# Radiative corrections to the hadronic vacuum polarization contribution to the muonium hyperfine interval

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Muonium is a purely leptonic system, but the accuracy of the theoretical QED prediction for its hyperfine interval is limited in part by hadronic effects. Here we consider radiative corrections to the leading hadronic vacuum polarization term, since their size is larger than the uncertainty in the calculation of the leading hadronic term itself. The total hadronic contribution of relative order of $\alpha$ in comparison with the leading hadronic term is found to be 4.97(19) Hz. The hadronic uncertainty sets an "ultimate" limit on the use of the $1s$ hyperfine interval in muonium as a QED variable.

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## I. INTRODUCTION

Quantum electrodynamics (QED) is a theory which, in principle, allows us to make theoretical predictions with an unlimited accuracy in the case of "purely leptonic" quantities. Unfortunately, such quantities do not exist in nature. Many theoretical predictions can be made for simple atoms. The most important atomic contributions can be obtained by considering one or a few electrons in an external field created by the nucleus. Ordinary atoms have nuclei, which are not pointlike, i.e., structureless particles. The effects of nuclear structure could not be treated *ab initio*. To take them into account one has to utilize experimental data on the nuclear structure and apply models.

There are two atoms available for experimental studies, muonium and positronium, that do not have a hadronic nucleus. In this paper we consider a muonium atom, which consists of an electron and a muon, where a heavier particle ($m_\mu/m_e \simeq 207$) plays the role of a nucleus. The most accurate QED tests can be performed on the hyperfine-structure (HFS) interval in the ground state. The history of studies of such a quantity is a long one and various QED contributions have been studied for decades. A review can be found in [1–3].

Even in such a purely leptonic two-body system, there are nonleptonic contributions present. The situation is pretty similar to that with the anomalous magnetic moment of the muon (see, e.g., [4,5]). Various contributions have QED diagrams with photon lines. The full photon propagator includes the vacuum-polarization loop and therefore, once we have a contribution with a photon line, there should be radiative corrections to it due to the vacuum-polarization effects. The latter have contributions due to all charged particles including hadrons. The leading hadronic contributions come from the hadronic vacuum polarization (hVP); see Figs. 1 and 2.

The hadronic contributions to the muonium HFS interval [6–10] and to the $g_\mu - 2$ (see, e.g., [4,5,11–14]) have reached the accuracy better than 1% and therefore their uncertainty is comparable to radiative corrections to the leading hadronic terms. In this paper, we study radiative corrections to the leading hadronic contribution to the HFS interval in the ground state of muonium.

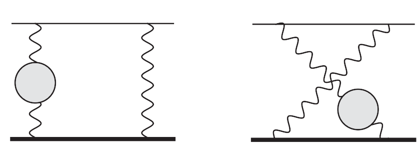

FIG. 1. Leading hadronic contribution to Mu HFS.

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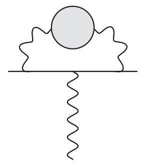

FIG. 2. Leading hadronic contribution to the anomalous magnetic moment of the muon [$a_\mu = (g_\mu - 2)/2$)].

This interval has been known with a high accuracy [15,16]

$$\nu_{\mathrm{HFS}}(\mathrm{exp}) = 4\,463\,302.776(51)\ \mathrm{kHz}, \tag{1}$$

and the progress seems possible. (In our paper we calculate the energy HFS interval and apply the relativistic units in which $\hbar = c = 1$, but all the numerical results are given in the frequency units, i.e., in terms of the related value of $\Delta E/h$.)

There are different estimations of the QED computational uncertainty, which vary from 70 to 200 Hz (see [2,3] for details.). Meanwhile the uncertainty in the leading term for the HFS interval is as large as 500 Hz and is determined by the accuracy of our (experimental) knowledge of the muon-to-electron mass ratio.

