Massive graviton geons

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We find vacuum solutions such that massive gravitons are confined in a local spacetime region by their gravitational energy in asymptotically flat spacetimes in the context of the bigravity theory. We call such self-gravitating objects massive graviton geons. The basic equations can be reduced to the Schrödinger-Poisson equations with the tensor "wave function" in the Newtonian limit. We obtain a nonspherically symmetric solution with $j = 2, \ell = 0$ as well as a spherically symmetric solution with $j = 0, \ell = 2$ in this system where j is the total angular momentum quantum number and ℓ is the orbital angular momentum quantum number, respectively. The energy eigenvalue of the Schrödinger equation in the nonspherical solution is smaller than that in the spherical solution. We then study the perturbative stability of the spherical solution and find that there is an unstable mode in the quadrupole mode perturbations which may be interpreted as the transition mode to the nonspherical solution. The results suggest that the nonspherically symmetric solution is the ground state of the massive graviton geon. The massive graviton geons may decay in time due to emissions of gravitational waves but this timescale can be quite long when the massive gravitons are nonrelativistic and then the geons can be long-lived. We also argue possible prospects of the massive graviton geons: applications to the ultralight dark matter scenario, nonlinear (in) stability of the Minkowski spacetime, and a quantum transition of the spacetime.

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I. INTRODUCTION

Recent observations found gravitational waves from black hole mergers in which a few percent of the energy of the system is radiated by the gravitational waves [1–4]. In this way, it is well known that the gravitational waves have their energy and change the background geometry. Therefore, it seems possible that the gravitational waves are gravitationally bounded in a local region of the spacetime by their own gravitational energy. This time-dependent self-gravitating object is called a gravitational *geon*, short for *gravitational-electromagnetic entity*, which was introduced by Wheeler [5]. Although a gravitational geon can be constructed in asymptotically flat spacetimes [6,7], the geon is not exactly periodic in time [8–15], and it has a finite lifetime. However, the instability of geons could be remedied in asymptotically anti–de Sitter (AdS) spacetimes since the AdS boundary can confine particles and waves as in a box. Gravitational geons in asymptotically AdS spacetimes are constructed in [16–18].

In the present paper, we consider asymptotically flat spacetimes and construct a gravitational geon of massive gravitons. The extra ingredient in the present analysis is a mass of a graviton. Although a graviton in general relativity (GR) is massless, theories with a massive graviton have received much attention. For instance, models with extra dimensions predict the existence of massive gravitons as well as a massless graviton as Kaluza-Klein modes. In other examples, the massive graviton has been discussed to give rise to the cosmic accelerating expansion (see [19,20] for reviews and references therein) and recently discussed as a candidate of dark matter [21–32].

Massive bosonic fields can form self-gravitating objects which are called *oscillatons* in a real massive scalar field [33], *boson stars* in a complex massive scalar field [34,35], and *Proca stars* in a complex massive vector field [36] (see also [37,38]). Hence, we can naturally expect that the massive graviton also yields self-gravitating solutions which we call *massive graviton geons*. Indeed, the paper [27] showed the coherent oscillations of the massive graviton lead to the Jeans instability and then the massive gravitons could be self-gravitated. However, the analysis [27] is based on the linear perturbation theory around the homogeneous background. It has not been cleared whether or not the massive gravitons indeed yield a nonperturbative localized object.

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We study nontrivial vacuum solutions to the bigravity theory. The bigravity theory contains a massless graviton and a massive graviton [39]. The energy-momentum tensor of the massive graviton was derived in [24,27] in the similar way to general relativity [40]. We find that the basic equations in the Newtonian limit can be reduced to the Schrödinger-Poisson equations with the tensor "wave function." Due to the tensorial feature, the solutions in this system have different properties from the scalar case. We construct two types of the massive graviton geons: the monopole geon and the quadrupole geon. The monopole geon is the spherically symmetric configuration of the massive graviton which corresponds to the eigenstate of the zero *total* angular momentum. On the other hand, the quadrupole geon is not spherically symmetric, instead, it is given by the quadrupole modes of the spherical harmonics. The quadrupole geon is the eigenstate of the zero *orbital* angular momentum. Although the field configuration of the quadrupole geon is not spherical, it yields the spherically symmetric energy density and then the gravitational potential of the quadrupole geon is spherically symmetric. We also discuss the stability of the geons. The monopole geon is unstable against quadrupole mode perturbations. We then expect a transition from the monopole geon to the quadrupole one. The quadrupole geon would be a ground state of the massive graviton geons.

The paper is organized as follows. In Sec. II, we introduce the bigravity theory and take the Newtonian limit with the self-gravity. We numerically construct the massive graviton geons in Sec. III. We find that not only the spherically symmetric configuration of the massive graviton but also a nonspherically symmetric configuration yields spherically symmetric energy density distributions. The perturbative stability of the monopole geon is studied in Sec. IV. We give a summary and discuss some physical aspects of the massive graviton geons in Sec. V. In Appendix A, we summarize multipole expansions of the tensor field and the definitions of the angular momenta. We also find an octupole configuration of the massive graviton geon in Appendix B.

II. NEWTONIAN LIMIT OF BIGRAVITY

We assume the existence of a massive graviton as well as a massless graviton. The gravitational action is given by [39]

$$S = \frac{1}{2\kappa_g^2}\int d^4x\sqrt{-g}R(g) + \frac{1}{2\kappa_f^2}\int d^4x\sqrt{-f}\mathcal{R}(f) - \frac{m_G^2}{\kappa^2}\int d^4x\sqrt{-g}\mathcal{U}(g,f), \tag{2.1}$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are two dynamical metrics, and $R(g)$ and $\mathcal{R}(f)$ are their Ricci scalars. The parameters κ_g^2 and κ_f^2 are the corresponding gravitational constants, while κ is defined by $\kappa^2 = \kappa_g^2 + \kappa_f^2$. Just for simplicity, to admit the Minkowski spacetime as a vacuum solution, we restrict the potential $\mathcal{U}$ as the form [41,42]

$$\mathcal{U} = \sum_{i=2}^{4} c_i\mathcal{U}_i(\mathcal{K}), \tag{2.2}$$

$$\begin{aligned}\mathcal{U}_2(\mathcal{K}) &= -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\mathcal{K}^{\mu}{}_{\alpha}\mathcal{K}^{\nu}{}_{\beta},\\ \mathcal{U}_3(\mathcal{K}) &= -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma}\mathcal{K}^{\mu}{}_{\alpha}\mathcal{K}^{\nu}{}_{\beta}\mathcal{K}^{\rho}{}_{\gamma},\\ \mathcal{U}_4(\mathcal{K}) &= -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\alpha}\mathcal{K}^{\nu}{}_{\beta}\mathcal{K}^{\rho}{}_{\gamma}\mathcal{K}^{\sigma}{}_{\delta},\end{aligned} \tag{2.3}$$

with

$$\mathcal{K}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\nu}, \tag{2.4}$$

where $(\sqrt{g^{-1}f})^{\mu}{}_{\nu}$ is defined by the relation

$$\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} = g^{\mu\rho}f_{\rho\nu}. \tag{2.5}$$

We can set $c_2 = -1$ by using the normalization of the parameter m_G. Then $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$ is a vacuum solution of the bigravity, and the parameter m_G describes the mass of the massive graviton propagating on the Minkowski background.

We assume that the curvature scales of the spacetimes are smaller than the graviton mass

$$|\partial^2 g_{\mu\nu}| \ll m_G^2, \qquad |\partial^2 f_{\mu\nu}| \ll m_G^2. \tag{2.6}$$

This means that we consider only weak gravity limit.[1]

We perform the perturbations up to second orders around Minkowski spacetime and discuss a bound state of perturbed gravitational field. The perturbed gravitational fields ($\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$) are rewritten into two modes, that is,

$$\text{massless mode}: \frac{1}{1+\alpha}(\delta g_{\mu\nu} + \alpha\delta f_{\mu\nu}), \tag{2.7}$$

$$\text{massive mode}: \frac{\alpha}{1+\alpha}(\delta g_{\mu\nu} - \delta f_{\mu\nu}). \tag{2.8}$$

where $\alpha := \kappa_g^2/\kappa_f^2$. Those two modes are split into four components by those frequencies: the low-frequency

[1]The inequalities (2.6) also lead to that we do not need to take into account the Vainshtein mechanism [43]. For instance, the exterior region of a localized object with a mass M yields $|\partial^2 g_{\mu\nu}| \sim GM/r^3$ and then (2.6) suggests the region outside the Vainshtein radius $r_V := (GM/m_G^2)^{1/3}$.

massless mode $\delta\overset{(0)}{g}_{\mu\nu}$, the low-frequency massive mode $M_{\mu\nu}$, the high-frequency massless mode $h_{\mu\nu}$, and the high-frequency massive mode $\varphi_{\mu\nu}$, i.e.,

$$\text{massless mode: } \delta\overset{(0)}{g}_{\mu\nu} + h_{\mu\nu}/M_{\rm pl} \tag{2.9}$$

$$\text{massive mode: } M_{\mu\nu} + \varphi^{\mu\nu}/M_G, \tag{2.10}$$

where

$$M_{\rm pl} \coloneqq \frac{\kappa}{\kappa_g\kappa_f}, \qquad M_G \coloneqq \frac{\kappa}{\kappa_g^2}. \tag{2.11}$$

Here the high frequency means that the frequency is same or larger than the graviton mass m_G, while the low frequency is that much smaller than m_G. Intuitively, $\delta\overset{(0)}{g}_{\mu\nu}$ and $M_{\mu\nu}$ represent the Newtonian potential and the Yukawa potential, respectively, while the high-frequency modes $h_{\mu\nu}$ and $\varphi_{\mu\nu}$ are the propagating massless and massive spin-2 fields (see [27] for more details).

