Exact derivation of the Hawking effect in canonical formulation

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The Hawking effect is one of the most extensively studied topics in modern physics, yet it remains relatively underexplored within the framework of canonical quantization. The key difficulty lies in the fact that the Hawking effect is principally understood using the relation between the ingoing modes which leave past null infinity and the outgoing modes which arrive at future null infinity. Naturally, these modes are described using advanced and retarded null coordinates instead of the usual Schwarzschild coordinates. However, null coordinates do not lead to a true Hamiltonian that describes the evolution of these modes. In order to overcome these hurdles in a canonical formulation, we introduce here a set of near-null coordinates which allows one to perform an exact Hamiltonian-based derivation of the Hawking effect. This derivation opens up an avenue to explore the Hawking effect using different canonical quantization methods such as polymer quantization.

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I. INTRODUCTION

The *Hawking effect* [1] is one of the most remarkable results obtained by employing quantum field theory in curved spacetime [2–4] where an asymptotic observer in the future finds a thermal emission from a black hole. The thermal emissions are usually associated with systems with a very large number of microstates. However, a classical black hole which is a solution of Einstein's general relativity [5–8] can be described by only a handful of parameters. This puzzling aspect often leads one to believe that the study of the Hawking effect might allow one to understand the possible microstates of a black hole which are expected to arise from a possible, yet unknown, quantum theory of gravity. This has led to an extensive set of studies on the Hawking effect in different contexts [9–37].

However, despite being one of the most extensively studied topics in modern physics, the study of the Hawking effect itself remains relatively underexplored within the canonical quantization framework. The key reason behind the difficulty in canonical formulation is the basic tenet through which one realizes the thermal emission. In particular, the thermal nature of the Hawking quanta is realized using the relation between the modes which leave the past null infinity as ingoing null rays and the modes which arrive at the future null infinity as outgoing null rays. As expected, instead of the regular Schwarzschild coordinates, the usage of the advanced and retarded *null coordinates* then becomes quite crucial in the derivation of the Hawking effect. However, null coordinates do not lead to a true Hamiltonian that describes the evolution of these modes incongruous to our need for studying the system (nevertheless, see Refs. [38–41]). This in turn creates hurdles for performing an extensive study of the Hawking effect using the canonical quantization framework.

In the context of polymer quantization of matter field, recently it has been argued that the Unruh effect [42–44] may get altered significantly due to the existence of a new length scale akin to the Planck length [45–47]. Polymer quantization [48,49] is a canonical quantization method which is used in loop quantum gravity [50–52]. Given the similarity of techniques employed in the study of the Unruh effect and the Hawking effect, naturally one then asks whether polymer quantization would also alter the Hawking effect.

Therefore, it has become imperative to pursue the study of the Hawking effect using the framework of canonical quantization. In this article, we introduce one such framework. In particular, we introduce here a set of *near-null coordinates* that allows one to closely follow the basic tenets of the Hawking effect and to perform an exact Hamiltonian-based derivation of it. In an earlier canonical attempt using the Lemaître coordinates by Melnikov and Weinstein [53], the Hawking effect is understood indirectly through the property of Green's function rather than the expectation value of the associated number operator. To the best of our knowledge, there does not yet exist any exact derivation of the thermal spectrum for Hawking radiation in canonical formulation.

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In Sec. II, we briefly review the key aspects of the standard derivation of the Hawking effect [1]. In particular, a massless, free scalar field is considered for describing the Hawking quanta. Additionally, a collapsing shell of matter is considered of which the eventual collapse leads to the formation of the black hole. The corresponding black hole spacetime is taken to be the Schwarzschild spacetime. Furthermore, one considers a set of two observers: one at the past null infinity and the other observer at the future null infinity. The observer at the past null infinity considers a set of ingoing modes which are specified by the *advanced* null coordinate. On the other hand, the observer at the future null infinity studies the outgoing modes which are specified by the *retarded* null coordinate. By using the relation between the advanced and retarded null coordinates, one computes the relevant Bogoliubov transformation coefficients. This in turn allows one to express the vacuum expectation value of the number operator associated with the Hawking quanta. The spectrum of these quanta turns out to be thermal in nature. The corresponding temperature is proportional to the *surface gravity* at the Schwarzschild horizon and is referred to as the Hawking temperature.

In Sec. III, we begin by describing the properties of the matter Hamiltonian for a massless, free scalar field in a general globally hyperbolic spacetime. In order to derive the Hawking effect within a canonical formulation, then we introduce a pair of near-null coordinates. These new coordinates are used by a set of two different observers mainly in the asymptotic regions near past and future null infinities respectively. Subsequently, we derive the relation between the intervals along the spatial hypersurfaces which are used by these two observers.

In order to perform canonical quantization of the scalar field, we consider the Fourier modes of the field. Later, we compute the Bogoliubov coefficients that relate the different Fourier modes of the two different observers. These coefficients are then used to compute the vacuum expectation value of the Hamiltonian operator for the Fourier modes as seen by the observer near future null infinity in the vacuum state of the observer near past null infinity. We identify the Hawking radiation as the characteristic outgoing radiation of which the existence is tied with the nonzero values of the surface gravity at the event horizon. This leads to an exact expression for the Hawking formula and the corresponding Hawking temperature.

II. HAWKING RADIATION

In the standard derivation of the Hawking effect [1], one considers a collapsing shell of matter of which the eventual collapse leads to the formation of the black hole. In addition, one also considers a massless scalar field to describe the Hawking quanta (see Refs. [54,55] for studies with massive field). However, the detailed dynamics of the collapsing shell of matter is not important for the derivation of the Hawking radiation.

A. Schwarzschild spacetime

We consider the resultant spacetime, after the collapse of the matter shell to a black hole, to be described by the Schwarzschild geometry. In particular, the spacetime metric for an observer in the asymptotic future is given by

$$ds^2 = -\Omega dt^2 + \Omega^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2, \quad (1)$$

where $\Omega = (1 - r_s/r)$ and $r_s = 2GM$ is the Schwarzschild radius associated with the metric. Here, we use *natural units* such that $c = \hbar = 1$. We note that the metric (1) can also be used by an observer in the asymptotic past by taking the limit $r_s \to 0$, when there was no black hole.

