

# Ambiguity-free completion of the equations of motion of compact binary systems at the fourth post-Newtonian order

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We present the first complete (i.e., ambiguity-free) derivation of the equations of motion of two nonspinning compact objects up to the 4PN (post-Newtonian) order, based on the Fokker action of point particles in harmonic coordinates. The last ambiguity parameter is determined from first principle, by resorting to a matching between the near-zone and far-zone fields, and a consistent computation of the 4PN tail effect in  $d$  dimensions. Dimensional regularization is used throughout for treating IR divergences appearing at 4PN order, as well as UV divergences due to the modeling of the compact objects as point particles.

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## I. INTRODUCTION

The recent detection of gravitational waves (GW) generated by inspiralling and merging black-hole or neutron-star binaries [1,2] highlights the importance of the problems of motion and radiation for systems of compact objects in general relativity. Analytical relativity, based on the post-Newtonian (PN) approximation, i.e., a formal expansion when the speed of light  $c \rightarrow \infty$ , plays a key role in the development of high-accuracy GW templates to be used in the signal analysis of detectors. The templates are cross-correlated with the detector's output, and the correlation builds up when a good match occurs between a particular template and the real signal [3,4]. This technique is highly sensitive to the phase evolution of the signal, which, in PN templates of compact binary coalescence, is computed from the energy balance between the decay of the binary's energy and the GW flux. For isolated binary systems, the orbit will have circularized by radiation reaction at the time when the signal enters the detectors' bandwidth, so we expect that for the current generation of detectors, there is no need to invoke the balance of orbital angular momentum.

For low mass compact binaries, such as double neutron-star systems [2], the detectors are mostly sensitive to the inspiral phase prior to the final coalescence; in that case the currently known analytical PN templates are accurate

enough for detection (at least for moderate spins). For higher masses, like in black-hole binary systems, one must somehow connect the PN templates to the numerical relativity (NR) results describing the final merger and ringdown phases. The hybrid inspiral-merger-ringdown (IMR) waveforms [5] are constructed by matching the PN and NR waveforms in an overlapping time interval; the effective-one-body (EOB) waveforms [6] are based on resummation techniques extending the domain of validity of the PN approximation. The IMR and EOB waveforms constitute key techniques in the data analysis (both on-line and off-line) of the recent black-hole events [1].

The two basic ingredients in the theoretical PN analysis correspond to the two sides of the energy balance equation obeyed by the binary's orbital frequency and phase. The GW flux on the right-hand side is obtained by solving the wave generation problem; the state of the art is the 3.5PN approximation beyond the quadrupole formula (i.e., formal order  $\sim c^{-7}$ ; see [7] for a review), the 4.5PN coefficient being also known [8]. The energy function on the left-hand side follows from the conservative dynamics or equations of motion; after one century of works on the problem of motion (see for instance [9–18] and references therein) and the completion of the 3PN dynamics [19–25], the state of the art is the 4PN approximation beyond the Newtonian force.

Calculations at the 4PN order have been undertaken by means of three methods: (i) the Arnowitt-Deser-Misner (ADM) Hamiltonian formalism [26–29], which led to complete results but for the appearance of one “ambiguity” parameter; (ii) the Fokker Lagrangian in harmonic coordinates [30–33], which is complete at the exception, until

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recently, of one equivalent ambiguity parameter<sup>1</sup>; (iii) the effective field theory (EFT) [34–37], which yielded partial results up to now (the terms  $\propto G^4$  still being uncomputed) and is expected to be free of ambiguities [38].

The ambiguity parameters in the ADM formalism and in the Fokker action have been computed by resorting to perturbative gravitational self-force (GSF) determinations of the so-called redshift variable [39,40]. An analytic GSF calculation provided the 4PN coefficient in the redshift [41]; then, the first law of compact binary mechanics [42–44] enabled one to deduce the corresponding 4PN coefficients in the conserved energy and periastron advance for circular orbits, in the small mass-ratio limit, which was sufficient to fix the ambiguities. The final result for the 4PN Fokker Lagrangian is given in Sec. V of [30] with some  $G^4$  terms corrected in Appendix A of [31].<sup>2</sup> It is fully equivalent to the final result of the ADM Hamiltonian given in Appendix A of [28].

