Baryon chiral perturbation theory combined with the $1/N_c$ expansion in SU(3): Framework

I. P. Fernando[1,2,*] and J. L. Goity[1,2,†]

[1]Department of Physics, Hampton University, Hampton, Virginia 23668, USA
[2]Thomas Jefferson National Accelerator Facility, Newport News, Virginia 23606, USA

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Baryon chiral perturbation theory combined with the $1/N_c$ expansion is implemented for three flavors. Baryon masses, vector charges and axial vector couplings are studied to one-loop and organized according to the ξ-expansion, in which the $1/N_c$ and the low-energy power countings are linked according to $1/N_c = \mathcal{O}(\xi) = \mathcal{O}(p)$. The renormalization to $\mathcal{O}(\xi^3)$ necessary for the mentioned observables is provided, along with applications to the baryon masses and axial couplings as obtained in lattice QCD calculations.

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I. INTRODUCTION

The low-energy effective theory for baryons is a recurrent topic in low-energy QCD, which has evolved through different approaches and improvements. The original version of baryon chiral perturbation theory (ChPT) [1] gave rise to different versions of baryon effective field theories based on effective chiral Lagrangians [2–4], starting with the relativistic version [5,6] or baryon ChPT (BChPT), followed by the nonrelativistic version based in an expansion in the inverse baryon mass [7–10] or heavy baryon ChPT (HBChPT), and by manifestly Lorentz covariant versions based on the IR regularization scheme [11–13], which allow for an explicit implementation of the low-energy power counting. In all those versions of the baryon effective theory a consistent low-energy expansion can be implemented. A key issue, which became apparent quite early, was the convergence of the low-energy expansion. Being an expansion that progresses in steps of $\mathcal{O}(p)$, in contrast to the expansion in the pure Goldstone boson sector where the steps are $\mathcal{O}(p^2)$, it is natural to expect a slower rate of convergence. However, a key factor affecting the convergence has to do with the relatively small mass gap between the spin 1/2 and 3/2 baryons. In the context of BChPT, it was realized in [14] that the inclusion of the spin 3/2 degrees of freedom improves the convergence of the one-loop contributions to certain observables such as the π-N scattering amplitude and the axial currents and magnetic moments. There have been since then numerous works including spin 3/2 baryons [15–24]. The explanation of those improvements was obtained through the study of baryons in the large N_c limit of QCD [25], where in that limit a dynamical spin-flavor symmetry emerges [26–29], which requires the inclusion of the higher spin baryons in the effective theory and leads to a better behaved low-energy expansion. In the large N_c limit, baryons behave very differently than mesons [30], in particular because their masses scale like $\mathcal{O}(N_c)$ (they are the heavy sector of QCD) and the π-baryon couplings are $\mathcal{O}(\sqrt{N_c})$. Those properties were shown to demand, for consistency with π-baryon scattering at large N_c, that at large N_c baryons must respect the mentioned dynamical contracted spin-flavor symmetry $SU(2N_f)$, N_f being the number of light flavors [26–29], which is broken by effects ordered in powers of $1/N_c$ and in powers of the quark mass differences. The inclusion of the consistency requirements of the large N_c limit into the effective theory came naturally through a combination of the $1/N_c$ expansion and HBChPT [31], which is the framework followed in the present work. The study of one-loop corrections in that framework was first carried out in Refs. [31–33] and more recently in [34,35]. In the combined theory, the $1/N_c$ and chiral expansions do not commute [36]: the reason is the baryon mass splitting scale of $\mathcal{O}(1/N_c)$ ($\Delta - N$ mass difference), for which it becomes necessary to specify its order in terms of the low-energy expansion. Thus the $1/N_c$ and chiral expansions must be linked. Particular emphasis will be given to the specific linking in which the baryon mass splitting is taken to be $\mathcal{O}(p)$ in the chiral expansion, and which will be called the ξ-expansion. Following Refs. [31–34], in the present work the framework for HBChPT $\times\, 1/N_c$ is extended to three flavors. The renormalization necessary for the baryon masses, and the vector charges and axial-vector currents is implemented to one-loop, i.e., $\mathcal{O}(\xi^3)$. As it had been done in the case of two flavors [34], the present work gives all results at generic values of N_c, i.e., all formulas presented have been derived for general N_c, and therefore detailed analyses of N_c dependencies can be carried out.

*ishara@jlab.org
†goity@jlab.org

The significant progress in lattice QCD (LQCD) calculations of baryon observables [37–39] provides opportunities for further testing and understanding low-energy effective theories of baryons, which in turn can serve to understand the LQCD results themselves. The determination of the quark mass dependence of the various low-energy observables, such as masses, axial couplings, magnetic moments, electromagnetic polarizabilities, etc., are of key importance for testing the effective theory, in particular its range of validity in quark masses, as well as for the determination of its low-energy constants (LECs). Lattice results for N and Δ as well as hyperon masses [40–48] (results of the last reference are used in the present work), the axial coupling g_A of the nucleon [49–54] and a subset of the axial couplings of the octet and decuplet baryons [55] at varying quark masses can be analyzed with the effective theory, as presented in this work.

This work is organized as follows. In Sec. II, the framework for the combined $1/N_c$ and HBChPT expansions is described. Section III presents the evaluation of the baryon masses to $\mathcal{O}(\xi^3)$, Sec. IV presents the corrections to the vector charges, and Sec. V the corrections to the axial couplings. In both Secs. III and V, applications to LQCD results are presented. Finally, a summary is given in Sec. VI. Several appendixes present useful material needed in the calculations, namely, Appendix A on spin-flavor algebra, Appendix B on tools to build the chiral Lagrangians, Appendix C on the one-loop integrals, and Appendix D on reduction formulas of composite operators.

II. COMBINED BARYON CHIRAL PERTURBATION THEORY AND 1/N$_c$ EXPANSION FOR THREE FLAVORS

In this section, the framework for the combined $1/N_c$ and chiral expansions in baryons is presented in some detail along similar lines as in the original works [31–33] and the more recent work [34,35]. The symmetries that constrain the effective Lagrangian in the chiral and large N_c limits are chiral $SU_L(N_f) \times SU_R(N_f)$, which is a Noether symmetry, and contracted dynamical spin-flavor symmetry $SU(2N_f)$ [26–29].[1] N_f is the number of light flavors, where in this work $N_f = 3$. In the limit $N_c \to \infty$, the spin-flavor symmetry requires baryon states to fill degenerate multiplets of $SU(6)$. In particular, the ground state (GS) baryons belong into a symmetric $SU(6)$ multiplet. At finite N_c the spin-flavor symmetry is broken by effects suppressed by powers of $1/N_c$, and the mass splittings in the GS multiplet between the states with spins $S+1$ and S are proportional to $(S+1)/N_c$. The effects of finite N_c are then implemented as an expansion in $1/N_c$ in the effective Lagrangian. Because baryon masses are proportional to N_c, it becomes natural to use the framework of HBChPT [7,56], where the expansion in inverse powers of the baryon mass becomes part of the $1/N_c$ expansion. The framework used here follows that of Refs. [31,32,34].

The dynamical contracted $SU(2N_f)$ symmetry results from the requirement of large N_c consistency of baryon observables [26–29],[2] in particular the requirement that the Born contribution to the Goldstone boson-baryon (GB-baryon) scattering amplitude be finite as $N_c \to \infty$. The constraint emerges because the GB-baryon coupling is $\mathcal{O}(\sqrt{N_c})$, and therefore cancellations between crossed diagrams must occur. The 35 generators of $SU(6)$ and their commutation relations are the following:

$$\begin{aligned}
&S^i:\ \mathrm{SU}(2)\ \text{spin generators},\\
&T^a:\ \mathrm{SU}(3)\ \text{flavor generators},\\
&G^{ia}:\ \text{spin-flavor generators}\\
&[S^i, S^j] = i\epsilon^{ijk} S^k\\
&[T^a, T^b] = i f^{abc} T^c\\
&[S^i, T^a] = 0,\quad [S^i, G^{ja}] = i\epsilon^{ijk} G^{ka},\quad [T^a, G^{ib}] = i f^{abc} G^{ic}\\
&[G^{ia}, G^{jb}] = \frac{i}{4}\delta^{ij} f^{abc} T^c + \frac{i}{6}\delta^{ab}\epsilon^{ijk} S^k + \frac{i}{2}\epsilon^{ijk} d^{abc} G^{kc}.
\end{aligned} \tag{1}$$

The generators G^{ia} have coherent matrix elements, i.e., matrix elements that scale as N_c between baryons of spin $S = \mathcal{O}(N_c^0)$. These generators are the ones that represent the spatial components of axial-vector currents at the leading order in the $1/N_c$ expansion. A contracted $SU(6)$ symmetry, which is the actual dynamical symmetry in large N_c, is generated by the Algebra where G^{ia} is replaced by $X^{ia} \equiv G^{ia}/N_c$. The ground state baryons belong to the totally symmetric spin-flavor irreducible representation with N_c spin-flavor indices, and consist of states with spin $S = 1/2, \ldots, N_c/2$ (assuming N_c to be odd). For a given spin S, the corresponding $SU(3)$ multiplet is $(p, q) = (2S, \frac{1}{2}(N_c - 2S))$ in the usual Young tableau notation. For $N_c = 3$, the states are the physical $S = 1/2$ octet and $S = 3/2$ decuplet.

In HBChPT, the baryon field, denoted by $\mathbf{B}$, represents the spin-flavor multiplet where its components are sorted out by spin and flavor, that is, the entries in $\mathbf{B}$ have well defined spin, and therefore they are in irreducible representations of $SU(3)$.

Implementing chiral symmetry follows the well known scheme of the nonlinear realization on the matter fields. Representing the Goldstone boson octet by

$$u = e^{i\pi^a T^a / F_\pi}, \tag{2}$$

the nonlinear transformation law is implemented,

[1] See also Appendix A.

[2] See also Appendix A.

$$Ruh^\dagger(L,R,u) = h(L,R,u)uL^\dagger, \tag{3}$$

where L (R) is a transformation of $SU_L(3)$ ($SU_R(3)$). $h(L,R,u)$ is then a $SU(3)$ flavor transformation. One can, therefore, define the usual chiral transformations on the baryon fields according to

$$(L,R)\colon\ \mathbf{B} = h(L,R,u)\mathbf{B}, \tag{4}$$

where obviously the nonlinear transformation h acts on the different components of $\mathbf{B}$ with the corresponding $SU(3)$ irreducible representation. Chiral transformations do not commute with $SU(6)$, but they leave the commutation relations unchanged. The chiral covariant derivative $D_\mu\mathbf{B}$ is then given by

$$\begin{aligned} D_\mu\mathbf{B} &= \partial_\mu\mathbf{B} - i\Gamma_\mu\mathbf{B},\\ \Gamma_\mu &= \frac{1}{2}(u^\dagger(i\partial_\mu + r_\mu)u + u(i\partial_\mu + l_\mu)u^\dagger), \end{aligned} \tag{5}$$

where $l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$ are gauge sources. Another building block is the axial Maurer-Cartan one-form:

$$\begin{aligned} u_\mu &= u^\dagger(i\partial_\mu + r_\mu)u - u(i\partial_\mu + l_\mu)u^\dagger,\\ (L,R)\colon\ u_\mu &= h(L,R,u)u_\mu h^\dagger(L,R,u). \end{aligned} \tag{6}$$

Both Γ_μ and u_μ belong to the $SU(3)$ Algebra, and are written in the general form $X = X^aT^a$. When acting on the different components of the field $\mathbf{B}$, T^a is obviously taken in the corresponding $SU(3)$ irreducible representation.

The scalar and pseudoscalar densities are collected into

$$\begin{aligned} \chi &= 2B_0(s + ip)\\ \chi_\pm &\equiv u^\dagger\chi u^\dagger \pm u\chi^\dagger u\\ \chi_\pm^0 &= \langle\chi_\pm\rangle\\ \tilde{\chi}_\pm &\equiv \chi_\pm^a T^a, \end{aligned} \tag{7}$$

where s and p are the scalar and pseudoscalar sources, and eventually s is set to be the quark mass matrix.

The field strengths associated with the gauge sources are

$$\begin{aligned} F_L^{\mu\nu} &= \partial^\mu\ell^\nu - \partial^\nu\ell^\mu - i[\ell^\mu,\ell^\nu],\\ F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]\\ F_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu\dagger}u. \end{aligned} \tag{8}$$

Since contracted $SU(6)$ is not a Noether symmetry, its role in the effective Lagrangian is to primarily constrain couplings. For instance, at the leading order, one such constraint is that the GB-baryon couplings are determined by a single coupling $\mathring{g}_A$. The effective Lagrangian will be explicitly invariant under rotations and chiral transformations and the QCD discrete symmetries P and T. The Lagrangian consists of terms which are the product of tensors containing the GB and source fields (chiral tensor operators) with terms which are composite spin-flavor tensor operators built with products of $SU(6)$ generators. The N_c power assigned to a term in the Lagrangian is determined by the spin-flavor operator according to N_c^{1-n}, where n is the number of factors of $SU(6)$ generators involved in the operator. In general, the chiral tensor operators carry hidden N_c dependencies through the factors of $1/F_\pi$ accompanying the GB field operators, where $F_\pi = \mathcal{O}(\sqrt{N_c})$. Matrix elements of the spin-flavor operators carry additional N_c dependencies, as is the case of operators where factors of the generators G^{ia} appear, which lead to additional factors of N_c in the matrix elements. Following this approach, the Lagrangian terms are organized in powers of the chiral and $1/N_c$ expansions. The $1/N_c$ expansion naturally leads to the HBChPT expansion, as the large mass of the expansion is taken to be the spin-flavor singlet component of the baryon masses, namely $M_0 = N_c m_0$ (m_0 can be considered here to be a LEC defined in the chiral limit and which will have itself an expansion in $1/N_c$).

Bases of spin-flavor tensor operators are built using the tools in Appendix A, and requires in general lengthy algebraic work. In the Appendix, only the bases needed in this work are provided.

In order to ensure the validity of the OZI rule for the quark mass dependency of baryon masses, namely, that the nonstrange baryon mass dependence on m_s is $\mathcal{O}(N_c^0)$, the following combination of the source χ_+ is defined:

$$\hat{\chi}_+ \equiv \tilde{\chi}_+ + N_c\chi_+^0, \tag{9}$$

which is $\mathcal{O}(N_c)$ but has dependence on m_s which is $\mathcal{O}(N_c^0)$ for al states where the strangeness is $\mathcal{O}(N_c^0)$.

For convenience, a scale Λ is introduced, which can be chosen to be a typical QCD scale, in order to render most of the LECs dimensionless. In the calculations, $\Lambda = m_\rho$ will be chosen.

The lowest-order Lagrangian is [31]

$$\mathcal{L}_{\mathbf{B}}^{(1)} = \mathbf{B}^\dagger\left(iD_0 + \mathring{g}_A u^{ia}G^{ia} - \frac{C_{\mathrm{HF}}}{N_c}\hat{S}^2 + \frac{c_1}{2\Lambda}\hat{\chi}_+\right)\mathbf{B}. \tag{10}$$

The kinetic term is $\mathcal{O}(pN_c^0)$, and the terms involving GBs (when the vector and axial vector sources are turned off) start with the Weinberg-Tomozawa term which is $\mathcal{O}(p/N_c)$. The second term gives in particular the axial vector current and the GB-baryon interaction. $\mathring{g}_A$ is the axial coupling in the chiral and large N_c limits (it has to be rescaled by a factor 5/6 to coincide with the usual axial coupling as defined for the nucleon, i.e., $g_A^N = g_A = \frac{5}{6}\mathring{g}_A$). Because the matrix elements of G^{ia} are $\mathcal{O}(N_c)$, the

GB-baryon coupling is $\mathcal{O}(\sqrt{N_c})$. This strong coupling at large N_c demands the constraints of $SU(6)$, which will allow for N_c consistency at higher orders in the effective theory. The third term gives the $SU(3)$ singlet mass splittings between baryons of different spins, and it is $\mathcal{O}(p^0/N_c)$. The fourth term gives the contributions of quark masses to the baryon masses, it is $\mathcal{O}(p^2 N_c)$ and gives $SU(3)$ breaking effects which are $\mathcal{O}(p^2 N_c^0)$. This indicates a first issue with the interchange of chiral and large N_c limits. As it becomes evident at the NLO due to the nonanalytic terms of loop corrections, the limits do not commute, and for that reason it becomes necessary to make a choice: the choice made here is that $1/N_c$ is counted as a quantity of order p: $1/N_c = \mathcal{O}(p) = \mathcal{O}(\xi)$, which is coined as the ξ expansion. The Lagrangian is now organized in powers of ξ. If the N_c dependencies of the matrix elements of the spin-flavor operators are disregarded, $\mathcal{L}_{\mathbf{B}}^{(1)}$ is $\mathcal{O}(\xi)$.