By defining the so-called *tortoise coordinate* $r_\star$ such that $dr_\star = \Omega^{-1} dr$, one may reduce the Schwarzschild metric (1) to the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \Omega[-dt^2 + dr_\star^2] + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2. \quad (2)$$

The choice of tortoise coordinate transforms the $t - r$ plane of the Schwarzschild geometry to become *conformally flat*. By a suitable choice of the constant of integration, $r_\star$ can be explicitly written as

$$r_\star = r + r_s \ln\left(\frac{r}{r_s} - 1\right). \quad (3)$$

For later convenience, we define the *advanced and retarded* null coordinates v and u respectively as

$$v = t + r_\star; \qquad u = t - r_\star. \quad (4)$$

The Schwarzschild metric (2) in terms of these null coordinates can then be expressed as

$$ds^2 = -\Omega du dv + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2. \quad (5)$$

B. Massless scalar field

In order to describe the Hawking quanta, we consider a minimally coupled, massless scalar field $\Phi(x)$, described by the action

$$S_\Phi = \int d^4x \left[-\frac{1}{2}\sqrt{-g} g^{\mu\nu} \nabla_\mu \Phi(x) \nabla_\nu \Phi(x)\right]. \quad (6)$$

The general solutions to the Klein-Gordon field equation $\Box\Phi(x) = 0$ in the Schwarzschild spacetime (2) can be expressed as

$$\Phi(x) = \sum_{\omega lm} \frac{c_{\omega lm}}{r} \tilde{f}_\omega(r_\star) e^{-i\omega(t \pm r_\star)} Y_{lm}(\theta, \phi), \quad (7)$$

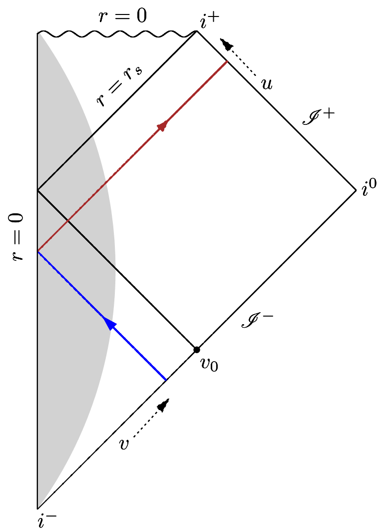

FIG. 1. The standard Penrose diagram which is used to describe the Hawking effect. The shaded region depicts the qualitative evolution of a collapsing shell of matter of which the collapse leads to the formation of the black hole. The ingoing null ray departs from the past null infinity $\mathcal{I}^-$, whereas the outgoing null ray arrives at the future null infinity $\mathcal{I}^+$.

where $c_{\omega lm}$ are some constants, $\tilde{f}_\omega(r_\star)$ are solutions to the *reduced* radial equation, and $Y_{lm}(\theta,\phi)$ are the regular *spherical harmonics*. In particular, at a large distance i.e. for $r \gg \omega^{-1}$, the function $\tilde{f}_\omega(r_\star)$ becomes a constant.

C. Creation and annihilation operators

In order to realize the Hawking effect, it is crucial to consider two asymptotic observers: one is at *past null infinity* $\mathcal{I}^-$, and the other is at *future null infinity* $\mathcal{I}^+$ (see Fig. 1). In particular, with respect to the past observer at $\mathcal{I}^-$, the scalar field operator can be expressed as

$$\hat{\Phi}(x) = \sum_\omega [f_\omega \hat{a}_\omega + f^*_\omega \hat{a}^\dagger_\omega], \tag{8}$$

where the set of *ingoing* solutions to field equation $\{f_\omega\}$ forms a complete family on $\mathcal{I}^-$ along with the inner product $(-i/2)\int_S d\Sigma^a(f_\omega \nabla_a f^*_{\omega'} - f^*_{\omega'}\nabla_a f_\omega) = \delta_{\omega\omega'}$ where $S=\mathcal{I}^-$. Further, in order to render the corresponding inner product *positive definite*, the only positive frequency modes $\{f_\omega\}$, with respect to a canonical affine parameter along $\mathcal{I}^-$, are chosen. The positive frequency ingoing solutions near the past null infinity $\mathcal{I}^-$ can be explicitly written as

$$f_\omega(v) = \frac{1}{\sqrt{2\pi\omega}} r^{-1} e^{-i\omega v} Y_{lm}(\theta,\phi). \tag{9}$$

The operators $\hat{a}^\dagger_\omega$ and $\hat{a}_\omega$ denote the creation and annihilation operators respectively. The corresponding vacuum state $|0_-\rangle$ is defined as

$$\hat{a}_\omega|0_-\rangle = 0. \tag{10}$$

Similarly, for an observer in the asymptotic future, the scalar field operator can be expressed as

$$\hat{\Phi}(x) = \sum_\omega [p_\omega \hat{b}_\omega + p^*_\omega \hat{b}^\dagger_\omega] + \sum_\omega [q_\omega \hat{c}_\omega + q^*_\omega \hat{c}^\dagger_\omega], \tag{11}$$

where field solutions $\{p_\omega\}$ are purely *outgoing* and given by

$$p_\omega(u) = \frac{1}{\sqrt{2\pi\omega}} r^{-1} e^{-i\omega u} Y_{lm}(\theta,\phi). \tag{12}$$

These solutions (12) have zero Cauchy data on the event horizon. On the other hand, field solutions $\{q_\omega\}$ have zero Cauchy data on the future null infinity $\mathcal{I}^+$. The operators $(\hat{b}^\dagger_\omega,\ \hat{b}_\omega)$ and $(\hat{c}^\dagger_\omega,\ \hat{c}_\omega)$ are the creation and annihilation operator pairs in the respective domain. The corresponding inner products are $(-i/2)\int_S d\Sigma^a(p_\omega\nabla_a p^*_{\omega'} - p^*_{\omega'}\nabla_a p_\omega) = \delta_{\omega\omega'}$ with the integration surface being $S=\mathcal{I}^+$ and $(-i/2)\int_S d\Sigma^a(q_\omega\nabla_a q^*_{\omega'} - q^*_{\omega'}\nabla_a q_\omega) = \delta_{\omega\omega'}$ where S is the event horizon. As earlier, the set of solutions $\{p_\omega\}$ is considered to contain only positive frequencies with respect to the canonical affine parameter along the null geodesic generator on $\mathcal{I}^+$.

D. Relation between null coordinates *v* and *u*

An essential input that leads to the emergence of the Hawking effect is the relation between the null coordinates of the asymptotic observers at the past and future null infinities. In particular, using the relation between an affine parameter interval along the future null infinity $\mathcal{I}^+$ and the corresponding interval on the past null infinity $\mathcal{I}^-$, one can show that

$$(v_0 - v) \approx -2r_s e^{-(u_0-u)/2r_s}, \tag{13}$$

where u^0 and v^0 denote some pivotal points on $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively. A way to understand the origin of the relation (13) is to use the following arguments. By considering a pivotal point v^0 on $\mathcal{I}^-$, an interval along $\mathcal{I}^-$ can be expressed as

$$(v^0 - v)_{|\mathcal{I}^-} = 2(r^0_\star - r_\star)_{|\mathcal{I}^-}, \tag{14}$$

where $r^0_\star$ is the tortoise coordinate corresponding to the point v^0. Similarly, we can express an interval along $\mathcal{I}^+$ as

$$(u^0 - u)_{|\mathcal{I}^+} = -2(r^0_\star - r_\star)_{|\mathcal{I}^+}, \tag{15}$$

where u^0 is a pivotal point on $\mathcal{I}^+$ and $r^0_\star$ is the corresponding value of the tortoise coordinate. However, there is a key difference between the coordinate $r_\star$ as used by each of these two observers. First, there was no black hole when the relevant ingoing modes departed from $\mathcal{I}^-$. So, for the observer at $\mathcal{I}^-$, we must take the $r_s \to 0$ limit in the expression of tortoise coordinate $r_\star$. This in turn reduces the interval (14) to