We shall focus on the scale beyond the Compton wavelength of the massive graviton (m_G^{-1}). In this scale, $M_{\mu\nu}$ can be ignored due to the Yukawa suppression. Since we are interested in a localized object composed of the massive graviton $\varphi_{\mu\nu}$, we assume the massless graviton $h_{\mu\nu}$ does not appear. As a result, the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ are approximated by

$$g_{\mu\nu} \simeq \overset{(0)}{g}_{\mu\nu} + \frac{\varphi_{\mu\nu}}{M_G}, \tag{2.12}$$

$$f_{\mu\nu} \simeq \overset{(0)}{g}_{\mu\nu} - \frac{\varphi_{\mu\nu}}{\alpha M_G}, \tag{2.13}$$

with $|\varphi_{\mu\nu}|/M_G \ll |\overset{(0)}{g}_{\mu\nu}|$ where $\overset{(0)}{g}_{\mu\nu} = \eta_{\mu\nu} + \delta\overset{(0)}{g}_{\mu\nu}$.

We then consider the perturbations up to second orders. Just the similar to the GR case [40], the high frequency modes $\varphi_{\mu\nu}$ give the source for the low frequency gravitational potential, when we take the Isaacson average. The low-frequency projection (the Isaacson average) is usually chosen as the spatial average (or the spacetime average) because of massless fields. However, here we consider the massive graviton, where momentum is much smaller compared with the rest mass energy m_G. Hence we perform the time average over the time interval $T = 2\pi/m_G$ and we do not take any spatial average.

Under this setting, the Einstein equations in bigravity without matter fluids are reduced into the Einstein-Klein-Gordon equations

$$\overset{(0)}{G}{}^{\mu\nu} = \frac{1}{M_{\rm pl}^2}\langle T_G^{\mu\nu}\rangle_{\rm low}, \tag{2.14}$$

$$(\overset{(0)}{\nabla}_\alpha\overset{(0)}{\nabla}{}^\alpha - m_G^2)\varphi_{\mu\nu} = 0 + \mathcal{O}(\varphi_{\mu\nu}^2), \tag{2.15}$$

with the constraints

$$\overset{(0)}{\nabla}_\mu\varphi^{\mu\nu} = 0 + \mathcal{O}(\varphi_{\mu\nu}^2), \qquad \varphi^\mu{}_\mu = 0 + \mathcal{O}(\varphi_{\mu\nu}^2), \tag{2.16}$$

where we have ignored the higher order corrections of $\varphi_{\mu\nu}$ to the Einstein-Klein-Gordon equations. The Einstein tensor $\overset{(0)}{G}_{\mu\nu}$ is constructed by the low-frequency background $\overset{(0)}{g}_{\mu\nu}$ while $T_G^{\mu\nu}$ is the energy-momentum tensor of the high-frequency perturbations $\varphi_{\mu\nu}$. The energy-momentum tensor of massive graviton is defined by

$$\begin{aligned} T_G^{\mu\nu} = &-\left(\overset{(0)}{g}{}^{\mu\alpha}\overset{(0)}{g}{}^{\nu\beta} - \frac{1}{2}\overset{(0)}{g}{}^{\mu\nu}\overset{(0)}{g}{}^{\alpha\beta}\right)\delta\overset{(2)}{R}_{\alpha\beta}[\varphi] \\ &- \frac{m_G^2}{8}(4\varphi^{\mu\alpha}\varphi^\nu{}_\alpha - \overset{(0)}{g}{}^{\mu\nu}\varphi^{\alpha\beta}\varphi_{\alpha\beta}) + \mathcal{O}(\varphi_{\mu\nu}^3), \end{aligned} \tag{2.17}$$

with

$$\begin{aligned} &\delta\overset{(2)}{R}_{\mu\nu}[\varphi] \\ &= \left[\frac{1}{4}\overset{(0)}{\nabla}_\mu\varphi^{\alpha\beta}\overset{(0)}{\nabla}_\nu\varphi_{\alpha\beta} + \overset{(0)}{\nabla}{}^\alpha\varphi^\beta{}_\nu\overset{(0)}{\nabla}_{[\alpha}\varphi_{\beta]\mu}\right. \\ &\quad + \frac{1}{2}\varphi^{\alpha\beta}(\overset{(0)}{\nabla}_\nu\overset{(0)}{\nabla}_\mu\varphi_{\alpha\beta} + \overset{(0)}{\nabla}_\alpha\overset{(0)}{\nabla}_\beta\varphi_{\mu\nu} - 2\overset{(0)}{\nabla}_\alpha\overset{(0)}{\nabla}_{(\mu}\varphi_{\nu)\beta}) \\ &\quad \left. + \left(\frac{1}{2}\overset{(0)}{\nabla}{}^\beta\varphi^\alpha{}_\alpha - \overset{(0)}{\nabla}_\alpha\varphi^{\alpha\beta}\right)\left(\overset{(0)}{\nabla}_{(\mu}\varphi_{\nu)\beta} - \frac{1}{2}\overset{(0)}{\nabla}_\beta\varphi_{\mu\nu}\right)\right]. \end{aligned} \tag{2.18}$$

The time average of this energy-momentum tensor gives the source of the low-frequency background $\overset{(0)}{g}_{\mu\nu}$. We have used the notations such that the suffixes on $\varphi_{\mu\nu}$ are raised and lowered by $\overset{(0)}{g}_{\mu\nu}$ and $\overset{(0)}{\nabla}_\mu$ is the covariant derivative with respect to $\overset{(0)}{g}_{\mu\nu}$.

We then take the Newtonian limit: the nonrelativistic limit of the massive graviton as well as the weak gravitational field approximation as for the background. The background metric is given by

$$\overset{(0)}{g}_{\mu\nu}dx^\mu dx^\nu = -(1+2\Phi)dt^2 + (1+2\Psi)\gamma_{ij}dx^i dx^j, \tag{2.19}$$

whereas the massive graviton is expressed by

$$\varphi_{\mu\nu} = \begin{pmatrix} \psi_{00} & \psi_{0i} \\ * & \frac{\psi_{\rm tr}}{3}\gamma_{ij} + \psi_{ij} \end{pmatrix}\frac{1}{\sqrt{2}}e^{-im_G t} + \text{c.c.}, \tag{2.20}$$

where $\{\Phi, \Psi, \psi_{00}, \psi_{0i}, \psi_{\rm tr}, \psi_{ij}\}$ are slowly varying functions of (t, x^i), e.g. $\partial_t^2\psi_{ij} \ll m_G\partial_t\psi_{ij}$, and ψ_{ij} is traceless,

$\psi^i{}_i = 0$. The three-dimensional Euclidean metric is denoted by γ_{ij} and ∂_i is the derivative with respect to γ_{ij}. We shall retain the three-dimensional covariance thus γ_{ij} and ∂_i are not necessary to be δ_{ij} and the partial derivatives, respectively. The indices i, j are raised and lowered by γ_{ij}. The constraint equations (2.16) yield

$$\psi_{\rm tr} = \psi_{00}, \qquad \psi_{00} = \frac{i}{m_G}\partial^i\psi_{i0}, \qquad \psi_{0i} = \frac{i}{m_G}\partial^j\psi_{ij}, \tag{2.21}$$

which indicate the inequalities

$$|\psi_{00}|, |\psi_{\rm tr}| \ll |\psi_{0i}| \ll |\psi_{ij}|. \tag{2.22}$$

Using the time average $\langle\cdots\rangle_{\rm low}$, we obtain

$$\langle T_G^{\mu\nu}\rangle_{\rm low} \simeq {\rm diag}\left[\frac{m_G^2}{4}\psi^*_{ij}\psi^{ij}, 0, 0, 0\right], \tag{2.23}$$

where $*$ denotes the complex conjugate. The equations are reduced to the Poisson-Schrödinger equations

$$\Delta\Phi = \frac{m_G^2}{8M_{\rm pl}^2}\psi^*_{ij}\psi^{ij}, \tag{2.24}$$

$$i\frac{\partial}{\partial t}\psi_{ij} = \left(-\frac{\Delta}{2m_G} + m_G\Phi\right)\psi_{ij}, \tag{2.25}$$

where $\Delta = \partial_i\partial^i$. The spatial component of the Einstein equation leads to $\Psi = -\Phi$. Note that these equations (2.24) and (2.25) are invariant under the rescaling

$$\Phi \to \lambda^2\Phi, \qquad \psi_{ij} \to \lambda^2\psi_{ij}, \qquad |x^i| \to \lambda^{-1}|x^i|, \qquad t \to \lambda^{-2}t, \tag{2.26}$$

thus, the system is scale invariant. We also note that the "wave function" of the Schrödinger equation is a tensor field ψ_{ij} which yields differences from the scalar wave function case as we will see in the following sections.

III. SELF-GRAVITATING MASSIVE GRAVITONS

In this section, we study the periodical solutions with the localized massive gravitons, i.e., the bound states of the system (2.24) and (2.25). To find the bound states, we focus on some eigenstates of the angular momentum of ψ_{ij}. As detailed in Appendix A, the angular dependence of each eigenstate is given by the so-called pure-orbital spherical harmonics $(T^s_{j,j_z})_{ij}$, where j and j_z inside the parentheses are the total angular momentum quantum number and the total angular momentum projection quantum number. The total angular momentum j and the orbital angular momentum ℓ are related by $j = \ell + s$ with $s = 0, \pm1, \pm2$. We note that the label s does not refer to polarizations of the massive graviton. The polarization eigenstates are explicitly shown in Appendix A.

Note that the Laplace operator Δ is connected with the orbital angular momentum ℓ whereas the symmetry of the field configuration is determined by j and j_z. For instance, the spherically symmetric configuration is given by $j = j_z = 0$ but this mode does not correspond to the zero orbital angular momentum mode as shown just below.

In what follows, we shall find two types of the massive graviton geons: the monopole geon and the quadrupole geon which correspond to the zero *total* angular momentum mode $j = 0$ and the zero *orbital* angular momentum mode $\ell = 0$, respectively. We discuss them in order. We will summarize their properties in Table V.