In this article—a companion paper of Ref. [32]—we detail the resolution of the important issue of the remaining (“last”) ambiguity parameter in the 4PN Lagrangian [30–32]. The ambiguity is due to the presence of IR divergences in the Fokker action, which are in turn associated with GW tails propagating at infinity. The tails are secondary nonlinear waves caused by the backscattering of linear waves onto the space-time curvature generated by the total mass of the source. As we shall see, the solution of the problem of ambiguities lies in performing the proper matching between the near-zone field described by the PN approximation and the far-zone radiation field. As a result of the matching, a contribution due to tails arises precisely at the 4PN order [45,46] in the particle’s action and the conservative dynamics. Due to this tail effect, the dynamics is nonlocal in time; this entails subtleties in the derivation of the invariants of motion and periastron advance, which have been dealt with in Refs. [28,29,31,44].

Another crucial ingredient in our approach, as well as in the EFT, is dimensional regularization, as it cures both IR divergences and concomitant UV divergences due to the point-particles model adopted to describe the compact objects. Dimensional regularization was introduced as a mean to preserve the gauge invariance of quantum gauge field theories [47–49]. Here, we use it in the problem of classical interaction of point masses, as a way to preserve the diffeomorphism invariance of general relativity [23,25].

<sup>1</sup>Two ambiguity parameters were introduced in Ref. [31]. In a first version of Ref. [32], one combination of these ambiguity parameters could be determined, but an incomplete implementation of the  $\epsilon\eta$  regularization (see below) did not permit to conclude on the “last” ambiguity parameter. We have updated the work [32] to take into account the new results presented in the present article.

<sup>2</sup>See also Ref. [33] for the Lagrangian and equations of motion in the frame of the center of mass and for a recapitulation of our result.

We argue that dimensional regularization is the only known method to successfully solve the problem at the 4PN order.

## II. OVERVIEW OF THE CALCULATION

We start from the complete gravitation-plus-matter action  $S = S_g + S_m$ , where the gravitational (Einstein-Hilbert) part  $S_g$  is written in the Landau-Lifshitz form with the usual harmonic gauge-fixing term, and where  $S_m$  is the matter part appropriate for two point particles without spin or internal structure [see Eqs. (2.1)–(2.2) in Ref. [30]]. The gauge-fixed Einstein field equations (GFEE) deriving from  $S$  read

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (2.1)$$

where  $\square$  is the flat d’Alembertian operator and where

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}. \quad (2.2)$$

The field variable  $h^{\mu\nu} = |g|^{1/2}g^{\mu\nu} - \eta^{\mu\nu}$  is the gothic metric deviation from the (inverse) Minkowski metric  $\eta^{\mu\nu}$ , with  $g^{\mu\nu}$  standing for the inverse metric and  $g$  for the metric determinant, while  $T^{\mu\nu}$  is the stress-energy tensor of the particles and  $\Lambda^{\mu\nu}$  the nonlinear gravitational source term, at least quadratic in  $h^{\mu\nu}$  or its space-time derivatives. The constant  $G$  is related to the usual Newton constant  $G_N$  in 3 dimensions by  $G = G_N \ell_0^{d-3}$  where  $d$  is the space dimension and  $\ell_0$  an arbitrary scale.

We shall denote by  $\bar{h}^{\mu\nu}$  the PN field constructed by standard PN iteration of the GFEE (2.1); such PN solution is a functional of the particle’s worldlines  $\mathbf{y}_A$  (with  $A = 1, 2$ ). The Fokker action for the binary is obtained by replacing the PN solution  $\bar{h}^{\mu\nu}[\mathbf{y}_A]$  back into the original action  $S$ , thus defining  $S_F[\mathbf{y}_A] = S(\mathbf{y}_A, \bar{h}[\mathbf{y}_A])$ . This action describes the purely gravitational dynamics of the compact binary system; it is equivalent, in the “tree-level” approximation, to the effective action used by the EFT approach [50,51].

The PN-expanded field  $\bar{h}^{\mu\nu}$  is physically valid in the near zone of the matter system, which is of small extent with respect to the radiation wavelength. On the other hand, the multipole expansion, denoted  $\mathcal{M}(h^{\mu\nu})$ , holds all over the exterior of the system including the far (or wave) zone. As the multipole expansion is a solution of the GFEE (2.1), it is also a functional of the particle’s worldlines. Our approach is based on the matching between the two expansions in the overlapping region where both approximations are valid, namely the exterior part of the near zone, which always exists for PN sources, i.e., slowly moving and weakly stressed sources.

