum, while between the dotted red line and the solid blue line there is also an $SU(2)\times SU(2)\times U(1)$-symmetric local minimum. In the gray-shaded region at $\kappa/\lambda < -1$ the potential is not bounded from below.

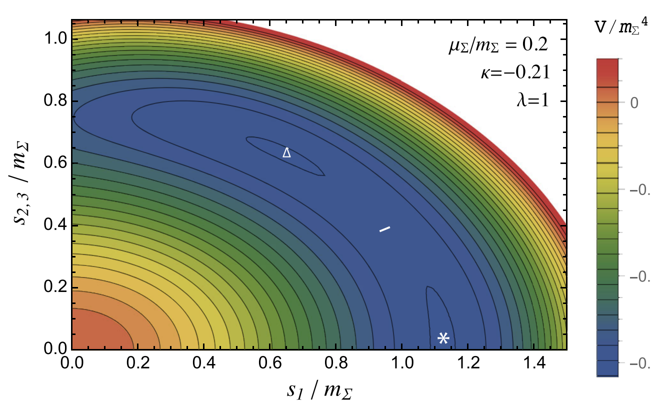

FIG. 2. Contours of the $V(\Sigma)$ potential in the (s_1/m_Σ, s_2/m_Σ) plane, along the $s_2 = s_3$ and $\alpha = 0$ direction. The depth of the potential is encoded in the colors: from dark blue representing the deepest potential, to bright red representing the highest. The potential is computed at a point in parameter space ($\mu_\Sigma/m_\Sigma = 0.2$, $\kappa = -0.21$, $\lambda = 1$) chosen such that the $SU(3)$-symmetric minimum (marked by a white $\triangle$) and the $SU(2)\times SU(2)\times U(1)$-symmetric minimum (marked by a white $*$) have equal depths, of $-0.632 m_\Sigma^4$. The saddle point between the two minima (marked by a white tilted line) has an $SU(2)\times U(1)$ symmetry and a depth of $-0.627 m_\Sigma^4$.

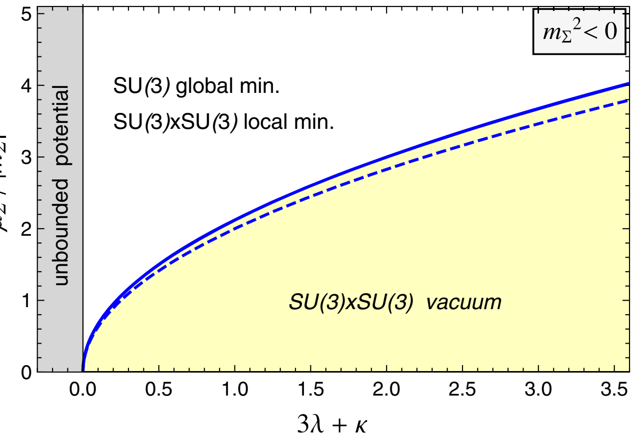

FIG. 3. Phase diagram for $m_\Sigma^2 < 0$, in the plane of $\mu_\Sigma/|m_\Sigma|$ versus $3\lambda + \kappa$. The global minimum is $SU(3)$ symmetric in the unshaded region and $SU(3) \times SU(3)$ symmetric in the yellow shaded region. Between the dashed blue line and the solid blue line there is also an $SU(3)$-symmetric local minimum, while in the unshaded region there is also an $SU(3) \times SU(3)$-symmetric local minimum. In the gray-shaded region at $3\lambda + \kappa < 0$ the potential is not bounded from below.

The region where $m_\Sigma > 0$ and $\lambda < 0$ has only the $SU(3)$-symmetric vacuum. In the phase diagram for $m_\Sigma^2 < 0$, shown in Fig. 3, there is competition between the $SU(3) \times SU(3)$ vacuum and the $SU(3)$ vacuum, as described by the inequalities (7.2) and (7.3). On the boundary between the two regions defined in (7.3), given by the solid blue line in Fig. 3, the two minima are degenerate. The saddle point that separates these two global minima corresponds to the negative-sign solution of Eq. (3.3).

In Fig. 4, we show the potential for a point with $m_\Sigma^2 < 0$, located on the boundary at $\mu_\Sigma/|m_\Sigma| = 1.8$, $\kappa = 0.12$, $\lambda = 0.2$, where the depth of the potential is the same at the two minima ($V = 0$), and at the saddle point it is given by $V = 0.52 m_\Sigma^4$.

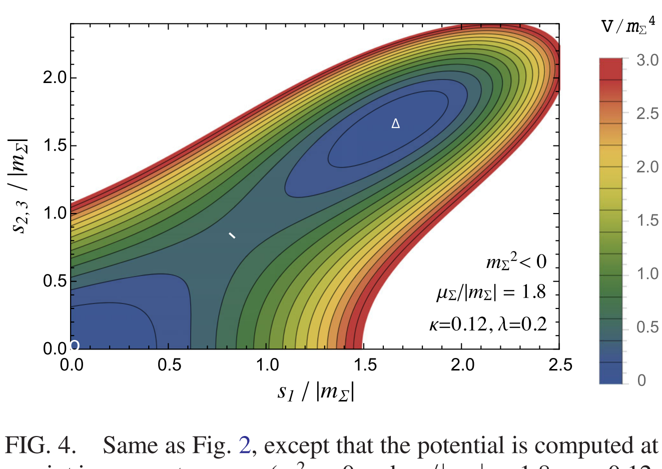

FIG. 4. Same as Fig. 2, except that the potential is computed at a point in parameter space ($m_\Sigma^2 < 0$ and $\mu_\Sigma/|m_\Sigma| = 1.8$, $\kappa = 0.12$, $\lambda = 0.2$) chosen such that the $SU(3)$-symmetric minimum (marked by a white $\triangle$) and the $SU(3) \times SU(3)$-symmetric minimum (marked by a white $\circ$) have equal depths. The saddle point between the two minima (marked by a white tilted line) has the same $SU(3)$ symmetry, and it corresponds to the negative-sign solution of Eq. (3.3).