$$(v^0 - v)_{|\mathcal{I}^-} = 2(r^0 - r)_{|\mathcal{I}^-} \equiv \Delta, \tag{16}$$

where Δ is taken to be a positive interval along the past null infinity $\mathcal{I}^-$. On the other hand, when the outgoing modes arrive at the future null infinity $\mathcal{I}^+$, the black hole horizon has already formed with nonzero Schwarzschild radius r_s. Therefore, using the radial coordinate r, the interval (15) along $\mathcal{I}^+$ can be expressed as

$$(u^0 - u)_{|\mathcal{I}^+} = \Delta + 2r_s \ln\left(1 + \frac{\Delta}{\Delta_0}\right), \tag{17}$$

where we have defined $\Delta_0 \equiv 2(r^0 - r_s)_{|\mathcal{I}^+}$ and have identified the interval $-2(r^0 - r)_{|\mathcal{I}^+}$ as Δ using the *geometric optics* approximation. By choosing the pivotal values $v^0 = -\Delta_0$ and $u^0 = v^0 - 2r_s \ln(-v^0/2r_s)$, we can simplify their relation as

$$-\frac{u}{2r_s} = -\frac{v}{2r_s} + \ln\left(-\frac{v}{2r_s}\right). \tag{18}$$

With the given choices of the pivotal values, the $\ln(-v/2r_s)$ term will dominate over the $(-v/2r_s)$ term in the region where $|v| \ll 2r_s$. It turns out that the relevant modes for Hawking radiation are precisely those modes which originate from the region $|v| \ll 2r_s$ on $\mathcal{I}^-$. Therefore, in this region, one can approximate the relation (18) as

$$v \approx -2r_s e^{-u/2r_s}. \tag{19}$$

The relation (19) can be identified with the relation (13) with suitable choices of the pivotal values. We shall use similar arguments for finding the analogous relation in canonical formulation.

E. Bogoliubov coefficients and number operator

Being a complete basis, one can express the outgoing modes p_ω in terms of the ingoing modes $\{f_\omega\}$ and $\{f_\omega^*\}$ as

$$p_\omega(u) = \sum_{\omega'} [\alpha_{\omega\omega'} f_{\omega'}(v) + \beta_{\omega\omega'} f_{\omega'}^*(v)]. \tag{20}$$

Due to the mixing of modes, the vacuum state $|0_-\rangle$ of the observer at the past null infinity $\mathcal{I}^-$ is no longer annihilated by the annihilation operator $\hat{b}_\omega$ of the observer at future null infinity $\mathcal{I}^+$ i.e. $\hat{b}_\omega|0_-\rangle \neq 0$. The expectation value of the number operator corresponding to the observer at the future null infinity $\mathcal{I}^+$, in the vacuum state corresponding to the observer at past null infinity $\mathcal{I}^-$, can be expressed as

$$N_\omega \equiv \langle 0_-|\hat{b}_\omega^\dagger \hat{b}_\omega|0_-\rangle = \sum_{\omega'} |\beta_{\omega\omega'}|^2. \tag{21}$$

The relation (20) between the modes of these two observers and the relation (19) between their coordinates are used to explicitly evaluate the Bogoliubov transformation coefficient $\beta_{\omega\omega'}$. This in turn leads the expectation value of the number operator to become

$$N_\omega = \frac{1}{e^{2\pi\omega/\varkappa} - 1}, \tag{22}$$

where $\varkappa = 1/(2r_s)$ is the surface gravity at the horizon. Equation (22) corresponds to the spectrum of blackbody radiation for bosons at the temperature $T_H = \varkappa/(2\pi k_B) = 1/(8\pi G M k_B)$. This phenomenon of blackbody radiation perceived by the observer at the future null infinity $\mathcal{I}^+$ in a black hole spacetime is referred to as the Hawking effect. The corresponding temperature T_H is called the Hawking temperature.

III. CANONICAL FORMULATION

A key structural step that leads to the derivation of the Hawking effect in the covariant formulation is the Bogoliubov transformation between the solutions of the field which are functions of advanced null coordinate v (i.e. ingoing modes) and the functions of retarded null coordinate u (i.e. outgoing modes). Furthermore, in order to evaluate these transformation coefficients explicitly, it is essential to have the relation between the null coordinates v and u along the past null infinity $\mathcal{I}^-$ and the future null infinity $\mathcal{I}^+$ respectively. However, despite being intuitively appealing, these coordinates, being null, pose challenges in the canonical formulation. In particular, null coordinates do not lead to a true Hamiltonian that describes the evolution of these modes. Therefore, one must look for other suitable coordinates to perform a Hamiltonian-based derivation of the Hawking effect.

A. General scalar field Hamiltonian

In this subsection, we briefly review few key aspects of a 3 + 1 spatiotemporal decomposition [56] of a general globally hyperbolic spacetime. After the decomposition, the invariant distance element can be expressed as

$$ds^2 = -N^2 dt^2 + q_{ab}(N^a dt + d\mathbf{x}^a)(N^b dt + d\mathbf{x}^b), \tag{23}$$

where q_{ab} denotes the *spatial metric*, N is the *lapse function*, and N^a is the *shift vector* associated with the foliation of the spacetime into the spatial hypersurfaces Σ_t, labeled by the time parameter t. The full scalar field Hamiltonian corresponding to the action (6) can be expressed as

$$H_\Phi = \int d^3x [N\mathcal{H} + N^a \mathcal{D}_a], \tag{24}$$

where the scalar Hamiltonian density $\mathcal{H}$ and the diffeomorphism generator $\mathcal{D}_a$ are given by

$$\mathcal{H} = \frac{\Pi^2}{2\sqrt{q}} + \frac{\sqrt{q}}{2} q^{ab}\partial_a\Phi\partial_b\Phi; \qquad \mathcal{D}_a = \Pi\partial_a\Phi. \quad (25)$$

The determinant of the spatial metric q_{ab} is denoted by q. The Poisson brackets between the field Φ and its conjugate momentum Π can be written as

$$\{\Phi(t,\mathbf{x}), \Pi(t,\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y}). \quad (26)$$

Using Hamilton's equation of motion, it is straightforward to check that the field momentum Π can be expressed as

$$\Pi = \frac{\sqrt{q}}{N}[\partial_t\Phi - N^a\partial_a\Phi]. \quad (27)$$

B. 1+1-dimensional reduced action

We have already noted that the Hawking effect is essentially connected with the structure of the Schwarzschild metric in the $t-r$ plane. Therefore, in order to simplify the analysis, here we reduce the $3+1$-dimensional scalar field action (6) to a $1+1$-dimensional action by integrating out the angular coordinates θ and ϕ. In particular, by considering the form of the general field solution (7), we make an ansatz for the scalar field of the form

$$\Phi(x^j,\theta,\phi) = \sum_{lm}\tilde{\varphi}_{lm}(x^j)Y_{lm}(\theta,\phi), \quad (28)$$

where $x^j = t, r_\star$. By substituting this general ansatz (28) of $\Phi(x^j,\theta,\phi)$ into the action (6) and integrating over the angular coordinates, we get the reduced action of the form

$$S_\Phi = \sum_{l,m}\int dtdr_\star\left[\frac{1}{2}(\partial_t\tilde{\varphi}_{lm})^2 - \frac{1}{2}(\partial_{r_\star}\tilde{\varphi}_{lm})^2 - \frac{\Omega}{2r^2}\{\Omega + l(l+1)\}(\tilde{\varphi}_{lm})^2 + \frac{\Omega}{r}\tilde{\varphi}_{lm}\partial_{r_\star}\tilde{\varphi}_{lm}\right]. \quad (29)$$