The matching is achieved using a variant of the general method of matched asymptotic expansions [52–54]. More precisely, we impose the matching equation which states that the PN (or near-zone) expansion of the multipolar field should be identical to the multipole (or far-zone) expansion of the PN field:

$$\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{M}(\bar{h}^{\mu\nu}). \quad (2.3)$$

The general solution of the GFEE satisfying the above relation is known: The multipolar field in the exterior region is determined as a functional of the source parameters through the explicit expressions of the multipole moments [25,55]; the PN-expanded field in the near zone reads (generalizing results from [56,57] to  $d$  dimensions)

$$\bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} \overline{\square_{\text{ret}}^{-1}}[r^\eta \bar{\tau}^{\mu\nu}] + \mathcal{H}^{\mu\nu}. \quad (2.4)$$

The second term,  $\mathcal{H}^{\mu\nu}$ , is a homogeneous solution of the wave equation and will be discussed later. The first term is a particular retarded solution of the GFEE (2.1) when PN expanded in the near zone. It is defined from the retarded Green's function of the wave operator in  $d+1$  space-time dimensions as [32]

$$\begin{aligned} \overline{\square_{\text{ret}}^{-1}}[r^\eta \bar{\tau}^{\mu\nu}] &= -\frac{\tilde{k}}{4\pi} \int d^d \mathbf{x}' |\mathbf{x}'|^\eta \\ &\times \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \frac{\bar{\tau}^{\mu\nu}(\mathbf{x}', t-z|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|^{d-2}}, \end{aligned} \quad (2.5)$$

where the overbar refers to the PN expansion (see, notably, Appendix A in [32]),  $\tilde{k} = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}}}$  ( $\Gamma$  being the Eulerian function) and

$$\gamma_{\frac{1-d}{2}}(z) = \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)} (z^2-1)^{\frac{1-d}{2}}, \quad (2.6)$$

with the normalization condition  $\int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) = 1$ .

We have introduced in (2.5) a factor  $r^\eta$  multiplying the PN source term. Such a factor is similar to the regulator  $r^B$  entering the general solution of the matching equation in 3 dimensions [55–57]. However, an important difference is that, here, we do not need to take a “finite part” after integration (as we do in 3 dimensions). Indeed, the regulator  $r^\eta$  is inserted into the solution in  $d=3+\varepsilon$  dimensions so that it acts “on the top” of dimensional regularization. Our prescription is thus simply that we must consider first the limit  $\eta \rightarrow 0$  for any generic dimension  $d$  (i.e., avoiding integral values of  $d$ ) and check that, although divergences  $\propto 1/\eta$  can occur in individual terms, this limit is *finite* for the sum of terms we consider. Only afterwards do we apply the limit  $\varepsilon \rightarrow 0$  and look for the presence of poles  $1/\varepsilon$ . This regularization will be called the “ $\varepsilon\eta$ ” regularization.

The contribution of the particular solution [i.e., the first term in (2.4)] to the Fokker Lagrangian has been computed in Ref. [30]. The PN order to which one must truncate the metric to be inserted so as to control the Lagrangian up to a given  $n$ PN order is determined by the method “ $n+2$ ”

(see Sec. IVA in [30]): Focusing on the conservative dynamics, i.e., neglecting dissipative odd PN contributions, the various metric components, in the guise  $\bar{h} = (\bar{h}^{00ii}, \bar{h}^{0i}, \bar{h}^{ij})$  with the notation

$$\bar{h}^{00ii} = \frac{2}{d-1} [(d-2)\bar{h}^{00} + \bar{h}^{ii}], \quad (2.7)$$

are to be inserted into the action up to the orders  $(c^{-n-2}, c^{-n-1}, c^{-n-2})$  inclusively when  $n$  is even, and up to the orders  $(c^{-n-1}, c^{-n-2}, c^{-n-1})$  inclusively when  $n$  is odd. At the 4PN order, this means that the metric components are required up to the orders  $(c^{-6}, c^{-5}, c^{-6})$ . We parametrize the metric with the help of certain potentials defined in  $d$  dimensions; the most important are  $V$ ,  $V_i$  and  $\hat{W}_{ij}$ , which enter at lowest order:

$$\bar{h}^{00ii} = -\frac{4}{c^2} V + \mathcal{O}(c^{-4}), \quad (2.8a)$$

$$\bar{h}^{0i} = -\frac{4}{c^3} V_i + \mathcal{O}(c^{-5}), \quad (2.8b)$$

$$\bar{h}^{ij} = -\frac{4}{c^4} \left( \hat{W}_{ij} - \frac{1}{2} \delta_{ij} \hat{W}_{kk} \right) + \mathcal{O}(c^{-6}). \quad (2.8c)$$

[See Eq. (4.14) in [30] for the complete parametrization to the desired accuracy  $(c^{-6}, c^{-5}, c^{-6})$ .] The PN potentials obey a sequence of iterated flat space-time wave equations in  $d$  dimensions. Defining the particles' mass, current and stress densities as  $\sigma = \frac{2}{d-1} [(d-2)T^{00} + T^{ii}]/c^2$ ,  $\sigma_i = T^{0i}/c$ , and  $\sigma_{ij} = T^{ij}$ , we have

$$\square V = -4\pi G \sigma, \quad \square V_i = -4\pi G \sigma_i, \quad (2.9)$$

together with the more complicated nonlinear potential

$$\square \hat{W}_{ij} = -4\pi G \left( \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) - \frac{d-1}{2(d-2)} \partial_i V \partial_j V. \quad (2.10)$$