The status of the muonium spectroscopy as well as proposals for a new series of experiments are described in [17–20]. One may expect more accurate results in the close future. At present, the experimental accuracy is above the uncertainty of the hadronic contribution, but the situation may hopefully change soon. The QED uncertainty is also above the level of radiative corrections to the leading hadronic contributions. However, the QED contributions can be determined, in principle, more accurately depending on purely theoretical efforts. In contrast to that, any calculation of the hadronic contributions depends on the existence of the data and sets a kind of an ultimate limit for a theoretical prediction.

## II. TWO-PHOTON EXCHANGE AND THE LEADING HADRONIC CONTRIBUTION TO MUONIUM HFS INTERVAL

The diagrams for the leading hadronic VP contribution to the HFS interval in muonium are presented in Fig. 1. They are related to two-photon-exchange diagrams. Currently the hadronic contributions are evaluated by the data-driven dispersion approach resulting in an accuracy of 0.5% or even better for the dominant contribution of the $\rho$-meson final state (see [6–10] and [4,5,11–14] for details). The "first-principles-based" lattice calculations, although showing good progress, are not yet able to provide the subpercent accuracy needed to solve the problem of the muon anomalous magnetic moment [21,22].

Within the dispersion-relation approach, the hadronic VP is described with help of a substitution in the photon propagator

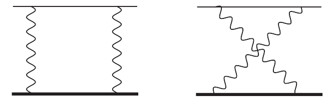

FIG. 3. The two-photon-exchange skeleton diagrams.

$$\frac{1}{k^2} \to \frac{\alpha}{\pi}\int \frac{ds\rho(s)}{k^2+s}, \tag{2}$$

where $k$ is the photon four-momentum and the parameter $s$ is the dispersion variable for the hVP.

The spectral density $\rho(s)$ can be, in principle, determined experimentally. In terms of the cross section of $e^+e^-$ annihilation into hadrons it takes the form

$$\rho(s) = \frac{R(s)}{3s}, \tag{3}$$

where

$$R(s) = \frac{\sigma(e^+e^- \to \gamma \to \mathrm{hadrons})}{4\pi\alpha^2/3s}. \tag{4}$$

Technically, the dispersion parameter $s$ plays a role of the photon mass squared and to take advantage of the dispersion presentation (3) we have to evaluate the skeleton diagrams (see Fig. 3) with one massless and one massive photon.

The contribution that is linear in hadronic vacuum polarization can be presented as an integral [23]

$$\Delta E(\mathrm{hVP\!:lead}) = 2\frac{\alpha(Z\alpha)}{\pi^2}\frac{m_e m_\mu}{m_\mu^2 - m_e^2}\tilde{E}_F \int ds K_{\mathrm{Mu}}(s)\rho(s), \tag{5}$$

where

$$K_{\mathrm{Mu}}(s) = \int_0^\infty dk L_0(k)\int \frac{1}{k^2+s}, \tag{6}$$

and

$$K_{\mathrm{Mu}}(s) = -\left(\frac{s}{4m_\mu^2}+2\right)\sqrt{1-\frac{4m_\mu^2}{s}}\ln\frac{1+\sqrt{1-\frac{4m_\mu^2}{s}}}{1-\sqrt{1-\frac{4m_\mu^2}{s}}} + \left(\frac{s}{4m_\mu^2}+\frac{3}{2}\right)\ln\frac{s}{m_\mu^2} - \frac{1}{2} \tag{7}$$

and

$$\tilde{E}_F = \frac{8}{3}(Z\alpha)^4\frac{m_e^2}{m_\mu}\left[\frac{m_R}{m_e}\right]^3. \tag{8}$$

The latter is the Fermi energy, the leading nonrelativistic contribution to the hyperfine interval in muonium. It is

TABLE I. Recent results for the leading hVP contribution to the muonium HFS interval (see Fig. 1).

| $\Delta E_{\mathrm{hVP:lead}}$ | Ref. |
|---|---|
| 240(7) Hz | [7] |
| 233(3) Hz | [8] |
| 232.5(2.5) Hz | [9] |
| 232.7(1.4) Hz | [10] |

customary to distinguish two similar values, the Fermi energy with the Dirac values of the magnetic moments, $\tilde{E}_F$, and a "full" nonrelativistic value with the actual value of the nuclear (muon) magnetic moment

$$E_F = \tilde{E}_F(1 + a_\mu).$$

Here, $Z$ stands for the nuclear charge and in muonium $Z = 1$, but it is customary to keep it in order to distinguish the effects due to radiative photons (with $\alpha$) and exchange photons (with $Z\alpha$).