The Hawking effect is understood using the relation between the modes of scalar fields as seen by the observers at the past and future null infinities. Therefore, with respect to these two observers, one can simplify the reduced action (29) by dropping the terms which explicitly contain inverse powers of r and are comparatively smaller at large radial distances (i.e. $r\to\infty$). The remaining terms in the simplified action then become independent of l and m. One may redefine the reduced scalar field $\varphi \propto \tilde{\varphi}_{lm}$, such that the simplified action can be viewed as a scalar field action in a $(1+1)$-dimensional Schwarzschild spacetime, given by

$$S_\varphi = \int d^2x\left[-\frac{1}{2}\sqrt{-\bar{g}}\bar{g}^{ij}\partial_i\varphi\partial_j\varphi\right], \quad (30)$$

where $\bar{g}_{ij}$ is the corresponding $(1+1)$-dimensional metric. We shall use this reduced action (30) for further computations.

C. Near-null coordinates

In the canonical formulation, the field dynamics can be viewed as the "time evolution" of the field modes on the "spatial hypersurfaces." Clearly, the advanced and retarded null coordinates are not suitable in the canonical formulation, and one must look for coordinates which are not null. First, we note that the ingoing field solutions (9) have a phase factor of the form $e^{-i\omega v}$. Along a given ingoing null trajectory, the advanced null coordinate v is constant. However, one may use the retarded null coordinate u to parametrize the propagation along the trajectory. In other words, ingoing field solutions $f_\omega(v)$, using the relation $v = u + 2r_\star$, can be viewed as if $f_\omega(u) = e^{-i\omega u}f_\omega(0)$ where u changes along the trajectory. Remarkably, this form can be compared with the time evolution of a Schrodinger wave function $\psi_\omega(\tau) = e^{-i\omega\tau}\psi_\omega(0)$ corresponding to a mechanical system with energy ω and the time coordinate τ. Furthermore, we know that a massless, free scalar field can be mapped into a set of quantum mechanical harmonic oscillators by using Fourier transformation. Therefore, these insights suggest that, in order to realize the Hawking effect in a canonical formulation and yet to mimic the methods of Bogoliubov transformations between the null coordinates, we could define a timelike coordinate by slightly deforming the retarded null coordinate u and define a spacelike coordinate by slightly deforming the advanced null coordinate v for an appropriate observer near the past null infinity $\mathcal{I}^-$. Similar arguments can be made for an observer near the future null infinity $\mathcal{I}^+$ using the outgoing field solutions.

Following the above insights, we define a new set of coordinates to be used by an observer near the past null infinity $\mathcal{I}^-$ as

$$\tau_- = t - (1-\epsilon)r_\star; \qquad \xi_- = -t - (1+\epsilon)r_\star, \quad (31)$$

where ϵ is a real-valued parameter such that τ_- and ξ_- are timelike and spacelike coordinates respectively. We use the notion where a constant time surface implies a spacelike surface. Here, we choose the parameter ϵ to be small and positive such that $\epsilon \gg \epsilon^2$. This choice of parameter mimics the basic tenets of the Hawking effect very closely. However, in principle, one can choose the values of the parameter in the domain $0 < \epsilon < 2$ which preserves the timelike characteristic of the coordinate τ_-. In any case, the final result will be independent of the explicit values of ϵ. We refer to this observer as $\mathbb{O}^-$.

Similarly, for an observer near the future null infinity $\mathcal{I}^+$, we define another set of timelike and spacelike coordinates τ_+ and ξ_+ as

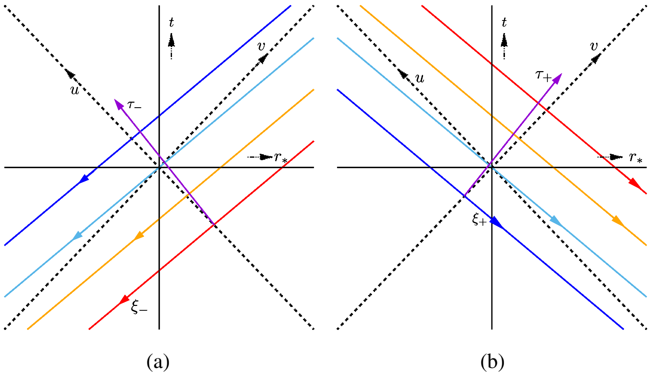

FIG. 2. (a) The near-null coordinates for the observer $\mathbb{O}^-$ consist of the spacelike coordinate ξ_- and the timelike coordinate τ_-. (b) Similarly, the near-null coordinates for the observer $\mathbb{O}^+$ consist of the spacelike coordinate ξ_+ and the timelike coordinate τ_+. In both plots, v and u are advanced and retarded null coordinates respectively.

$$\tau_+ = t + (1-\epsilon)r_\star; \qquad \xi_+ = -t + (1+\epsilon)r_\star. \tag{32}$$

As earlier, we refer this observer as $\mathbb{O}^+$. We may note from Eqs. (31) and (32) that one can algebraically transform these two sets of the coordinates from one to another by simply substituting $r_\star \to -r_\star$. Further, for later convenience, we have chosen the directions of ξ_- and ξ_+ to be the opposite of the directions of v and u coordinates respectively when $\epsilon = 0$ is set (see Fig. 2 and Fig. 3).