The construction of higher-order Lagrangians is accomplished making use of the tools provided in Appendixes A and B. In this work, the Lagrangians of $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^3)$ are needed. Throughout, the spin-flavor operators appearing in the effective Lagrangians will be scaled by the appropriate powers of $1/N_c$ in such a way that all LECs are of zeroth order in N_c. The $1/N_c$ power of a Lagrangian term with n_π pion fields is given by [57], $n - 1 - \kappa + \frac{n_\pi}{2}$, where the spin-flavor operator is n-body (n is the number of factors of $SU(6)$ generators appearing in the operator), and κ takes into account the N_c dependency of the spin-flavor matrix elements. The last term, $n_\pi/2$, stems from the factor $(1/F_\pi)^{n_\pi}$ carried by any term with n_π GB fields.

For convenience, the following definitions are used:

$$\delta\hat{m} \equiv \frac{C_{\rm HF}}{N_c}\hat{S}^2 - \frac{c_1}{2\Lambda}\hat{\chi}_+$$
$$i\tilde{D}_0 \equiv iD_0 - \delta\hat{m}. \tag{11}$$

Note that $\delta\hat{m}$ gives rise to mass splittings between baryons which are $\mathcal{O}(1/N_c)$ or $\mathcal{O}(p^2)$.

With this, the $\mathcal{O}(\xi^2)$ Lagrangian is given by[3]:

$$\begin{aligned}\mathcal{L}_{\mathbf{B}}^{(2)} = \mathbf{B}^\dagger\Bigg(&\left(-\frac{1}{2N_c m_0}+\frac{w_1}{\Lambda}\right)\vec{D}^2+\left(\frac{1}{2N_c m_0}-\frac{w_2}{\Lambda}\right)\tilde{D}_0^2+\frac{c_2}{\Lambda}\chi_+^0\\
&+\frac{C_1^A}{N_c}u^{ia}S^iT^a+\frac{C_2^A}{N_c}\epsilon^{ijk}u^{ia}\{S^j,G^{ka}\}\\
&+\kappa_0\epsilon^{ijk}F_{+ij}^0S^k+\kappa_1\epsilon^{ijk}F_{+ij}^aG^{ka}+\rho_0F_{-0i}^0S^i+\rho_1F_{-0i}^aG^{ia}\\
&+\frac{\tau_1}{N_c}u_0^aG^{ia}D_i+\frac{\tau_2}{N_c^2}u_0^aS^iT^aD_i+\frac{\tau_3}{N_c}\nabla_iu_0^aS^iT^a+\tau_4\nabla_iu_0^aG^{ia}+\cdots\Bigg)\mathbf{B},\end{aligned} \tag{12}$$

where additional terms not explicitly displayed are not needed in the present work. Note that there are also $\mathcal{O}(\xi^2)$ terms stemming from the $1/N_c$ suppressed terms in the LECs of the lower-order Lagrangian. Similar comments apply to the higher-order Lagrangians. Such terms require knowledge of the physics at $N_c > 3$ to be determined, which can in principle be obtained using LQCD results at varying N_c [58,59].

Similarly, the $\mathcal{O}(\xi^3)$ Lagrangian needed here is given by

$$\begin{aligned}\mathcal{L}_{\mathbf{B}}^{(3)} = \mathbf{B}^\dagger\Bigg(&\frac{c_3}{N_c\Lambda^3}\hat{\chi}_+^2+\frac{h_1\Lambda}{N_c^3}\hat{S}^4+\frac{h_2}{N_c^2\Lambda}\hat{\chi}_+\hat{S}^2+\frac{h_3}{N_c\Lambda}\chi_+^0\hat{S}^2+\frac{h_4}{N_c\Lambda}\chi_+^a\{S^i,G^{ia}\}\\
&+\frac{C_3^A}{N_c^2}u^{ia}\{\hat{S}^2,G^{ia}\}+\frac{C_4^A}{N_c^2}u^{ia}S^iS^jG^{ja}\\
&+\frac{D_1^A}{\Lambda^2}\chi_+^0u^{ia}G^{ia}+\frac{D_2^A}{\Lambda^2}\chi_+^au^{ia}S^i+\frac{D_3^A(d)}{\Lambda^2}d^{abc}\chi_+^au^{ib}G^{ic}+\frac{D_3^A(f)}{\Lambda^2}f^{abc}\chi_+^au^{ib}G^{ic}\\
&+g_E[D_i,F_{+i0}]+\alpha_1\frac{i}{N_c}\epsilon^{ijk}F_{+0i}^aG^{ia}D_k+\beta_1\frac{i}{N_c}F_{-ij}^aG^{ia}D_j+\cdots\Bigg)\mathbf{B}\end{aligned} \tag{13}$$

[3]The notation for the LECs used here differs from the ones used in ordinary BChPT due to the unification of terms demanded by the $1/N_c$ expansion. The notation aims at distinguishing classes of terms in the Lagrangian, e.g., spin-independent mass terms, spin-dependent mass terms, axial-vector couplings, etc. The identification of some of the LECs with those used in ordinary versions of BChPT are straightforward.

In this work, some terms $\mathcal{O}(\xi^4)$ are needed for subtracting UV divergencies, but they are beyond the order of the present calculations and can be consistently eliminated. Through the calculation of the one-loop corrections to the self energies and the vector and axial vector currents, the β functions associated with the LECs that affect those quantities are determined.

The terms in the effective Lagrangian are constrained in their N_c dependence by the requirement of the consistency of QCD at large N_c. This constraint is in the form of a lower bound in the power in $1/N_c$ for each term in the Lagrangian. This leads in particular to constraints on the N_c dependencies of the ultra-violet (UV) divergencies of loop corrections, which have to be subtracted by the corresponding counterterms in the Lagrangian. The UV divergencies are necessarily polynomials in low momenta p (derivatives), in $\chi_\pm$ and other sources, and in $1/N_c$ (modulo factors of $1/\sqrt{N_c}$ due to $1/F_\pi$ factors in terms where GBs are attached). Therefore, the structure of counterterms is independent of any linking between the $1/N_c$ and chiral expansions. For this reason, in order to determine the UV divergencies, the large N_c and low-energy limits can be taken independently. For a connected diagram with n_B external baryon legs, n_π external GB legs, n_i vertices of type i which have n_{B_i} baryon legs and n_{π_i} GB legs, and L loops, the following topological relations hold [60,61]:

$$L = 1 + I_\pi + I_B - \sum n_i,$$
$$2I_B + n_B = \sum n_i n_{B_i}, \qquad 2I_\pi + n_\pi = \sum n_i n_{\pi_i}, \qquad (14)$$

where I_π is the number of GB propagators and I_B the number of baryon propagators.

The chiral or low-energy order of a diagram, where ν_{p_i} is the chiral power of the vertex of type i, is then given by [61]:

$$\nu_p = 2 - \frac{n_B}{2} + 2L + \sum_i n_i \left(\nu_{p_i} + \frac{n_{B_i}}{2} - 2\right), \qquad (15)$$

Note that n_{B_i} is equal to 0 or 2 in the single baryon sector.

On the other hand, the $1/N_c$ power of a connected diagram is determined by looking only at the vertices: the order in $1/N_c$ of a vertex of type i is given by $\nu_{O_i} + \frac{n_{\pi_i}}{2}$, where ν_{O_i} is the order of the spin-flavor operator. Thus, the $1/N_c$ power of a diagram, upon use of the third Eq. (14), is given by

$$\nu_{\frac{1}{N_c}} = \frac{n_\pi}{2} + I_\pi + \sum n_i \nu_{O_i}, \qquad (16)$$

where n_π is the number of external pions, and ν_{O_i} the $1/N_c$ order of the spin-flavor operator of the vertex of type i. Since ν_{O_i} can be negative (due to factors of G^{ia} in vertices), there are individual diagrams with $\nu_{\frac{1}{N_c}}$ negative and violating large N_c consistency. When the latter occurs, there must be other diagrams that cancel those violating terms. This will be clearly seen in the calculations presented here.

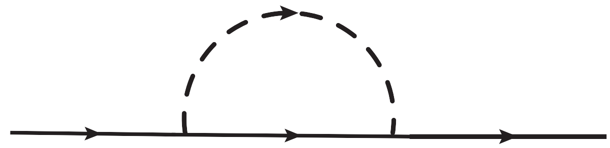

FIG. 1. One-loop contribution to baryon self energy.

One can determine now the nominal counting of the one-loop contributions to the baryon masses and currents. The LO baryon masses are $\mathcal{O}(N_c)$, with hyperfine mass splittings that are $\mathcal{O}(1/N_c)$ and $SU(3)$ symmetry breaking mass splittings that are $\mathcal{O}(p^2)$. The one-loop correction shown in Fig. 1 has: $(L=1, n_B=2, n_\pi=0, n_1=2, \nu_{O_1}=-1, n_{B_1}=2, \nu_{p_1}=1)$ giving $\nu_p=3$ as it is well known, and $\nu_{\frac{1}{N_c}}=-1$. Since there is only one possible diagram, this will be consistent if it contributes $\mathcal{O}(N_c)$ to the spin-flavor singlet component of the masses, it must contribute at $\mathcal{O}(1/N_c)$ or higher to the hyperfine splittings, and at $\mathcal{O}(N_c^0)$ to $SU(3)$ breaking. Indeed, this will be shown to be the case. For the vector and axial-vector currents, the one-loop diagrams are depicted in Figs. 2 and 3, respectively. Taking as example the axial currents, at tree level it is $\mathcal{O}(N_c)$, and the sum of the diagrams cannot scale as a higher power of N_c. Performing the counting for the individual diagrams one obtains: $\nu_p(j)=2$ for $j=1,\ldots,4$, and $\nu_{\frac{1}{N_c}}(j)=-2$, $j=1,\ 2,\ 3$ and $\nu_{\frac{1}{N_c}}(4)=0$. Thus a cancellation must occur of the $\mathcal{O}(N_c^2)$ terms when the contributions to the axial currents by the different diagrams are added, as it will be shown to be the case.

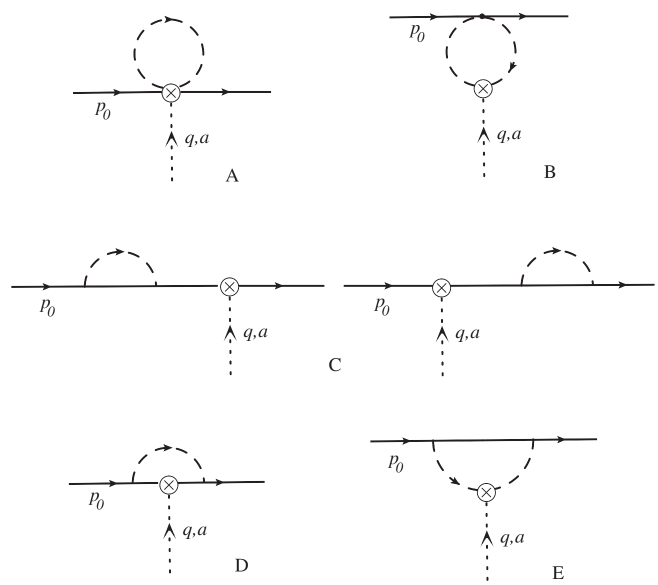

FIG. 2. Diagrams contributing to the 1-loop corrections to the vector charges.

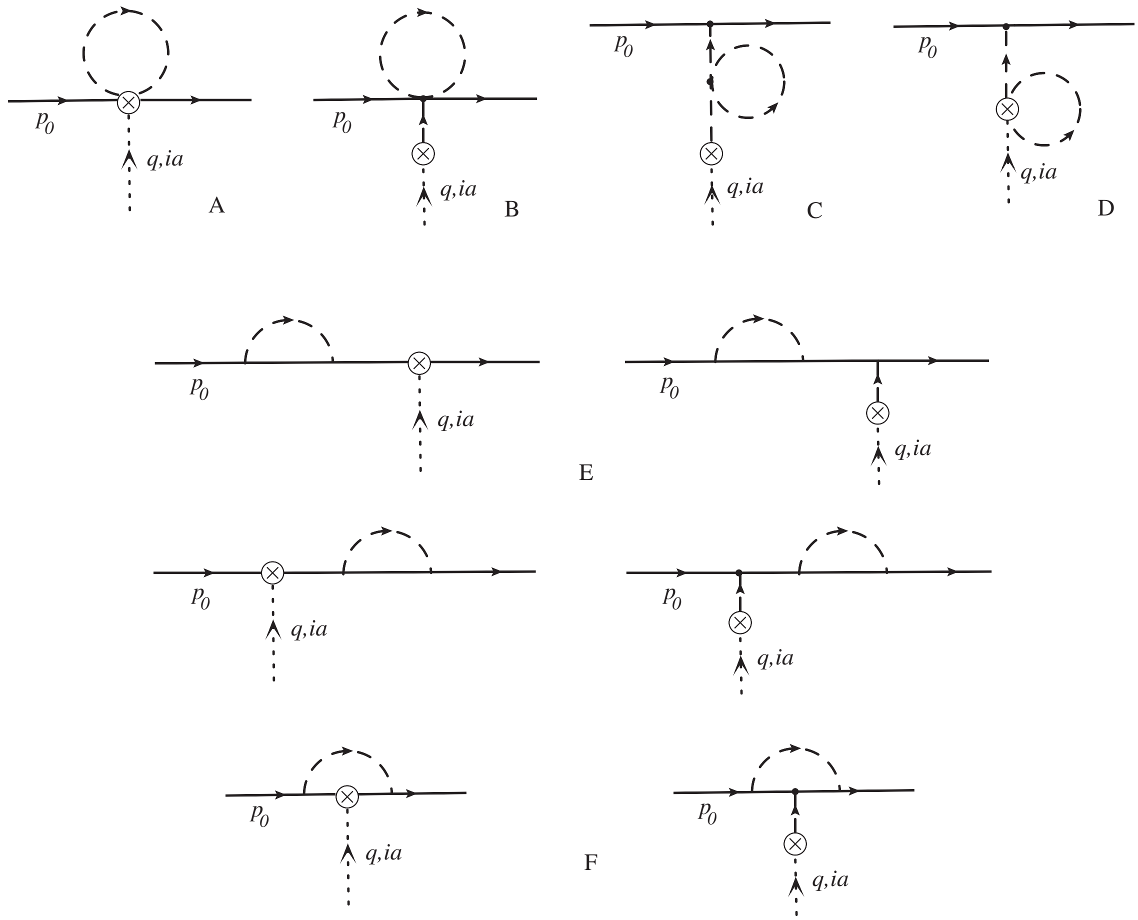

FIG. 3. Diagrams contributing to the 1-loop corrections to the axial vector currents.

One can consider the case of two-loop diagrams, in particular diagrams where the same GB-baryon vertex Eq. (10) appears four times. For the self-energy, the chiral power is $\nu_p(j) = 5$, and individual diagrams give $\nu_{\frac{1}{N_c}} = -2$. Thus a cancellation among the different diagrams must therefore occur. A comment is here in order: in Refs. [34,59], the wave function renormalization factor was included in defining the baryon mass, but that is not correct as in includes an incomplete inclusion of the two-loop contributions. In all cases, and as shown in this work, the diagrams that invoke the wave function renormalization factors play a key role in such cancellations.

Using the linked power counting ξ, $\mathcal{O}(1/N_c) = \mathcal{O}(p) = \mathcal{O}(\xi)$, the ξ order of a given Feynman diagram will then be equal to $\nu_p + \nu_{\frac{1}{N_c}}$ as given by Eqs. (15) and (16), which upon use of the topological formulas Eq. (14) leads to

$$\nu_\xi = 1 + 3L + \frac{n_\pi}{2} + \sum_i n_i(\nu_{O_i} + \nu_{p_i} - 1). \tag{17}$$

The ξ-power counting of the UV divergencies is obvious from the earlier discussion. At one-loop the masses have $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^3)$ counterterms, while the axial currents will have $\mathcal{O}(\xi)$ and $\mathcal{O}(\xi^2)$ counterterms. To two loops there are in addition $\mathcal{O}(\xi^4)$ and $\mathcal{O}(\xi^5)$, and $\mathcal{O}(\xi^3)$ and $\mathcal{O}(\xi^4)$ counterterms for masses and axial currents, respectively. The noncommutativity of limits is manifested in the finite terms where the GB masses and/or momenta, and $\delta\hat{m}$ appear combined in nonanalytic terms, and are therefore sensitive to the linking of the two expansions. The ξ expansion corresponds to not expanding such terms at all.