An important factor in (6) is the leptonic factor, that for the skeleton two-photon-exchange diagram, which is responsible for the leading hVP contribution, is of the form

$$L_0(k) = 4\left[\left(\sqrt{4m_\mu^2 + k^2} - k\right) + \frac{k}{2}\left(-\frac{k}{4m_\mu^2}\sqrt{4m_\mu^2 + k^2} + \frac{k^2}{4m_\mu^2} + \frac{1}{2}\right)\right]. \quad (9)$$

An application of that or a similar expression with a realistic sophisticated description of the available experimental data on $R$ leads to results on the leading hVP term with an uncertainty at the level of 1% and even better (for earlier less accurate results see references in [7–10]). The most accurate results are summarized in Table I.

We note that most of the radiative corrections to the leading hadronic term are linear in hVP and can be presented in a similar form. To find those linear contributions we have to evaluate the radiative corrections to the leptonic factor $L(k)$. To study the radiative corrections to the leading hVP term which are suppressed by a factor of $\alpha \simeq 1/137$, we do not need a sophisticated description of $R$; a simple model will be sufficient.

## III. A MODEL FOR THE HADRONIC VP SPECTRAL FUNCTION $\rho(s)$

The model we use includes the contribution of meson resonances and a background. The dispersion density is defined as

$$\rho_{\mathrm{mod}}(s) = \rho_{\mathrm{res}}(s) + \rho_{\mathrm{b}}(s). \quad (10)$$

The resonances ($\rho$, $\omega$, $\phi$) are described with $\delta$ functions

TABLE II. Parameters of the $\rho$, $\omega$, and $\phi$ mesons [24].

| Resonance | $m_{\mathrm{res}}$, [MeV] | $4\pi/f_{\mathrm{res}}^2$ |
|---|---|---|
| $\rho$ | 775.26 | 1.61 |
| $\omega$ | 782.65 | 0.136 |
| $\phi$ | 1019.46 | 0.221 |

$$\rho_{\mathrm{res}}(s) = \sum_{\rho,\omega,\phi} \rho_{\mathrm{res}}^i(s), \quad (11)$$

where

$$\rho_{\mathrm{res}}^i(s) = \frac{4\pi^2}{f_{\mathrm{res}}^2}\delta(s - m_{\mathrm{res}}) \quad (12)$$

and their parameters are presented in Table II (cf. [24]).

The background contributions within the model are

$$\rho_{\mathrm{b}}(s) = \frac{R_{\mathrm{b}}(s)}{3s}, \quad (13)$$

where

$$R_b = \begin{cases} 2.3, & \text{for } 1.0\ \mathrm{GeV} < \sqrt{s} \le 3.7\ \mathrm{GeV}, \\ 3.8, & \text{for } \sqrt{s} > 3.7\ \mathrm{GeV}. \end{cases} \quad (14)$$

The model described above is our base model and it apparently has its limitations. To control the accuracy of our calculations we also consider a supporting model, which involves a somewhat more realistic description of the resonances (an asymmetric Lorentzian profile with the effective width $\Gamma$ depending on $s$) and a somewhat more advanced background (with a more complicated behavior at threshold between 1.0 and 1.5 GeV).

To estimate the quality of the base model, we calculate now the leading hVP contributions to the HFS interval in the ground state of muonium and to the anomalous magnetic moment of the muon $a_\mu$. The analytic expression for the former has been already discussed above [see (5)]. The result is

$$\Delta E^{\mathrm{model}}(\mathrm{hVP:lead}) = 239.9\ \mathrm{Hz}, \quad (15)$$

which should be compared to the complete result (see Table I). The deviation is about 5%. The application of the supporting model shifts the result by approximately 10%. The dominant contribution comes from the $\rho$ meson (see Table III).