A. Monopole geons

We first consider spherically symmetric bound states of Eqs. (2.24) and (2.25) which we call the monopole geon. The spherical symmetry leads to

$$\psi_{ij}dx^idx^j = \sqrt{16\pi}\psi_0(r)e^{-iEt}(T^{-2}_{0,0})_{ij}dx^idx^j = \sqrt{\frac{2}{3}}\psi_0(r)e^{-iEt}(2dr^2 - r^2d\Omega^2), \tag{3.1}$$

where E is the "energy eigenvalue" of the bound state. Since we are interested in the bound state, E is a real constant and ψ_0 is a real function of r. The Poisson-Schrödinger equations are then given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G m_G^2\psi_0^2, \tag{3.2}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_0}{dr}\right) = 2\left(\frac{3}{r^2} + m_G^2\Phi\right)\psi_0 - 2m_G E\psi_0, \tag{3.3}$$

where $G = 1/8\pi M_{\rm pl}^2$. The mass of the geon is defined by

$$M := 4\pi\int_0^\infty dr r^2 m_G^2\psi_0^2. \tag{3.4}$$

The mass M is rescaled under the scaling (2.26) as

$$M \to \lambda M. \tag{3.5}$$

The asymptotic forms of the solutions are analytically found as

$$\Phi \to -\frac{GM}{r}, \tag{3.6}$$

$$\psi_0 \to Cr^{-1}W_{(-2\tilde{E})^{-1/2},\frac{5}{2}}\left(2\sqrt{-2\tilde{E}}\tilde{r}\right) \to C'\frac{e^{-\sqrt{-2\tilde{E}}\tilde{r}}}{r^{1-1/\sqrt{-2\tilde{E}}}}, \tag{3.7}$$

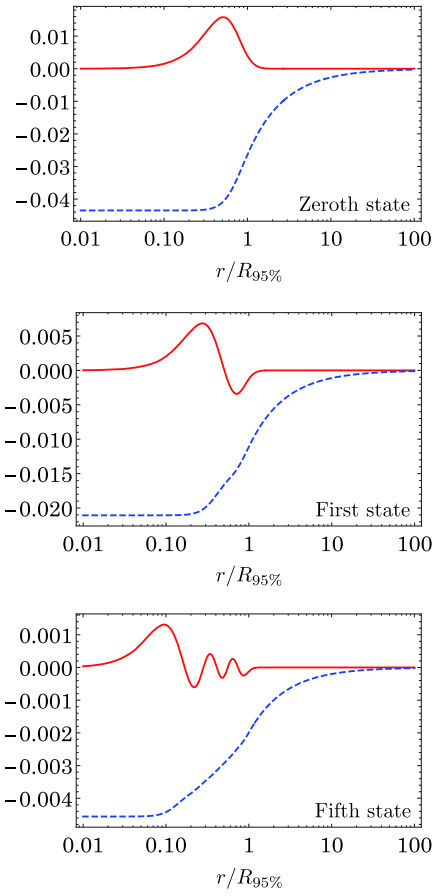

FIG. 1. The solutions to the Poisson-Schrödinger equations of the spherically symmetric bound state $j = 0$. The red curves show the configuration of the massive graviton $\tilde{\psi}_0$ and the blue dashed curves correspond to the gravitational potential $\tilde{\Phi}$. The solutions are specified by the number of nodes of $\tilde{\psi}_0$.

with

$$\tilde{E} = \frac{E}{(GM)^2 m_G^3}, \qquad \tilde{r} = GMm_G^2 r, \tag{3.8}$$

where $W_{\kappa,\mu}(z)$ is the Whittaker function, and C and C' are integration constants. We notice that the dimensionless variables $\tilde{E}$ and $\tilde{r}$ are invariant under the scaling (2.26).

Some numerical solutions to the Poisson-Schrödinger equations are shown in Fig. 1 where we have used the dimensionless scale-invariant variables:

$$\tilde{\Phi} = \frac{\Phi}{(GMm_G)^2}, \qquad \tilde{\psi}_0 = \frac{\psi_0/M_{\rm pl}}{(GMm_G)^2}. \tag{3.9}$$

The boundary condition is assumed to be $d\Phi/dr$, $d\psi_0/dr \to 0$ as $r \to 0$, and Φ and ψ_0 vanish at infinity. The solutions are parametrized by the number of nodes. We call the solution with n nodes the n-th eigenstate of the monopole geon because the energy eigenvalue increases as n increases as shown in Table I. In Table I, we summarize the scale-invariant energy eigenvalues $\tilde{E}$ and the 95% mass radii $\tilde{R}_{95\%}$, below which 95% of the mass energy is included. Since the 95% mass radius increases in the higher-node state, the lowest-node (the lowest-energy) state gives the most compact object.

TABLE I. Energy eigenvalues and 95% mass radii of monopole geon.

Node	$\tilde{E}$	$\tilde{R}_{95\%}$
$n = 0$	−0.02708	37.72
$n = 1$	−0.01140	89.85
$n = 2$	−0.006302	161.9
$n = 3$	−0.003995	254.3
$n = 4$	−0.002757	367.4
$n = 5$	−0.002016	500.0

The orbital angular momentum ℓ does not vanish in the $j = 0$ mode since the zero total angular momentum in the tensor field is realized only when $\ell = +2$ and $s = -2$. As a result, the orbital angular momentum term $3/m_G r^2$ exists in Eq. (3.3). The regularity at the center of the monopole geon leads to that ψ_0 vanishes at $r = 0$. As a result, the monopole geon has a shell-like field configuration as shown in Fig. 1. On the other hand, the Newtonian potential Φ is a scalar field, which has no orbital angular momentum for a spherical configuration, and then Φ is finite at the center.

B. Quadrupole geons

Next we consider a nonspherically symmetric configuration of the massive graviton and find its bound state. We assume a quadrupole configuration of ψ_{ij}, which is given by

$$\psi_{ij} = \sqrt{16\pi} \sum_{j_z=-2}^{2} \psi_{2,j_z}(r) e^{-iEt} (T^{+2}_{2,j_z})_{ij}, \tag{3.10}$$

where $(T^{+2}_{2,j_z})_{ij}$ is given by the spherical harmonics of the quadrupole mode. We call this bound state (3.10) the quadrupole geon. As shown in Appendix A, $(T^{+2}_{2,j_z})_{ij}$ has special properties which lead to

$$\psi^*_{ij}\psi^{ij} = 4 \sum_{j_z=-2}^{2} \psi^2_{2,j_z}, \tag{3.11}$$

$$\Delta\psi_{ij} = e^{-iEt} \sum_{j_z=-2}^{2} \frac{\sqrt{16\pi}}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi_{2,j_z}}{dr}\right)(T^{+2}_{2,j_z})_{ij}. \tag{3.12}$$

As a result, the energy density of ψ_{ij} turns to be spherically symmetric. We then assume $\Phi = \Phi(r)$. The basic equations are now given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G m_G^2 \sum_{j_z=-2}^{2} \psi_{2,j_z}^2, \tag{3.13}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_{2,j_z}}{dr}\right) = 2m_G(m_G\Phi - E)\psi_{2,j_z}. \tag{3.14}$$

The orbital angular momentum ℓ vanishes in the quadrupole geon although that is not the case in the monopole geon. Note that the orbital angular momentum vanishes only when $j = 2$ and $s = +2$.

The variables ψ_{2,j_z} are the solutions to the Sturm-Liouville equations with the same eigenvalue E under the boundary conditions $d\psi_{2,j_z}/dr \to 0$ at the center and $\psi_{2,j_z} \to 0$ at infinity. Hence, the functional forms of ψ_{2,j_z} are uniquely determined by some function $\psi_{2E}(r)$ for each E and then ψ_{2,j_z} are expressed by

$$\psi_{2,j_z} = a_{j_z}\psi_{2E}, \tag{3.15}$$

where a_{j_z} are constants which are normalized as

$$\sum_{j_z} |a_{j_z}|^2 = 1. \tag{3.16}$$

Then, the equations are reduced to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G m_G^2 \psi_{2E}^2, \tag{3.17}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_{2E}}{dr}\right) = 2m_G(m_G\Phi - E)\psi_{2E}, \tag{3.18}$$

which are exactly the same equations of the Poisson-Schrödinger system for the scalar field with the $j = 0$ mode.

The solutions to Eqs. (3.17) and (3.18) have been already investigated, e.g. in [44–49]. The energy eigenvalues and the explicit forms of ψ_{2E} are shown in Table. II and Fig. 2. Compared to those of the monopole geon, the lowest energy eigenvalue of the quadrupole geon ($\tilde{E} = -0.1628$) is much lower than that of the monopole geon ($\tilde{E} = -0.0278$). Since the lower energy state should be more stable than the higher state, the monopole geon may transit to the quadrupole geon. In the next section, we will indeed find that the monopole geon is unstable against quadrupole mode perturbations.

TABLE II. Energy eigenvalues and 95% mass radii of quadrupole geon.

Node	$\tilde{E}$	$\tilde{R}_{95\%}$
$n = 0$	−0.1628	7.830
$n = 1$	−0.03080	36.00
$n = 2$	−0.01250	85.04
$n = 3$	−0.006747	154.4

Whereas the radial dependence of the quadrupole geon is uniquely determined by ψ_{2E} for each E, the angular dependence is not unique because the expansion coefficients a_{j_z} of five tensor spherical harmonics $(T^{+2}_{2,j_z})_{ij}$ are arbitrary although there is one constraint (3.16). Each energy eigenstate of the quadrupole geon is degenerated. Needless to say, the scalar bosonic field with $j = 0$ has no degeneracy. Although the equations (3.17) and (3.18) are the same, there are different features between the gravitons and the scalar bosons.

Another difference from the bosonic case is that the field configuration is not spherically symmetric. The quadrupole geons have anisotropic pressures although they can be ignored to discuss the bound states. As a result, the quadrupole geons may emit gravitational waves. We will return some consequences of the gravitational wave emission in Sec. V.

Before ending this section, we comment on the polarizations of the geons. The massive graviton has spin-0, spin-1, and spin-2 polarization modes. The monopole geon is an eigenstate of the spin-0 polarization while the quadrupole geon is not an eigenstate of the polarizations. The relation between the polarization eigenstates and the angular momentum eigenstates is given by Eq. (A26) in Appendix A which explicitly shows that $(T^{+2}_{2,j_z})_{ij}$ is not the polarizations eigenstate.