III. BARYON MASSES

In this section, the baryon masses are analyzed to order ξ^3, or next-to-next-to-leading order (NNLO), in the limit of exact isospin symmetry. To that order, one must include the one-loop contribution depicted in Fig. 1 with the vertices from $\mathcal{L}_{\mathbf{B}}^{(1)}$ given in Appendix B. The contribution to the self-energy is then given by

$$\delta\Sigma_{1-\mathrm{loop}} = i\frac{\overset{\circ}{g}{}_A^2}{F_\pi^2}\sum_{a=1}^{8}\sum_n G^{ia}\mathcal{P}_n G^{ia}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \times J(1,0,M_a^2-(p_0-\delta m_n)^2,1,p_0-\delta m_n), \tag{18}$$

where n indicates the possible intermediate baryon states in the loop, $\mathcal{P}_n$ are the corresponding spin-flavor projection operators, the loop integral J is given in Appendix C, δm_n is the residual mass of the baryon in the propagator, i.e. $\delta\hat{m}$ in Eq. (11) evaluated for that state n, M_a is the mass of the Goldstone boson in the loop (throughout the Gell-Mann-Okubo (GMO) mass relation $M_\eta^2 = (4M_K^2 - M_\pi^2)/3$ is used), and p_0 is the energy of the external baryon. In the ξ expansion, the $SU(3)$ breaking effects in δm_n are $\mathcal{O}(\xi^2)$, and thus they can be neglected, i.e., one can simply use $\delta\hat{m} \to \frac{C_{\rm HF}}{N_c}\hat{S}^2$ which is $\mathcal{O}(\xi)$. In the specific evaluation of $\delta\Sigma_{1-\rm loop}$ for a given baryon state denoted by "*in*", $p_0 = \delta m_{\rm in} + \mathfrak{p}_0$, where $\mathfrak{p}_0$ is the kinetic energy $\mathcal{O}(p^2/N_c)$. The noncommutativity of the $1/N_c$ and chiral expansions of course resides in the nonanalytic terms of the loop integral through their dependence on the ratios of the small scales $(\delta m_n - \delta m_{\rm in})/M_a$. Notice that when the one-loop integrals are written in terms of the residual momentum $\mathfrak{p}_0$, they do not depend on the spin-flavor singlet piece of $\delta\hat{m}$. $\mathfrak{p}_0$ is naturally associated with $i\tilde{D}^0$. The one-loop contribution to the wave function renormalization factor is given by $\delta Z_{1-\rm loop} = \frac{\partial}{\partial\mathfrak{p}_0}\delta\Sigma_{1-\rm loop}\big|_{\mathfrak{p}_0\to 0}$. Appendixes A and D provide all the necessary elements for the evaluation of the spin-flavor matrix elements in Eq. (18). The explicit final expressions for the self energy are straightforwardly calculated using those elements, and are not given explicitly because they are too lengthy.

The correction to the baryon mass is given by setting $\mathfrak{p}_0 = 0$ in the self-energy correction, and the mass of the baryon state $|S, YI\rangle$ then reads

$$m_{\mathbf{B}}(S,Y,I) = N_c m_0 + \frac{C_{\rm HF}}{N_c} S(S+1) - \frac{c_1}{2\Lambda}((N_c + 2\mathcal{S})M_\pi^2 - 2\mathcal{S}M_K^2) + \delta m_{\mathbf{B}}^{1-\text{loop}+\text{CT}}(S,Y,I), \tag{19}$$

where $\mathcal{S}$ is the strangeness, $\delta m_{\mathbf{B}}^{1-\text{loop}+\text{CT}}(S,Y,I)$ is the contribution from the one-loop diagram in Fig. 1 and CT denotes counterterm contributions. From both types of contributions, there are $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^3)$ terms, and the calculation is exact to the latter order, as can be deduced from the previous discussion on power counting. Note that in LO the LEC $C_{\rm HF}$ is equal to the hyperfine splitting $M_\Delta - M_N$ in the real world $N_c = 3$.

The ultraviolet divergent pieces of the self energy can be brought to have the following form:

$$\begin{aligned}\delta\Sigma_{1-\rm loop}^{\rm UV} = \frac{\lambda_\epsilon}{(4\pi)^2}\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\Bigg(&\mathfrak{p}_0 M_a^2 G^{ia}G^{ia} + \frac{1}{2}M_a^2[[\delta\hat{m}, G^{ia}], G^{ia}] - \frac{2}{3}\mathfrak{p}_0{}^3 \\ &-\mathfrak{p}_0{}^2[[\delta\hat{m}, G^{ia}], G^{ia}] - \mathfrak{p}_0[[\delta\hat{m}, [\delta\hat{m}, G^{ia}]], G^{ia}] - \frac{1}{3}[[\delta\hat{m}, [\delta\hat{m}, [\delta\hat{m}, G^{ia}]]], G^{ia}]\Bigg),\end{aligned} \tag{20}$$

where $\lambda_\epsilon \equiv 1/\epsilon - \gamma + \log 4\pi$. Using the $SU(3)$ singlet and octet components of the quark masses, m^0 and m^a, the meson mass-squared matrix can be written as:

$$M^{2\,ab} = 2B_0\left(\delta^{ab}m^0 + \frac{1}{2}d^{abc}m^c\right), \tag{21}$$

and therefore,

$$M_a^2 W^{aa} = M^{2\,ab}W^{ab}, \tag{22}$$

for any symmetric $\mathbf{8}\times\mathbf{8}$ tensor W. In terms of M_π and M_K, one has $m^0 = \frac{1}{3}(2\hat{m} + m_s) = \frac{2M_K^2 + M_\pi^2}{6B_0}$ and $m^a = \delta^{8a}\frac{2}{\sqrt{3}}(\hat{m} - m_s) = \delta^{8a}\frac{2(-M_K^2 + M_\pi^2)}{\sqrt{3}B_0}$.

In order to obtain from Eq. (21) the counterterms necessary to renormalize the mass and wave function, one uses the results in Appendix D. The explicit UV divergent and polynomial (in $1/N_c$, m_q, $\mathfrak{p}_0$) terms of the self-energy are the given by

$$\delta\Sigma^{\rm poly} = -\frac{1}{(4\pi)^2}\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\left\{\left(\frac{7}{3}+\lambda_\epsilon\right)B_0\frac{C_{\rm HF}}{N_c}\left(\left(\frac{3}{4}N_c(N_c+6)-7\hat{S}^2\right)m^0\right.\right.$$
$$\left.+\left(-2\{S^i,G^{ia}\}+\frac{3}{4}(N_c+3)T^a\right)m^a\right)$$
$$+\left(\frac{8}{3}+\lambda_\epsilon\right)\frac{C_{\rm HF}^3}{N_c^3}\left(-N_c(N_c+6)+\frac{1}{3}(36-5N_c(N_c+6))\hat{S}^2+12\hat{S}^4\right)$$
$$+\mathfrak{p}_0\left((1+\lambda_\epsilon)B_0\left(\left(-\frac{3}{8}N_c(N_c+6)+\frac{5}{6}\hat{S}^2\right)m^0+\left(\frac{7}{12}\{S^i,G^{ia}\}-\frac{3}{8}(N_c+3)T^a\right)m^a\right)\right.$$
$$\left.\left.+(2+\lambda_\epsilon)\frac{C_{\rm HF}^2}{N_c^2}\left(\frac{3}{2}N_c(N_c+6)+(-18+N_c(N_c+6))\hat{S}^2-4\hat{S}^4\right)\right)\right\}, \quad (23)$$

where terms of higher powers in $\mathfrak{p}_0$ have been disregarded. A few observations on $\delta\Sigma^{\rm poly}$ are in order: (1) the contributions to the spin-flavor singlet component of the masses is $\mathcal{O}(p^2N_c^0)$ and proportional to $C_{\rm HF}$, the spin-symmetry breaking is $\mathcal{O}(1/N_c^2)$, and the $SU(3)$ breaking is $\mathcal{O}(p^2/N_c)$, (2) the UV divergencies in the mass are produced by the contribution of the partner baryon in the loop, i.e. baryon of different spin, and is therefore determined by the mass splitting, i.e., by $C_{\rm HF}$, and (3) the contributions to δZ are suppressed by powers of $1/N_c$, but with two exceptions, namely, there is a spin-flavor singlet contribution proportional to m^0 which is $\mathcal{O}(N_c)$ and a term proportional to m^a which is $\mathcal{O}(N_c^0)$. The term $\mathcal{O}(N_c)$ in δZ is of key importance for the mechanism of cancellations of $1/N_c$ power counting violating terms, as it is shown later in the analysis of the one-loop contributions to the currents.

The counterterms for renormalizing the masses and wave functions are $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^3)$ (all contributions $\mathcal{O}(\xi^4)$ are consistently dropped) and involve terms that appear in $\mathcal{L}_{\mathbf{B}}^{(1)}$ with higher-order terms in $1/N_c$ in the LECs and terms in $\mathcal{L}_{\mathbf{B}}^{(2,3)}$. To renormalize, the LECs are written as: $X=X(\mu)+\frac{1}{(4\pi)^2}\beta_X\lambda_\epsilon$, where μ is the renormalization scale and the beta-functions β_X necessary to renormalize the masses are given in Table I. The reader can easily work out the renormalization of the wave functions.

TABLE I. β functions for mass renormalization.

LEC	$F_\pi^2\beta/g_A^2$
m_0	$-\frac{N_c+6}{N_c^3}C_{\rm HF}^3$
$C_{\rm HF}$	$\frac{36-5N_c(N_c+6)}{3N_c^2}C_{\rm HF}^3$
c_1	$-\frac{3}{8}\frac{N_c+3}{N_c}\Lambda C_{\rm HF}$
c_2	$\frac{3}{16}(2N_c+9)\Lambda C_{\rm HF}$
c_3	0
h_1	$-\frac{12}{\Lambda}C_{\rm HF}^3$
h_2	0
h_3	$\frac{7}{4}\Lambda C_{\rm HF}$
h_4	$\frac{1}{2}\Lambda C_{\rm HF}$

Finally, the nonanalytic contributions to $\delta\Sigma$ are

$$\delta\Sigma^{\rm NA} = -\frac{1}{(4\pi)^2}\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\sum_n G^{ia}\mathcal{P}_nG^{ia}$$
$$\times\left((p_0-\delta m_n)\left(M_a^2-\frac{2}{3}(p_0-\delta m_n)^2\right)\log\frac{M_a^2}{\mu^2}\right.$$
$$+\frac{2}{3}(M_a^2-(p_0-\delta m_n)^2)^{\frac{3}{2}}$$
$$\left.\times\left(\pi+2\arctan\left(\frac{p_0-\delta m_n}{\sqrt{M_a^2-(p_0-\delta m_n)^2}}\right)\right)\right). \quad (24)$$

At tree level, and up to order ξ^3, baryon masses satisfy the GMO and Equal Spacing (ES) relations, which hold unchanged at arbitrary N_c. The deviations from these relations are given by the nonanalytic terms in the self-energy; i.e., they are calculable to the one-loop order, and in the strict large N_c limit they are $\mathcal{O}(p^3/N_c)$ and $\mathcal{O}(p^2/N_c^2)$. The calculated deviations compare to the observed ones as follows: GMO: $(3m_\Lambda+m_\Sigma)-2(m_N+m_\Xi)=\Delta_{\rm GMO}=$ Th: $(g_A^N/F_\pi)^2\times 2.42\,10^5$ MeV^3 vs Exp: 25.8 MeV, and ES: $m_{\Xi^*}-2m_{\Sigma^*}+m_\Delta=\Delta_{\rm ES}=(g_A^N/F_\pi)^2\times(-3.72\,10^4)$ MeV^3 vs -4 ± 7 MeV, where for the theoretical evaluation $C_{\rm HF}=m_\Delta-m_N$ was used. Note that using the physical $g_A^N=1.267\pm 0.004$ and $F_\pi=93$ MeV, the value of $\Delta_{\rm GMO}$ turns out to be significantly larger than the physical one. When studying the axial couplings, it will be found that the LO value of the axial coupling is smaller than the physical one. In fact, $\Delta_{\rm GMO}$ could be used in determining the ratio g_A^N/F_π at LO. Expanding $\Delta_{\rm GMO}$ in the strict large N_c limit one obtains:

$$\Delta_{\rm GMO} = -\left(\frac{\overset{\circ}{g}_A}{4\pi F_\pi}\right)^2 \left(\frac{2\pi}{3}\left(M_K^3 - \frac{1}{4}M_\pi^3 - \frac{2}{\sqrt{3}}\left(M_K^2 - \frac{1}{4}M_\pi^2\right)^{\frac{3}{2}}\right) + \frac{C_{\rm HF}}{2N_c}\left(4M_K^2 \log\left(\frac{4M_K^2 - M_\pi^2}{3M_K^2}\right) - M_\pi^2 \log\left(\frac{4M_K^2 - \frac{1}{3}M_\pi^2}{3M_\pi^2}\right)\right)\right) + \mathcal{O}(1/N_c^3). \tag{25}$$

For the physical M_K and M_π, the shown expansion is within 30% of the exact result, and the expansion gives a good approximation for $N_c > 5$. Note the large cancellations that appear within the first line and within the second line of the equation, and also the tendency to cancel between the first and second lines. In the physical case and not expanding in $1/N_c$, it is found that the numerical dependency of $\Delta_{\rm GMO}$ on $C_{\rm HF}$ is not very significant. One also observes that only 43% of $\Delta_{\rm GMO}$ is contributed by the octet baryons in the loop, and thus the decuplet contribution is very important. $\Delta_{\rm GMO}$ is therefore an important observable for assessing whether the decuplet baryons ought to be included or not in the effective theory; as indicated earlier, this however depends on the value the LO $\overset{\circ}{g}_A$, which to be independently determined requires the analysis of other observables, namely the axial currents. Along the same lines $\Delta_{\rm ES}$ can be analyzed, although in this case the experimental uncertainty is rather large.

Disregarding the term proportional to h_2 in $\mathcal{L}_{\mathbf{B}}^{(3)}$ Eq. (13), which gives $SU(3)$ breaking in the hyperfine splittings, one additional relation follows, first found by Gürsey and Radicati [62], namely:

$$\Delta_{\rm GR} = m_{\Xi^*} - m_{\Sigma^*} - (m_\Xi - m_\Sigma) = 0, \quad \text{Exp: } 21 \pm 7\ \text{MeV}, \tag{26}$$

which relates $SU(3)$ breaking in the octet and decuplet, and which is valid for arbitrary N_c. The deviation from that relation (26) is due to $SU(3)$ breaking effects in the hyperfine interaction that splits **8** and **10** baryons, and such deviation starts with the term proportional to h_2 which is $\mathcal{O}(p^2/N_c)$. In addition, the one-loop contributions to it are free of UV divergencies and the nonanalytic terms when expanded in the large N_c limit give contributions $\mathcal{O}(1/N_c^2)$. To one-loop:

$$\begin{aligned}\Delta_{\rm GR} &= \frac{h_2}{\Lambda}\frac{12}{N_c}M_K^2 + \left(\frac{\overset{\circ}{g}_A}{4\pi F_\pi}\right)^2\left(\frac{2\pi}{9}M_K^3 + \frac{(9N_c-43)\pi}{72}\left(M_K^2 - \left(\frac{3C_{\rm HF}}{N_c}\right)^2\right)^{\frac{3}{2}}\right. \\ &\quad - \frac{N_c-3}{24}\left[3\left(M_K^2 - \left(\frac{5C_{\rm HF}}{N_c}\right)^2\right)^{\frac{3}{2}}\left(\pi - 2\arctan\frac{5C_{\rm HF}}{N_c\sqrt{M_K^2 - (\frac{5C_{\rm HF}}{N_c})^2}}\right)\right. \\ &\quad \left.\left. + 10\left(M_K^2 - \left(\frac{3C_{\rm HF}}{N_c}\right)^2\right)^{\frac{3}{2}}\arctan\frac{3C_{\rm HF}}{N_c\sqrt{M_K^2 - (\frac{3C_{\rm HF}}{N_c})^2}} + \frac{240}{N_c^3}C_{\rm HF}^3\log M_K^2\right]\right) - (M_K \to M_\pi) \\ &= \frac{h_2}{\Lambda}\frac{12}{N_c}(M_K^2 - M_\pi^2) + \frac{3\pi}{N_c}\left(\frac{\overset{\circ}{g}_A C_{\rm HF}}{4\pi F_\pi}\right)^2(M_K - M_\pi) + \mathcal{O}\left(\frac{\log(M_K/M_\pi)}{N_c^3}\right),\end{aligned} \tag{27}$$

where the last line corresponds to strictly expanding in the large N_c limit. For the physical M_π, M_K, and $C_{\rm HF}$, the $1/N_c$ expansion of $\Delta_{\rm GR}$ is, however, only reasonable for $N_c > 8$: clearly the nonanalytic dependency in $1/N_c$ is important, showing the need for the combined ξ expansion in the physical case, similarly to what occurs for $\Delta_{\rm GMO}$. Still, the understanding of the smallness of the deviation is connected with the $1/N_c$ expansion. Finally, it is important to emphasize, as indicated earlier, that all the relations are not explicitly dependent on N_c, and their deviations are suppressed by powers of $1/N_c$ at large N_c.

The σ-terms are obtained following the Hellman-Feynman theorem, $\sigma_{Bm_q} \equiv m_q \partial m_B/\partial m_q$, where m_q can be taken to be $\hat{m}, m_s$, or the $SU(3)$ singlet and octet components of the quark masses, namely $m^0 = (2\hat{m} + m_s)/3$ and $m^8 = 2/\sqrt{3}(\hat{m} - m_s)$. Naturally they will satisfy the same relations discussed above for the masses. In particular, σ terms associated with the same m_q are related via those relations and their deviations are calculable as described before for the masses. In addition to the GMO and ES relations, the following tree level $\mathcal{O}(\xi^3)$ relations hold,

TABLE II. Results for LECs: the ratio $\overset{\circ}{g}_A/F_\pi = 0.0122$ MeV^{-1} is fixed by using $\Delta_{\rm GMO}$. The first row is the fit to LQCD octet and decuplet baryon masses [48] including results for $M_\pi \leq 303$ MeV (dof = 50), and second row is the fit including also the physical masses (dof = 58). Throughout the $\mu = \Lambda = m_\rho$.

