Ladder operators for the Klein-Gordon equation with a scalar curvature term

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Recently, Cardoso, Houri and Kimura constructed generalized ladder operators for massive Klein-Gordon scalar fields in space-times with conformal symmetry. Their construction requires a closed conformal Killing vector, which is also an eigenvector of the Ricci tensor. Here, a similar procedure is used to construct generalized ladder operators for the Klein-Gordon equation with a scalar curvature term. It is proven that a ladder operator requires the existence of a conformal Killing vector, which must satisfy an additional property. This property is necessary and sufficient for the construction of a ladder operator. For maximally symmetric space-times, the results are equivalent to those of Cardoso, Houri and Kimura.

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I. INTRODUCTION

In two recent papers [1,2], Cardoso, Houri and Kimura constructed ladder operators for the Klein-Gordon equation in manifolds possessing closed conformal Killing vectors, which are, in addition, eigenvectors of the Ricci tensor. More precisely, they constructed a first-order operator $\mathcal{D}$ such that, if Φ is a solution of

$$(\Box - m^2)\Phi = 0, \tag{1}$$

then $\mathcal{D}\Phi$ satisfies

$$(\Box - m^2 - \delta m^2)\mathcal{D}\Phi = 0. \tag{2}$$

Most of their examples involve maximally symmetric space-times, where the above conditions are satisfied. Because maximally symmetric space-times have constant curvature, one may wonder whether the mass term in Eq. (1) could be replaced by a scalar curvature term, which is quite natural from a geometrical point of view.

Here, I will consider the Klein-Gordon equation

$$(\Box + \chi R)\Phi = 0, \tag{3}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian (or Laplacian for a Riemannian manifold) and χ is a constant, which I will call the "eigenvalue," with some abuse of nomenclature. The scalar curvature R is assumed to be nonvanishing. I will investigate, under which conditions there exists a first-order ladder operator $\mathcal{D}$ that maps a solution of Eq. (3) to a solution of

$$(\Box + \chi' R)\mathcal{D}\Phi = 0. \tag{4}$$

If it exists, the ladder operator $\mathcal{D}$ and the new eigenvalue will be determined.

II. LADDER OPERATORS FROM CONFORMAL KILLING VECTORS

A. Properties of conformal Killing vectors

Because conformal Killing vectors (CKVs) will play a crucial role in the construction of the ladder operator $\mathcal{D}$, I will start by recalling some of their basic properties.

Consider a Riemannian or pseudo-Riemannian manifold of dimension n admitting a CKV ζ,

$$\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu = 2Qg_{\mu\nu}, \qquad Q = \frac{1}{n}\nabla_\mu\zeta^\mu. \tag{5}$$

Several identities derive from Eq. (5). It is straightforward to obtain

$$\Box\zeta^\mu = -(n-2)\nabla^\mu Q - R^{\mu\nu}\zeta_\nu. \tag{6}$$

Differentiating this once more, one gets

$$\nabla_\mu\Box\zeta^\mu = -(n-2)\Box Q - \frac{1}{2}\zeta^\mu\nabla_\mu R - RQ. \tag{7}$$

However, the left-hand side of Eq. (7) can also be written as

$$\nabla_\mu\Box\zeta^\mu = [\nabla_\mu, \Box]\zeta^\mu + n\Box Q = \frac{1}{2}\zeta^\mu\nabla_\mu R + RQ + n\Box Q. \tag{8}$$

Comparing Eqs. (7) and (8), one obtains the identity

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$$\Box Q = \frac{1}{1-n}\left(RQ + \frac{1}{2}\zeta^\mu \nabla_\mu R\right). \tag{9}$$

When acting on a scalar, the following commutation relation holds:

$$[\Box, \zeta^\mu \nabla_\mu] = 2Q\Box - (n-2)(\nabla^\mu Q)\nabla_\mu. \tag{10}$$

B. Equations for ladder operators

Consider the first-order operator

$$\mathcal{D} = \eta^\mu \nabla_\mu + V, \tag{11}$$

where η and V are some vector and scalar, respectively. If $\mathcal{D}$ is a ladder operator in the sense of Eqs. (3) and (4), then there must exist another first-order operator $\mathcal{D}' = \eta'^\mu \nabla_\mu + V'$ such that

$$(\Box + \chi' R)\mathcal{D} - \mathcal{D}'(\Box + \chi R) = 0. \tag{12}$$

Hence, the problem to be solved is to establish under which conditions one can find $\mathcal{D}$, $\mathcal{D}'$ and the new eigenvalue χ', given the eigenvalue χ.