IV. LINEAR FLUCTUATIONS AROUND THE MONOPOLE GEON

In the previous section, we find two types of geon solutions; monopole and quadrupole. Since the binding energy $(-\tilde{E})$ of the quadrupole geon is much larger than that of the monopole one, the monopole geon is expected to be unstable. In order to confirm it, we study the perturbations around the monopole geon. As long as the perturbations do not spoil the Newtonian approximation, we can use the equations (2.24) and (2.25). Hence, we consider the configurations

$$\Phi = \Phi_0 + \delta\Phi, \qquad \psi_{ij} = \psi_{0,ij} + \delta\psi_{ij}, \tag{4.1}$$

where Φ_0 and $\psi_{0,ij}$ are the solutions of the n-th state of the monopole geon. The perturbations $\delta\psi_{ij}$ can be decomposed into the odd parity perturbations and the even parity perturbations,

$$\delta\psi_{ij} = \delta\psi_{ij}^{(\mathrm{even})} + \delta\psi_{ij}^{(\mathrm{odd})}, \tag{4.2}$$

and the different parity modes are not coupled with each other.

There is no odd parity mode of the gravitational potential. The odd parity perturbations $\delta\psi_{ij}^{(\mathrm{odd})}$ are obtained by only solving the Schrödinger equation with the potential $\Phi = \Phi_0$ and then the problem can be reduced to the

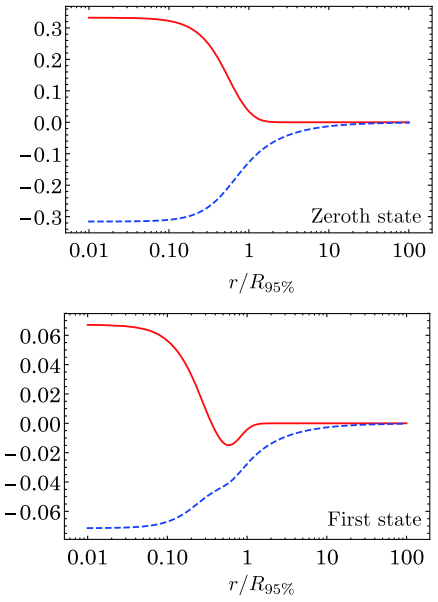

FIG. 2. The configurations of ψ_2 (red curves) and Φ (blue dashed curves) in the case of the quadrupole mode.

eigenvalue problem of a self-adjoint operator. Hence, the eigenvalues, i.e., the frequencies of $\delta\psi_{ij}^{(\mathrm{odd})}$, are real and then there is no growing mode (and no decaying mode). The monopole geons are linearly stable against the odd parity perturbations.

We then study the even parity perturbations. We introduce a different basis of the multipole expansion form $(T^s_{j,j_z})_{ij}$ which is called the pure-spin spherical harmonics denoted by $(Y^A_{j,j_z})_{ij}$ with $A = S_0, E_1, E_2, B_1, B_2$.[2] The label A characterizes the polarization states of the massive spin-2 field and the parity (see Appendix A). Then, the even parity perturbations can be expressed by

$$\delta\Phi = \sqrt{4\pi}\sum_{j\geq 0}\sum_{|j_z|\leq j}\delta\phi_{j,j_z}(t,r)Y_{j,j_z}, \tag{4.3}$$

$$\begin{aligned}\delta\psi_{ij}^{(\mathrm{even})} &= \sqrt{16\pi}e^{-iEt}\sum_{j\geq 0}\sum_{|j_z|\leq j}\delta\xi_{j,j_z}(t,r)(Y^{S_0}_{j,j_z})_{ij}\\ &+ \sqrt{16\pi}e^{-iEt}\sum_{j\geq 1}\sum_{|j_z|\leq j}\delta\chi_{j,j_z}(t,r)(Y^{E_1}_{j,j_z})_{ij}\\ &+ \sqrt{16\pi}e^{-iEt}\sum_{j\geq 2}\sum_{|j_z|\leq j}\delta\sigma_{j,j_z}(t,r)(Y^{E_2}_{j,j_z})_{ij},\end{aligned} \tag{4.4}$$

where Y_{j,j_z} is the scalar spherical harmonics. Since the different modes of j and j_z are not coupled due to the background spherical symmetry, we just omit the suffixes j and j_z hereafter.

The perturbed Poisson equation yields

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\delta\phi) - \frac{j(j+1)}{r^2}\delta\phi = 4\pi G\psi_0(\delta\xi + \delta\xi^*), \tag{4.5}$$

while the Schrödinger equation with $j \geq 2$ gives

$$\begin{aligned}&-\frac{1}{2m_G r}\frac{\partial^2}{\partial r^2}(r\delta\xi) + \left[\frac{j(j+1)+6}{2m_G r^2} + m_G U\right]\delta\xi\\ &+ m_G\psi_0\delta\phi + \frac{\sqrt{3j(j+1)}}{m_G r^2}\delta\chi = i\frac{\partial}{\partial t}\delta\xi,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&-\frac{1}{2m_G r}\frac{\partial^2}{\partial r^2}(r\delta\chi) + \left[\frac{j(j+1)+4}{2m_G r^2} + m_G U\right]\delta\chi\\ &+ \frac{\sqrt{3j(j+1)}}{m_G r^2}\delta\xi + \frac{\sqrt{(j-1)(j+2)}}{m_G r^2}\delta\sigma = i\frac{\partial}{\partial t}\delta\chi,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&-\frac{1}{2m_G r}\frac{\partial^2}{\partial r^2}(r\delta\sigma) + \left[\frac{(j-1)(j+2)}{2m_G r^2} + m_G U\right]\delta\sigma\\ &+ \frac{\sqrt{(j-1)(j+2)}}{m_G r^2}\delta\chi = i\frac{\partial}{\partial t}\delta\sigma,\end{aligned} \tag{4.8}$$

where $U = \Phi_0 - E/m$. We note $j = 0$ and $j = 1$ of the perturbations are exceptional modes. The variables $\delta\chi$ and $\delta\sigma$ are undefined for the $j = 0$ mode in which (4.7) and (4.8) are not obtained. For the $j = 1$ mode, the variable $\delta\sigma$ is undefined and the equation (4.8) does not exist. However, Eq. (4.6) [and Eq. (4.7)] is correct even for the $j = 0$ mode (and the $j = 1$ mode) since the coefficients in front of the undefined variable vanish.

We find solutions of the form

$$\delta\phi = \frac{W}{r}e^{-i\omega t} + \frac{W^*}{r}e^{i\omega^* t}, \tag{4.9}$$

$$\delta\xi = \frac{1}{r}(\xi_A + \xi_B)e^{-i\omega t} + \frac{1}{r}(\xi_A^* - \xi_B^*)e^{i\omega^* t}, \tag{4.10}$$

$$\delta\chi = \frac{1}{r}(\chi_A + \chi_B)e^{-i\omega t} + \frac{1}{r}(\chi_A^* - \chi_B^*)e^{i\omega^* t}, \tag{4.11}$$

$$\delta\sigma = \frac{1}{r}(\sigma_A + \sigma_B)e^{-i\omega t} + \frac{1}{r}(\sigma_A^* - \sigma_B^*)e^{i\omega^* t}, \tag{4.12}$$

where $\{W, \xi_{A,B}, \chi_{A,B}, \sigma_{A,B}\}$ are complex functions of r and ω is a complex constant. The regularity conditions at the center lead to the boundary condition $\{W, \xi_{A,B}, \chi_{A,B}, \sigma_{A,B}\} \to 0$. At infinity we assume the boundary condition $\delta\xi, \delta\psi, \delta\chi, \delta\sigma \to 0$. Except for the $j = 0$ mode, all variables $\{\delta\phi, \delta\xi, \delta\chi, \delta\sigma\}$ decay faster than r^{-1}. Hence,

[2]We have used two types of the orthonormal sets of the tensor harmonics $(T^s_{j,j_z})_{ij}$ and $(Y^A_{j,j_z})_{ij}$ which are related by the orthogonal transformation (A26). Although $(T^s_{j,j_z})_{ij}$ are useful to discuss the bound state solutions, the perturbation equations are given by a simpler form by using $(Y^A_{j,j_z})_{ij}$.

we can assume the boundary condition $\{W, \xi_{A,B}, \chi_{A,B}, \sigma_{A,B}\} \to 0$ at infinity. However, $\delta\phi$ of the $j = 0$ mode decays as r^{-1}, i.e., $W \to W_0$ (constant). We cannot assume $W \to 0$ at infinity in the case of $j = 0$; instead, we should assume $dW/dr \to 0$.

The Poisson equation yields that W is formally given by

$$W = -8\pi G \mathcal{O}_\phi^{-1} \psi_0 \xi_A, \tag{4.13}$$

where $\mathcal{O}_\phi^{-1}$ is the inverse operator of $\mathcal{O}_\phi$, which is defined by

$$\mathcal{O}_\phi = -\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2}. \tag{4.14}$$