TABLE III. The $\rho$-meson contributions to the leading hVP terms in the muonium HFS interval and in $a_\mu$.

| Term | $\rho$-Meson Contribution |
|---|---|
| $\Delta E_{\mathrm{HFS}}$ | 64% |
| $a_\mu$ | 70% |

A similar situation is with a calculation of the leading hVP contribution to $a_\mu$ [11] (see Fig. 2)

$$\Delta a_\mu(\text{hVP: lead}) = \frac{\alpha^2}{\pi^2}\int ds K_a(s)\rho(s), \tag{16}$$

where

$$K_{\rm a} = -\left(\frac{s^2}{2m_\mu^4} - \frac{2s}{m_\mu^2} + 1\right)\frac{1}{\sqrt{1-\frac{4m_\mu^2}{s}}}\ln\frac{1+\sqrt{1-\frac{4m_\mu^2}{s}}}{1-\sqrt{1-\frac{4m_\mu^2}{s}}} + \left(\frac{s^2}{2m_\mu^4} - \frac{s}{m_\mu^2}\right)\ln\frac{s}{m_\mu^2} - \frac{s}{m_\mu^2} + \frac{1}{2}. \tag{17}$$

The result of the application of the model reads

$$\Delta a_\mu^{\rm model}(\text{hVP: lead}) = 6.92\times 10^{-8}. \tag{18}$$

The alternative model shifts the result by 3% and the difference between the value in (18) and that obtained with full $R$ is approximately twice smaller. The dominant contribution to $\Delta a_\mu(\text{hVP: lead})$ comes from the $\rho$ meson (see Table III).

The calculation of these two quantities, which are known with the accuracy at the level of 1%, is helpful to estimate the uncertainty of our base model. The difference between the results obtained in the base model and the supporting one is of the order of a few percent. The difference between the results of our base model and those found with the "complete" description of the data for contributions to the muonium HFS interval [6–10] (see Table I) and to the anomalous magnetic moment of the muon [4,5,11–14] is somewhat smaller.

It is important that in all the results the dominant contribution to the hVP terms comes from the $\rho$ meson. With a complete consideration, the $\rho$ is the main source of the uncertainty. We conclude that if in the case of the evaluation of the corrections the contribution of the $\rho$ meson dominates, the accuracy of the model could be safely estimated as 10%.

Speaking about the accuracy of the calculation of the leading hVP term (see Table I) we note that there have been more efforts to estimate the leading hVP contribution to $a_\mu$. The $\rho$ meson not only produces a dominant contribution to the central value, but is also responsible for most of the uncertainty for the leading hVP terms in the muonium HFS interval and $a_\mu$. The $\rho$ contributions are comparable in them. That means that whatever progress is achieved for the leading hVP contribution to $a_\mu$, we should expect that an equally accurate evaluation is possible for the HFS interval. The accuracy at the level of a portion of percent is quite achievable. That is not an accuracy of a particular calculation for muonium [10]. That is also an accuracy which we should expect by adopting the model used for various calculations for $a_\mu$ (see, e.g., [12]) and applying them to muonium.

For both muonium HFS and $a_\mu$, the dominant contribution (70% or more) comes from the $\rho$ meson, which is known to very high accuracy of about 0.5%. The multibody hadronic final states have been measured much less accurately but now worse than 10%. A comparison of our approximate model with the sum of the measured cross sections shows that conservatively one can estimate the accuracy of our approach as better than 5%.

The dominance of the $\rho$-meson contribution also means that the momentum exchange is essentially higher than the muon scale

$$\frac{4m_\mu^2}{m_\rho^2} \simeq 0.076, \tag{19}$$

which is helpful for various estimations, allowing us to use numerous asymptotics.

## IV. RADIATIVE CORRECTIONS TO THE LEADING HVP TERM

The radiative corrections to the leading hVP term are presented in Fig. 4. All but the last [Fig. 4(g)] are QED radiative corrections and they are linear in hadronic vacuum polarizations. The last contribution is quadratic.