D. Relation between spatial coordinates ξ_- and ξ_+

In order to explicitly realize the Hawking effect in covariant formulation, one needs to compute Bogoliubov transformation coefficient between the ingoing field solutions (i.e. functions of v) and outgoing field solutions (i.e. functions of u). The key input that is required to evaluate these coefficients is the relation (19) between the null coordinates of two different observers. We show here that one can derive an analogous relation between the spatial coordinates ξ_- and ξ_+. As earlier, let us consider a pivotal point ξ_-^0 on a $\tau_- =$ constant surface as seen by the observer $\mathbb{O}^-$ near the past null infinity. Then, an interval along this surface can be expressed as

$$(\xi_- - \xi_-^0)_{|\tau_-} = 2(r_\star^0 - r_\star)_{|\tau_-} = 2(r^0 - r)_{|\tau_-} \equiv \Delta, \tag{33}$$

where r^0 corresponds to the pivotal value ξ_-^0 and the interval Δ is positive. In Eq. (33), we have used the fact that for the observer near the past null infinity there was no black hole i.e. $r_s \to 0$. For the observer $\mathbb{O}^+$, we can express an interval on a $\tau_+ =$ constant surface near future null infinity as

$$(\xi_+ - \xi_+^0)_{|\tau_+} = \Delta + 2r_s \ln\left(1 + \frac{\Delta}{\Delta_0}\right), \tag{34}$$

where $\Delta_0 \equiv 2(r_0 - r_s)_{|\tau_+}$ and using geometric optics approximation we have identified the interval $2(r - r^0)_{|\tau_+}$ also as Δ. By choosing the pivotal values $\xi_+^0 = \xi_-^0 + 2r_s \ln(\Delta_0/2r_s)$ and $\xi_-^0 = \Delta_0$, we can simplify the relation between the spatial coordinates ξ_- and ξ_+ as

$$\xi_+ = \xi_- + 2r_s \ln\left(\frac{\xi_-}{2r_s}\right). \tag{35}$$

The modes which are seen as Hawking radiation to the observer $\mathbb{O}^+$ propagate out from a region where $|\xi_-| \ll 2r_s$ on a constant τ_- surface as seen by the observer $\mathbb{O}^-$ near the past null infinity. For these modes, Eq. (35) can be approximated as

$$\xi_- \approx 2r_s e^{\xi_+/2r_s}. \tag{36}$$

Equation (36) plays a role similar to the relation (19) between the advanced and retarded null coordinates.

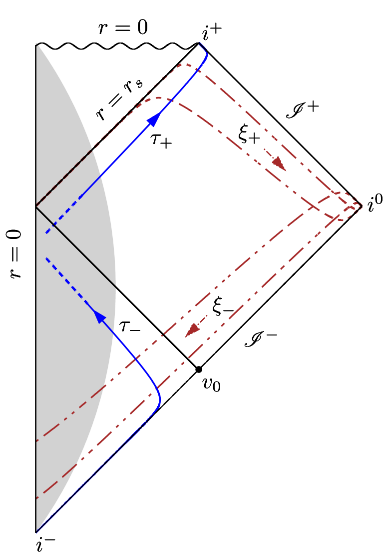

FIG. 3. The spatial hypersurfaces and the temporal directions in near-null coordinates drawn on a Penrose diagram together with a collapsing shell of matter.

E. Field Hamiltonian for observer $\mathbb{O}^-$

We have seen that the Hawking effect is crucially connected with the structure of the Schwarzschild metric in the $t - r$ plane. This $(1+1)$-dimensional metric, when viewed from the coordinate system used by the observer $\mathbb{O}^-$, appears to be of the form

$$ds^2 = g^-_{\mu\nu} dx^\mu dx^\nu = \frac{\epsilon\Omega}{2}\left[-d\tau_-^2 + d\xi_-^2 + \frac{2}{\epsilon} d\tau_- d\xi_-\right]. \tag{37}$$

In the spacetime geometry (37), the reduced action (30) for a minimally coupled massless scalar field can also be expressed as

$$S_\varphi = \int d\tau_- d\xi_- \left[-\frac{1}{2}\sqrt{-g^0} g^{0\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right], \qquad (38)$$

where $g^-_{\mu\nu} = (\epsilon\Omega/2) g^0_{\mu\nu}$. The metric $g^0_{\mu\nu}$ although has off-diagonal terms but being flat it allows one to use the machinery of the Fock quantization.

Following the general form (24), the scalar field Hamiltonian in $(1+1)$ dimensions, as seen by the observer $\mathbb{O}^-$, can be expressed as

$$H_\varphi = \int d\xi_- \frac{1}{\epsilon}\left[\left\{\frac{\Pi^2}{2} + \frac{1}{2}(\partial_{\xi_-}\varphi)^2\right\} + \Pi \partial_{\xi_-}\varphi\right], \qquad (39)$$

where the lapse function $N = 1/\epsilon$, the shift vector $N^1 = 1/\epsilon$, and the determinant of the spatial metric $q = 1$. The Poisson bracket between the field φ and its conjugate momentum Π can be written as

$$\{\varphi(\tau_-, \xi_-), \Pi(\tau_-, \xi'_-)\} = \delta(\xi_- - \xi'_-). \qquad (40)$$

Using the equations of motion, the field momentum Π can be expressed as

$$\Pi(\tau_-, \xi_-) = \epsilon(\partial_{\tau_-}\varphi) - (\partial_{\xi_-}\varphi). \qquad (41)$$

F. Field Hamiltonian for observer $\mathbb{O}^+$

With respect to the observer $\mathbb{O}^+$, the metric for the $(1+1)$-dimensional spacetime is given by

$$ds^2 = g^+_{\mu\nu} dx^\mu dx^\nu = \frac{\epsilon\Omega}{2}\left[-d\tau_+^2 + d\xi_+^2 + \frac{2}{\epsilon} d\tau_+ d\xi_+\right]. \qquad (42)$$

As earlier, by performing a conformal transformation of the metric as $g^+_{\mu\nu} = (\epsilon\Omega/2) g^0_{\mu\nu}$, one can express the scalar field Hamiltonian for the observer $\mathbb{O}^+$ as

$$H_\varphi = \int d\xi_+ \frac{1}{\epsilon}\left[\left\{\frac{\Pi^2}{2} + \frac{1}{2}(\partial_{\xi_+}\varphi)^2\right\} + \Pi \partial_{\xi_+}\varphi\right], \qquad (43)$$

where the lapse function $N = 1/\epsilon$, the shift vector $N^1 = 1/\epsilon$, and the determinant of the spatial metric $q = 1$. The corresponding Poisson bracket is given by

$$\{\varphi(\tau_+, \xi_+), \Pi(\tau_+, \xi'_+)\} = \delta(\xi_+ - \xi'_+). \qquad (44)$$

Similar to the expression (41), the field momentum for the observer $\mathbb{O}^+$ can be expressed as

$$\Pi(\tau_+, \xi_+) = \epsilon(\partial_{\tau_+}\varphi) - (\partial_{\xi_+}\varphi). \qquad (45)$$

G. Fourier modes for observer $\mathbb{O}^-$

In order to perform the canonical quantization of the scalar field, we follow the method as used in Ref. [57]. First, we define the Fourier modes for the scalar field with respect to the observer $\mathbb{O}^-$, as

$$\varphi = \frac{1}{\sqrt{V_-}} \sum_k \tilde{\phi}_k e^{ik\xi_-}; \qquad \Pi = \frac{1}{\sqrt{V_-}} \sum_k \sqrt{q}\, \tilde{\pi}_k e^{ik\xi_-}, \qquad (46)$$

where $\tilde{\phi}_k = \tilde{\phi}_k(\tau_-)$, $\tilde{\pi}_k = \tilde{\pi}_k(\tau_-)$ are the complex-valued mode functions. The spatial volume $V_- = \int d\xi_- \sqrt{q}$ is formally divergent. Therefore, to avoid dealing with explicitly divergent quantity, we choose a fiducial box with finite volume as

$$V_- = \int_{\xi_-^L}^{\xi_-^R} d\xi_- \sqrt{q} = \xi_-^R - \xi_-^L \equiv L_-. \qquad (47)$$

We shall see later that with the given definition of the Fourier modes (46) the fiducial volume V_- will drop out from the expression of the Hamiltonian of the modes. Given the finiteness of the fiducial volume, the Kronecker delta can be expressed as

$$\int d\xi_- \sqrt{q} e^{i(k-k')\xi_-} = V_- \delta_{k,k'}, \qquad (48)$$

whereas the Dirac delta can be written as

$$\sum_k e^{ik(\xi_- - \xi'_-)} = V_- \delta(\xi_- - \xi'_-)/\sqrt{q}. \qquad (49)$$

Equations (48) and (49) together imply that wave vector $k \in \{k_r\}$ where $k_r = 2\pi r/L_-$ with r being a nonzero integer.