The perturbed equations for the $j = 0$ mode are given by

$$\begin{pmatrix} 0 & \mathcal{O}_\psi \\ \mathcal{O}_\psi - 8\pi G m_G \psi_0 \mathcal{O}_\phi^{-1} \psi_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix} = \omega \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix}, \tag{4.15}$$

the equations for $j = 1$ are

$$\begin{pmatrix} 0 & \mathcal{O}_\psi & 0 & \frac{\sqrt{6}}{m_G r^2} \\ \mathcal{O}_\psi - 8\pi G m_G \psi_0 \mathcal{O}_\phi^{-1} \psi_0 & 0 & \frac{\sqrt{6}}{m_G r^2} & 0 \\ 0 & \frac{\sqrt{6}}{m_G r^2} & 0 & \mathcal{O}_\chi \\ \frac{\sqrt{6}}{m_G r^2} & 0 & \mathcal{O}_\chi & 0 \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \\ \chi_A \\ \chi_B \end{pmatrix} = \omega \begin{pmatrix} \xi_A \\ \xi_B \\ \chi_A \\ \chi_B \end{pmatrix}, \tag{4.16}$$

and the equations for the general modes $j \geq 2$ are written as

$$\begin{pmatrix} 0 & \mathcal{O}_\psi & 0 & \frac{\sqrt{3j(j+1)}}{m_G r^2} & 0 & 0 \\ \mathcal{O}_\psi - 8\pi G m_G \psi_0 \mathcal{O}_\phi^{-1} \psi_0 & 0 & \frac{\sqrt{3j(j+1)}}{m_G r^2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3j(j+1)}}{m_G r^2} & 0 & \mathcal{O}_\chi & 0 & \frac{\sqrt{(j-1)(j+2)}}{m_G r^2} \\ \frac{\sqrt{3j(j+1)}}{m_G r^2} & 0 & \mathcal{O}_\chi & 0 & \frac{\sqrt{(j-1)(j+2)}}{m_G r^2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{(j-1)(j+2)}}{m_G r^2} & 0 & \mathcal{O}_\sigma \\ 0 & 0 & \frac{\sqrt{(j-1)(j+2)}}{m_G r^2} & 0 & \mathcal{O}_\sigma & 0 \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \\ \chi_A \\ \chi_B \\ \sigma_A \\ \sigma_B \end{pmatrix} = \omega \begin{pmatrix} \xi_A \\ \xi_B \\ \chi_A \\ \chi_B \\ \sigma_A \\ \sigma_B \end{pmatrix}, \tag{4.17}$$

where

$$\mathcal{O}_\psi = -\frac{1}{2m_G}\frac{d^2}{dr^2} + \frac{j(j+1)+6}{2m_G r^2} + m_G U, \tag{4.18}$$

$$\mathcal{O}_\chi = -\frac{1}{2m_G}\frac{d^2}{dr^2} + \frac{j(j+1)+4}{2m_G r^2} + m_G U, \tag{4.19}$$

$$\mathcal{O}_\sigma = -\frac{1}{2m_G}\frac{d^2}{dr^2} + \frac{(j-1)(j+2)}{2m_G r^2} + m_G U. \tag{4.20}$$

Once we find one solution with the eigenvalue ω, we can easily obtain the solutions with the eigenvalues $-\omega, \omega^*$ and $-\omega^*$. Supposing $\{W, \xi_A, \xi_B, \cdots\}$ with ω is a solution, the solution with $-\omega$ is given by $\{W, \xi_A, -\xi_B, \cdots\}$. The solutions with eigenvalues ω^* and $-\omega^*$ are also easily constructed from the definitions of the variables (4.9)–(4.12).

We use the spectral method and then the problem is reduced to the eigenvalue problem of the matrix. Since the infinity cannot be treated in numerical calculations, we choose a range $(0, 10R_{95\%})$ for r for which we have confirmed the numerical results are sufficiently converged. When all eigenvalues are real, the solution is stable; while if there exists an imaginary part of ω, there are always exponentially growing modes because $\{\omega, -\omega, \omega^*, -\omega^*\}$ are the eigenvalues of the perturbations.

In Tables III and IV, we show some eigenvalues of $\mathrm{Re}(\omega) \geq 0$, $\mathrm{Im}(\omega) \geq 0$, where the rescaled ω is defined by

$$\tilde{\omega} = \frac{\omega}{(GM)^2 m_G^3}. \tag{4.21}$$

We note that the $j = 0$ mode perturbations have a trivial solution such that $\omega = 0$ and

TABLE III. The seven lowest eigenvalues $\tilde{\omega}_i$ of the $j=0$ mode perturbations with $\mathrm{Re}(\tilde{\omega}_i)$, $\mathrm{Im}(\tilde{\omega}_i) \geq 0$ and $|\tilde{\omega}_0| < |\tilde{\omega}_1| < |\tilde{\omega}_2| < \cdots$.

	Zeroth state	First state	Second state
$\tilde{\omega}_0$	0	0	0
$\tilde{\omega}_1$	0.005127	0.001389	0.0005677
$\tilde{\omega}_2$	0.01326	$0.003794 + 0.0007111i$	$0.001571 + 0.0003633i$
$\tilde{\omega}_3$	0.01699	0.005369	0.002495
$\tilde{\omega}_4$	0.01936	0.006537	$0.002879 + 0.0001624i$
$\tilde{\omega}_5$	0.02095	0.007374	0.003445
$\tilde{\omega}_6$	0.02197	0.008009	0.003839

TABLE IV. The seven lowest eigenvalues $\tilde{\omega}_i$ of perturbations around the zeroth state of the monopole geon.

	$j=1$	$j=2$	$j=3$
$\tilde{\omega}_0$	0.00000	0.00000	0.004440
$\tilde{\omega}_1$	0.004674	$0.0005155i$	0.004918
$\tilde{\omega}_2$	0.00622	0.008190	0.005600
$\tilde{\omega}_3$	0.01078	0.008469	0.01133
$\tilde{\omega}_4$	0.01132	0.008660	0.01189
$\tilde{\omega}_5$	0.01551	0.01070	0.01346
$\tilde{\omega}_6$	0.01581	0.01358	0.01559

$$\xi_B \propto \psi_0/r, \qquad \xi_A = 0, \qquad \delta\phi = 0, \tag{4.22}$$

which corresponds to a phase shift of the background solution, i.e., $\psi_{0,ij} + \delta\psi_{ij} = e^{i\epsilon}\psi_{0,ij}$ with a small constant ϵ. We also find almost zero eigenvalues in the $j=1$ and $j=2$ modes perturbations which are not trivial solutions.

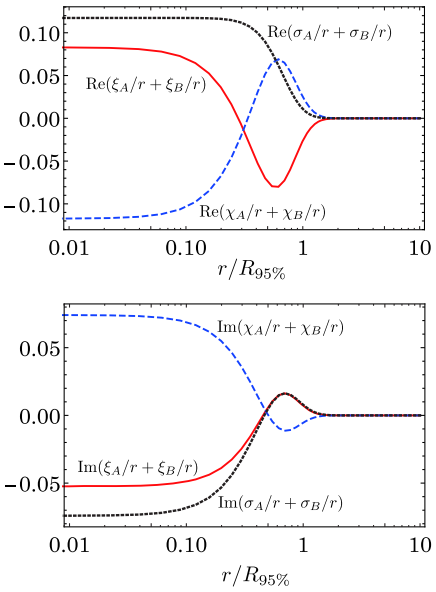

FIG. 3. The solution to the $j=2$ mode perturbation equations with $\tilde{\omega} = 0.00052i$ (the unstable perturbation around the monopole geon). The amplitudes of the solutions are scale free because the equations are linear.

We first discuss the $j=0$ mode perturbations. We find the n-th state has the n different unstable modes as shown in Table III. The unstable modes could be interpreted as the transition modes to the lower energy eigenstate from the higher one. Only the zeroth state of the monopole geon is linearly stable against the $j=0$ mode perturbation.

Henceforth, we shall only show the results of the perturbations around the zeroth state. Our numerical calculations show that the zeroth state is linearly stable against the $j \neq 2$ mode perturbations. However, we find there exists an unstable mode in the $j=2$ mode perturbations as shown in Table IV. The explicit forms of the unstable mode are depicted in Fig. 3.

As a result, the zeroth state of the monopole geon is unstable against the $j=2$ mode perturbation which is consistent with the energy argument. However, it is not so manifest whether the unstable mode indeed represents the transition to the zeroth state of the quadrupole geon from the zeroth state of the monopole geon. The quadrupole geon is purely given by the $j=2$ mode and the gravitational potential is exactly spherically symmetric. We expect that the $j \neq 2$ modes of ψ_{ij} and the nonspherical part of Φ decay in time and then the configurations of Φ and ψ_{ij} will converge towards the zeroth state of the quadrupole geon. However, we need to fully solve (2.24) and (2.25) for the monopole geon with perturbations in order to discuss this process, which we leave for a future work.

Furthermore, the stability of the quadrupole geon is an open question. Since the field configuration is not spherically symmetric in this case, different modes of the multipole expansion should be coupled and the stability analysis becomes very difficult. We hope that the quadrupole geon is linearly stable since it is the only mode with the zero orbital angular momentum and thus the lowest energy eigenstate of (2.24) and (2.25).

V. SUMMARY AND DISCUSSIONS

We find vacuum solutions such that the massive gravitons are localized by their gravitational energy in the context of the bigravity theory, which we have called the massive graviton geon by following Wheeler's idea [5]. In the Newtonian limit, the vacuum equations of the bigravity

TABLE V. Properties of the massive graviton geons.

		Monopole geon	Quadrupole geon
Angular momentum		$j=0,\ell=2$	$j=2,\ell=0$
Configuration	Angular	spherical	quadrupole
	Radial	shell (Fig. 1)	ball (Fig. 2)
Lowest energy		$\tilde{E}=-0.027$	$\tilde{E}=-0.16$
Degeneracy		1	5
Polarization		scalar	not polarized

theory can be reduced to the Poisson-Schrödinger equations with the tensor "wave function." We numerically construct two types of the massive graviton geons: the spherically symmetric configuration of the massive graviton (the monopole geon) and the configuration is represented by the quadrupole modes of the spherical harmonics (the quadrupole geon). We summarize their properties in Table V. We find the energy density distribution is spherically symmetric not only for the monopole geon but also for the quadrupole geon. The lower energy eigenstate should be more stable than the higher eigenstate. Indeed, the perturbation analysis around the monopole geon reveals that there exists an unstable mode in the quadrupole mode perturbations which is consistent with the energy argument.

Since the system is scale invariant, we can construct a massive graviton geon with any size of $R_{95\%}$ as long as $R_{95\%} \gg m_G^{-1}$. The most stable and compact massive graviton geon would be given by the zeroth state of the quadrupole geon whose mass and typical amplitude are

$$GM \simeq \frac{7.8}{m_G^2 R_{95\%}}, \tag{5.1}$$

$$|\varphi_{\mu\nu}|/M_G \simeq 0.3 \times \alpha^{1/2}(GMm_G)^2. \tag{5.2}$$

If the size of the geon becomes smaller such as $R_{95\%} \sim m_G^{-1}$, we cannot ignore the relativistic effects and the present approximations are no longer valid. Nevertheless, we may evaluate the maximum mass of the massive graviton geon from the expression (5.1) as follows: if $R_{95\%} \lesssim m_G^{-1}$, $M \gtrsim (Gm_G)^{-1}$, then the Schwarzschild radius $2GM$ becomes larger than the size of the massive graviton geon $R_{95\%}$, which may become a black hole in a relativistic situation. Then we expect that the massive graviton geon exists only when $M \lesssim M_{\max}$, where

$$M_{\max} \sim (Gm_G)^{-1} \sim 1\ M_\odot \left(\frac{10^{-10}\ \mathrm{eV}}{m_G}\right). \tag{5.3}$$

This mass bound of the geon is consistent with the stability analysis of the Schwarzschild black hole in the bigravity. The papers [50,51] showed that the Schwarzschild black hole solution is stable when $2GM \gtrsim m_G^{-1}$ while it turns to be unstable when $2GM \lesssim m_G^{-1}$, in which case we may expect a hairy black hole surrounded by time-dependent massive gravitons is formed.