In the conservative dynamics, these potentials are generated by the standard symmetric propagator; this corresponds to the first term in Eq. (2.4), with the retarded inverse  $d$ 'Alembertian operator  $\overline{\square_{\text{ret}}^{-1}}$  replaced by the symmetric one. The resulting conservative dynamics is characterized by an equal amount of incoming and outgoing radiation. In the language of EFT, where the perturbative expansion is achieved with Feynman diagrams, the conservative sector is defined by diagrams that have no external graviton lines—the so-called “radiative” gravitons [51]. At the 4PN order, in the conservative sector, a process appears in which the graviton is emitted and then reabsorbed by the particles, and interacts with the total particles' mass through a “potential” graviton. This is

the tail effect, which has been computed in the context of EFT in Refs. [35,36].

### III. THE LAST AMBIGUITY PARAMETER

In our formalism, the computation of the last ambiguity parameter is achieved by means of a consistent derivation of the tail effect at the 4PN order in  $d$  dimensions, following the rules of the  $\epsilon\eta$  regularization. This effect is described by the second term in Eq. (2.4), which—as a consequence of the matching equation (2.3)—is a specific homogeneous solution of the wave equation, regular when  $r \rightarrow 0$ . Hence it is of the form

$$\mathcal{H}^{\mu\nu}(\mathbf{x}, t) = \sum_{\ell=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{1}{c^{2j}} \Delta^{-j} \hat{x}_L f_L^{(2j)\mu\nu}(t), \quad (3.1)$$

where the superscript  $(2j)$  refers to time derivatives,  $L = i_1 \cdots i_\ell$  is a multi-index made of  $\ell$  spatial indices,  $\hat{x}_L$  is the symmetric trace-free (STF) product of  $\ell$  spatial vectors  $x^i$ , and the  $\ell$  summations on the dummy spatial indices  $L$  are omitted. The  $j$ th iterated inverse Poisson operator  $\Delta^{-j}$  acts on  $\hat{x}_L$  as

$$\Delta^{-j} \hat{x}_L = \frac{\Gamma(\ell + \frac{d}{2})}{\Gamma(\ell + j + \frac{d}{2})} \frac{r^{2j} \hat{x}_L}{2^{2j} j!}. \quad (3.2)$$

Most importantly, the function  $f_L^{\mu\nu}(t)$  depends on the multipole expansion  $\mathcal{M}(\Lambda^{\mu\nu})$  of the gravitational source term in the GFEE (2.1). This reflects the fact that the PN-expanded solution in the near zone is sensitive, via the matching equation (2.3), to the boundary conditions obeyed by the radiation field, in particular the no-incoming radiation condition at past null infinity. We have shown that [32]

$$\begin{aligned} f_L^{\mu\nu}(t) &= \frac{(-)^{\ell+1} \tilde{k}}{4\pi\ell!} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \\ &\times \int d^d \mathbf{x}' |\mathbf{x}'|^\eta \hat{\partial}_L \left[ \frac{\mathcal{M}(\Lambda^{\mu\nu})(\mathbf{y}, t - zr'/c)}{r'^{d-2}} \right]_{\mathbf{y}=\mathbf{x}'}, \end{aligned} \quad (3.3)$$

where  $\hat{\partial}_L$  denotes the STF projection of a product of  $\ell$  partial derivatives  $\partial/\partial x'^i$ , being understood that the vector  $\mathbf{y}^i$  is to be treated as a constant when differentiating and replaced by  $x'^i$  only afterwards. Observe that Eq. (3.3) is also defined with the  $\epsilon\eta$  regularization.