The scalar field Hamiltonian (39) for the observer $\mathbb{O}^-$ can be expressed in terms of the Fourier modes as $H_\varphi = \sum_k (\mathcal{H}_k^- + \mathcal{D}_k^-)/\epsilon$ where the Hamiltonian density $\mathcal{H}_k^-$ for the kth mode is

$$\mathcal{H}_k^- = \frac{1}{2}\tilde{\pi}_k \tilde{\pi}_{-k} + \frac{1}{2}|k|^2 \tilde{\phi}_k \tilde{\phi}_{-k} \qquad (50)$$

and the diffeomorphism generator $\mathcal{D}_k^-$ is

$$\mathcal{D}_k^- = -\frac{ik}{2}(\tilde{\pi}_k \tilde{\phi}_{-k} - \tilde{\pi}_{-k}\tilde{\phi}_k). \qquad (51)$$

The Poisson bracket between the Fourier modes of the scalar field and the conjugate field momentum is given by

$$\{\tilde{\phi}_k, \tilde{\pi}_{-k'}\} = \delta_{k,k'}. \qquad (52)$$

H. Fourier modes for observer $\mathbb{O}^+$

In parallel to the case of the observer $\mathbb{O}^-$, we define the Fourier modes for the scalar field as seen by the observer $\mathbb{O}^+$ as

$$\varphi = \frac{1}{\sqrt{V_+}} \sum_\kappa \tilde{\phi}_\kappa e^{i\kappa\xi_+}; \qquad \Pi = \frac{1}{\sqrt{V_+}} \sum_\kappa \sqrt{q}\, \tilde{\pi}_\kappa e^{i\kappa\xi_+}, \tag{53}$$

where $\tilde{\phi}_\kappa = \tilde{\phi}_\kappa(\tau_+)$, $\tilde{\pi}_\kappa = \tilde{\pi}_\kappa(\tau_+)$ are the complex-valued mode functions. The spatial volume V_+ is also chosen to be that of a fiducial box with finite volume, given by

$$V_+ = \int_{\xi_+^L}^{\xi_+^R} d\xi_+ \sqrt{q} = \xi_+^R - \xi_+^L \equiv L_+. \tag{54}$$

Using the appropriate representations of the Kronecker delta and the Dirac delta, as in Eqs. (48) and (49), the scalar field Hamiltonian (43) can be expressed in terms of the Fourier modes as $H_\varphi = \sum_\kappa (\mathcal{H}_\kappa^+ + \mathcal{D}_\kappa^+)/\epsilon$ where the Hamiltonian density is

$$\mathcal{H}_\kappa^+ = \frac{1}{2} \tilde{\pi}_\kappa \tilde{\pi}_{-\kappa} + \frac{1}{2} |\kappa|^2 \tilde{\phi}_\kappa \tilde{\phi}_{-\kappa}, \tag{55}$$

the diffeomorphism generator is

$$\mathcal{D}_\kappa^+ = -\frac{i\kappa}{2} (\tilde{\pi}_\kappa \tilde{\phi}_{-\kappa} - \tilde{\pi}_{-\kappa} \tilde{\phi}_\kappa), \tag{56}$$

and the corresponding Poisson bracket is given by

$$\{\tilde{\phi}_\kappa, \tilde{\pi}_{-\kappa'}\} = \delta_{\kappa,\kappa'}. \tag{57}$$

I. Relation between Fourier modes

In order to establish the relation between the Hamiltonian densities of Fourier modes for these two observers, we need to find the relation between the individual modes of the field and their conjugate momenta. Being a scalar, the matter field can be expressed as $\varphi(\tau_-, \xi_-) = \varphi(\tau_+, \xi_+)$ where the coordinates can be viewed as $\tau_- = \tau_-(\tau_+, \xi_+)$ and $\xi_- = \xi_-(\tau_+, \xi_+)$ in general. However, the relation between the field momenta (41) and (45) is slightly involved. First, we note that the observer $\mathbb{O}^-$ deals with the field modes which are ingoing modes. For these modes, $v = t + r_\star = (\epsilon/2)\tau_- - (1 - \epsilon/2)\xi_-$ is *constant*. Similarly, the relevant modes for the observer $\mathbb{O}^+$ are the outgoing modes. For these modes, $u = t - r_\star = (\epsilon/2)\tau_+ - (1 - \epsilon/2)\xi_+$ is constant. Therefore, the temporal derivative term in the expression of the field momentum is not independent and can be related to the spatial derivative term. This property holds true for both the observers. This in turn leads to a relation between the field momenta as $\Pi(\tau_+, \xi_+) = (\partial\xi_-/\partial\xi_+)\Pi(\tau_-, \xi_-)$. We use these relations between the field and the conjugate momentum, as seen by two different observers, to establish the relation between the Fourier modes. In particular, by choosing spatial hypersurfaces for the observers $\mathbb{O}^-$ and $\mathbb{O}^+$, labeled by fixed τ_-^0 and τ_+^0 respectively, we can relate the Fourier modes of the field as

$$\tilde{\phi}_\kappa = \sum_k \tilde{\phi}_k F_0(k, -\kappa), \tag{58}$$

where $\tilde{\phi}_\kappa = \tilde{\phi}_\kappa(\tau_+^0)$ and $\tilde{\phi}_k = \tilde{\phi}_k(\tau_-^0)$. Similarly, the Fourier modes for field momenta can also be related as

$$\tilde{\pi}_\kappa = \sum_k \tilde{\pi}_k F_1(k, -\kappa), \tag{59}$$

where $\tilde{\pi}_\kappa = \tilde{\pi}_\kappa(\tau_+^0)$ and $\tilde{\pi}_k = \tilde{\pi}_k(\tau_-^0)$. The coefficient function $F_m(k, \kappa)$ is given by

$$F_m(k, \kappa) = \frac{1}{\sqrt{V_- V_+}} \int d\xi_+ e^{m\xi_+/2r_s} e^{ik\xi_- + i\kappa\xi_+}, \tag{60}$$

where $m = 0, 1$. The coefficient functions $F_m(k, \kappa)$ play a role similar to the Bogoliubov coefficients in the covariant formulation.