$\chi^2_{\rm dof}$	m_0 [MeV]	$C_{\rm HF}$ [MeV]	c_1	c_2	h_2	h_3	h_4
0.47	221(26)	215(46)	−1.49(1)	−0.83(5)	0.03(3)	0.61(8)	0.59(1)
0.64	191(5)	242(20)	−1.47(1)	−0.99(3)	0.01(1)	0.73(3)	0.56(1)

$$\begin{aligned}
\sigma_{Nm_s} &= \frac{m_s}{8\hat{m}}(-4(N_c-1)\sigma_{N\hat{m}} + (N_c+3)\sigma_{\Lambda\hat{m}} + 3(N_c-1)\sigma_{\Sigma\hat{m}})\\
\sigma_{\Lambda m_s} &= \frac{m_s}{8\hat{m}}(-4(N_c-3)\sigma_{N\hat{m}} + (N_c-5)\sigma_{\Lambda\hat{m}} + 3(N_c-1)\sigma_{\Sigma\hat{m}})\\
\sigma_{\Sigma m_s} &= \frac{m_s}{8\hat{m}}(-4(N_c-3)\sigma_{N\hat{m}} + (N_c+3)\sigma_{\Lambda\hat{m}} + (3N_c-11)\sigma_{\Sigma\hat{m}})\\
\sigma_{\Delta m_s} &= \frac{m_s}{8\hat{m}}(-4(N_c-1)\sigma_{\Delta\hat{m}} - 5(N_c-3)(\sigma_{\Lambda\hat{m}} - \sigma_{\Sigma\hat{m}}) + 4N_c\sigma_{\Sigma^*\hat{m}})\\
\sigma_{\Sigma^* m_s} &= \frac{m_s}{8\hat{m}}(-(N_c-3)(4\sigma_{\Delta\hat{m}} + 5\sigma_{\Lambda\hat{m}} - 5\sigma_{\Sigma\hat{m}}) + 4(N_c-2)\sigma_{\Sigma^*\hat{m}}). \qquad (28)
\end{aligned}$$

Several of these relations are poorly satisfied. The deviations are calculable and given by the nonanalytic contributions to one-loop. In the physical case $N_c = 3$, those deviations are numerically large for the first, third, and fourth relations above. This in particular affects the nucleon strangeness σ term, and thus indicates that its estimation from arguments based on tree level relations is subject to important corrections [63]. In terms of the octet components of the quark masses, in addition to GMO and ES relations one finds:

$$\sigma_{Nm^8} = \frac{(N_c+3)\sigma_{\Lambda m^8} + 3(N_c-1)\sigma_{\Sigma m^8}}{4(N_c-3)} \qquad (29)$$

$$\sigma_{\Delta m^8} = \frac{-5(N_c-3)\sigma_{\Lambda m^8} + 5(N_c-3)\sigma_{\Sigma m^8} + 4N_c\sigma_{\Sigma^* m^8}}{4(N_c-3)}, \qquad (30)$$

where it can be readily checked that they are well defined for $N_c \to 3$ as the numerators on the RHS are proportional to $(N_c - 3)$. These relations are violated at large N_c as $\mathcal{O}(p^3 N_c^0)$. For both relations in the limit $N_c \to \infty$, one finds LHS − RHS $= \frac{N_c}{128\pi}(\frac{\overset{\circ}{g}_A}{F_\pi})^2 (M_K - M_\pi)\times (M_K^2 - M_\pi^2) + \mathcal{O}(1/N_c)$. Thus they are not as precise as the GMO and ES relations.

Finally, if the LEC constant h_3 vanishes, one extra tree-level relation related to Eq. (26) follows, namely,

$$\sigma_{\Xi^* m^8} - \sigma_{\Sigma^* m^8} - (\sigma_{\Xi m^8} - \sigma_{\Sigma m^8}) = 0 \qquad (31)$$

which is only violated at large N_c as $\mathcal{O}(1/N_c^2)$, and thus expected to be very good.

To complete this section, fits to the octet and decuplet baryon masses including results from LQCD are presented. This in particular allows for exploring the range of validity of the calculation as the quark masses are increased. The mass formula for the fit is[4]:

$$\begin{aligned}
m_B = N_c m_0 &+ \frac{C_{\rm HF}}{N_c}\hat{S}^2 - \frac{c_1}{2\Lambda}\hat{\chi}_+ - \frac{c_2}{\Lambda}\chi_+^0 - \frac{c_3}{N_c\Lambda^3}\hat{\chi}_+^2\\
&- \frac{h_2}{N_c^2\Lambda}\hat{\chi}_+\hat{S}^2 - \frac{h_3}{N_c\Lambda}\chi_+^0\hat{S}^2 - 2\frac{h_4}{N_c\Lambda}\tilde{\chi}_+^a S^i G^{ia}\\
&+ \delta m_B^{1-\rm loop}, \qquad (32)
\end{aligned}$$

where, in the isospin symmetry limit, $\chi_+^0 \to 4B_0 m^0$, $\tilde{\chi}_+^a \to 8B_0\delta^{a8}m^8$, and $\hat{\chi}_+ \to 4B_0(m^8 T^8 + N_c m^0)$. The fits at $N_c = 3$ cannot obviously give the N_c dependence of LECs. LECs of terms that depend on quark masses can be more completely determined by fits that include the LQCD results for different quark masses, e.g., c_2 and the various $h's$. For this reason, such combined fits are presented here, in Table II and in Fig. 4. Also, some LECs are redundant at $N_c = 3$, and are thus set to vanish for the fit. The constant c_3 is also set to vanish as it turns out to be of marginal importance for the fit. A test of mass relations is shown in Table III.

The study of the fits show that at fixed $M_K \sim 500$ MeV, the physical plus LQCD results up to $M_\pi \sim 300$ MeV can

[4] A useful formula for the term proportional to h_4 is [64]: $S^i G^{i8} = \frac{1}{\sqrt{3}}(\frac{3}{4}\hat{I}^2 - \frac{1}{4}\hat{S}^2 - \frac{1}{48}N_c(N_c+6) + \frac{1}{8}(N_c+3)Y - \frac{3}{16}Y^2) = \frac{1}{16\sqrt{3}}(12\hat{I}^2 - 4\hat{S}^2 + 3\mathcal{S}(2-\mathcal{S}))$, where $\mathcal{S}$ is the strangeness. This term is responsible for the tree-level mass splitting between Λ and Σ.

TABLE III. Deviations from mass relations in MeV. Here $\Delta_{\mathrm{ES1}} = m_{\Xi^*} - 2m_{\Sigma^*} + m_{\Delta}$ and $\Delta_{\mathrm{ES2}} = m_{\Omega^-} - 2m_{\Xi^*} + m_{\Sigma^*}$.

M_π	M_K	Δ_{GMO}		Δ_{GR}		Δ_{ES1}		Δ_{ES2}	
[MeV]		Exp/LQCD	Th	Exp/LQCD	Th	Exp/LQCD	Th	Exp/LQCD	Th
139	497	31 ± 42	46	23 ± 30	38	-6 ± 30	−14	-9 ± 30	−14
213	489	75 ± 70	33	0 ± 72	29	-40 ± 97	−11	9.2 ± 83	−11
246	499	124 ± 77	30	-7 ± 75	25	-46 ± 101	−11	23 ± 86	−11
255	528	133 ± 89	37	-12 ± 94	26	-32 ± 125	−14	29 ± 108	−14
261	524	139 ± 99	35	24 ± 103	25	-29 ± 138	−13	-3 ± 119	−13
302	541	77 ± 87	32	-14 ± 94	23	-30 ± 125	−13	46 ± 108	−13

be fitted with natural size LECs. The LEC h_2 which enters in Δ_{GR} is best determined by fixing it using Δ_{GR} in the physical case, and then the rest of the LECs are determined by the overall fit. In this way, the deviations of the mass relations are one of the predictions of the effective theory, and can therefore be used as a test of LQCD calculations. At present the errors in the LQCD calculations are relatively large, and thus such a test is not yet very significant.

IV. VECTOR CURRENTS: CHARGES

In this section, the one-loop corrections to the vector current charges are calculated. The analysis is similar to that carried out in [65], except that in that reference higher-order terms in $1/N_c$ in the GB-baryon vertices were included. In the ξ expansion and the order considered here, such higher-order terms are not required. At lowest order the charges are simply given by the generators T^a, the one-loop corrections are UV finite,

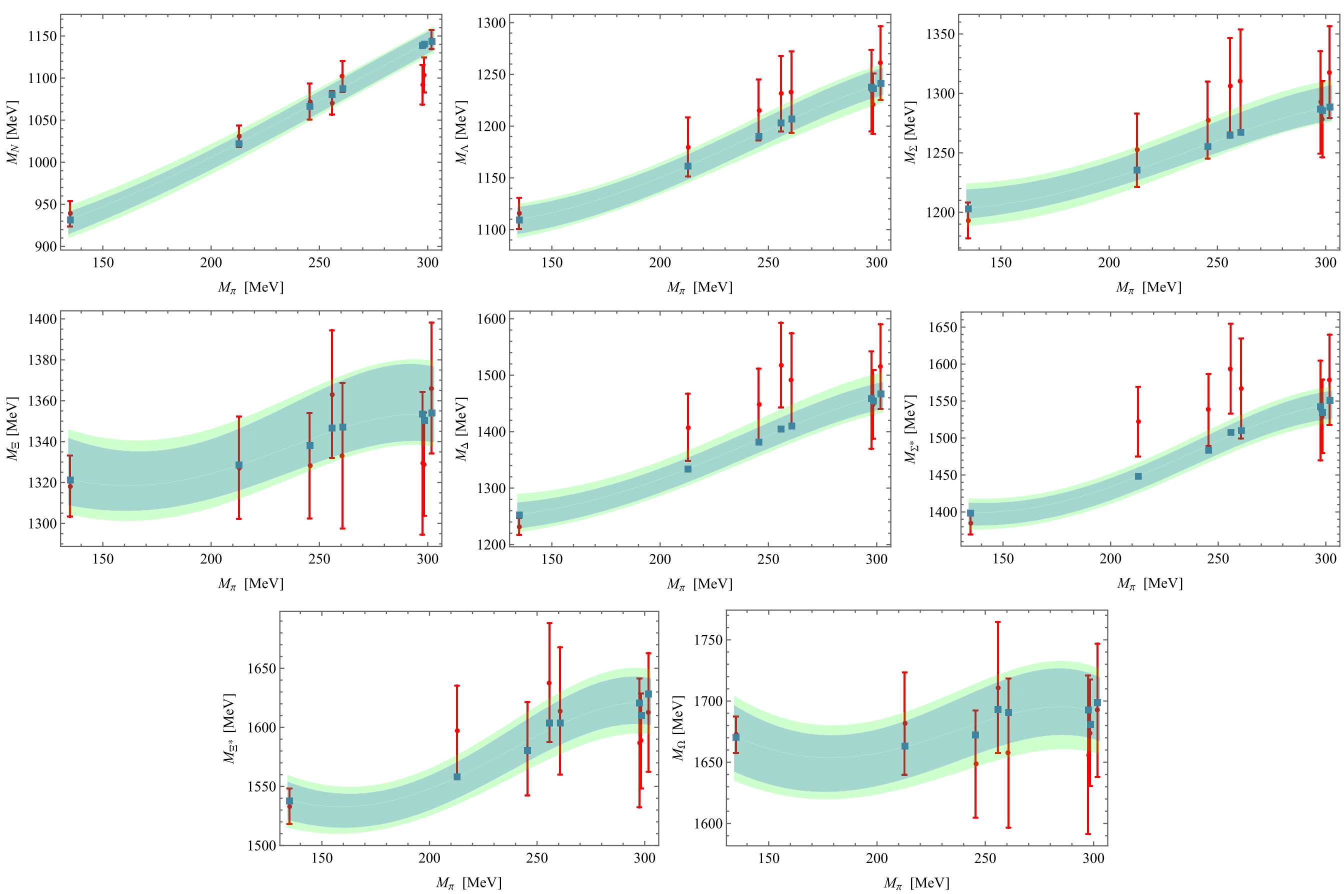

FIG. 4. Baryon masses vs M_π obtained from the combined fit (second row of Table II). The bands correspond to the 67% and 95% confidence intervals. The red points with error bars are from the LQCD calculations [48], and the squares are the theoretical values for the values of M_π and M_K of the corresponding data point.

and since up to $\mathcal{O}(\xi^3)$ the Ademollo-Gatto theorem (AGT) is satisfied, the corrections to the charges are unambiguously given at one-loop.

The one-loop diagrams are shown in Fig. 2, and the corrections to the charges are obtained by evaluating the diagrams at $q \to 0$. In that limit, the UV divergencies as well as the finite polynomial terms in quark masses and $\delta\hat{m}$ cancel in each of the two sets of diagrams, $A+B$, and $C+D+E$, as required by the AGT. The results for the diagrams are the following:

$$
\begin{aligned}
A &= -\frac{i}{2F_\pi^2} f^{abc} f^{bcd} T^d I(0,1,M_b) \\
B &= \frac{i}{4F_\pi^2} f^{abc} f^{bcd} T^d (q^{02} K(q,M_b,M_c) + 4q^0 K^0(q,M_b,M_c) + 4K^{00}(q,M_b,M_c)) \\
C &= \frac{1}{2}\{T^a, \delta\hat{Z}_{1-\mathrm{loop}}\} \\
D &= i\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \sum_{n_1,n_2} G^{ib}\mathcal{P}_{n_2} T^a \mathcal{P}_{n_1} G^{jb} \frac{1}{q_0 - \delta m_{n_2} + \delta m_{n_1}} (H_{ij}(p_0 - \delta m_{n_1}, M_b) - H_{ij}(p_0 + q_0 - \delta m_{n_2}, M_b)) \\
E &= \left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 f^{abc} \sum_n G^{ib}\mathcal{P}_n G^{jc} H_{ij0}(p_0 - \delta m_n, q, M_b, M_c),
\end{aligned}
\tag{33}
$$

where the integrals K, K^μ, $K^{\mu\nu}$, H_{ij} and H_{ij0} are given in Appendix C. Since the temporal component of the current can only connect baryons with the same spin, q_0 is equal to the $SU(3)$ breaking mass difference between them plus the kinetic energy transferred by the current, which are all $\mathcal{O}(p^2)$, and can be neglected: the limit $q_0 \to 0$ must then be taken in the end. Diagram D indeed requires a careful handling of that limit in the cases when the denominator vanishes. The same is the case for diagram F in the axial-vector currents in next section. The $U(1)$ baryon number current is used to check the calculation: only diagrams $C+D$ contribute, and as required cancel each other.