The radiative corrections were in part considered in [23,25], where some terms were evaluated with a large fractional uncertainty (essentially about 10%). In this section we intend to reach the level of 10% (see above) and below in this section we discuss a further improvement in the accuracy.

To discuss the radiative corrections to the leptonic factor $L_0(k)$ in (9), we note that all contributions similar to those presented in Fig. 4, but with a substitution of the hadronic vacuum polarization by the muonic one, have been calculated [26–29] and the results were considered in detail. The muonic vacuum polarization was taken there into account by a substitution

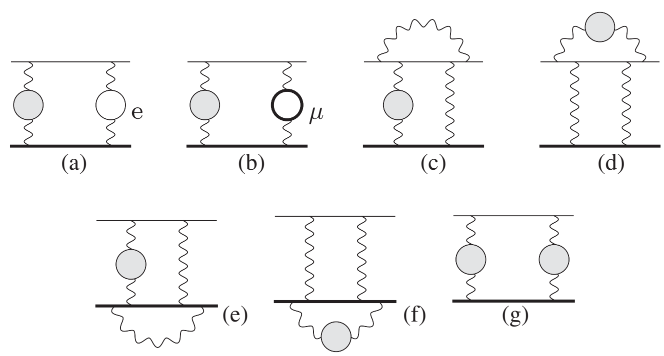


FIG. 4. Characteristic diagrams for the higher-order two-photon-exchange hVP contributions.

$$\frac{1}{k^2} \to \int_0^1 dv \frac{v^2(1-v^2/3)}{4m_\mu^2 + k^2(1-v^2)}, \tag{20}$$

which is a dispersion presentation as well as the substitution for the hadronic vacuum polarization in (2). The similarity of these two substitutions means that the $k$ integration for the leading term and for the QED radiative corrections to the leading term is the same for the hVP terms and muonic VP terms. The integral over the dispersion density is different. The radiative corrections to the integrand can be presented in terms of the corrections to the leptonic factor $L_0(k)$ in (9), which do not depend on the dispersion parameter. We adopt those corrections from the calculations with the muonic VP.

In particular, let us mention two of them.

To take into account the single-electron vacuum polarization [see Fig. 4(a)] we have to multiply the expression $L_0(k)$ by

$$\int_0^1 dv \frac{v^2(1-v^2/3)k^2}{4m_e^2 + k^2(1-v^2)}. \tag{21}$$

The combinatorial coefficient is 6. Another useful presentation of the same factor is

$$\frac{1}{3}\ln\frac{m_\mu^2}{m_e^2} + \frac{1}{3}\ln\frac{k^2}{m_\mu^2} - \frac{5}{9}, \tag{22}$$

which is sufficient for the hadronic contributions at $k \gg m_e$. The second presentation allows us to explicitly extract a logarithmically enhanced contribution.

The radiative correction for the diagrams of Fig. 4(b) is at $k \gg m_e$ described by a factor of [27]

$$-\frac{23}{12}.$$

A more accurate formula is also available [27].

The results of the calculations of the hVP contributions are presented in Table IV. Even if the asymptotics for $k \gg m_e$ (similar to those described above) are available, we applied completely accurate expressions for the radiative corrections. Certain marginal differences between the exact formulas and the asymptotic ones (essentially below the accuracy of the model) have been observed. We estimate the accuracy of the results linear in hVP as 10% (for the final result).