The quadrupole geons may emit gravitational waves. We note, however, that the low-frequency part of $T_G^{\mu\nu}$ has no oscillation due to the phase cancellation up to the second order. Hence, we study the backreaction of the massive graviton $\varphi_{\mu\nu}$ on the high-frequency massless gravitons $h_{\mu\nu}$. We find T_G^{00} is static and spherically symmetric, while the spatial components are given by

$$T_G^{ij} \simeq m_G^2\left(-\psi^{ik}\psi^j{}_k + \frac{3}{4}\gamma^{ij}\psi^{kl}\psi_{kl}\right)e^{-2im_G t} + \text{c.c.}, \tag{5.4}$$

which clearly has the oscillating anisotropic stress and then the quadrupole geons emit the gravitational waves $h_{\mu\nu}$. The gravitational waves observed sufficiently far from the geon are given by

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{r}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}P_{ij,kl}\int d^3\boldsymbol{x}' T_G^{kl}(t,\boldsymbol{x}')e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'} + \text{c.c.}, \tag{5.5}$$

where k is the wave vector of the gravitational waves with $\boldsymbol{k}^2 = (2m_G)^2$ and $P_{ij,kl}$ is the transverse-traceless projection operator. Note that the pure-orbital spherical harmonics with $j=2$ and $s=+2$ are just constant matrices in the Cartesian coordinates (see Appendix A). Hence, the integration in (5.5) is

$$\int d^3\boldsymbol{x}' T_G^{kl}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'} \propto \int d^3\boldsymbol{x}'\psi_{2E}^2(r')e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'} = \frac{2\pi}{m_G}\int dr' r'\psi_{2E}^2(r')\sin[2m_G r']. \tag{5.6}$$

As a result, the gravitational wave emission is suppressed (probably exponentially) for the nonrelativistic geons, i.e., $\partial_r\psi_{2E} \ll m_G\psi_{2E}$. The nonrelativistic quadrupole geons are long-lived objects even if they have the oscillating part of the anisotropic stress.

The gravitational potential of the quadrupole geon has the same structure as the case of a massive scalar field of the $j=0$ mode because Eqs. (3.17) and (3.18) are the same ones as those in the scalar case (see e.g., [44–49]). Hence, the quadrupole geon can represent a central part of a dark matter halo if the mass of the graviton is about 10^{-22} eV just as the case with the ultralight scalar dark matter (see [49] for example and references therein).

Because the matter fields are coupled with $g_{\mu\nu}$ given by (2.12), the massive graviton $\varphi_{\mu\nu}$ can be observed as localized "gravitational waves" with the frequency and amplitude of

$$f \simeq m_G/2\pi \sim 10^{-8}\left(\frac{m_G}{10^{-22}\ \mathrm{eV}}\right)\ \mathrm{Hz}, \tag{5.7}$$

$$\frac{|\varphi_{\mu\nu}|}{M_G} \sim 10^{-7}\alpha^{1/2}\left(\frac{M}{10^9\ M_\odot}\right)^2. \tag{5.8}$$

In the case $\alpha = (M_{\rm pl}/M_G)^2 \sim 1$, i.e., $M_{\rm pl} \sim M_G$, this oscillation has a large amplitude; thus, such a possibility is already excluded by the pulsar timing array (PTA) observations [52]. One possibility to be a consistent scenario is introducing the hierarchy of the mass scales $M_G \gtrsim 10^8 M_{\rm pl}$ ($\alpha^{1/2} \lesssim 10^{-8}$). Another possibility is introducing the Z_2 symmetry of the massive graviton [27]; the theory is symmetric under $\varphi_{\mu\nu} \to -\varphi_{\mu\nu}$. The symmetry leads to $M_G = M_{\rm pl}$ and the matter fields are coupled with the metric [53]

$$\begin{aligned} g_{{\rm eff}\mu\nu} &= \frac{1}{4}\left[g_{\mu\nu} + 2g_{\mu\alpha}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\nu} + f_{\mu\nu}\right] \\ &= \overset{(0)}{g}{}_{\mu\nu} - \frac{1}{4M_{\rm pl}^2}\varphi_{\mu\alpha}\varphi^{\alpha}{}_{\nu} + \cdots. \end{aligned} \tag{5.9}$$

In this case, the amplitude of the oscillation is given by $|\varphi_{\mu\alpha}\varphi^{\alpha}{}_{\nu}/4M_{\rm pl}^2| \sim 10^{-15}$, which can be detectable by the future observations of PTA. We note that the ultralight scalar dark matter predicts oscillations of the gravitational potential because the coherent oscillation of dark matter leads to the oscillation of the pressure [54]. In the tensor case, we also expect the oscillation of the gravitational potential [see (5.4)].

To discuss the dark matter scenario, we have to take into account other observational constraints, such as the Yukawa force constraints and the gravitational wave constraints, on the graviton mass. Supposing the linear theory is a good approximation in the observational scales, the mass range 10^{-23} eV $\lesssim m_G \lesssim 10^{-4}$ eV is excluded when $M_{\rm pl} \sim M_G$ (see [55–58] for examples). However, if $M_G \gg M_{\rm pl}$, the interaction of the massive graviton becomes weak and then the constraints are relaxed (For instance, the lunar laser ranging experiment can give a graviton mass constraint only if $(M_{\rm pl}/M_G)^2 \gtrsim 10^{-11}$ [59]). Furthermore, the observational constraints cannot be applied to the Z_2 symmetric model since the Z_2 symmetry prohibits the Yukawa interaction $M_G^{-1}\varphi_{\mu\nu}T^{\mu\nu}$. As a result, the graviton mass constraints cannot be applied to the graviton dark matter models with the hierarchy $M_G \gtrsim 10^8 M_{\rm pl}$ or with the Z_2 symmetry.

The gravitational geons are recently discussed in the context of the nonlinear instability of the AdS spacetime [16–18,60]. The energy eigenvalue of the massive graviton geon is negative which suggests that the massive graviton geons can be produced from fluctuations around the Minkowski vacuum. Does it imply the instability of Minkowski spacetime if a graviton is massive? We expect that is not the case, because the mass energy of the produced geon is not negative although the energy eigenvalue is negative. When all fluctuation energy is converted into the geons, the production of geons will stop. This is just like a structure formation by Jeans instability.

Nonetheless, extended analysis may be required to conclude the nonlinear (in)stability of the Minkowski spacetime with massive gravitons. We have also to study a quantum mechanical (in)stability against quantum fluctuations.

Since the ground state of the massive graviton geon would be the quadrupole geon, the produced geons should eventually transit to the quadrupole geon. Although the resultant quadrupole geon will emit the gravitational waves, the gravitational wave emission is drastically suppressed if the produced geon is nonrelativistic. As a result, we may expect that the geon may not decay soon and the localized structure would remain.

Although we have only discussed the classical features of the Poisson-Schrödinger equations (2.24) and (2.25), one may quantize the massive graviton regarding ψ_{ij} as indeed the wave function of the massive graviton. The "Hamiltonian operator" could be defined by

$$\hat{H} := -\frac{\Delta}{2m_G} + m_G\Phi. \tag{5.10}$$

The massive graviton geon may be interpreted as a Bose-Einstein condensate of the massive gravitons just by the analogy to the massive scalar field and then it can be seen as a macroscopic object. If the massive graviton geon can transit quantum mechanically from one state to another, this describes a quantum transition of spacetime since the spacetime metric is given by (2.12) [or (5.9)]. On the other hand, introducing a matter field and supposing the gravitational potential is dominated by the matter with a mass $M_\star$, i.e., $\Phi = -GM_\star/r$, the Schrödinger equation is exactly solved in a way similar to the quantization of the Hydrogen atom. The nonrelativistic massive graviton may be quantized by the method of the canonical quantization.

Finally, we notice that interesting phenomena must exist in relativistic extensions of our study. For instance, we need to discuss the relativistic system to obtain the precise value of the maximum mass of the geon. In the Newtonian limit, the quadrupole geon is degenerated because the projection angular momentum j_z does not contribute to the equations of the geons. However, in the relativistic system, it is no longer true. We may expect that the degeneracy is resolved. Furthermore, the Vainshtein mechanism is irrelevant to the nonrelativistic geons while it should be important for the relativistic geons. All these investigations require treatments beyond the Newtonian approximation.

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APPENDIX A: PURE-SPIN AND PURE-ORBITAL SPHERICAL HARMONICS

As is well known, the angular dependence of a quantity (which can be either a scaler, a vector, or a tensor) can be expanded in terms of the spherical harmonics. Here, we focus on a symmetric and traceless three-dimensional tensor t_{ij} and consider a multipole expansion of it. We introduce two types of the orthonormal bases of the multipole expansion which are called the pure-spin spherical harmonics $(Y^A_{j,j_z})_{ij}$ and the pure-orbital spherical harmonics $(T^s_{j,j_z})_{ij}$, respectively[3] (see [61,62] for more details). Note that the suffix j inside the parentheses represents the total angular momentum while the indices outside the parentheses are the spatial indices of the tensor.