The UV divergent and polynomial pieces contributed by the diagrams are the following:

$$
\begin{aligned}
A^{\mathrm{poly}} &= \frac{\lambda_\epsilon + 1}{(4\pi)^2}\frac{1}{2F_\pi^2} f^{abc} f^{bcd} M_b^2 T^d \\
B^{\mathrm{poly}} &= -\frac{\lambda_\epsilon + 1}{(4\pi)^2}\frac{1}{2F_\pi^2} f^{abc} f^{bcd} T^d \left(M_b^2 + \frac{1}{6}\vec{q}^2\right) \\
C^{\mathrm{poly}} &= \frac{1}{(4\pi)^2}\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \frac{1}{2}\{T^a, (\lambda_\epsilon+1) M_b^2 G^{ib} G^{ib} - 2(\lambda_\epsilon+2) G^{ib}[\delta\hat{m},[\delta\hat{m},G^{ib}]]\} \\
D^{\mathrm{poly}} &= \frac{1}{(4\pi)^2}\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \frac{1}{3}\sum_{n_1,n_2} G^{ib}\mathcal{P}_{n_2} T^a \mathcal{P}_{n_1} G^{ib} \frac{1}{q_0 - \delta m_{n_2} + \delta m_{n_1}} \\
&\quad \times \{(p_0 - \delta m_{n_1})(3(\lambda_\epsilon+1)M_b^2 - 2(\lambda_\epsilon+2)(p_0-\delta m_{n_1})^2 - \{p_0 \to p_0 + q_0, \delta m_{n_1} \to \delta m_{n_2}\})\} \\
&= \frac{1}{(4\pi)^2}\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \frac{1}{3}\{-3(\lambda_\epsilon+1)M_b^2 G^{ib} T^a G^{ib} + 2(\lambda_\epsilon+2)([\delta\hat{m},[\delta\hat{m},G^{ib}]]T^a G^{ib} \\
&\quad + G^{ib} T^a [\delta\hat{m},[\delta\hat{m},G^{ib}]] - [\delta\hat{m},G^{ib}] T^a [\delta\hat{m},G^{ib}])\} \\
E^{\mathrm{poly}} &= -\frac{1}{(4\pi)^2}\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \frac{i}{6} f^{abc}\sum_n G^{ib}\mathcal{P}_n G^{jc}\{\lambda_\epsilon(2q^i q^j + q^2 g^{ij}) + q^2 g^{ij} - 3g^{ij}((\lambda_\epsilon+1)(M_b^2+M_c^2) \\
&\quad - (\lambda_\epsilon+2)(\delta m_{\mathrm{in}} - 2\delta m_n + \delta m_{\mathrm{out}})^2)\} \\
&= -\frac{1}{(4\pi)^2}\left(\frac{\overset{\circ}{g}_A}{F_\pi}\right)^2 \frac{i}{6}\{((2q^i q^j + q^2 g^{ij})\lambda_\epsilon - q^2 g^{ij})[T^a, G^{ib}]G^{jb} + 3(\lambda_\epsilon+1)M_b^2[[T^a,G^{ib}],G^{jb}] \\
&\quad - 3(\lambda_\epsilon+2)([[T^a,G^{ib}],[\delta\hat{m},[\delta\hat{m},G^{jb}]]] + [[\delta\hat{m},G^{ib}],[T^a,[\delta\hat{m},G^{ib}]]])\},
\end{aligned}
\tag{34}
$$

where in the evaluations $p_0 \to \delta m_{\rm in}$ and $p_0 + q_0 \to \delta m_{\rm out}$. Combining the polynomial pieces and using that $[\delta\hat{m}, T^a] = [\delta\hat{m}, \hat{G}^2] = [\delta\hat{m}, G^{ib}T^aG^{ib}] = 0$ lead to the result:

$$(A+B)^{\rm poly} = -\frac{\lambda_\epsilon + 1}{(4\pi)^2}\frac{\vec{q}^2}{4F_\pi^2}T^a$$
$$(C+D+E)^{\rm poly} = \frac{\lambda_\epsilon - 3}{(4\pi)^2}\left(\frac{\mathring{g}_A}{4F_\pi}\right)^2\vec{q}^2T^a \tag{35}$$

As required by the AGT, when $q \to 0$ the UV divergencies and polynomial terms vanish for all the $SU(3)$ vector charges of the baryon spin-favor multiplet. The calculation of the finite nonanalytic contributions has been carried out in previous work [65], and will not be revisited here.

The only counterterm required is the one proportional to g_E in Eq. (13), where $\beta_{g_E} = \frac{1}{(4F_\pi)^2}(4 - \mathring{g}_A^2)$, and which provides the only analytic contribution to the octet and decuplet charge radii up to the order of the calculation. More details will be presented elsewhere in a study of the form factors of the vector currents. In the context of the charge form factors, studies implementing the $1/N_c$ expansion for extracting the long distance charge distribution of the nucleon has been carried out in Refs. [66–69].

V. AXIAL COUPLINGS

The axial vector currents are studied to one-loop. At the tree level the axial vector currents have two contributions, namely the contact term and the GB pole ones, and reads

$$A^{\mu a} = \mathring{g}_A G^{ja}\left(g^\mu_j - \frac{q^\mu q_j}{q^2 - M_a^2}\right). \tag{36}$$

In the nonrelativistic limit, or equivalently large N_c limit, the time component of the axial vector current is suppressed with respect to the spatial components. The couplings associated with the latter are analyzed below to $\mathcal{O}(\xi^2)$.

At the leading order, the axial couplings are all given in terms of $\mathring{g}_A$. For $N_c = 3$: $F = \mathring{g}_A/3$, $D = \mathring{g}_A/2$, and the axial coupling in the decuplet baryons is $\mathcal{H} = \mathring{g}_A/6$.

The one-loop diagrams contributing at that order are shown in Fig. 3.

The matrix elements of interest for the axial currents are $\langle \mathbf{B}'|A^{ia}|\mathbf{B}\rangle$ evaluated at vanishing external 3-momentum. The axial couplings $g_A^{\mathbf{BB}'}$ are conveniently defined by

$$\langle \mathbf{B}'|A^{ia}|\mathbf{B}\rangle = g_A^{\mathbf{BB}'}\frac{6}{5}\langle \mathbf{B}'|G^{ia}|\mathbf{B}\rangle, \tag{37}$$

which are $\mathcal{O}(N_c^0)$. The $\mathcal{O}(N_c)$ of the matrix elements of the axial currents is due to the operator G^{ia}. The factor $6/5$ mentioned earlier is included so that g_A^{NN} at $N_c = 3$ exactly corresponds to the usual nucleon g_A, which has the value 1.267 ± 0.004 [70].

The results for the one-loop diagrams are the following:

$$A = -g^\mu_i \frac{\mathring{g}_A}{2F_\pi^2} f^{abc}f^{cdb}G^{id}I(0,1,M_b)$$
$$B = \frac{\mathring{g}_A}{6F_\pi^2}\frac{q^\mu q_i}{q^2 - M_a^2} f^{abc}f^{cdb}G^{id}I(0,1,M_b)$$
$$C = \frac{2\mathring{g}_A}{3F_\pi^2}\frac{q^\mu q_i}{q^2 - M_d^2} f^{abc}f^{cdb}G^{id}I(0,1,M_b)$$
$$D = -\frac{\mathring{g}_A}{3F_\pi^2}\frac{q^\mu q_i}{q^2 - M_a^2} f^{abc}f^{cdb}G^{id}I(0,1,M_b)$$
$$E = \frac{1}{2}\mathring{g}_A\left(g^\mu_i - \frac{q^\mu q_i}{q^2 - M_a^2}\right)\{G^{ia}, \delta\hat{Z}_{1-\rm loop}\}$$
$$F = i\left(g^\mu_i - \frac{q^\mu q_i}{q^2 - M_a^2}\right)\mathring{g}_A\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\sum_{n_1,n_2} G^{jb}\mathcal{P}_{n_2}G^{ia}\mathcal{P}_{n_1}G^{kb}\frac{1}{q_0 - \delta m_{n_2} + \delta m_{n_1}}$$
$$\times (H_{jk}(p_0 - \delta m_{n_1}, M_b) - H_{jk}(p_0 + q_0 - \delta m_{n_2}, M_b)) \tag{38}$$

The corresponding polynomial terms of these one-loop contributions are

$$A^{\rm poly} = \frac{1}{(4\pi)^2}\frac{\mathring{g}_A}{2F_\pi^2}(\lambda_\epsilon+1)g_i^\mu f^{abc}f^{bcd}G^{id}M_b^2$$

$$B^{\rm poly} = -\frac{1}{(4\pi)^2}\frac{\mathring{g}_A}{6F_\pi^2}(\lambda_\epsilon+1)\frac{q^\mu q_i}{q^2-M_a^2}f^{abc}f^{bcd}G^{id}M_b^2$$

$$C^{\rm poly} = -\frac{1}{(4\pi)^2}\frac{2\mathring{g}_A}{3F_\pi^2}(\lambda_\epsilon+1)\frac{q^\mu q_i}{q^2-M_d^2}f^{abc}f^{bcd}G^{id}M_b^2$$

$$D^{\rm poly} = \frac{1}{(4\pi)^2}\frac{\mathring{g}_A}{3F_\pi^2}(\lambda_\epsilon+1)\frac{q^\mu q_i}{q^2-M_a^2}f^{abc}f^{bcd}G^{id}M_b^2$$

$$E^{\rm poly} = \frac{1}{(4\pi)^2}\frac{1}{2}\mathring{g}_A\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\left(g_i^\mu-\frac{q^\mu q_i}{q^2-M_a^2}\right) \times\{G^{ia},(\lambda_\epsilon+1)M_b^2G^{jb}G^{jb}-2(\lambda_\epsilon+2)G^{jb}[\delta\hat{m},[\delta\hat{m},G^{jb}]]\}$$

$$F^{\rm poly} = -\frac{1}{(4\pi)^2}\mathring{g}_A\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\left(g_i^\mu-\frac{q^\mu q_i}{q^2-M_a^2}\right)\bigg((\lambda_\epsilon+1)M_b^2G^{jb}G^{ia}G^{jb} -\frac{2}{3}(\lambda_\epsilon+2)(G^{jb}G^{ia}[\delta\hat{m},[\delta\hat{m},G^{jb}]]+[\delta\hat{m},[\delta\hat{m},G^{jb}]]G^{ia}G^{jb}-[\delta\hat{m},G^{jb}]G^{ia}[\delta\hat{m},G^{jb}])\bigg). \tag{39}$$

The conservation of the axial currents is readily checked in the chiral limit. At this point it is important to check the cancellation of the N_c power counting violating terms shown in the polynomial terms of diagrams E and F. Such terms cancel in the sum, as it is easy to show using the results displayed in Appendix D for the axial vector currents. One obtains:

$$(E+F)^{\rm poly} = \frac{1}{(4\pi)^2}\mathring{g}_A\left(\frac{\mathring{g}_A}{F_\pi}\right)^2\left(g_i^\mu-\frac{q^\mu q_i}{q^2-M_a^2}\right) \times\Bigg((\lambda_\epsilon+1)\frac{1}{6}B_0\left(23m^0G^{ia}+\frac{11}{4}d^{abc}m^bG^{ic}+\frac{5}{3}m^aS^i\right)+(\lambda_\epsilon+2)\frac{C_{\rm HF}^2}{N_c^2}\Bigg(\left(1-\frac{N_c(N_c+6)}{3}\right)G^{ia} +\frac{11}{6}(N_c+3)S^iT^a-\frac{8}{3}\{\hat{S}^2,G^{ia}\}-\frac{4}{3}S^i\{S^j,G^{ja}\}+\frac{11}{6}\hat{S}^2G^{ia}\hat{S}^2\Bigg)\Bigg) \tag{40}$$

The quark mass dependent UV divergencies are $\mathcal{O}(m_q/N_c)$, and the quark mass independent ones give a term proportional to G^{ia}, i.e., to the LO term but suppressed by a factor $1/N_c$, while the rest of the terms are $\mathcal{O}(1/N_c^2)$ or higher. The cancellation mechanism clearly requires the contributions from the wave function renormalization factors (diagrams E), and it is rather subtle as it requires an explicit and lengthy calculation starting from Eq. (39). To obtain the counterterms, the relations given in Appendix D are used. The counterterms are contained in the Lagrangians $\mathcal{L}_{\mathbf{B}}^{(1,2,3)}$, and the corresponding β functions are the ones shown in Table IV. In addition to $\mathring{g}_A$, there are seven LECs that are necessary to renormalize the axial vector couplings for generic N_c. For $N_c=3$, the terms proportional to $C^A_{1,2,3}$ are linearly dependent and one can be eliminated. At $N_c=3$, after considering isospin symmetry, there are thirty four axial couplings associated with the axial currents mediating transitions in the spin-flavor multiplet of baryons. This means that there are twenty seven relations among those couplings that must be satisfied at the order of the present calculation. Such relations are straightforward to derive with the results provided here, and they should eventually become one good test for their LQCD calculations. It should be noted that in general the relations dependent on N_c explicitly.

TABLE IV. β functions for counterterms contributing to the axial-vector currents.

LEC	$F_\pi^2\beta$	LEC	$F_\pi^2\beta/\Lambda^2$
$\mathring{g}_A$	$\mathring{g}_A^3\frac{C_{\rm HF}^2}{3}$	D_1^A	$-\frac{1}{48}\mathring{g}_A(36+23\mathring{g}_A^2)$
C_1^A	$-\frac{11}{6}\mathring{g}_A^3C_{\rm HF}^2\frac{N_c+3}{N_c}$	D_2^A	$-\frac{5}{144}\mathring{g}_A^3$
C_2^A	$\frac{1}{2}\mathring{g}_A^3C_{\rm HF}^2\frac{1-2N_c}{N_c}$	$D_3^A(d)$	$-\frac{1}{192}\mathring{g}_A(36+11\mathring{g}_A^2)$
C_3^A	$\frac{8}{3}\mathring{g}_A^3C_{\rm HF}^2$	$D_3^A(f)$	0
C_4^A	$\frac{8}{3}\mathring{g}_A^3C_{\rm HF}^2$		

The one-loop corrections to the axial currents are such that they do not contribute to the Goldberger-Treiman discrepancies (GTD) [71]. The discrepancies are given by terms in the Lagrangian of $\mathcal{O}(\xi^3)$, namely:

$$\mathcal{L}_{\mathbf{B}}^{(3)} = \cdots + i\mathbf{B}^\dagger(g_{\mathrm{GTD}}[\nabla^i, \tilde{\chi}_-^a]G^{ia} + g_{\mathrm{GTD}}^0 \partial^i \chi_-^0 S^i)\mathbf{B}. \quad (41)$$

As noted in [71] there are three LECs determining the spin 1/2 GTD in $SU(3)$. The $1/N_c$ expansion shows that those LECs are actually determined by the two shown above, which also determine the GTDs of the decuplet baryons.

The following observations are important: if the nonanalytic contributions to the corrections to the axial couplings are disregarded, the corrections $\mathcal{O}(N_c)$ and $\mathcal{O}(N_c^0)$ to the matrix elements in $S=1/2$ and $3/2$ baryons due to the counterterms are as expected $\mathcal{O}(p^2)$, i.e., proportional to quark masses. On the other hand the terms independent of quark masses are $\mathcal{O}(1/N_c)$, i.e., spin symmetry breaking is suppressed by $\mathcal{O}(1/N_c^2)$ with respect to the leading order, as it was noted long ago [72]. This indicates that the effects of spin-symmetry breaking are more suppressed than the $SU(3)$ symmetry breaking ones [32,33,73]. It is important to note that at tree level NNLO the axial couplings satisfy some N_c independent relations. For the case of $\Delta Y = 0$ couplings within the baryon octet and decuplet, in the $I=1$ case the first relation below follows, and in the $I=0$ (η channel) case there are GMO and ES relations, namely:

$$\left(\frac{g_A}{g_V}\right)^{\pi\Delta} + \frac{3}{5}\left(\frac{g_A}{g_V}\right)^{\pi\Xi^*} - \frac{8}{5}\left(\frac{g_A}{g_V}\right)^{\pi\Sigma^*} = 0$$
$$2(g_A^{\eta N} + g_A^{\eta\Xi}) - 3g_A^{\eta\Lambda} - g_A^{\eta\Lambda} = 0$$
$$g_A^{\eta\Sigma^*} - g_A^{\eta\Delta} = g_A^{\eta\Xi^*} - g_A^{\eta\Sigma^*} = g_A^{\eta\Omega} - g_A^{\eta\Xi^*} \quad (42)$$

These relations are only violated by finite nonanalytic terms. Additional relations are straightforward to derive for other couplings, such as those involving the $\Delta Y = \pm 1$ and the octet to decuplet off diagonal ones. Such relations will be a good tool to check results obtained in LQCD calculations of the axial couplings.

At LO and using $(\frac{g_A}{g_V})^{\pi N} = 1.267 \pm 0.004$ for the nucleon, it follows that $(\frac{g_A}{g_V})^{KN\Lambda} = 0.760$, $(\frac{g_A}{g_V})^{KN\Sigma} = -0.253$, and $(\frac{g_A}{g_V})^{K\Sigma\Xi} = (\frac{g_A}{g_V})^{\pi N}$, to be compared with the ones obtained from semileptonic hyperon decays [74] 0.718 ± 0.015, -0.340 ± 0.017 and 1.32 ± 0.20, respectively. The NLO $SU(3)$ breaking corrections are evidently necessary. On the other hand, the coupling $g_A^{N\Delta}$ is at LO equal to g_A, while its phenomenological value extracted from the width of the Δ assuming a vanishing GTD is equal to 1.235 ± 0.011 [34,35], which shows a remarkably small breaking of the spin-symmetry. This seems to be in line with what was discussed above, namely that spin symmetry breaking is suppressed with respect to $SU(3)$ breaking by one extra order in $1/N_c$. In the following subsections, the results for the axial couplings are confronted with recent LQCD calculations.

A. Fits to LQCD results

While LQCD calculations of the axial coupling of the nucleon have a long history, calculations involving hyperons and including the decuplet baryons are very recent. Indeed, the first such calculations were carried out by C. Alexandrou *et al.* [55], where the axial couplings associated with the two neutral $\Delta\mathcal{S} = 0$ currents for transitions within the octet and within the decuplet baryons were obtained. They used a twisted mass Wilson action adapted to $2+1+1$ flavors (the calculation includes charmed baryons). The results in [55] show the a similar recurring issue in LQCD calculations of the nucleon's axial coupling, which turn out to be from 5% to 10% smaller than the physical value. Recent calculations of g_A^N have been able to give consistent results [75], but those calculations are still missing for hyperons and the baryon decuplet.