By direct calculation, one finds

$$\begin{aligned}(\Box + \chi' R)\mathcal{D} - \mathcal{D}'(\Box + \chi R) &= [\eta^\mu - \eta'^\mu]\nabla_\mu \Box + [2(\nabla^\nu \eta^\mu)\nabla_\nu \nabla_\mu + (V - V')\Box] \\ &\quad + [(\chi' \eta^\mu - \chi \eta'^\mu)R + 2\nabla^\mu V + R^\mu{}_\nu \eta^\nu + \Box \eta^\mu]\nabla_\mu \\ &\quad + [\Box V + (\chi' V - \chi V')R - \chi \eta'^\mu \nabla_\mu R]. \end{aligned} \tag{13}$$

To satisfy Eq. (12), the terms collected in brackets on the right-hand side of Eq. (13) must vanish separately. The term in front of the third-order derivative simply yields

$$\eta'^\mu = \eta^\mu. \tag{14}$$

The second-order term vanishes, if and only if η is a CKV, $\eta^\mu = \zeta^\mu$, and

$$V' = V + 2Q, \tag{15}$$

where Q was defined in Eq. (5). Thus, Eqs. (14) and (15) determine $\mathcal{D}'$, if $\mathcal{D}$ can be found.

Using Eqs. (14) and (15) as well as the identities (6) and (9), the two terms in the second and third lines of Eq. (13) give rise to the following two equations:

$$(\chi' - \chi)R\zeta^\mu + 2\nabla^\mu V - (n-2)\nabla^\mu Q = 0, \tag{16}$$

$$(\chi' - \chi)RV + \Box V + 2\chi(n-1)\Box Q = 0. \tag{17}$$

Obviously, these always allow for the trivial solution $Q = V = 0$, $\chi' = \chi$, in which ζ^μ is a Killing vector. In this trivial case, the operator $\mathcal{D}$ is a symmetry operator. Henceforth, we shall assume nonzero Q.

Taking the divergence of Eq. (16) and using again Eq. (9), one finds

$$2\Box V - [n-2+2(n-1)(\chi'-\chi)]\Box Q + (\chi'-\chi)(n-2)RQ = 0. \tag{18}$$

To proceed, let us introduce

$$\tilde{V} = V + \gamma Q, \tag{19}$$

where γ is a constant. A short calculation shows that, if γ is chosen such that

$$\chi' - \chi = \frac{(n-2)\chi}{\gamma} - \frac{n-2+\gamma}{n-1}, \tag{20}$$

then Eqs. (17) and (18) can be combined into an equation involving only $\tilde{V}$,

$$[(n-2+2\gamma)\Box + (n-2)(\chi' - \chi)R]\tilde{V} = 0. \tag{21}$$

Note that γ should be considered as a parameter, from which the new eigenvalue χ' is determined via Eq. (20). To proceed further, one must distinguish the cases $n \neq 2$ and $n = 2$.

C. Case $n \neq 2$

With some hindsight, one can introduce a new constant α by

$$\chi' - \chi = -\frac{(n-2+2\gamma)\alpha}{n-1}, \tag{22}$$

and define, for the sake of brevity,

$$W = \frac{2\tilde{V}}{n-2+2\gamma}. \tag{23}$$

With Eqs. (19), (22) and (23), Eqs. (16) and (21) take the forms

$$\left(\nabla^\mu Q + \frac{\alpha}{n-1} R\zeta^\mu\right) = \nabla^\mu W, \tag{24}$$

$$\left(\Box - \frac{n-2}{n-1}\alpha R\right) W = 0, \tag{25}$$

respectively. Using Eqs. (20) and (22), γ is determined in terms of α and χ as one of

$$\gamma = -\frac{(n-2)(1-\alpha)}{2(1-2\alpha)}\left[1 \pm \sqrt{1 + \frac{4(n-1)(1-2\alpha)\chi}{(n-2)(1-\alpha)^2}}\right]. \tag{26}$$

We will comment on the case $\alpha = 1/2$, which seems to be special, in a moment.

At this point, we can make the following observation. If, for a CKV ζ^μ, $\nabla^\mu Q$ is proportional to $R\zeta^\mu$, i.e., if

$$\nabla^\mu Q + \frac{\alpha}{n-1} R\zeta^\mu = 0 \tag{27}$$

holds for some constant α, then Eqs. (24) and (25) can be trivially solved by $W = 0$. This implies that, with γ given by Eq. (26), the scalar V in the ladder operator (11) is $V = -\gamma Q$, and the new eigenvalue χ' follows from Eq. (22). Therefore, Eq. (27) is a sufficient condition for the existence of a ladder operator. In the following, we show that the property (27) of the CKV is also a necessary condition.