In practice, the multipolar field  $\mathcal{M}(h^{\mu\nu})$  is computed by means of the so-called multipolar-post-Minkowskian (MPM) algorithm [55,58],

$$\mathcal{M}(h^{\mu\nu}) = h_{\text{MPM}}^{\mu\nu}. \quad (3.4)$$

The MPM field represents the most general solution of the vacuum GFEE outside the matter source. It consists of a formal post-Minkowskian (or post-linear) expansion

$$h_{\text{MPM}}^{\mu\nu} = \sum_{n=1}^{+\infty} G^n h_n^{\mu\nu}, \quad (3.5)$$

with each post-Minkowskian coefficient  $h_n^{\mu\nu}$  given in the form of a multipole expansion. The MPM algorithm starts from the most general multipolar solution of the linearized GFEE [59],

$$h_1^{00} = -\frac{4G}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \tilde{I}_L, \quad (3.6a)$$

$$h_1^{0i} = \frac{4G}{c^3} \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_{L-1} \tilde{I}_{iL-1}^{(1)}, \quad (3.6b)$$

$$h_1^{ij} = -\frac{4G}{c^4} \sum_{\ell=2}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_{L-2} \tilde{I}_{ijL-2}^{(2)}, \quad (3.6c)$$

where the mass-type multipole moments are denoted  $I_L(t)$  with, in particular, for the monopole case  $\ell = 0$ ,  $I = M$  representing the constant ADM mass; moreover, the tilde over the moments means

$$\tilde{I}_L(t, r) = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) I_L(t - zr/c), \quad (3.7)$$

which, in the monopole case, reduces to  $\tilde{M}(r) = \tilde{k} M r^{2-d}$ . For our purpose, we ignore the corresponding current-type multipole moments, which could be defined in  $d$  dimensions by means of “mixed Young tableaux” (see [60] for a discussion).

In order to determine the dominant tail effect at the 4PN order, we shall consider the quadratic interaction between the ADM mass  $M$  and the varying mass quadrupole moment  $I_{ij}(t)$ . Thus, we shall focus on the source term of the vacuum GFEE corresponding to that interaction:  $M \times I_{ij}$ . Now, the full source term reads in general

$$N_{\text{MPM}}^{\mu\nu} = \mathcal{M}(\Lambda^{\mu\nu}) = \sum_{n=2}^{+\infty} G^n N_n^{\mu\nu}, \quad (3.8)$$

with  $\mathcal{M}(\Lambda^{\mu\nu}) = \frac{16\pi G}{c^d} \mathcal{M}(\tau^{\mu\nu})$ , since the multipole expansion is a formal vacuum solution of the GFEE. Each MPM coefficient in (3.8) admits the decomposition

$$N_n^{\mu\nu}(\mathbf{x}, t) = \sum_{\ell=0}^{+\infty} \hat{n}_L N_{nL}^{\mu\nu}(r, t), \quad (3.9)$$

where  $\hat{n}_L$  is the STF product of  $\ell$  unit vectors  $n^i = x^i/r$ . Plugging (3.9) into (3.3), we obtain a related post-Minkowskian expansion  $f_L^{\mu\nu} = \sum_n G^n f_{nL}^{\mu\nu}$  with [32]



$$f_{nL}^{\mu\nu} = -\frac{1}{d+2\ell-2} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}-\ell}(z) \times \int_0^{+\infty} dr' r'^{-\ell+1+\eta} N_{nL}^{\mu\nu}(r', t-zr'/c). \quad (3.10)$$

To order  $n=2$ , for the interaction  $M \times I_{ij}$ , the source term is a sum of the type

$$N_{2,L}^{\mu\nu} = \sum r^{-k-2\varepsilon} \int_1^{+\infty} dy y^p \gamma_{-1-\frac{\varepsilon}{2}}(y) F_L^{\mu\nu}(t-yr/c), \quad (3.11)$$

where the sum ranges over integers  $k, p$ , and the function  $F_L^{\mu\nu}$  is made of the product of  $M$  and components of  $I_{ij}$ ; we have posed  $\varepsilon = d-3$ . In that case, the expression (3.10) becomes

$$f_{2,L}^{\mu\nu} = \sum \frac{(-)^{\ell+k} C_\ell^{p,k}}{2\ell+1+\varepsilon \Gamma(\ell+k-1+2\varepsilon-\eta)} \frac{\Gamma(2\varepsilon-\eta)}{\Gamma(\ell+k-1+2\varepsilon-\eta)} \times \int_0^{+\infty} d\tau \tau^{-2\varepsilon+\eta} F_L^{(\ell+k-1)\mu\nu}(t-\tau). \quad (3.12)$$

Interestingly, we could factorize out two of the three independent integrations in (3.10)–(3.11) into a single (though nontrivial looking) dimensionless coefficient

$$C_\ell^{p,k} = \int_1^{+\infty} dy y^p \gamma_{-1-\frac{\varepsilon}{2}}(y) \times \int_1^{+\infty} dz (y+z)^{\ell+k-2+2\varepsilon-\eta} \gamma_{-\ell-1-\frac{\varepsilon}{2}}(z). \quad (3.13)$$

The computation of this coefficient in analytic closed form is described in the Appendix D of [32].