In this subsection, the results [55], are fitted with the effective theory. The LECs that can be fitted with these results are $\mathring{g}_A, \delta\mathring{g}_A$ (which is a $1/N_c$ correction to $\mathring{g}_A$ and needed for a counterterm), and $C_{1,3}^A, D_{1,2,3}^A$. Using the definition of couplings in Eq. (37), the results shown above for the UV divergencies of the one-loop contributions imply that $\delta g_A^{aBB'}(UVdiv)/g_A^{aBB'} = \mathcal{O}(C_{\mathrm{HF}}/N_c) + \mathcal{O}(m_q/N_c)$. At LO, $g_A^{aBB'} = g_A^N = 1.267 \pm 0.004$. The relations between the couplings $g_A^{aBB'}$ and the ones displayed in [55] are the following,

$$\langle B_8|A^{i=03}|B_8\rangle = \frac{1}{2}g_A^{B_8}$$
$$\langle B_{10}|A^{i=03}|B_{10}\rangle = \frac{1}{6}g_A^{B_{10}}$$
$$\langle B_8|A^{i=08}|B_8\rangle = \frac{1}{2\sqrt{3}}g_8^{B_8}$$
$$\langle B_{10}|A^{i=08}|B_{10}\rangle = \frac{1}{6\sqrt{3}}g_8^{B_{10}}, \quad (43)$$

where $B_{8,10}$ is an octet (decuplet) baryon with spin projection $+1/2$, and the couplings on the RHS are those used in [55] and displayed in Tables IV and V of that reference. The LQCD results are given for several values of M_π by keeping m_s approximately fixed. The values of M_π for the different cases are given in Table I of [55], and the corresponding M_K is determined using the physical masses by the LO relation: $M_K^2 = M_{K\,\mathrm{phys}}^2 + \frac{1}{2}(M_\pi^2 - M_{\pi\,\mathrm{phys}}^2)$, which corresponds to keeping m_s fixed. While for general N_c, the nine terms associated with the LECs in Table IV are linearly independent, at $N_c = 3$, the term associated with C_2^A becomes linearly dependent with the LO term, and thus its effects are absorbed into $\delta\mathring{g}_A$. In the case of the LQCD

TABLE V. LECs obtained by fitting to the LQCD results presented in Tables IV and V of Ref. [55]. The results correspond to making the choices $\Lambda = \mu = m_\rho$. In the NLO full fits $C_{\mathrm{HF}} = 250$ MeV, and $\mathring{g}_A$ is given as input, displaying fits for three different values.

Fit	χ^2_{dof}	$\mathring{g}_A$	$\delta\mathring{g}_A$	C_1^A	C_2^A	C_3^A	C_4^A	D_1^A	D_2^A	D_3^A	D_4^A
LO	3.9	1.35	...	...	...	...	...	...	...	...	...
NLO Tree	0.91	1.42	...	−0.18	...	...	...	...	0.009	...	...
NLO Full	1.08	1.02	0.15	−1.11	0.	1.08	0.	−0.56	−0.02	−0.08	0.
	1.13	1.04	0.08	−1.17	0.	1.15	0.	−0.59	−0.02	−0.09	0.
	1.19	1.06	0.	−1.23	0.	1.21	0.	−0.62	−0.03	−0.09	0.

results being fitted here, there is an additional linear dependency, namely that of the term C_4^A which becomes linearly dependent with the term C_3^A. So the fit will involve seven NLO LECs in addition to $\mathring{g}_A$. The results of the fits are shown in Table V. The LO fit, which involves only fitting the LO value of $\mathring{g}_A$, shows a remarkably good approximation to the full set of the LQCD results. This is clearly aided by the very small dependency on M_π of the LQCD results. It also shows the very good approximate spin-flavor symmetry that relates axial couplings in the octet and decuplet. The LO fit implies that $g_A^N = 1.13$ for the physical pion mass. A fit where only tree contributions are included up to the NNLO gives a very precise description of the LQCD results. Indeed, turning off some of the LECs as indicated in Table V provides a consistent fit, and corresponds in this case to $g_A^N = 1.15$. Note that in this case $\delta\mathring{g}_A$, which is required to cancel an UV divergency proportional to the leading term, can be turned off, as it is only required when the loop contributions are included.

The full NLO fit is more complicated. Although the implemented consistency with the $1/N_c$ expansion gives an important reduction of the nonanalytic contributions, these are still significant. The most significant issue in this case becomes the determination of the LO $\mathring{g}_A$. If it is used as a fitting parameter, then the fit naturally drives it down to small values, suppressing the nonanalytic contributions. Such a situation is unrealistic, and therefore an strategy is needed. The problem originates in the need to renormalize $\mathring{g}_A$, as there is an UV divergency proportional to the LO term of the axial current. This is performed using $\delta\mathring{g}_A$, which is suppressed by one power in $1/N_c$ with respect to $\mathring{g}_A$. Fixing both the LO $\mathring{g}_A$ and the counterterm would thus require information at different values of N_c, which is not accessible at present. One possible approach is to fix $\mathring{g}_A$ to the value obtained with the LO fit, and then fit the higher-order LECs. This however fails because the resulting fit has too large a χ^2. Another strategy is to input several different values of $\mathring{g}_A$, and determine an approximate range for it based of obtaining a χ^2 that is acceptable. Finally, a different strategy can be used involving additional observables: for instance, as mentioned earlier, the value for $\mathring{g}_A$ could be obtained by matching to Δ_{GMO}, giving a value for $\mathring{g}_A/F_\pi$, which in Δ_{GMO} should be taken at LO. In that case, and in the physical case one obtains $\mathring{g}_A \sim 1.15$ when $F_\pi = 93$ MeV. This however cannot be used for the present LQCD results, because they have the mentioned issue of extrapolating to too low of a value for g_A^N at the physical point. In that case a correspondingly smaller value should be used, namely $\mathring{g}_A \sim 1.05$ or so. The NLO fit with such an input for $\mathring{g}_A$ is almost consistent, and is shown in Table V for three different input values. The extrapolation of those fits to the physical M_π give a rather low value, $g_A^N \sim 0.97$. This value is increased if only the $LQCD$ results in [55] for the nucleon are included, namely $g_A^N \sim 1.05$. The effective theory is also checked to fit the most recent results on g_A^N [75], where the LQCD result agrees with the physical value. Clearly, it is necessary to await additional lattice calculations of the octet and decuplet axial couplings in order to have a thorough test of the effective theory vis-á-vis LQCD.

Ultimately, in order to have the LECs in BChPT $\times\, 1/N_c$ fully determined, a global analysis involving LQCD calculations of a complete set of observables is necessary. This requires the LQCD determination of the quark mass dependencies of the observables, and also the possibility of results for different values of N_c, which is a more difficult task, but which has already been initiated with the baryon masses for two flavors [58], and which has been analyzed with the effective theory [59].

VI. SUMMARY

Chiral symmetry and the expansion in $1/N_c$ are two fundamental aspects of QCD. The former is known to play a crucial role in light hadrons, and there are multiple indications that the latter is also important, in particular for baryons. In the context of effective theories, it is therefore crucial to incorporate those two aspects of QCD consistently. This is possible with the combined chiral and $1/N_c$ expansions. In the present work that framework for baryons in $SU(3)$ was implemented using the ξ-expansion. The renormalization to one-loop for baryon masses and currents were presented for generic N_c, and LQCD results for masses and axial couplings were analyzed. This work serves as a basis for further applications, where it is expected that the improved convergence of the effective theory will have a significant impact, which should be particularly important in the case of three flavors.

In the case of three flavors, there are numerous parameter free relations that hold at tree level NNLO in the ξ expansion, such as GMO, ES, and various other relations for σ terms and axial couplings. Those relations have calculable corrections given solely by the nonanalytic loop contributions, thus providing useful tests for the accuracy of the effective theory and also serving as control tests of LQCD results through those same relations.

It is important to emphasize the importance of the decuplet in the effective theory, which has a key role in taming the nonanalytic contributions and thus improving the convergence, as it is clearly manifested in particular in the axial couplings. This improvement in the behavior of the effective theory when it is made consistent with the $1/N_c$ expansion permeates other observables, such as the mass relations and vector charges, as well as virtually any other observable, such as in pion-nucleon scattering, in Compton scattering, etc.

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APPENDIX A: SPIN-FLAVOR ALGEBRA AND OPERATOR BASES

The $4N_f^2-1$ generators of the spin-flavor group $SU(2N_f)$ consist of the three spin generators S^i, the N_f^2-1 flavor $SU(N_f)$ generators T^a, and the remaining $3(N_f^2-1)$ spin-flavor generators G^{ia}. The commutation relations are

$$\begin{gathered}[S^i,S^j]=i\epsilon^{ijk}S^k,\quad [T^a,T^b]=if^{abc}T^c,\quad [T^a,S^i]=0,\\ [S^i,G^{ja}]=i\epsilon^{ijk}G^{ka},\qquad [T^a,G^{ib}]=if^{abc}G^{ic},\\ [G^{ia},G^{jb}]=\frac{i}{4}\delta^{ij}f^{abc}T^c+\frac{i}{2N_f}\delta^{ab}\epsilon^{ijk}S^k+\frac{i}{2}\epsilon^{ijk}d^{abc}G^{kc}.\end{gathered} \tag{A1}$$

In representations with N_c indices (baryons), the generators G^{ia} have matrix elements $\mathcal{O}(N_c)$ on states with $S=\mathcal{O}(N_c^0)$. A contracted $SU(6)$ algebra is defined by the generators $\{S^i,I^a,X^{ia}\}$, where $X^{ia}=G^{ia}/N_c$. In large N_c, the generators X^{ia} become semiclassical as $[X^{ia},X^{jb}]=\mathcal{O}(1/N_c)$, and have matrix elements $\mathcal{O}(1)$ between baryons.

The symmetric irreducible representation of $SU(6)$ with N_c Young boxes decomposes into the following $SU(2)_{\rm spin}\times SU(3)$ irreducible representations: $[S,(p,q)]=[S,(2S,\frac{1}{2}(N_c-2S))]$, $S=1/2,\ldots,N_c/2$ (assumed N_c is odd). The baryon states are then denoted by $|SS_3,YII_3\rangle$. Clearly the spin S of the baryons determines its $SU(3)$ irreducible representation.

Some useful details about the contents of $SU(3)$ multiplets are in order. For a given irreducible representation (p,q), the range of hypercharge is

$$Y_{\min}(p,q)=-\frac{2p+q}{3}\le Y\le Y_{\max}(p,q)=\frac{p+2q}{3} \tag{A2}$$

Defining:

$$\begin{aligned}\bar{Y}(p,q)&=Y_{\max}(p,q)-q\\ \bar{Y}'(p,q)&=Y_{\min}(p,q)+q,\end{aligned} \tag{A3}$$

where $\bar{Y}>\bar{Y}'$ if $p>q$, and viceversa. The possible isospin values for a given Y are as follows:

$$\text{if } p\ge q\colon\ I(Y)=\begin{cases}\text{if}\quad Y\ge\bar{Y}\colon\ \frac{1}{2}(p-Y_{\max}+Y),\ldots,\frac{1}{2}(p+Y_{\max}-Y)\\ \text{if}\quad \bar{Y}'\le Y<\bar{Y}\colon\ \frac{1}{2}(p-Y_{\max}+Y),\ldots,\frac{1}{2}(p+Y_{\max}+Y-2\bar{Y})\\ \text{if}\quad Y_{\min}\le Y<\bar{Y}'\colon\ \frac{1}{2}(q+Y_{\min}-Y),\ldots,\frac{1}{2}(q+Y-Y_{\min})\end{cases}$$

$$\text{if } q\ge p\colon\ I(Y)=\begin{cases}\text{if}\quad Y\ge\bar{Y}'\colon\ \frac{1}{2}(p-Y_{\max}+Y),\ldots,\frac{1}{2}(p+Y_{\max}-Y)\\ \text{if}\quad \bar{Y}\le Y<\bar{Y}'\colon\ \frac{1}{2}(p+2\bar{Y}'-Y_{\max}-Y),\ldots,\frac{1}{2}(p+Y_{\max}-Y)\\ \text{if}\quad Y_{\min}\le Y<\bar{Y}\colon\ \frac{1}{2}(q+Y_{\min}-Y),\ldots,\frac{1}{2}(q+Y-Y_{\min})\end{cases}$$

1. Matrix elements of spin-flavor generators

The $SU(6)$ algebra involved in the calculations is quite lengthy and laborious, and therefore it is useful to provide basic details that are of help in implementing it. Here the matrix elements of the $SU(6)$ generators are summarized. Detailed presentations are given in Refs. [64,76]. In general the matrix elements of a $SU(2)_{\rm spin}\times SU(3)$ tensor operator between baryons of the form $|SS_3, RYII_3\rangle$, where R is the irreducible representation of $SU(3)$ to which the state belongs, will be given according to the Wigner-Eckart theorem in terms of reduced matrix elements and Clebsch-Gordan coefficients as follows:

$$\langle S'S'_3, R'Y'I'I'_3|O^{\ell\ell_3}_{\tilde R\tilde Y\tilde I\tilde I_3}|SS_3, RYII_3\rangle = \frac{1}{\sqrt{2S'+1}\sqrt{\dim R'}}\langle SS_3, \ell\ell_3|S'S'_3\rangle \times\sum_\gamma \langle S',R'||O^\ell_{\tilde R}||S,R\rangle_\gamma \left\langle \begin{matrix} R & \tilde R \\ YII_3 & \tilde Y\tilde I\tilde I_3\end{matrix}\middle| \begin{matrix} R' \\ Y'I'I'_3\end{matrix}\right\rangle_\gamma, \tag{A4}$$

where for $R'=(p,q)$, $\dim(R')=\frac{1}{2}(p+1)(q+1)(2+p+q)$, and γ is a multiplicity index associated to R' when the $SU(3)$ Clebsch-Gordan series of the direct product $R\otimes\tilde R\to R'$ contains R' more than once. Matrix elements of the spin-flavor generators between baryon states in the spin-flavor symmetric representation are then given by

$$\begin{aligned}
\langle S'S'_3, Y'I'I'_3|S^m|SS_3, YII_3\rangle &= \delta_{SS'}\delta_{YY'}\delta_{II'}\delta_{I_3I'_3}\sqrt{S(S+1)}\langle SS_3, 1m|S'S'_3\rangle\\
\langle S'S'_3, Y'I'I'_3|T^{yii_3}|SS_3, YII_3\rangle &= \delta_{SS'}\delta_{S_3S'_3}\frac{1}{\sqrt{\dim(2S,\frac{1}{2}(N_c-2S))}}\langle S||T||S\rangle\\
&\quad\times\left\langle \begin{matrix} (2S,\frac{1}{2}(N_c-2S)) & (1,1) \\ YII_3 & yii_3\end{matrix}\middle| \begin{matrix} (2S,\frac{1}{2}(N_c-2S)) \\ Y'I'I'_3\end{matrix}\right\rangle_{\gamma=1}\\
\langle S'S'_3, Y'I'I'_3|G^{m,yii_3}|SS_3, YII_3\rangle &= \frac{\langle SS_3, 1m|S'S'_3\rangle}{\sqrt{2S'+1}\sqrt{\dim(2S,\frac{1}{2}(N_c-2S))}}\\
&\quad\times\sum_{\gamma=1,2}\langle S'||G||S\rangle_\gamma\left\langle \begin{matrix} (2S,\frac{1}{2}(N_c-2S)) & (1,1) \\ YII_3 & yii_3\end{matrix}\middle| \begin{matrix} (2S,\frac{1}{2}(N_c-2S)) \\ Y'I'I'_3\end{matrix}\right\rangle_\gamma
\end{aligned}\tag{A5}$$

In the conventions of Ref. [64], for states $(p=2S, q=\frac{1}{2}(N_c-2S)$, the reduced matrix elements of the $SU(6)$ generators read [64]:

$$\begin{aligned}
\langle S'||T||S\rangle &= \delta_{SS'}\frac{\sqrt{(2S+1)(N_c-2S+2)(N_c+2S+4)(N_c(N_c+6)+12S(S+1))}}{4\sqrt{6}}\\
\langle S'||G||S\rangle_{\gamma=1} &= \begin{cases} \text{if}\quad S=S'+1: & -\frac{\sqrt{(4S^2-1)((N_c+2)^2-4S^2)((N_c+4)^2-4S^2)}}{8\sqrt{2}}\\ \text{if}\quad S=S'-1: & -\frac{\sqrt{(4S(S+2)+3)(N_c-2S)(N_c-2S+2)(N_c+2S+4)(N_c+2S+6)}}{8\sqrt{2}}\\ \text{if}\quad S=S': & \frac{(N_c+3)(2S+1)\sqrt{S(S+1)(N_c-2S+2)(N_c+2S+4)}}{\sqrt{6N_c(N_c+6)+12S(S+1)}}\end{cases}\\
\langle S'||G||S\rangle_{\gamma=2} &= -\delta_{SS'}\frac{(2S+1)\sqrt{(N_c-2S)(N_c+2S+6)((N_c+2)^2-4S^2)((N_c+4)^2-4S^2)}}{8\sqrt{2}\sqrt{N_c(N_c+6)+12S(S+1)}}
\end{aligned}\tag{A6}$$

In the case of the generators G and T, $\gamma=1,2$ because R' appears twice in the Clebsch Gordan series of $R\otimes\bar R\to R'$ when $R=R'=(p,q)$ and $\bar R=(1,1)$. In the case of T, the matrix element for $\gamma=2$ vanishes by definition.