We first introduce the orbital angular momentum operators $\hat{\mathbf{L}}_I$ and the total angular momentum operators $\hat{\mathbf{J}}_I$ which satisfy the commutator relations

$$[\hat{\mathbf{L}}_I, \hat{\mathbf{L}}_J] = i\sum_K \epsilon_{IJK}\hat{\mathbf{L}}_K, \tag{A1}$$

$$[\hat{\mathbf{J}}_I, \hat{\mathbf{J}}_J] = i\sum_K \epsilon_{IJK}\hat{\mathbf{J}}_K, \tag{A2}$$

$$[\hat{\mathbf{J}}_I, \hat{\mathbf{L}}_J] = i\sum_K \epsilon_{IJK}\hat{\mathbf{L}}_K, \tag{A3}$$

where I, J, K are the indices of the Cartesian coordinates, $I, J, K = (x, y, z)$. The operators are explicitly given by

$$\begin{aligned}\hat{\mathbf{L}}_I &\coloneqq -i(\mathbf{r}\times\partial/\partial\mathbf{r})_I,\\ &= -i\xi^i_I\partial_i,\end{aligned} \tag{A4}$$

$$\hat{\mathbf{J}}_I \coloneqq -i\mathcal{L}_{\xi_I}, \tag{A5}$$

where ξ^i_I is the rotational Killing vectors around $I = x, y, z$ axes and $\mathcal{L}_{\xi_I}$ are the Lie derivatives with respect to them. We recall that the indices i, j, k are not necessary to be those of the Cartesian coordinates. For instance, the Killing vectors are written by

$$\xi^i_x = (0, -\sin\phi, -\cot\theta\cos\phi), \tag{A6}$$

$$\xi^i_y = (0, \cos\phi, -\cot\theta\sin\phi), \tag{A7}$$

$$\xi^i_z = (0, 0, 1), \tag{A8}$$

in the spherical coordinates, $i, j, k = (r, \theta, \phi)$. The spin angular momentum operators $\hat{\mathbf{S}}_I$ are defined so that $\hat{\mathbf{J}}_I = \hat{\mathbf{L}}_I + \hat{\mathbf{S}}_I$. The magnitude squared of the operators are defined by the sum of the squared operators, e.g.,

$$\hat{\mathbf{L}}^2 = \hat{\mathbf{L}}^2_x + \hat{\mathbf{L}}^2_y + \hat{\mathbf{L}}^2_z. \tag{A9}$$

We note

$$\hat{\mathbf{S}}^2 t_{ij} = 2\times(2+1)t_{ij}, \tag{A10}$$

which explicitly shows that the symmetric and traceless tensor is a spin-2 field.

We assume that some tensor spherical harmonics $(Y_{j,j_z})_{ij}$ are the eigenstates of the operators $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_z$:

$$\hat{\mathbf{J}}^2(Y_{j,j_z})_{ij} = j(j+1)(Y_{j,j_z})_{ij}, \tag{A11}$$

$$\hat{\mathbf{J}}_z(Y_{j,j_z})_{ij} = j_z(Y_{j,j_z})_{ij}, \tag{A12}$$

where $j = 0, 1, 2, \cdots$ and $j_z = 0, \pm1, \ldots, \pm j$. However, these conditions do not determine the tensor spherical harmonics completely. We can further assume a property of them.

The pure-spin spherical harmonics $(Y^A_{j,j_z})_{ij}$ are understood as polarization eigenstates of a spherical wave of a spin-2 field where the polarization states are labeled by $A = S_0, E_1, E_2, B_1, B_2$. The propagating direction of the spherical wave is given by the unit radial vector $n^i = r^i/|r|$. The spin-2 polarization modes E_2, B_2 are transverse to the propagating direction,

$$n^i(Y^{E_2}_{j,j_z})_{ij} = n^i(Y^{B_2}_{j,j_z})_{ij} = 0, \tag{A13}$$

the spin-1 polarization modes E_1, B_1 are mixed longitudinal and transverse

$$\begin{aligned} n^i(Y^{E_1}_{j,j_z})_{ij} \neq 0, \qquad n^i(Y^{B_1}_{j,j_z})_{ij} \neq 0,\\ n^i n^j(Y^{E_1}_{j,j_z})_{ij} = n^i n^j(Y^{B_1}_{j,j_z})_{ij} = 0,\end{aligned} \tag{A14}$$

and the spin-0 polarization mode S_0 is the longitudinal mode

$$n^i(Y^{S_0}_{j,j_z})_{ij} \neq 0, \qquad n^i n^j(Y^{S_0}_{j,j_z})_{ij} \neq 0. \tag{A15}$$

The modes E_1 and E_2 are called the electric-type parity, i.e., the even parity according to $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$ while B_1 and B_2 are the magnetic-type parity, i.e., the odd parity. The pure-spin spherical harmonics are orthonormal

[3] If t_{ij} is not traceless, the trace part can be expanded in terms of $Y_{j,j_z}\gamma_{ij}/\sqrt{3}$ which corresponds to the spin-zero state, i.e., $\hat{\mathbf{S}}^2(Y_{j,j_z}\gamma_{ij}) = 0$.

$$\int d\Omega (Y^{A}_{j,j_z})^{*}_{ij} (Y^{A'}_{j',j'_z})^{ij} = \delta_{AA'}\delta_{jj'}\delta_{j_z j'_z}, \tag{A16}$$

where the complex conjugate is given by $(Y^{A}_{j,j_z})^{*}_{ij} = (-1)^{j_z}(Y^{A}_{j,-j_z})_{ij}$. In the spherical coordinates, they are explicitly given by

$$\begin{aligned}
(Y^{S_0}_{j,j_z})_{ij} &= \frac{1}{\sqrt{6}}\begin{pmatrix} 2Y_{j,j_z} & 0 \\ * & -r^2 Y_{j,j_z}\hat{\gamma}_{ab} \end{pmatrix}, \\
(Y^{E_1}_{j,j_z})_{ij} &= \begin{pmatrix} 0 & r(Y_{j,j_z})_b \\ * & 0 \end{pmatrix}, \\
(Y^{E_2}_{j,j_z})_{ij} &= \begin{pmatrix} 0 & 0 \\ * & r^2(Y_{j,j_z})_{ab} \end{pmatrix}, \\
(Y^{B_1}_{j,j_z})_{ij} &= \begin{pmatrix} 0 & r\epsilon_b{}^c(Y_{j,j_z})_c \\ * & 0 \end{pmatrix}, \\
(Y^{B_2}_{j,j_z})_{ij} &= \begin{pmatrix} 0 & 0 \\ * & r^2\epsilon_{(a}{}^c(Y_{j,j_z})_{b)c} \end{pmatrix},
\end{aligned} \tag{A17}$$

with

$$(Y_{j,j_z})_a = -\frac{1}{\sqrt{2j(j+1)}}\hat{D}_a Y_{j,j_z}, \tag{A18}$$

$$\begin{aligned}
(Y_{j,j_z})_{ab} = &\sqrt{\frac{2}{(j-1)j(j+1)(j+2)}} \\
&\times \left(\hat{D}_a\hat{D}_b - \frac{1}{2}\hat{D}^2\hat{\gamma}_{ab}\right) Y_{j,j_z},
\end{aligned} \tag{A19}$$

and the spherical harmonics Y_{j,j_z} where a, b, c are the indices in the unit two-sphere $d^2\Omega = \hat{\gamma}_{ab}dx^a dx^b = d^2\theta + \sin^2\theta d\phi^2$ and $\hat{D}_a$ is the covariant derivative with respect to $\hat{\gamma}_{ab}$. The two-dimensional Levi-Civita tensor is denoted by ϵ_{ab}. Clearly from the explicit forms, the modes E_1, B_1 are defined only for $j \geq 1$ and the modes E_2, B_2 are defined only for $j \geq 2$. In Table VI, we summarize which tensor spherical harmonics are defined in each total angular momentum.

The helicity eigenstates $(Y^{h}_{j,j_z})_{ij}$ are defined by

$$\mathbf{n}\cdot\hat{\mathbf{S}}(Y^{h}_{j,j_z})_{ij} = h(Y^{h}_{j,j_z})_{ij}, \tag{A20}$$

where $\mathbf{n}$ is the unit radial vector in the Cartesian coordinates and $h = 0, \pm 1, \pm 2$. The helicity eigenstates are given by linear combinations of the pure-spin spherical harmonics as

TABLE VI. Defined harmonics in each total angular momentum.

	Pure-spin (helicity)	Pure-orbital (orbital angular momentum)
$j = 0$	$A = S_0$ $(h = 0)$	$s = -2$ $(\ell = 2)$
$j = 1$	$A = S_0, E_1, B_1$ $(h = 0, \pm 1)$	$s = 0, -1, -2$ $(\ell = 1, 2, 3)$
$j \geq 2$	All modes are defined $(h = 0, \pm 1, \pm 2)$	$(\ell = 0, 1, 2, \ldots)$

$$\begin{aligned}
(Y^{0}_{j,j_z})_{ij} &= (Y^{S_0}_{j,j_z})_{ij}, \\
(Y^{+1}_{j,j_z})_{ij} &= \frac{1}{\sqrt{2}}[(Y^{E_1}_{j,j_z})_{ij} + i(Y^{B_1}_{j,j_z})_{ij}], \\
(Y^{-1}_{j,j_z})_{ij} &= \frac{1}{\sqrt{2}}[(Y^{E_1}_{j,j_z})_{ij} - i(Y^{B_1}_{j,j_z})_{ij}], \\
(Y^{+2}_{j,j_z})_{ij} &= \frac{1}{\sqrt{2}}[(Y^{E_2}_{j,j_z})_{ij} + i(Y^{B_2}_{j,j_z})_{ij}], \\
(Y^{-2}_{j,j_z})_{ij} &= \frac{1}{\sqrt{2}}[(Y^{E_2}_{j,j_z})_{ij} - i(Y^{B_2}_{j,j_z})_{ij}],
\end{aligned} \tag{A21}$$

which also indicates that $A = S_0$, $A = E_1, B_1$, and $A = E_2, B_2$ represent the polarization states of the spin-2 field, respectively. In the $j = 0$ mode, there exists only the helicity-zero mode and there are the $h = 0, \pm 1$ modes in the $j = 1$ modes. The helicity-two modes appear when $j \geq 2$. In a massless spin-2 field, the helicity-zero mode and the helicity-one modes are not dynamical degrees of freedom whereas all modes are dynamical in the massive case.