TABLE IV. The higher-order hadronic contributions to the muonium 1$s$ HFS interval. The classification of the terms follows Fig. 4.

| Term | $\Delta E_{\rm hVP}$ [Hz] | $\rho$-Meson Contribution |
|---|---|---|
| $a$ | 6.60 | 62% |
| $b$ | 0.70 | 55% |
| $c$ | −1.05 | 64% |
| $d$ | −0.38 | 61% |
| $e$ | −0.58 | 61% |
| $f$ | −0.31 | 61% |
| $a-f$ | 4.4(4) | 61% |

The model we use is not very accurate. However, the uncertainty can be nevertheless improved. Comparing Tables III and IV we note that the fractional contribution of the $\rho$ meson is approximately the same. That is an indication that the calculation of the leading hVP term and any QED radiative correction to it discussed above are correlated. By comparing the results of the base and supporting models we check that for $\Delta E_{\rm hvp:i}$ $(i = a, b, \ldots, f)$ the deviation of one model from the other is essentially larger (in fractional units) than the deviation of the ratio $\Delta E_{\rm hvp:i}/\Delta E_{\rm hvp:lead}$. The result with an improved estimation of the uncertainty is summarized in Table V.

TABLE V. The hVP contributions to the HFS interval in muonium in fractional units (with respect to the leading hVP term).

| Term | $\Delta E$ in Units of $\frac{\alpha}{\pi}\cdot\Delta E_{\rm hvp:lead}$ |
|---|---|
| $a$ | $\ln\frac{m_\mu^2}{m_e^2} - \frac{5}{3} + 1.475(35)$ |
| $b$ | 1.272(65) |
| $c$ | −1.876(1) |
| $d$ | −0.687(75) |
| $e$ | −1.057(20) |
| $f$ | −0.566(10) |
| $a-f$ | $\ln\frac{m_\mu^2}{m_e^2} - 2.70(10)$ |
| | $= 7.97(10)$ |

## V. OTHER HADRONIC CONTRIBUTIONS IN ORDER $\alpha E_{\rm hVP:lead}$

There is a radiative correction [see Fig. 4(g)] which is not a QED radiative correction to the leading hVP term. The contribution is quadratic in hVP effects. The calculation does not present a problem, while using the dispersion presentation for the hadronic vacuum polarization, and the result of the direct calculations is

$$\Delta E({\rm hVP{:}g}) = 0.58(11)\ {\rm Hz}. \tag{23}$$

We estimate the uncertainty as 20% (because it is quadratic in hVP).

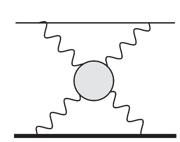

FIG. 5. A characteristic diagram for the hadronic light-by-light scattering contribution.

TABLE VI. The hVP contributions to the muonium HFS interval. The factor 1/2 for the $b$ term arises because of the difference in combinatorial coefficients; that is, 0.464 Hz that should be compared to 0.68 Hz. The leading hadronic term is taken from [8]. The uncertainty of the hadronic terms follows the results of Table V and the leading result from [8]. We consider the uncertainties as uncorrelated.

| Term | $\Delta E_{\rm hVP}$ [Hz] | $\Delta E_{\mu\rm VP}$ [Hz] |
|---|---|---|
| Leading | 233(3) [8] | 253.217 |
| $a$ | 5.89(2) | 6.158 [26] |
| $b$ | 0.68(3) | $1/2 \times 0.464$ |
| $c$ | −1.015(5) | −1.109 [27] |
| $d$ | −0.37(4) | −0.352 [28] |
| $e$ | −0.57(1) | −1.060 [27] |
| $f$ | −0.303(5) | −0.257 [29] |
| $g$ | 0.58(3) | 0.232 |
| hLbL | −0.0065(10) [30] | −0.495 [31] |

The accuracy can be improved by presenting the contribution in units of the leading hVP term as

$$\Delta E(\mathrm{hVP\!:g}) = 1.054(50)\frac{\alpha}{\pi}\Delta E_{\mathrm{hvp:lead}}. \quad (24)$$

The result is more accurate than the previous one, since the model dependence is partly removed once we compare the contribution of interest to $\Delta E_{\mathrm{hvp:lead}}$. That is a consequence of the domination of the $\rho$-meson contribution in $\Delta E(\mathrm{hVP\!:g})$.