2. Bases of spin-flavor composite operators

Here the bases of 2- and 3-body spin-flavor operators along with important operator relations relevant to this work are given.

TABLE VI. 2-body identities for the $SU(6)$ generators acting on the irreducible representation $(N_c, 0, 0, 0, 0, 0)$.

Relation	$SU(2)_{\rm spin} \times SU(3)$
$2\hat{S}^2 + 3\hat{T}^2 + 12\hat{G}^2 = \frac{5}{2}N_c(N_c+6)$	$(\ell = 0, \mathbf{1})$
$d^{abc}\{G^{ia}, G^{ib}\} + \frac{2}{3}\{S^i, G^{ic}\} + \frac{1}{4}d^{abc}\{T^a, T^b\} = \frac{2}{3}(N_c+3)T^c$	(0, $\mathbf{8}$)
$\{T^a, G^{ia}\} = \frac{2}{3}(N_c+3)S^i$	(1, $\mathbf{1}$)
$\frac{1}{3}\{S^i, T^a\} + d^{abc}\{T^b, G^{ic}\} - \epsilon^{ijk}f^{abc}\{G^{jb}, G^{kc}\} = \frac{4}{3}(N_c+3)G^{ia}$	(1, $\mathbf{8}$)
$-12\hat{G}^2 + 27\hat{T}^2 - 32\hat{S}^2 = 0$	(0, $\mathbf{1}$)
$d^{abc}\{G^{ib}, G^{ic}\} + \frac{9}{4}d^{abc}\{T^b, T^c\} - \frac{10}{3}\{S^i, G^{ia}\}$	(0, $\mathbf{8}$)
$4\{G^{ia}, G^{ib}\}^{\mathbf{27}} = \{T^a, T^b\}^{\mathbf{27}}$	(0, $\mathbf{27}$)
$d^{abc}\{T^b, G^{ic}\} = \frac{1}{3}(\{S^i, T^a\} - \epsilon^{ijk}f^{abc}\{G^{jb}, G^{kc}\})$	(1, $\mathbf{8}$)
$\epsilon^{ijk}\{G^{ja}, G^{kb}\}^{\mathbf{10}+\overline{\mathbf{10}}} = (f^{acd}d^{bce}\{T^d, G^{ie}\})^{\mathbf{10}+\overline{\mathbf{10}}}$	(1, $\mathbf{10}+\overline{\mathbf{10}}$)
$\{G^{ia}, G^{ja}\}^{\ell=2} = \frac{1}{3}\{S^i, S^j\}^{\ell=2}$	(2, $\mathbf{1}$)
$d^{abc}\{G^{ia}, G^{jb}\}^{\ell=2} = \frac{1}{3}\{S^i, G^{ja}\}^{\ell=2}$	(2, $\mathbf{8}$)

There are operator relations which are valid for matrix elements in the symmetric irreducible representation of $SU(6)$. The first ones are relations for 2-body operators [57], and are shown in Table VI. The relations in Ref. [57] are for general N_f, and the correspondence for $N_f = 3$ given here is as follows (left Ref. [57], right Table VI): $0 \to \mathbf{1}$, $\bar{s}s \to \mathbf{27}$, $\bar{a}s + \bar{s}a \to \mathbf{10} + \overline{\mathbf{10}}$, while there is no term $\bar{a}a$ for $N_f = 3$.

The following identities follow from Table VI, namely from the (0, $\mathbf{1}$) relations:

$$\hat{G}^2 = \frac{1}{4}\left(\frac{3}{4}N_c(N_c+6) - \frac{5}{3}S^2\right)$$
$$\hat{T}^2 = \frac{1}{4}\left(\frac{N_c(N_c+6)}{3} + 4\hat{S}^2\right), \qquad \text{(A7)}$$

from the (0, $\mathbf{8}$) relations:

$$d^{abc}\{G^{ib}, G^{ic}\} = \frac{3}{4}(N_c+3)T^a - \frac{7}{6}\{S^i, G^{ia}\}$$
$$d^{abc}\{T^b, T^c\} = -\frac{(N_c+3)}{3}T^a + 2\{S^i, G^{ia}\}, \qquad \text{(A8)}$$

TABLE VII. 2-body basis operators.

2-body operator	$(\ell, \mathbf{R})$
$\hat{S}^2$	(0, $\mathbf{1}$)
$\{S^i, S^j\}^{\ell=2}$	(2, $\mathbf{1}$)
$\{S^i, T^a\}$	(1, $\mathbf{8}$)
$\{S^i, G^{ia}\}$	(0, $\mathbf{8}$)
$\epsilon^{ijk}\{S^j, G^{ka}\}$	(1, $\mathbf{8}$)
$\{S^i, G^{ja}\}^{\ell=2}$	(2, $\mathbf{8}$)
$\{T^a, G^{ib}\}^{\mathbf{10}+\overline{\mathbf{10}}}$	(1, $\mathbf{10}+\overline{\mathbf{10}}$)
$\{T^a, T^b\}^{\mathbf{27}}$	(0, $\mathbf{27}$)
$\{G^{ia}, G^{jb}\}^{(2,\mathbf{27})}$	(2, $\mathbf{27}$)
$\{T^a, G^{ib}\}^{\mathbf{27}}$	(1, $\mathbf{27}$)

and from the (1, $\mathbf{8}$) relations:

$$\epsilon^{ijk}f^{abc}\{G^{ia}, G^{jb}\} = (S^kT^c - (N_c+3)G^{kc})$$
$$d^{abc}\{T^a, G^{ib}\} = 2d^{abc}T^aG^{ib} = \frac{1}{3}(S^iT^c + (N_c+3)G^{ic})$$
$$f^{abc}\{T^b, G^{ic}\} = \epsilon^{ijk}\{S^j, G^{ka}\}, \qquad \text{(A9)}$$

while the rest of the identities are explicit in Table VI. Making use of these relations, the basis of 2-body operators can be chosen to be as shown in Table VII:

Making use of the basis of 2-body operators, some lengthy work leads to building the basis of 3-body operators with $\ell = 0$, 1. That basis is displayed in Table VIII:

TABLE VIII. Operators of interest in the 3-body basis up to $\ell = 1$.

3-body operator	$(\ell, \mathbf{R})$
$T^a\hat{S}^2$	(0, $\mathbf{8}$)
$\{T^a, \{S^i, G^{ib}\}\}^{\mathbf{10}+\overline{\mathbf{10}}}$	(0, $\mathbf{10}+\overline{\mathbf{10}}$)
$\{T^a, \{S^i, G^{ib}\}\}^{\mathbf{27}}$	(0, $\mathbf{27}$)
$S^i\hat{S}^2$	(1, $\mathbf{1}$)
$\{T^a, \{T^b, T^c\}^{\mathbf{27}}\}$	(0, $\mathbf{8}\otimes\mathbf{27}$)
$S^i\{T^a, T^b\}^{\mathbf{27}}$	(1, $\mathbf{27}$)
$\{S^j, \{G^{ia}, G^{jb}\}^{(2,\mathbf{27})}\}$	(1, $\mathbf{27}$)
$\{\hat{S}^2, G^{ia}\}$	(1,$\mathbf{8}$)
$\epsilon^{ijk}\{S^j, \{T^a, G^{kb}\}\}^{\mathbf{10}+\overline{\mathbf{10}}}$	(1, $\mathbf{10}+\overline{\mathbf{10}}$)
$\epsilon^{ijk}\{S^j, \{T^a, G^{kb}\}\}^{\mathbf{27}}$	(1, $\mathbf{27}$)
$\{G^{ia}, \{T^b, T^c\}^{\mathbf{27}}\}$	(1, $\mathbf{8}\otimes\mathbf{27}$)
$\{G^{ia}, \{S^j, G^{jb}\}\}$	(1, $\mathbf{8}\otimes\mathbf{8}$)

APPENDIX B: BUILDING BLOCKS FOR THE EFFECTIVE LAGRANGIANS

In the symmetric representations of $SU(6)$ the baryon spin-flavor multiplet consists of the baryon states in the $SU(3)$ irreducible representations $(p=2S, q=\frac{1}{2}(N_c-2S))$, where S is the baryon spin. This permits a straightforward implementation of the nonlinear realization of chiral $SU_L(3)\times SU_R(3)$ on the spin-flavor multiplet. The baryon spin-flavor multiplet is given by the field $\mathbf{B}$, where the components of the field have well defined spin, and therefore also are in irreducible representations of $SU(3)$.

Defining as usual the Goldstone boson fields π^a, $a=1,\ldots,8$, through the unitary parametrization $u=\exp(i\frac{\pi^a T^a}{F_\pi})$ (note that in the fundamental representation $T^a=\lambda^a/2$, with λ^a the Gell-Mann matrices), for any isospin representation one defines a nonlinear realization of chiral symmetry according to [3,4]:

$$(L,R):\ u=u'=Ruh^\dagger(L,R,u)=h(L,R,u)uL^\dagger, \tag{B1}$$

where (L,R) is a $SU_L(3)\times SU_R(3)$ transformation. This equation defines h, and since h is a $SU(3)$ transformation itself, it can be written as $h=\exp(ic^aT^a)$. The chiral transformation on the baryon multiplet $\mathbf{B}$ is then given by

$$(L,R):\ \mathbf{B}=\mathbf{B}'=h(L,R,u)\mathbf{B}. \tag{B2}$$

On the other hand, spin-flavor transformations of interest are the contracted ones, namely those generated by $\{S^i, I^a, X^{ia}=\frac{1}{N_c}G^{ia}\}$. While the isospin transformations act on the pion fields in the usual way, and the spin transformations must be perfomed along with the corresponding spatial rotations. The transformations generated by X^{ia} are defined to only act on the baryons.

The effective baryon Lagrangian can be expressed in the usual way as a series of terms which are $SU_L(3)\times SU_R(3)$ invariant (upon introduction of appropriate sources; see for instance [77] for details). The fields in the effective Lagrangian are the Goldstone bosons parametrized by the unitary $SU(3)$ matrix field u and the baryons given by the symmetric $SU(6)$ multiplet $\mathbf{B}$.

The building blocks for the effective theory consist of low-energy operators composed in terms of the GB fields, derivatives and sources (chiral tensors), and spin-flavor composite operators (spin-flavor tensors).

The low-energy operators are the usual ones, namely,

$$\begin{aligned}
D_\mu &= \partial_\mu - i\Gamma_\mu,\\
\Gamma_\mu &= \Gamma_\mu^\dagger = \frac{1}{2}(u^\dagger(i\partial_\mu + r_\mu)u + u(i\partial_\mu + \ell_\mu)u^\dagger),\\
u_\mu &= u_\mu^\dagger = u^\dagger(i\partial_\mu + r_\mu)u - u(i\partial_\mu + \ell_\mu)u^\dagger,\\
\chi &= 2B_0(s+ip), \qquad \chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u,\\
F_L^{\mu\nu} &= \partial^\mu\ell^\nu - \partial^\nu\ell^\mu - i[\ell^\mu,\ell^\nu],\\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu],
\end{aligned} \tag{B3}$$

where D_μ is the chiral covariant derivative, s and p are scalar and pseudoscalar sources, and ℓ_μ and r_μ are gauge sources. It is convenient to define the $SU(3)$ singlet and octet components of $\chi^\pm$ using the fundamental $SU(3)$ irreducible representation, namely:

$$\begin{aligned}
\chi_\pm^0 &= \frac{1}{3}\langle\chi_\pm\rangle\\
\tilde\chi_\pm &= \chi_\pm - \chi_\pm^0 = \tilde\chi_\pm^a\frac{\lambda^a}{2}
\end{aligned} \tag{B4}$$

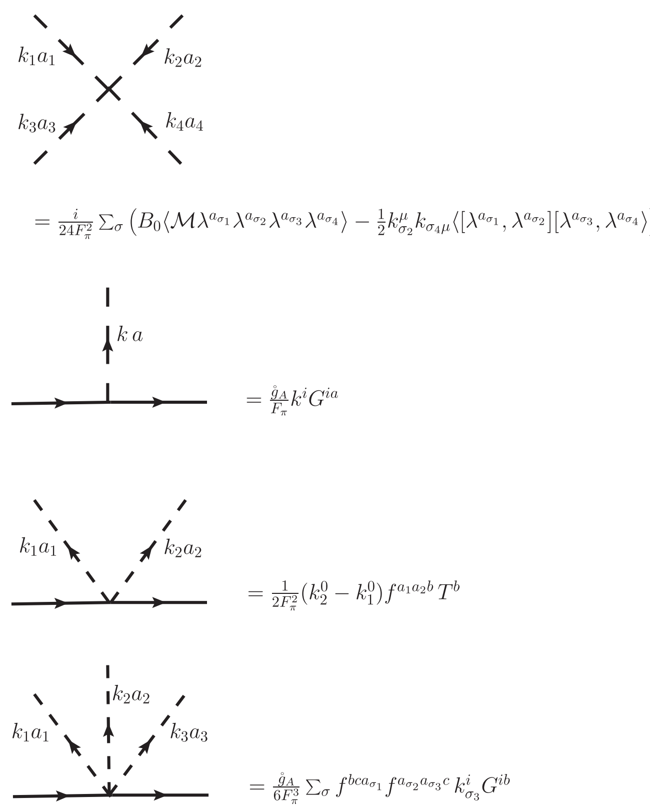

FIG. 5. Interaction vertices from the LO Lagrangians. $\mathcal{M}$ is the quark mass matrix. $\sum_\sigma$ indicates sum over the corresponding permutations.

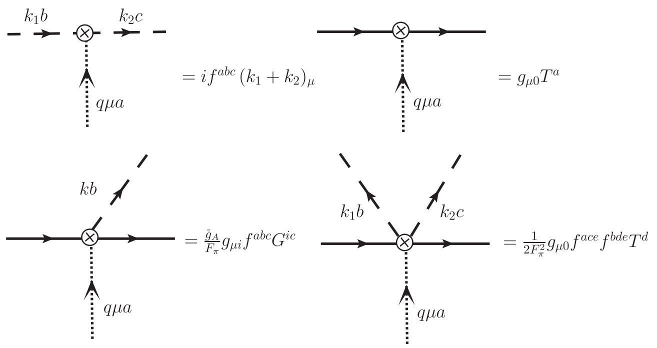

FIG. 6. Vertices involving the vector currents from the LO Lagrangians.

Displaying explicitly the quark masses,

$$\chi_+ = 4B_0 \mathcal{M}_q + \cdots. \tag{B5}$$

The three quark mass combinations, namely $SU(3)$ singlet, isosinglet, and isotriplet are, respectively, defined to be

$$m^0 = \frac{1}{3}(m_u + m_d + m_s), \qquad m^8 = \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s),$$
$$m^3 \equiv (m_u - m_d). \tag{B6}$$

The spin-flavor operators were discussed in Appendix A.

The leading-order equations of motion are used in the construction of the higher-order terms in the Lagrangian, namely, $iD_0\mathbf{B} = (\frac{C_{\rm HF}}{N_c}S(S+1) + \frac{c_1}{2\Lambda}\hat{\chi}_+)\mathbf{B}$, and $\nabla_\mu u^\mu = \frac{i}{2}\chi_-$.

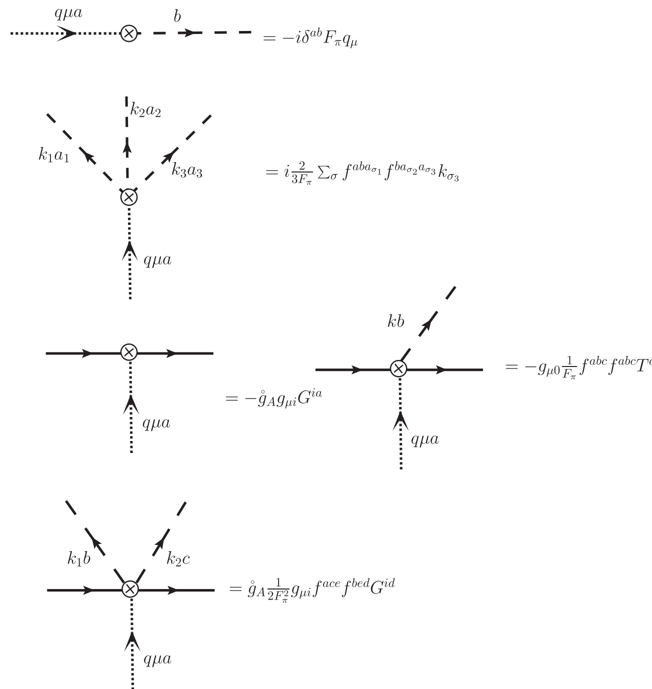

FIG. 7. Vertices involving the axial-vector currents from the LO Lagrangians.

1. Interaction vertices and currents at LO

The interaction vertices and the currents derived from the LO Lagrangian and needed for the one-loop calculations are given here for convenience. The interactions are depicted in Fig. 5, the vector currents in Fig. 6 and the axial-vector currents in Fig. 7.