J. Regularization of Bogoliubov coefficients

Like the standard Bogoliubov coefficients, these coefficient functions are also formally divergent, and one needs to employ some regularization techniques to render them finite. Here, we follow the regularization techniques which are similar to the one used in Ref. [45]. First, one introduces a nonoscillatory regulator with a small parameter δ in the expression (60), as follows,

$$F_m^\delta(k, \kappa) = \frac{1}{\sqrt{V_- V_+}} \int d\xi_+ \left[\frac{e^{(m+\delta)\xi_+/2r_s}}{d_m}\right] e^{ik\xi_- + i\kappa\xi_+}, \tag{61}$$

where $d_m = (1 - i\delta m/2r_s\kappa)$. One can check that regulated expression (61) reduces to the exact expression (60) when the parameter δ is removed i.e. $\lim_{\delta\to 0} F_m^\delta(k, \kappa) = F_m(k, \kappa)$.

In order to evaluate the integral (61), one performs a change of variable as $z \equiv |k|\xi_-$. This in turn leads to

$$F_m^\delta(\pm|k|, \kappa) = \frac{(2r_s)^{-\beta-m} |k|^{-\beta-m-1}}{d_m \sqrt{V_- V_+}} I_\pm(\beta + m), \tag{62}$$

where $\beta = (2i\kappa r_s + \delta - 1)$. The integral $I_\pm(\beta) = \int dz e^{\pm iz} z^\beta$ can be explicitly expressed in terms of Gamma function $\Gamma(\beta + 1)$ by analytic continuation in either the upper or lower half of the complex plane, depending on the sign of k. The evaluated expression is given by

$$I_\pm(\beta) = e^{\pm i\pi(\beta+1)/2} \Gamma(\beta + 1). \tag{63}$$

In order to express the integral (63) in terms of a *complete* Gamma function, one needs to add two boundary terms $\Delta I^L = \int_0^{|k|\xi_-^L} dz e^{\pm iz} z^\beta$ and $\Delta I^R = \int_{|k|\xi_-^R}^\infty dz e^{\pm iz} z^\beta$. Both of these terms vanish identically when one removes the volume regulator by taking the limit $\xi_-^L \to 0$ and $\xi_-^R \to \infty$. For later convenience, we write down two key relations between the different forms of $F_m^\delta(\pm|k|,\kappa)$, for different values of the parameter, as

$$F_0^\delta(-|k|,\kappa) = e^{2\pi r_s\kappa - i\delta\pi} F_0^\delta(|k|,\kappa), \tag{64}$$

and

$$F_1^\delta(\pm|k|,\kappa) = \mp \frac{\kappa}{|k|} F_0^\delta(\pm|k|,\kappa). \tag{65}$$

K. Consistency relation between the regulators

We have introduced two sets of regulators so far. One set is to regulate the volumes of the flat spatial slices through the means of finite (ξ_-^L, ξ_-^R) and (ξ_+^L, ξ_+^R). The relation (36) implies that they are related among themselves. Second, we have introduced the parameter δ as the integral regulator. However, it turns out that these two sets of regulators cannot be chosen independently in order to ensure the consistency of the Poisson brackets for both the observers.

In particular, the requirement that both Poisson brackets $\{\tilde{\phi}_k, \tilde{\pi}_{-k'}\} = \delta_{k,k'}$ and $\{\tilde{\phi}_\kappa, \tilde{\pi}_{-\kappa'}\} = \delta_{\kappa,\kappa'}$ are simultaneously satisfied leads to a consistency condition on the coefficient functions as

$$\sum_{k>0}[F_0(k,-\kappa)F_1(-k,\kappa) + F_0(-k,-\kappa)F_1(k,\kappa)] = 1. \tag{66}$$

In terms of the regulated expression (61) of $F_m(k,\kappa)$, Eq. (66) demands

$$\frac{(\kappa r_s/\pi)|\Gamma(2i\kappa r_s + \delta)|^2}{(e^{2\pi\kappa r_s} - e^{-2\pi\kappa r_s})^{-1}} = \frac{(V_+/2r_s)(4\pi r_s/V_-)^{2\delta}}{\zeta(1+2\delta)}, \tag{67}$$

where the *Riemann zeta function* $\zeta(1+2\delta) = \sum_{r=1}^\infty r^{-1(1+2\delta)}$. Using the Gamma function identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, the zeta function identity $\lim_{s\to 0}[s\zeta(1+s)] = 1$, and Eq. (36), it is straightforward to show that the volume regulator ξ_-^L and integral regulator δ should be varied together as $\xi_-^L/2r_s \simeq 2\pi e^{-1/2\delta}$. In other words, the consistency condition (66) implies that these two regulators are not independent of each other.

L. Relation between Hamiltonian densities and diffeomorphism generators

Using the relations (58), (59) between the Fourier modes of the field and their conjugate momenta, we can establish the relation between the corresponding Hamiltonian densities and the diffeomorphism generators. Furthermore, by using the relations (64) and (65), we can express the Hamiltonian density $\mathcal{H}_\kappa^+$ as

$$\mathcal{H}_\kappa^+ = h_\kappa^1 + (e^{4\pi r_s\kappa} + 1)\sum_{k>0}\left(\frac{\kappa}{k}\right)^2 |F_0(k,\kappa)|^2 \mathcal{H}_k^-, \tag{68}$$

where $h_\kappa^1 = \sum_{k\neq k'}[\frac{1}{2}F_1(k,-\kappa)F_1(-k',\kappa)\tilde{\pi}_k\tilde{\pi}_{-k'} + \frac{1}{2}|\kappa|^2 \times F_0(k,-\kappa)F_0(-k',\kappa)\tilde{\phi}_k\tilde{\phi}_{-k'}]$. Similarly, we can express the diffeomorphism generators of the Fourier modes corresponding to the observer $\mathbb{O}^+$ as

$$\mathcal{D}_\kappa^+ = d_\kappa^1 + (e^{4\pi r_s\kappa} + 1)\sum_{k>0}\left(\frac{\kappa}{k}\right)^2 |F_0(k,\kappa)|^2 \mathcal{D}_k^-, \tag{69}$$

where $d_\kappa^1 = \sum_{k\neq k'}(-i\kappa/2)[F_1(k,-\kappa)F_0(-k',\kappa)\tilde{\pi}_k\tilde{\phi}_{-k'} - F_1(-k,\kappa)F_0(k',-\kappa)\tilde{\pi}_{-k}\tilde{\phi}_{k'}]$. We note that both the terms h_κ^1 and d_κ^1 are linear in Fourier modes or their conjugate momenta. Therefore, the vacuum expectation values of the corresponding operators in the quantum theory will vanish for these terms.