To show this, consider Eqs. (24) and (25). The divergence of Eq. (24) can be used to eliminate $\Box W$ from Eq. (25), which yields

$$W = Q + \frac{(n-1)(1-2\alpha)}{(n-2)\alpha} \frac{1}{R} \Box Q. \tag{28}$$

In passing, we note that the case $\alpha = 1/2$ cannot be a solution, because Eq. (28) would imply $W = Q$, which is inconsistent with Eq. (24). Substituting Eq. (28) back into Eq. (24) and taking again the divergence gives

$$\Box \frac{1}{R} \Box Q + \frac{2(n-2)\alpha^2}{(n-1)(1-2\alpha)} \Box Q - \frac{(n-2)^2\alpha^2}{(n-1)^2(1-2\alpha)} RQ = 0. \tag{29}$$

This can be rewritten as

$$(\Box + a_1 R) \frac{1}{R} (\Box + a_2 R) Q = 0, \tag{30}$$

where $a_{1,2}$ are given by

$$a_{1,2} = \frac{(n-2)\alpha}{(n-1)(1-2\alpha)} [\alpha \pm (\alpha - 1)]. \tag{31}$$

Therefore, Q must satisfy either

$$\left[\Box + \frac{(n-2)\alpha}{(n-1)(1-2\alpha)} R\right] Q = 0 \quad \text{or} \quad \left[\Box - \frac{(n-2)\alpha}{(n-1)} R\right] Q = 0. \tag{32}$$

In the first case, Eq. (28) gives $W = 0$, which is the solution discussed above. In the second case, one gets $W = 2\alpha Q$, so that Eq. (24) becomes

$$\nabla^\mu Q + \frac{\alpha}{(n-1)(1-2\alpha)} R\zeta^\mu = 0, \tag{33}$$

which is again of the form (27), with $\tilde{\alpha} = \alpha/(1-2\alpha)$. Hence, we have shown that the property (27) of the CKV is a necessary and sufficient condition for the existence of a ladder operator.

It is interesting to note that, by virtue of the identity (9), the condition (27) implies

$$\zeta^\mu \nabla_\mu R = 2\beta R Q, \qquad \left(\Box + \frac{1+\beta}{n-1} R\right) Q = 0, \tag{34}$$

where

$$\beta = \frac{n\alpha - 1}{1 - 2\alpha}, \qquad \alpha = \frac{1+\beta}{n+2\beta}. \tag{35}$$

Finally, it may also be useful to express Eq. (26) in terms of β,

$$\gamma = -\frac{1}{2}(n-1+\beta) \pm \frac{1}{2}\sqrt{(n-1+\beta)^2 + 4(n-1)(n+2\beta)\chi}. \tag{36}$$

D. Case $n = 2$

For $n = 2$, there is no need to introduce α, because Eq. (20) reduces to

$$\chi' - \chi = -\gamma, \tag{37}$$

and Eqs. (16) and (17) become, with Eq. (19),

$$-\gamma R\zeta^\mu + 2\nabla^\mu \tilde{V} - 2\gamma\nabla^\mu Q = 0, \tag{38}$$

$$-\gamma R\tilde{V} + \gamma^2 RQ + \Box\tilde{V} + (2\chi - \gamma)\Box Q = 0. \tag{39}$$

The divergence of Eq. (38) implies

$$\Box\tilde{V} = 0, \tag{40}$$

so that Eq. (39) gives

$$\tilde{V} = \gamma Q + \frac{2\chi - \gamma}{\gamma} \frac{1}{R} \Box Q. \tag{41}$$

Proceeding as in the case $n \neq 2$, one can show that Eqs. (40) and (41) allow only the solutions

$$\Box Q = 0 \quad \text{or} \quad \left(\Box + \frac{\gamma^2}{2\chi - \gamma} R\right) Q = 0. \tag{42}$$

In the first case, Eq. (41) gives $\tilde{V} = \gamma Q$, which means $V = 0$ from Eq. (19). This, in turn, implies $\gamma = 0$ from Eq. (38), i.e., $\chi' = \chi$. Thus, if $\Box Q = 0$, which is equivalent to the statement that $R\zeta^\mu$ must be divergence free, then $\mathcal{D} = \zeta^\mu \nabla_\mu$

maps a solution of Eq. (3) to another solution with the same eigenvalue.