It is interesting to study the mass spectra of heavy scalars at different points on the phase diagrams. As the parameters are changed such that an $SU(3)$-symmetric vacuum approaches the blue dashed line of Fig. 1, the octet scalar becomes lighter [see Eq. (3.7)], until it vanishes on that line. Similarly, as an $SU(2) \times SU(2) \times U(1)$-symmetric vacuum is moved to the right or up on Fig. 1, the mass of the lighter bifundamental scalar decreases [see Eq. (4.8)], until it vanishes on the red dotted line. Finally, as an $SU(3)$-symmetric vacuum is moved down or to the right on Fig. 3, one of the singlet scalars becomes lighter [see Eq. (3.8)], until it vanishes on the blue dashed line.

Note that the inequalities (7.2), (7.3) and (7.6) do not explicitly refer to the cases where some parameters vanish. The reason for that is that the analysis in those cases becomes sensitive to loop corrections.[4] For example, $\lambda = 0$ at tree level makes the vertical axis ill defined in Fig. 1, but one-loop corrections would generate a nonzero λ. Likewise, $m_\Sigma = 0$ is not stable against loops. By contrast, the $\mu_\Sigma = 0$ limit is protected by a global $U(1)_\Sigma$ symmetry, as discussed in Sec. II.

For some regions of the phase diagrams (Figs. 1 and 3), the one-loop effective potential could change the quantitative, or even qualitative, features of the locations of different symmetries. The corrections from the one-loop effective potential become especially important when there are light pseudo-NGBs in the mass spectrum. In analogy to the analysis performed in grand unified models [26], there is a color-octet NGB in the limit of $\kappa \to 0$ and $\mu_\Sigma \to 0$. The properties of this color octet will be influenced by the loop corrections to the potential induced by its gauge interactions. The actual phase diagram around the origin of Fig. 1 will likely be modified. Similarly, along the dashed blue and dotted red lines, there are massless scalars, whose properties are sensitive to loop corrections. Computing the effect of loop corrections on the locations of the dashed blue and dotted red lines in these phase diagrams is beyond the scope of this paper. Nevertheless, the bulk behaviors of the phase diagrams obtained from the tree-level analysis should be overall reliable.

Lastly, we emphasize that although the global minimum of the potential will eventually be the vacuum, the universe might be stuck for a while in the shallower local minimum. Thus, a local minimum may be a viable vacuum provided that it is longer lived than the age of the universe, and that the thermal history allows the universe to settle in it.

[4]The computation of the one-loop effective potential is a mature subject (see, e.g., [24]), and even the three-loop effective potential has been recently computed for a general renormalizable theory [25].

VIII. CONCLUSIONS

We have analyzed the vacuum structure of an $SU(3) \times SU(3)$-symmetric renormalizable theory with a bifundamental scalar field. The parameter space is four dimensional, with two quartic couplings and two mass parameters. One of the latter, which is the coefficient of a cubic term in the potential, is not present in $SU(N) \times SU(N)$-symmetric theories for $N \neq 3$.

There are three possible types of vacua, with different symmetry properties: $SU(3)$, $SU(2) \times SU(2) \times U(1)$ and $SU(3) \times SU(3)$. Depending on which of these is a global minimum, and whether there are also some local minima, the parameter space is divided into seven regions. These are described by Eqs. (7.2), (7.3) and (7.6). Remarkably, the phase diagram of the theory can be fully displayed in two-dimensional plots, namely Figs. 1 and 3. We have also computed the mass spectrum of physical spin-0 particles in the different vacua and identified the scalars whose masses vanish on the boundaries of various regions on the phase diagram.

The cubic term in the potential, of coefficient $\mu_\Sigma > 0$, plays an important role in the selection of the possible vacua. Even when the bifundamental scalar has a positive squared mass, i.e., $m_\Sigma^2 < 0$ in the notation of Eq. (2.1), a nontrivial VEV is developed for μ_Σ above a coupling-dependent value (see Fig. 3), breaking the symmetry down to $SU(3)$. For a negative squared mass (or equivalently $m_\Sigma > 0$), as μ_Σ increases, the region with an $SU(3)$-symmetric vacuum is enlarged, while the region with an $SU(2) \times SU(2) \times U(1)$-symmetric vacuum is reduced (see Fig. 1).

The vacuum structure of this theory is useful for various model building applications, including in the contexts of the ReCoM [8,9] and trinification [15–17], fermion mass hierarchies [18,19], or chiral symmetry breaking in strongly coupled gauge theories [27]. In addition, it opens new possibilities for nonstandard cosmology, such as color breaking in the early universe followed by color restoration at a lower temperature [28]. In particular, the presence of two minima of different symmetry properties, which for a range of parameters are nearly degenerate and separated by a shallow saddle point (see Fig. 2), may lead to exotic cosmological or astrophysical phenomena.

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