APPENDIX C: LOOP INTEGRALS

The one-loop integrals needed in this work are provided here. The definition $\widetilde{d^dk} \equiv d^dk/(2\pi)^d$ is used.

The scalar and tensor one-loop integrals are

$$I(n,\alpha,\Lambda) \equiv \int \widetilde{d^dk}\frac{k^{2n}}{(k^2-\Lambda^2)^\alpha} = i(-1)^{n-\alpha}\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(n+\frac{d}{2})\Gamma(\alpha-n-\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(\alpha)}(\Lambda^2)^{n-\alpha+\frac{d}{2}}$$
$$I^{\mu_1,\ldots,\mu_{2n}}(\alpha,\Lambda) \equiv \int \widetilde{d^dk}\frac{k_{\mu_1}\cdots k_{\mu_{2n}}}{(k^2-\Lambda^2)^\alpha} = i(-1)^{n-\alpha}\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{4^n n!}\frac{\Gamma(\alpha-n-\frac{d}{2})}{\Gamma(\alpha)}(\Lambda^2)^{n-\alpha+\frac{d}{2}} \times \sum_\sigma g_{\mu_{\sigma_1}\mu_{\sigma_2}}\cdots g_{\mu_{\sigma_{2n-1}}\mu_{\sigma_{2n}}}$$
$$= \frac{1}{4^n n!}\frac{\Gamma(\frac{d}{2})}{\Gamma(n+\frac{d}{2})}I(n,\alpha,\Lambda)\sum_\sigma g_{\mu_{\sigma_1}\mu_{\sigma_2}}\cdots g_{\mu_{\sigma_{2n-1}}\mu_{\sigma_{2n}}}, \tag{C1}$$

where σ are the permutations of $\{1,\ldots,2n\}$.

The Feynman parametrizations needed when heavy propagators are in the loop are as follows:

$$\frac{1}{A_1\cdots A_m B_1\cdots B_n} = 2^m\Gamma(m+n)\int_0^\infty d\lambda_1\cdots d\lambda_m\int_0^1 d\alpha_1\cdots d\alpha_n\delta(1-\alpha_1-\cdots-\alpha_n)$$
$$\times\frac{1}{(2\lambda_1 A_1+\cdots+2\lambda_m A_m+\alpha_1 B_1+\cdots+\alpha_n B_n)^{m+n}}, \tag{C2}$$

where the A_i are heavy particle static propagators denominators, and the B_i are relativistic ones.

The integration over a Feynman parameter λ is of the general form:

$$J(C_0, C_1, \lambda_0, d, \nu) \equiv \int_0^\infty (C_0 + C_1(\lambda - \lambda_0)^2)^{-\nu+\frac{d}{2}} d\lambda, \tag{C3}$$

which satisfies the recurrence relation:

$$J(C_0, C_1, \lambda_0, d, \nu) = \frac{-\lambda_0(C_0 + C_1\lambda_0^2)^{1-\nu+\frac{d}{2}} + (3 + d - 2\nu)J(C_0, C_1, \lambda_0, d, \nu - 1)}{(d - 2\nu + 2)C_0}$$
$$J(C_0, C_1, \lambda_0, d, \nu) = C_0 \frac{d - \nu}{d - 2\nu + 1} J(C_0, C_1, \lambda_0, d, \nu + 1) + \frac{\lambda_0}{d - 2\nu + 1} (C_0 + C_1\lambda_0^2)^{\frac{d}{2}-\nu}. \tag{C4}$$

Integrals with factors of λ in the numerator are obtained by using

$$J(C_0, C_1, \lambda_0, d, \nu, n = 1) \equiv \int_0^\infty (\lambda - \lambda_0)^{n=1} (C_0 + C_1(\lambda - \lambda_0)^2)^{-\nu+\frac{d}{2}} d\lambda$$
$$= -\frac{1}{2C_1(\frac{d}{2} + 1 - \nu)} (C_0 + C_1\lambda_0^2)^{\frac{d}{2}+1-\nu}, \tag{C5}$$

and the recurrence relations

$$J(C_0, C_1, \lambda_0, d, \nu, n) = \frac{1}{C_1} (J(C_0, C_1, \lambda_0, d, \nu - 1, n - 1) - C_0 J(C_0, C_1, \lambda_0, d, \nu, n - 2)). \tag{C6}$$

For convenience, in some of the calculations for the currents, the following integral is defined:

$$\tilde{J}(C_0, C_1, \lambda_0, d, \nu, n) \equiv J(C_0, C_1, \lambda_0, d, \nu, n) + \lambda_0 J(C_0, C_1, \lambda_0, d, \nu) \tag{C7}$$

For the calculations in this work, the following integrals are needed at $d = 4 - 2\epsilon$:

$$J(C_0, C_1, \lambda_0, d, 3) = \frac{1}{\sqrt{C_0 C_1}} \left(\frac{\pi}{2} + \arctan\left(\lambda_0 \sqrt{\frac{C_1}{C_0}} \right) \right)$$
$$J(C_0, C_1, \lambda_0, d, 2) = \frac{1}{d - 3} (\lambda_0 (C_0 + C_1\lambda_0^2)^{\frac{d}{2}-2} + (d - 4) C_0 J(C_0, C_1, \lambda_0, d, 3))$$
$$J(C_0, C_1, \lambda_0, d, 1) = \frac{1}{d - 1} (\lambda_0 (C_0 + C_1\lambda_0^2)^{\frac{d}{2}-1} + (d - 2) J(C_0, C_1, \lambda_0, d, 2)) \tag{C8}$$

1. Specific integrals

Here a summary of relevant one-loop integrals for the calculations in this work is provided for the convenience of the reader.

(1) Loop integrals involving only relativistic propagators

$$
\begin{aligned}
I(0,1,M) &= -\frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right)M^{d-2} \\
I(0,2,M) &= \frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)M^{d-4} \\
I(1,1,M) &= \frac{i}{(4\pi)^{\frac{d}{2}}}\frac{d}{2}\Gamma\left(-\frac{d}{2}\right)M^{d} \\
I(1,2,M) &= -\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)M^{d-2} \\
K(q,M_a,M_b) &\equiv \int \widetilde{d^dk}\frac{1}{(k^2-M_a^2+i\epsilon)((k+q)^2-M_b^2+i\epsilon)} = \int_0^1 d\alpha I(0,2,\Lambda(\alpha)) \\
K^{\mu}(q,M_a,M_b) &\equiv \int \widetilde{d^dk}\frac{k^{\mu}}{(k^2-M_a^2+i\epsilon)((k+q)^2-M_b^2+i\epsilon)} = \int_0^1 d\alpha(\alpha-1)q^{\mu}I(0,2,\Lambda(\alpha)) \\
K^{\mu\nu}(q,M_a,M_b) &\equiv \int \widetilde{d^dk}\frac{k^{\mu}k^{\nu}}{(k^2-M_a^2+i\epsilon)((k+q)^2-M_b^2+i\epsilon)} \\
&= \int_0^1 d\alpha\left((1-\alpha)^2q^{\mu}q^{\nu}I(0,2,\Lambda(\alpha)) + \frac{g^{\mu\nu}}{d}I(1,2,\Lambda(\alpha))\right), \qquad \text{(C9)}
\end{aligned}
$$

where:

$$
\Lambda(\alpha) = \sqrt{\alpha M_a^2 + (1-\alpha)M_b^2 - \alpha(1-\alpha)q^2}
$$

(2) Loop integrals involving one heavy propagator

$$
\begin{aligned}
H(p^0,M) &\equiv \int \widetilde{d^dk}\frac{1}{(p^0-k^0+i\epsilon)(k^2-M^2+i\epsilon)} \\
&= \frac{2i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)J(M^2-p^{0^2},1,p^0,d,2) \\
H^{ij}(p^0,M) &\equiv \int \widetilde{d^dk}\frac{k^ik^j}{(p^0-k^0+i\epsilon)(k^2-M^2+i\epsilon)} \\
&= -\frac{i}{(4\pi)^{\frac{d}{2}}}g^{ij}\Gamma\left(1-\frac{d}{2}\right)J(M^2-p^{0^2},1,p^0,d,1) \\
H^{ij\mu}(p^0,M_a,M_b,q) &\equiv \int \widetilde{d^dk}\frac{k^i(k+q)^j(2k+q)^{\mu}}{(p^0-k^0+i\epsilon)(k^2-M_a^2+i\epsilon)((k+q)^2-M_b^2+i\epsilon)} \\
&= i\frac{4}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\left\{-\frac{1}{2}\Gamma\left(3-\frac{d}{2}\right)q^iq^j\alpha(1-\alpha)\right. \\
&\quad \times((1-2\alpha)q^{\mu}J(C_0,C_1,\lambda_0,d,3) - 2g^{\mu 0}\tilde{J}(C_0,C_1,\lambda_0,d,3,1)) \\
&\quad + \Gamma\left(2-\frac{d}{2}\right)((-(1-2\alpha)g^{ij}q^{\mu} + 2(\alpha g^{\mu i}q^j - (1-\alpha)g^{\mu j}q^i))J(C_0,C_1,\lambda_0,d,2) \\
&\quad \left. + 2g^{ij}g^{\mu 0}\tilde{J}(C_0,C_1,\lambda_0,d,2,1))\right\}, \qquad \text{(C10)}
\end{aligned}
$$

where:

$$
\begin{aligned}
C_0 &= \alpha M_a^2 + (1-\alpha) M_b^2 - p^{0^2} - 2(1-\alpha) p^0 q^0 - (1-\alpha)(\alpha q^2 + (1-\alpha) q^{0^2}) \\
C_1 &= 1 \\
\lambda_0 &= p^0 + (1-\alpha) q^0.
\end{aligned} \tag{C11}
$$

The polynomial pieces of the integrals are as follows:

$$
\begin{aligned}
H(p^0, M)^{\text{poly}} &= \frac{i}{(4\pi)^2} 2p^0(\lambda_\epsilon + 2) \\
H^{ij}(p^0, M)^{\text{poly}} &= \frac{i}{(4\pi)^2} \frac{p^0}{3} \left((3M^2 - 2p^{0^2})\lambda_\epsilon + 7M^2 - \frac{16}{3} p^{0^2} \right) \\
H^{ij0}(p^0, M_a, M_b, q)^{\text{poly}} &= \frac{i}{6(4\pi)^2} ((2q^i q^j + q^2 g^{ij})\lambda_\epsilon + q^2 g^{ij} - 3(\lambda_\epsilon + 1)(M_a^2 + M_b^2) g^{ij} \\
&\quad + 3(\lambda_\epsilon + 2)(2p^0 + q^0)^2 g^{ij}),
\end{aligned} \tag{C12}
$$

where the UV divergency is given by the terms proportional to $\lambda_\epsilon \equiv 1/\epsilon - \gamma + \log 4\pi$, where $d = 4 - 2\epsilon$.

APPENDIX D: USEFUL OPERATOR REDUCTIONS

The reductions of multi-body spin-flavor operators which appear in the polynomial contributions of the one-loop corrections to the self-energy and the currents require some lengthy work, and are therefore provided here. The reductions are only valid for matrix elements between states in the totally symmetric irreducible representation of $SU(6)$. In the following $\delta\hat{m}$ contains only the hyperfine term.

(1) Self-energy:

$$
\begin{aligned}
[[\delta\hat{m}, G^{ia}], G^{ia}] &= \frac{C_{\text{HF}}}{N_c} \left(\frac{7}{2} \hat{S}^2 - \frac{3}{8} N_c(N_c + 6) \right) \\
[[\delta\hat{m}, [\delta\hat{m}, G^{ia}]], G^{ia}] &= \left(\frac{C_{\text{HF}}}{N_c} \right)^2 \left(4\hat{S}^4 - (N_c(N_c + 6) - 18)\hat{S}^2 - \frac{3}{2} N_c(N_c + 6) \right) \\
[[\delta\hat{m}, [\delta\hat{m}, [\delta\hat{m}, G^{ia}]]], G^{ia}] &= \left(\frac{C_{\text{HF}}}{N_c} \right)^3 (36\hat{S}^4 - (5N_c(N_c + 6) - 36)\hat{S}^2 - 3N_c(N_c + 6)) \\
M_a^2 G^{ia} G^{ia} &= 2B_0 \left(m^0 \hat{G}^2 + m^a \left(-\frac{7}{24} \{S^i, G^{ia}\} + \frac{3}{16}(N_c + 3) T^a \right) \right) \\
M_a^2 [[\delta\hat{m}, G^{ia}], G^{ia}] &= 4 \frac{C_{\text{HF}}}{N_c} B_0 \left(\frac{8}{3} m^0 \hat{S}^2 + \frac{5}{12} m^a \{S^i, G^{ia}\} \right) - 4M_a^2 G^{ia} G^{ia}
\end{aligned} \tag{D1}
$$

(2) Vector currents:

$$
\begin{aligned}
G^{ia}[\delta\hat{m},[\delta\hat{m},G^{ia}]] &= \left(\frac{C_{\mathrm{HF}}}{N_c}\right)^2\left(\frac{3}{4}N_c(N_c+6)+\left(\frac{1}{2}Nc(N_c+6)-9\right)\hat{S}^2-2\hat{S}^4\right)\\
[\delta\hat{m},G^{ia}][\delta\hat{m},G^{ia}] &= -G^{ia}[\delta\hat{m},[\delta\hat{m},G^{ia}]]\\
G^{ib}T^a[\delta\hat{m},[\delta\hat{m},G^{ib}]] &= -[\delta\hat{m},G^{ib}]T^a[\delta\hat{m},G^{ib}]\\
&= \left(\frac{C_{\mathrm{HF}}}{N_c}\right)^2\left(3(N_c+3)S^iG^{ia}\right.\\
&\quad\left.+\left(\frac{3}{4}(N_c(N_c+6)-6)+\frac{1}{2}(N_c(N_c+6)-30)\hat{S}^2-2\hat{S}^4\right)T^a\right)\\
[[T^a,G^{ib}],[\delta\hat{m},[\delta\hat{m},G^{ib}]]] &= -[[T^a,[\delta\hat{m},G^{ib}]],[\delta\hat{m},G^{ib}]]\\
&= 2[\delta\hat{m},G^{ib}]T^a[\delta\hat{m},G^{ib}]-\{T^a,[\delta\hat{m},G^{ib}][\delta\hat{m},G^{ib}]\}\\
f^{abc}f^{bcd}M_b^2T^d &= 6B_0\left(m^0T^a+\frac{1}{4}d^{abc}m^bT^c\right)\\
M_b^2G^{ib}T^aG^{ib} &= 2B_0\left(m^0\left(\hat{G}^2-\frac{9}{8}\right)T^a+\frac{1}{2}m^b\left(\frac{1}{2}\{T^a,\frac{3}{8}(N_c+3)T^b-\frac{7}{24}S^iG^{ib}\}-\frac{3}{4}d^{abc}T^c\right)\right)\\
M_b^2[[T^a,G^{ib}],G^{ib}] &= \frac{9}{2}B_0\left(m^0T^a+\frac{1}{4}m^bd^{abc}T^c\right)
\end{aligned}
\tag{D2}
$$

(3) Axial-vector currents:

$$
\begin{aligned}
G^{jb}G^{ia}[\delta\hat{m},[\delta\hat{m},G^{jb}]]+\mathrm{H.c.} &= \left(\frac{C_{\mathrm{HF}}}{N_c}\right)^2\left(\frac{3}{2}N_c(N_c+6)G^{ia}+\left(\frac{1}{2}N_c(N_c+6)-14\right)\{\hat{S}^2,G^{ia}\}\right.\\
&\quad\left.-\{\hat{S}^2,\{\hat{S}^2,G^{ia}\}\}+\frac{3}{2}(N_c+3)S^iT^a+2S^iS^jG^{ja}\right)\\
[\delta\hat{m},G^{jb}]G^{ia}[\delta\hat{m},G^{jb}] &= \left(\frac{C_{\mathrm{HF}}}{N_c}\right)^2\left(-\frac{1}{2}\left(3+\frac{1}{2}N_c(N_c+6)\right)G^{ia}\right.\\
&\quad\left.+\frac{1}{2}\left(13-\frac{1}{2}N_c(N_c+6)\right)\{\hat{S}^2,G^{ia}\}+\frac{1}{2}\{\hat{S}^2,\{\hat{S}^2,G^{ia}\}\}-\frac{5}{4}(N_c+3)S^iT^a\right)\\
f^{acd}f^{bcd}M_c^2G^{ib} &= 6B_0\left(m^0\delta^{ab}+\frac{1}{4}m^cd^{abc}\right)G^{ib}\\
M_b^2G^{jb}G^{ia}G^{jb} &= \frac{1}{2}\{G^{ia},M_b^2G^{jb}G^{jb}\}-\frac{B_0}{12}\left(23m^0G^{ia}+m^b\left(\frac{5}{3}\delta^{ab}S^i+\frac{11}{4}d^{abc}G^{ic}\right)\right)
\end{aligned}
\tag{D3}
$$

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