The second case of Eq. (42) gives $\tilde{V} = 0$, so that Eqs. (38) and (39) give rise to the two conditions

$$R\zeta^\mu + 2\nabla^\mu Q = 0, \qquad \left(\Box + \frac{\gamma^2}{2\chi - \gamma} R\right) Q = 0. \tag{43}$$

Notice that these conditions are independent of each other, because the vector condition is divergence free. The eigenvalue shift γ is to be determined from the scalar condition, in the sense that, if Q satisfies

$$[\Box + (1+\beta)R]Q = 0 \tag{44}$$

for some β, then γ is one of

$$\gamma = -\frac{1}{2}(1+\beta) \pm \frac{1}{2}\sqrt{(1+\beta)^2 + 8(1+\beta)\chi}. \tag{45}$$

In the last two equations, we have adopted the notation of the general case, cf. Eqs. (34) and (36).

III. COROLLARY AND EXAMPLES

One can observe that, in all cases, in which a ladder operator exists, Q itself satisfies a Klein-Gordon equation of the form (3),

$$(\Box + \chi R)Q = 0, \qquad \chi = \frac{1+\beta}{n-1}, \tag{46}$$

where we have adopted the notation used in Eq. (34). Therefore, one can apply the results to construct the function

$$\Phi = \mathcal{D}Q = \zeta^\mu \nabla_\mu Q - \gamma Q^2, \tag{47}$$

where γ is one of the solutions of Eq. (36),

$$\gamma \in (1+\beta, -n-2\beta). \tag{48}$$

For $\gamma = 1+\beta$, a short calculation shows that $\chi' = 0$. Therefore, if Φ given in Eq. (47) is nonzero, then it must satisfy the massless Klein-Gordon equation.

The simplest examples of ladder operators are, of course, those of a maximally symmetric space-time. Maximally symmetric space-times have $R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu}$ with constant Ricci scalar R. This implies $\beta = 0$ from Eq. (34). Moreover, they possess closed CKVs, for which $\nabla_{[\mu}\zeta_{\nu]} = 0$. Using Eq. (6), one can easily show that

$$\nabla^\mu Q + \frac{1}{n(n-1)} R\zeta^\mu = 0, \tag{49}$$

which is Eq. (27) with $\alpha = 1/n$. Taking the case of AdS_n with unit radius, where $R = -n(n-1)$, and writing $-\chi R = m^2$, one recovers the results for AdS_n of [1].

As a nontrivial example, consider a spatially flat Friedmann-Lemaître-Robertson-Walker universe in $n = 4$ dimensions,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2). \tag{50}$$

The vector $\zeta = a\partial_t$ is a time-like (closed) CKV, with $Q = \dot{a}$, $\nabla Q = \ddot{a}\mathrm{d}t$. Moreover, the Ricci scalar is

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{51}$$

Taking, with hindsight,

$$a(t) = t^{-1/\beta}, \tag{52}$$

one can easily verify that Eq. (27) holds with α given by Eq. (35). In this example, $\Phi = \mathcal{D}Q = 0$.

IV. CONCLUSIONS

Ladder operators for the Klein-Gordon equation with a scalar curvature term have been considered. It has been shown that ladder operators require the existence of a CKV. Furthermore, ladder operators exist, if and only if the CKV satisfies an additional property. This property, for dimensions $n \neq 2$, is simply that $R\zeta^\mu$ must be proportional to $\nabla^\mu Q$. For $n = 2$, there are two cases, which have been discussed in detail. In all cases, the ladder operator has the form

$$\mathcal{D} = \zeta^\mu \nabla_\mu - \gamma Q, \tag{53}$$

with a constant γ that depends on the eigenvalue χ and a geometrical parameter that is involved in the additional property of the CKV.

The construction of the ladder operators is similar to Ref. [1], but appears to be somewhat more general, because the assumptions that the CKV be closed and an eigenvector of the Ricci tensor are replaced by the single requirement Eq. (27) (for $n \neq 2$). This simplification can be attributed to the use of a scalar curvature instead of a mass term in the Klein-Gordon equation. The results of Ref. [1] for maximally symmetric space-times have been recovered and a simple nontrivial example has been provided. It would be interesting to find more examples of CKVs satisfying Eq. (27), e.g., among those given in Refs. [3,4].

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