On the other hand, the pure-orbital spherical harmonics $(T^{s}_{j,j_z})_{ij}$ are defined by the eigenstates of the orbital angular momentum operator

$$\hat{\mathbf{L}}^2(T^{s}_{j,j_z})_{ij} = \ell(\ell+1)(T^{s}_{j,j_z})_{ij}, \tag{A22}$$

with the orthonormality

$$\int d\Omega (T^{s}_{j,j_z})^{*}_{ij}(T^{s'}_{j',j'_z})^{ij} = \delta_{ss'}\delta_{jj'}\delta_{j_z j'_z}. \tag{A23}$$

The possible total angular momenta of the spin-2 field are given by $j = \ell, \ell \pm 1, \ell \pm 2$ which are labeled by $s = 0, \pm 1, \pm 2$, i.e., $j = \ell + s$. We note that the label s does not represent the spin states since the pure-orbital spherical harmonics are no longer the eigenstates of the polarizations of the spin-2 field. The pure-orbital spherical harmonics are characterized by their angular momenta. We note that

$$\hat{\mathbf{L}}^2 = -r^2\Delta + r^i r^j \partial_i\partial_j + 2r^i\partial_i, \tag{A24}$$

thus, the pure-orbital spherical harmonics are also eigenstates of the Laplace operator. As a result, we obtain

$$\Delta[f(r)(T^s_{j,j_z})_{ij}] = \left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}f - \frac{\ell(\ell+1)}{r^2}f\right](T^s_{j,j_z})_{ij}, \tag{A25}$$

where f is a function of r. The pure-orbital spherical harmonics are explicitly constructed by the so-called spin function as well as the (scalar) spherical harmonics (see [61,62]). Here, we only show the relations between the pure-spin spherical harmonics and the pure-orbital spherical harmonics by which one can explicitly obtain the pure-orbital spherical harmonics from Eq. (A17). The pure-orbital spherical harmonics are obtained by the orthogonal transformation as

$$\begin{aligned}
(T^{-2}_{j,j_z})_{ij} &= \sqrt{\frac{3(j+1)(j+2)}{2(2j+1)(2j+3)}}(Y^{S_0}_{j,j_z})_{ij} + \sqrt{\frac{2j(j+2)}{(2j+1)(2j+3)}}(Y^{E_1}_{j,j_z})_{ij} + \sqrt{\frac{j(j-1)}{2(2j+1)(2j+3)}}(Y^{E_2}_{j,j_z})_{ij},\\
(T^{0}_{j,j_z})_{ij} &= -\sqrt{\frac{j(j+1)}{(2j-1)(2j+3)}}(Y^{S_0}_{j,j_z})_{ij} + \sqrt{\frac{3}{(2j-1)(2j+3)}}(Y^{E_1}_{j,j_z})_{ij} + \sqrt{\frac{3(j-1)(j+2)}{(2j-1)(2j+3)}}(Y^{E_2}_{j,j_z})_{ij},\\
(T^{+2}_{j,j_z})_{ij} &= \sqrt{\frac{3j(j-1)}{2(2j+1)(2j-1)}}(Y^{S_0}_{j,j_z})_{ij} - \sqrt{\frac{2(j-1)(j+1)}{(2j+1)(2j-1)}}(Y^{E_1}_{j,j_z})_{ij} + \sqrt{\frac{(j+1)(j+2)}{2(2j+1)(2j-1)}}(Y^{E_2}_{j,j_z})_{ij},\\
(T^{-1}_{j,j_z})_{ij} &= \sqrt{\frac{j+2}{2j+1}}(Y^{B_1}_{j,j_z})_{ij} + \sqrt{\frac{j-1}{2j+1}}(Y^{B_2}_{j,j_z})_{ij},\\
(T^{+1}_{j,j_z})_{ij} &= \sqrt{\frac{j-1}{2j+1}}(Y^{B_1}_{j,j_z})_{ij} - \sqrt{\frac{j+2}{2j+1}}(Y^{B_2}_{j,j_z})_{ij}.
\end{aligned} \tag{A26}$$

We note that (T^0_{j,j_z}) and (T^{-1}_{j,j_z}) are defined only for $j \geq 1$, whereas (T^{+2}_{j,j_z}) and (T^{+1}_{j,j_z}) are defined only for $j \geq 2$ (see Table VI). The relations (A26) can be used even for $j = 0$, 1 since the coefficients in front of the undefined variables in the right-hand side vanish when $j = 0$, 1.

The total angular momentum operators are given by the Lie derivatives. The symmetry of the field configuration is determined by the total angular momentum. For instance, the spherically symmetric mode is $j = j_z = 0$ and the axisymmetric modes are $j_z = 0$. On the other hand, the orbital angular momentum vanishes only for the $j = s = 2$ modes of the pure-orbital spherical harmonics which are not spherically symmetric.

We find that the $j = s = 2$ pure-orbital spherical harmonics further have a special property: orthonormal *without* integration

$$(T^{+2}_{2,j_z})^*_{ij}(Y^{+2}_{2,j'_z})^{ij} = \frac{1}{4\pi}\delta_{j_z j'_z}, \tag{A27}$$

which hold only for the $j = 2$ modes of $(T^{+2}_{j,j_z})_{ij}$. The explicit forms in the spherical coordinates are given by

$$(T^{+2}_{2,0})_{ij} = \frac{1}{\sqrt{16\pi}}\begin{pmatrix} \frac{1}{\sqrt{6}}(1+3\cos 2\theta) & -\frac{\sqrt{6}}{2}r\sin 2\theta & 0 \\ * & \frac{r^2}{\sqrt{6}}(1-3\cos 2\theta) & 0 \\ * & * & -\sqrt{\frac{2}{3}}r^2\sin^2\theta \end{pmatrix}, \tag{A28}$$

$$(T^{+2}_{2,1})_{ij} = \frac{e^{i\phi}}{\sqrt{16\pi}}\begin{pmatrix} -\sin 2\theta & -r\cos 2\theta & -ir\sin\theta\cos\theta \\ * & r^2\sin 2\theta & ir^2\sin^2\theta \\ * & * & 0 \end{pmatrix}, \tag{A29}$$

$$(T^{+2}_{2,2})_{ij} = \frac{e^{2i\phi}}{\sqrt{16\pi}}\begin{pmatrix} \sin^2\theta & r\sin\theta\cos\theta & ir\sin^2\theta \\ * & r^2\cos^2\theta & ir^2\sin\theta\cos\theta \\ * & * & -r^2\sin^2\theta \end{pmatrix}, \tag{A30}$$

and $j_z < 0$ are obtained by the relation $(T^{+2}_{2,-j_z})_{ij} = (-1)^{j_z}(T^{2}_{2,j_z})^*_{ij}$. On the other hand, in the Cartesian coordinates, they are just constant matrices

$$(T^{+2}_{2,0})_{ij} = \frac{1}{\sqrt{24\pi}}\begin{pmatrix} -1 & 0 & 0 \\ * & -1 & 0 \\ * & * & 2 \end{pmatrix}, \tag{A31}$$

$$(T^{+2}_{2,\pm1})_{ij} = \mp\frac{1}{\sqrt{16\pi}}\begin{pmatrix} 0 & 0 & 1 \\ * & 0 & \pm i \\ * & * & 0 \end{pmatrix}, \tag{A32}$$

$$(T^{+2}_{2,\pm2})_{ij} = \frac{1}{\sqrt{16\pi}}\begin{pmatrix} 1 & \pm i & 0 \\ * & -1 & 0 \\ * & * & 0 \end{pmatrix}. \tag{A33}$$

TABLE VII. Energy eigenvalues and 95% mass radii of octupole geon.

Node	$\tilde{E}$	$\tilde{R}_{95\%}$
$n = 0$	−0.05411	20.30
$n = 1$	−0.01760	60.23
$n = 2$	−0.008631	120.6
$n = 3$	−0.005109	201.4

APPENDIX B: OCTUPOLE GEONS

An octupole configuration also gives a spherically symmetric energy density distribution. We find

$$(T^{+2}_{3,2})^*_{ij}(T^{+2}_{3,2})^{ij} = (T^{+2}_{3,-2})^*_{ij}(T^{+2}_{3,-2})^{ij} = \frac{1}{4\pi}, \tag{B1}$$

$$(T^{+2}_{3,2})^*_{ij}(T^{+2}_{3,-2})^{ij} = 0, \tag{B2}$$

where

$$(T^{+2}_{3,2})_{ij} = \frac{e^{2i\phi}}{\sqrt{16\pi}}\begin{pmatrix} 3\cos\theta\sin^2\theta & r(1+3\cos 2\theta)\sin\theta/2 & 2ir\cos\theta\sin^2\theta \\ * & r^2(\cos\theta+3\cos 3\theta)/4 & ir^2\cos 2\theta\sin\theta \\ * & * & -r^2\cos\theta\sin^2\theta \end{pmatrix}. \tag{B3}$$

Hence, if the configuration of the massive graviton is given by

$$\psi_{ij} = \sqrt{16\pi}e^{-iEt}\psi_3(r)[a_+(T^{+2}_{3,2})_{ij} + a_-(T^{+2}_{3,-2})_{ij}], \tag{B4}$$

we obtain $\Phi = \Phi(r)$, where a_+ and a_- are constants with $a_+^2 + a_-^2 = 1$. The orbital angular momentum of this mode is $\ell = 1$.

The solutions can be found under the boundary conditions $\psi_3 \to 0$ both at the center and at infinity. We note $d\psi_3/dr$ does not vanish at the center because the $j = 3$ modes are not symmetric about the equatorial plane. On the other hand, the boundary conditions of the gravitational potential are $d\Phi/dr \to 0$ at the center and $\Phi \to 0$ at infinity.

A few lower energy eigenvalues and the 95% mass radii are summarized in Table VII. Since the orbital angular momentum is smaller than the monopole geon ($\ell = 2$) but larger than the quadrupole geon ($\ell = 0$), the energy eigenvalues are intermediate between them.

Although the energy eigenvalue of the octupole geon is smaller than that of the monopole geon, we could not find the unstable mode in the linear stability analysis of the $j = 3$ mode perturbations in Sec. IV. This is not surprising because the linear stability does not conclude the background is indeed stable. No unstable mode in the $j = 3$ mode linear perturbations implies that the monopole geon does not immediately transit to the octupole geon even if the transition is possible due to the nonlinear effects.

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