We have applied the formulas (3.12)–(3.13) to obtain the dominant tail effect in the metric at the 4PN order, which is given, according to the matching procedure, by the homogeneous (regular at  $r=0$ ) solution (3.1). The result can be expressed in terms of a logarithmic kernel involving the combination

$$L(\tau) \equiv \ln\left(\frac{c\sqrt{\bar{q}}\tau}{2\ell_0}\right) - \frac{1}{2\varepsilon}, \quad (3.14)$$

where  $\bar{q} = 4\pi e^{\gamma_E}$ , with  $\gamma_E$  being the Euler constant, and  $\ell_0$  the dimensional regularization scale. Note the appearance of a pole  $\propto 1/\varepsilon$ , which originates from the lower integration bound  $\tau \rightarrow 0$  in (3.12) and is thus a UV pole. Applying the latter precepts along with the  $\varepsilon\eta$  regularization and expanding the result at the 4PN order, we arrive at [with  $\mathcal{H}^{00ii} = \frac{2}{d-1}[(d-2)\mathcal{H}^{00} + \mathcal{H}^{ii}]$ :

$$\mathcal{H}^{00ii} = \frac{8G_N^2 M}{15c^{10}} x^{ij} \int_0^{+\infty} d\tau \left[ L(\tau) + \frac{61}{60} \right] I_{ij}^{(7)}(t-\tau) + \mathcal{O}(c^{-12}), \quad (3.15a)$$

$$\mathcal{H}^{0i} = -\frac{8G_N^2 M}{3c^9} x^j \int_0^{+\infty} d\tau \left[ L(\tau) + \frac{107}{120} \right] I_{ij}^{(6)}(t-\tau) + \mathcal{O}(c^{-11}), \quad (3.15b)$$

$$\mathcal{H}^{ij} = \frac{8G_N^2 M}{c^8} \int_0^{+\infty} d\tau \left[ L(\tau) + \frac{4}{5} \right] I_{ij}^{(5)}(t-\tau) + \mathcal{O}(c^{-10}). \quad (3.15c)$$

We have made the important verification that the homogeneous solution (3.15) is divergenceless up to the required order, i.e.,  $\partial_\nu \mathcal{H}^{0\nu} = \mathcal{O}(c^{-13})$  and  $\partial_\nu \mathcal{H}^{i\nu} = \mathcal{O}(c^{-12})$ . We have also verified that the first term in (2.4) is separately divergenceless (using the matching equation for the considered interaction  $M \times I_{ij}$ ). Thus, the complete PN solution satisfies the harmonic gauge condition up to that order:  $\partial_\nu \bar{h}^{\mu\nu} = 0$ .

Finally, we insert these results into the Fokker action in order to compute the tail contribution therein. The quadratic interactions yield compact-support expressions depending on the values of the homogeneous solution (3.15) at the locations of the particles. However, a cubic term with noncompact support also needs to be consistently included in the action at the 4PN order, so that [30]

$$S_F^{\text{tail}} = \sum_A m_A c^2 \int dt \left[ -\frac{1}{8} \mathcal{H}_A^{00ii} + \frac{1}{2c} \mathcal{H}_A^{0i} v_A^i - \frac{1}{4c^2} \mathcal{H}_A^{ij} v_A^i v_A^j \right] - \frac{1}{32\pi G d-2} \int dt \int d^d \mathbf{x} \mathcal{H}^{ij} \partial_i V \partial_j V. \quad (3.16)$$

This cubic term has a twofold origin: it comes from (i) a direct cubic term  $\sim h\partial h\partial h$  in the action, and (ii) the quadratic nonlinearity in the source of the potential  $\hat{W}_{ij}$  [see Eq. (2.10)]. Inserting Eqs. (3.15) into (3.16), we observe that the noncompact support piece elegantly combines with the other terms to give a simple expression quadratic in the time derivatives of the quadrupole moment  $I_{ij}$ . In the end, we get the tail contribution to the action:

$$S_F^{\text{tail}} = \frac{2G_N^2 M}{5c^8} \int_{-\infty}^{+\infty} dt I_{ij}^{(3)}(t) \times \int_0^{+\infty} d\tau \left[ L(\tau) + \frac{41}{60} \right] I_{ij}^{(4)}(t-\tau), \quad (3.17)$$

which can be rewritten in a manifestly time-symmetric way (under time reversal) by means of a Hadamard partie finie (Pf) integral as

$$S_{\text{F}}^{\text{tail}} = \frac{G_N^2 M}{5c^8} \text{Pf}_{\tau_0} \iint \frac{dt dt'}{|t-t'|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t'), \quad (3.18)$$

where  $\tau_0$  denotes the usual cutoff scale, here given by  $\tau_0 = \frac{2\ell_0}{c\sqrt{q}} \exp[\frac{1}{2\epsilon} - \frac{41}{60}]$ .