M. Reality condition on diffeomorphism generators

In order to represent the Hawking quanta, we have considered here a real-valued scalar field i.e. $\varphi^*(x) = \varphi(x)$. This property in turn imposes a *reality condition* on the complex-valued Fourier modes as $\tilde{\phi}_k^* = \tilde{\phi}_{-k}$. In general, we can express a complex-valued mode function as $\tilde{\phi}_k = \phi_k^r + i\phi_k^i$ where ϕ_k^r and ϕ_k^i both are real-valued functions. Similarly, we can express the Fourier modes of the conjugate field momentum as $\tilde{\pi}_k = \pi_k^r + i\pi_k^i$ which are also subjected to the reality condition $\tilde{\pi}_k^* = \tilde{\pi}_{-k}$. Therefore, unless one imposes this reality condition appropriately, there would be double counting of the degrees of freedom in terms of the real-valued mode functions.

In order to remove this double counting and also to express the total Hamiltonian $(\mathcal{H}_k^- + \mathcal{D}_k^-)$ in terms of the real-valued mode functions, here we make a *choice* by setting $\phi_k^i = 0$ and $\pi_k^i = 0$. The key advantages of this choice are that it brings $\mathcal{H}_k^-$ (50) to the form of a standard Hamiltonian of a simple harmonic oscillator with real-valued coordinate and it also makes the diffeomorphism generator term $\mathcal{D}_k^-$ (51) vanish identically. Therefore, by redefining the modes as $\phi_k \equiv \phi_k^r$ and $\pi_{-k} \equiv \pi_k^r$, we can reduce the Hamiltonian density and the Poisson bracket for the observer $\mathbb{O}^-$ as

$$\mathcal{H}_k^- = \frac{1}{2}\pi_k^2 + \frac{1}{2}|k|^2\phi_k^2; \qquad \{\phi_k, \pi_{k'}\} = \delta_{k,k'}. \tag{70}$$

We shall use the simplified form (70) of the Hamiltonian density for performing Fock quantization. We should mention here that one could also arrive at a similar conclusion by considering a general complex-valued scalar field *ab initio* [58,59] and imposing the reality condition only at the end.

N. Number operator using Hamiltonian density

The Fock quantization of the scalar field is achieved by essentially quantizing the real-valued Fourier modes ϕ_k by using the method of Schrodinger quantization. We have already seen that a massless, free scalar field can be viewed as a system of infinitely many decoupled harmonic oscillators. Therefore, the Fock space is essentially a direct product space of infinitely many quantum harmonic oscillators. If we denote the vacuum state for the kth mode as $|0_k\rangle$, then the Fock vacuum state for the observer $\mathbb{O}^-$ can be expressed as $|0_-\rangle = \prod_k \otimes |0_k\rangle$. Furthermore, the energy spectrum of the kth oscillator can be written as $\hat{\mathcal{H}}_k^-|n_k\rangle = (n+\frac{1}{2})|k||n_k\rangle$ where n denotes the energy levels.

We may recall that in covariant formulation the Hawking radiation is realized by computing the vacuum expectation value of the number operator corresponding to an observer at future null infinity $\mathcal{I}^+$, whereas the vacuum state is taken to be that of the observer at the past null infinity $\mathcal{I}^-$. Therefore, in order to realize the Hawking effect, here we consider the vacuum state to be $|0_-\rangle$ which is the vacuum state of the observer $\mathbb{O}^-$. On the other hand, we consider the matter field operators that correspond to the observer $\mathbb{O}^+$. For such a combination, the vacuum expectation value of the Hamiltonian density operator $\langle\hat{\mathcal{H}}_\kappa^+\rangle \equiv \langle 0_-|\hat{\mathcal{H}}_\kappa^+|0_-\rangle$ of a *positive* frequency mode i.e. $\kappa > 0$ can be expressed as

$$\frac{\langle\hat{\mathcal{H}}_\kappa^+\rangle}{\kappa} = \frac{e^{2\pi\kappa/\varkappa}+1}{e^{2\pi\kappa/\varkappa}-1}\left[\frac{1}{\zeta(1+2\delta)}\sum_{r=1}^{\infty}\frac{1}{r^{1+2\delta}}\frac{\langle\hat{\mathcal{H}}_{k_r}^-\rangle}{k_r}\right], \tag{71}$$

where we have used the properties of the vacuum state such that $\langle 0_k|\hat{\phi}_k|0_k\rangle = 0$ and $\langle 0_k|\hat{\pi}_k|0_k\rangle = 0$.

In the Fock quantization, usually one defines the number operator by defining the *creation* and *annihilation* operators of the respective modes. However, we show here that one can also extract the vacuum expectation value of the number operator directly from the vacuum expectation value of the Hamiltonian density operator corresponding to the given mode. This will also be useful for the situation in which the notion of creation and annihilation operators may not be readily available. So, we define the number density operator which represents the Hawking quanta as

$$\hat{N}_\kappa = [\hat{\mathcal{H}}_\kappa^+ - \lim_{\varkappa\to 0}\hat{\mathcal{H}}_\kappa^+]|\kappa|^{-1}. \tag{72}$$

The definition of number operator (72) makes it clear that the existences of these Hawking quanta are related to the nonzero values of the surface gravity $\varkappa$ at the horizon, as we have subtracted out the contribution to the Hamiltonian density due to the zero surface gravity.

In Fock quantization, $\langle\hat{\mathcal{H}}_{k_r}^-\rangle = \frac{1}{2}|k_r|$ for all Fourier modes. This in turns leads the vacuum expectation value of the number density operator (72) to become

$$N_\omega = \langle\hat{N}_{\kappa=\omega}\rangle = \frac{1}{e^{2\pi\omega/\varkappa}-1} = \frac{1}{e^{(4\pi r_s)\omega}-1}. \tag{73}$$

The expression (73) precisely corresponds to a blackbody radiation at the temperature $T_H = \varkappa/(2\pi k_B) = 1/(4\pi r_s k_B)$. The temperature T_H is known as the Hawking temperature. Clearly, it demonstrates that one could obtain the exact thermal spectrum for Hawking radiation using a Hamiltonian approach.

IV. DISCUSSION

In summary, we have presented an exact derivation of the Hawking effect using the canonical formulation. In the standard covariant derivation of the Hawking effect, the thermal nature of the Hawking radiation is realized using the key relation between the modes which leave the past null infinity as ingoing null rays and the modes which arrive at the future null infinity as outgoing null rays. This key relation in essence underlies the basic tenets of the Hawking effect. Naturally, to describe the Hawking effect in covariant formulation, it is quite convenient to use the advanced and retarded null coordinates. However, these null coordinates are not quite useful in the canonical formulation. In particular, null coordinates do not lead to a true Hamiltonian that describes the evolution of these modes. This in turn creates hurdles for a Hamiltonian-based derivation of the Hawking effect in a canonical quantization framework. Here, we overcome these hurdles by introducing a pair of near-null coordinates. These new coordinates, which are used by two different observers, are obtained by a slight deformation of the advanced and the retarded null coordinates. Being structurally close to the null coordinates, these new coordinates allow one to follow the basic tenets of the Hawking effect very closely. Therefore, the presented framework in this article opens up a rather new avenue to explore the Hawking effect using various canonical quantization methods such as the polymer quantization [60]. Besides, it would be interesting in its own right to pursue the canonical evolution, possibly using numerical methods, of the field modes as seen by the observers from the region near past null infinity to the region near future null infinity.

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