There are two more contributions in order $\alpha\Delta E_{\mathrm{hvp:lead}}$. One of them is due to hadronic light-by-light (hLbL) scattering (see Fig. 5). The result for the hLbL contribution is negligibly small [30]

$$\Delta E(\mathrm{hLbL}) = -0.0065(10)\ \mathrm{Hz}.$$

For all the contributions above, similar diagrams with a substitution of the hadronic closed loop by a muonic one are known and it may be interesting to compare their numerical contributions (see Table VI). In many situations, the muonic result can be considered as a very rough preliminary estimation, but still a reasonable one, of the related hadronic term. That is correct for all the hVP contributions, but fails for the hLbL term, which is essentially smaller compared to its muon counterpart.

There is a contribution in order $\alpha\Delta E_{\mathrm{hvp:lead}}$ for which its muonic VP analog is not known. That is a three-photon hVP contribution (see Fig. 6). The contribution does not seem to have any logarithmic enhancement as we estimate it as [25]

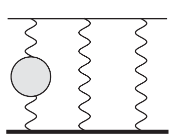

FIG. 6. A characteristic three-photon-exchange hVP diagram.

$$\Delta E(\mathrm{hVP\!:}3\gamma) = \pm 0.16\ \mathrm{Hz}.$$

## VI. CONCLUSION

To find the numerical value we rely on the result 233(3) Hz [8] for the leading hVP term and use the results in fractional units in Table V and in Eq. (24). We choose the result of [8] because, as we see from the situation with calculations of the leading hVP term to $a_\mu$, the results more accurate than 1% already exist; however, they have a large scatter. The uncertainty essentially below one percent for the leading term is possible; however, first, more accurate results for $a_\mu$ should be consistent. The results are summarized in Table VII.

The uncertainty due to three-photon contributions is the dominant one for the total uncertainty of the radiative corrections to the hadronic contribution in order $\alpha\Delta E_{\mathrm{hVP:lead}}$. A consideration of higher-order terms, such as $\alpha^2\Delta E_{\mathrm{hVP:lead}}$, shows that they are comparable to that uncertainty. For example, the leading $\alpha^2$ contribution (due to two-electronic VP) is in the leading logarithmic approximation

$$\frac{2\alpha^2}{3\pi^2}\ln^2\frac{m_\mu^2}{m_e^2}\Delta E_{\mathrm{hVP:lead}} \simeq 0.094\ \mathrm{Hz}.$$

The next-to-leading terms (with a single logarithm) are smaller, but numerically comparable to the leading $\alpha^2$ term and we assign the uncertainty of 50% to the leading logarithmic approximation for the $\alpha^2$ term.

The total result for the higher-order hadronic contributions is

$$\Delta E_{\mathrm{hadr:h.o}} = 4.97(19)\ \mathrm{Hz}, \quad (25)$$

which is consistent with previous less accurate estimations such as 7(2) Hz [23] and 5.0(15) Hz [25]. The uncertainty of the former calculations of the higher-order term is comparable to the uncertainty of the leading term in Table I and in case of any further progress in the leading term it would dominate within the hadronic sector.

TABLE VII. The total higher-order hadronic contributions to the $1s$ HFS interval in muonium.

| Term | $\Delta E_{\rm hVP}$ [Hz] | Ref. |
|---|---|---|
| Linear ($a-f$) | 4.31(6) | |
| Quadratic ($g$) | 0.57(3) | |
| hLbL | −0.0065(10) | [30] |
| $3\gamma$ hVP | ±0.16 | [25] |
| $\alpha^2$ | ±0.09(5) | |
| Total | 4.97(19) | |

The current uncertainty in the higher-order hadronic contributions in (25) does not have this problem. An improvement in the leading hVP term will immediately improve the complete hadronic uncertainty.

To conclude, we improved the calculation of the radiative corrections to the leading hadronic contribution to the 1*s* HFS interval in muonium compared to our previous paper [25]. The radiative corrections are definitely above the uncertainty of the calculation of the leading hVP term which is from 1 to 3 Hz [8–10] and could be likely improved to reach the uncertainty below 1 Hz. That opens an opportunity to check bound-state QED theory of the muonium HFS interval at the level of a part in $10^{10}$.

## ACKNOWLEDGMENTS

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