Equations (3.17)–(3.18) describe the conservative part of the tail effect at the 4PN order. It is shown in [32] that, modulo an unphysical shift of the particle’s worldlines, the UV pole present in (3.17)–(3.18) cancels out the corresponding IR pole entering the gravitational part of the Fokker action computed with the method  $n+2$ ; furthermore, the associated dimensional regularization scale  $\ell_0$  cleanly disappears from the final Lagrangian.

The result (3.17)–(3.18) closes our ambiguity-free derivation of the 4PN equations of motion. Indeed, we found in [32] that the “last” ambiguity parameter, say  $\kappa$ , which is equivalent to the ambiguity parameter of the Hamiltonian formalism [28], is precisely given by the numerical constant entering the tail term when evaluated in  $3+\epsilon$  dimensions, beyond the pole  $1/\epsilon$ . Now, the value we obtain for this constant in (3.17), i.e.,  $\kappa = \frac{41}{60}$ , is in perfect agreement with that determined in [31,32] so as to recover GSF calculations of the conserved energy and periastron advance for circular orbits in the small mass-ratio limit.

Let us point out (as remarked in [32]) that the latter value of  $\kappa$  is exactly the one found in the computation of the tail

effect through EFT methods [see Eq. (3.3) in [36]]. This confirms that the EFT Lagrangian, when it is completed by all the instantaneous (nontail) terms up to the 4PN order, will be ambiguity-free like ours, and in agreement with GSF calculations.

We also want to stress the nice correspondence between the EFT approach and our formalism. In the EFT, the tail effect is computed as a Feynman diagram with one graviton emitted and absorbed by the particles, and one “potential” graviton responsible for the interaction with the total mass  $M$ . In our work, the tail effect is the consequence of the second term in Eq. (2.4), which represents a crucial additional homogeneous solution imposed by the matching between the near and far zones. In this respect, it seems that the lack of a consistent matching between the near and far zones in the ADM Hamiltonian formalism [26–29], i.e., an analogue of our Eqs. (2.3)–(2.4), forces this formalism to be still plagued by one ambiguity parameter (denoted  $C$  in [28]).

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- [1] B. Abbott *et al.* (LIGO Scientific Collaboration and VIRGO Collaboration), *Phys. Rev. Lett.* **116**, 061102 (2016).
  - [2] B. Abbott *et al.* (LIGO Scientific Collaboration and VIRGO Collaboration), *Phys. Rev. Lett.* **119**, 161101 (2017).
  - [3] K. Thorne, *Gravitational Radiation* (Cambridge University Press, Cambridge, 1987), pp. 330–458.
  - [4] A. Buonanno and B. Sathyaprakash, in *General Relativity and Gravitation: A Centennial Perspective*, edited by A. Ashtekar, B. Berger, J. Isenberg, and M. MacCallum (Cambridge University Press, Cambridge, 2015), p. 513.
  - [5] P. Ajith, M. Hannam, S. Husa, Y. Chen, B. Bruegmann, N. Dorband, D. Mueller, F. Ohme, D. Pollney, C. Reisswig *et al.*, *Phys. Rev. Lett.* **106**, 241101 (2011).
  - [6] A. Buonanno and T. Damour, *Phys. Rev. D* **59**, 084006 (1999).
  - [7] L. Blanchet, *Living Rev. Relativity* **17**, 2 (2014).
  - [8] T. Marchand, L. Blanchet, and G. Faye, *Classical Quantum Gravity* **33**, 244003 (2016).
  - [9] H. Lorentz and J. Droste, *The Motion of a System of Bodies Under the Influence of their Mutual Attraction, According to Einstein’s Theory* (Nijhoff, The Hague, 1937), p. 330; *Versl. K. Akad. Wet. Amsterdam* **26**, 392 and 649 (1917).
  - [10] A. Einstein, L. Infeld, and B. Hoffmann, *Ann. Math.* **39**, 65 (1938).
  - [11] V. Fock, *Theory of Space, Time, and Gravitation* (Pergamon, London, 1959).
  - [12] S. Chandrasekhar and F. Esposito, *Astrophys. J.* **160**, 153 (1970).
  - [13] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, *Prog. Theor. Phys.* **51**, 1220 (1974).
  - [14] T. Damour and N. Deruelle, *C. R. Acad. Sc. Paris* **293**, 537 (1981).
  - [15] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland Company, Amsterdam, 1983), pp. 59–144.
  - [16] G. Schäfer, *Ann. Phys. (N.Y.)* **161**, 81 (1985).
  - [17] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, 1993).
  - [18] L. Blanchet, G. Faye, and B. Ponsot, *Phys. Rev. D* **58**, 124002 (1998).
  - [19] P. Jaranowski and G. Schäfer, *Phys. Rev. D* **57**, 7274 (1998).
  - [20] L. Blanchet and G. Faye, *Phys. Rev. D* **63**, 062005 (2001).
  - [21] T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Rev. D* **63**, 044021 (2001); **66**, 029901(E) (2002).
  - [22] V. de Andrade, L. Blanchet, and G. Faye, *Classical Quantum Gravity* **18**, 753 (2001).
  - [23] T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Lett. B* **513**, 147 (2001).

- [24] L. Blanchet and B. R. Iyer, *Classical Quantum Gravity* **20**, 755 (2003).
- [25] L. Blanchet, T. Damour, and G. Esposito-Farèse, *Phys. Rev. D* **69**, 124007 (2004).
- [26] P. Jaranowski and G. Schäfer, *Phys. Rev. D* **86**, 061503(R) (2012).
- [27] P. Jaranowski and G. Schäfer, *Phys. Rev. D* **87**, 081503(R) (2013).
- [28] T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Rev. D* **89**, 064058 (2014).
- [29] T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Rev. D* **93**, 084014 (2016).
- [30] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, *Phys. Rev. D* **93**, 084037 (2016).
- [31] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, *Phys. Rev. D* **95**, 044026 (2017).
- [32] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, *Phys. Rev. D* **96**, 104043 (2017).
- [33] L. Bernard, L. Blanchet, G. Faye, and T. Marchand, *arXiv*: 1711.00283.
- [34] S. Foffa and R. Sturani, *Phys. Rev. D* **87**, 064011 (2013).
- [35] S. Foffa and R. Sturani, *Phys. Rev. D* **87**, 044056 (2013).
- [36] C. R. Galley, A. K. Leibovich, R. A. Porto, and A. Ross, *Phys. Rev. D* **93**, 124010 (2016).
- [37] S. Foffa, P. Mastrolia, R. Sturani, and C. Sturm, *Phys. Rev. D* **95**, 104009 (2017).
- [38] R. Porto and I. Rothstein, *arXiv*:1703.06433.
- [39] S. Detweiler, *Phys. Rev. D* **77**, 124026 (2008).
- [40] L. Barack and N. Sago, *Phys. Rev. D* **83**, 084023 (2011).
- [41] D. Bini and T. Damour, *Phys. Rev. D* **87**, 121501(R) (2013).
- [42] A. Le Tiec, L. Blanchet, and B. Whiting, *Phys. Rev. D* **85**, 064039 (2012).
- [43] A. Le Tiec, *Phys. Rev. D* **92**, 084021 (2015).
- [44] L. Blanchet and A. Le Tiec, *Classical Quantum Gravity* **34**, 164001 (2017).
- [45] L. Blanchet and T. Damour, *Phys. Rev. D* **37**, 1410 (1988).
- [46] L. Blanchet, *Phys. Rev. D* **47**, 4392 (1993).
- [47] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44**, 139 (1972).
- [48] C. G. Bollini and J. J. Giambiagi, *Phys. Lett.* **40B**, 566 (1972).
- [49] P. Breitenlohner and D. Maison, *Commun. Math. Phys.* **52**, 11 (1977).
- [50] W. Goldberger and I. Rothstein, *Phys. Rev. D* **73**, 104029 (2006).
- [51] S. Foffa and R. Sturani, *Classical Quantum Gravity* **31**, 043001 (2014).
- [52] W. Burke and K. Thorne, in *Relativity*, edited by M. Carmeli, S. Fickler, and L. Witten (Plenum Press, New York, 1970), pp. 209–228.
- [53] W. Burke, *J. Math. Phys. (N.Y.)* **12**, 401 (1971).
- [54] J. Anderson, R. Kates, L. Kegeles, and R. Madonna, *Phys. Rev. D* **25**, 2038 (1982).
- [55] L. Blanchet, *Classical Quantum Gravity* **15**, 1971 (1998).
- [56] O. Poujade and L. Blanchet, *Phys. Rev. D* **65**, 124020 (2002).
- [57] L. Blanchet, G. Faye, and S. Nissanke, *Phys. Rev. D* **72**, 044024 (2005).
- [58] L. Blanchet and T. Damour, *Phil. Trans. R. Soc. A* **320**, 379 (1986).
- [59] K. Thorne, *Rev. Mod. Phys.* **52**, 299 (1980).
- [60] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, *Phys. Rev. D* **71**, 124004 (2005).