Eight dimensional QCD at one loop

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The Lagrangian for a non-Abelian gauge theory with an $SU(N_c)$ symmetry and a linear covariant gauge fixing is constructed in eight dimensions. The renormalization group functions are computed at one loop with the special cases of $N_c = 2$ and 3 treated separately. By computing the critical exponents derived from these in the large N_f expansion at the Wilson-Fisher fixed point it is shown that the Lagrangian is in the same universality class as the two dimensional non-Abelian Thirring model and quantum chromodynamics (QCD). As the eight dimensional Lagrangian contains new quartic gluon operators not present in four dimensional QCD, we compute in parallel the mixing matrix of *four* dimensional dimension 8 operators in pure Yang-Mills theory.

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I. INTRODUCTION

Non-Abelian gauge theories are established as the core quantum field theories which govern the particles of nature through the Standard Model. One sector, which is known as quantum chromodynamics (QCD), describes the strong force between fundamental quarks and gluons which leads to the binding of these quanta into the mesons and hadrons seen in Nature. QCD has rather distinct properties in comparison with the electroweak sector. For instance, at high energy quarks and gluons become effectively free particles due to the property of asymptotic freedom, [1,2]. While this attribute is essential to developing a field theoretic formalism which allows us to extract meaningful information from experimental data, it has an implicit sense that at lower energies quarks and gluons can never be treated as distinct particles in the same spirit as a free electron in quantum electrodynamics (QED) which is an Abelian gauge theory. The concept of a lack of low energy freedom is known as colour confinement or infrared slavery in contradistinction to the virtual freedom at ultraviolet scales. As it stands QCD has been studied in depth over many years. One area where there has been significant progress recently is in the evaluation of the fundamental renormalization group functions at very high loop order. For instance, following the one loop discovery of asymptotic freedom, [1,2], the two and three loop corrections to the β-function appeared within a decade [3–5]. Progress to the four loop term followed in the 1990s, [6,7], before a lull to the recent five loop explosion of all the renormalization group functions [8–15]. By this we mean the β-function was determined for the $SU(3)$ color group in [9] before this was extended to a general Lie group in [10]. The supporting five loop renormalization group functions were determined in [8,11–15]. While such multiloop QCD results are impressive in the extreme, in the overall scheme of things having independent checks on such calculations is useful. The recent five loop QCD β-function of [9] is relatively unique in this respect in that the independent computation of [10] followed quickly. Ordinarily such a task requires as much human and computer resources as the initial breakthrough which are not always immediately available.

For QCD there is a parallel method of verifying part of the perturbative series which is via the large N_f expansion where N_f is the number of massless quarks. For instance, the QCD β-function was determined at $O(1/N_f)$ in [16] which extended the QED result of [17]. Subsequently the quark mass anomalous dimension was found at $O(1/N_f^2)$ in [18]. The $1/N_f$ or large N_f expansion provides an alternative way of deducing certain coefficients in the perturbative series and the work of [16,18] extended the original method for spin-0 fields of [19,20] to the spin-1 case. However, the formalism for the gauge theory context derives from a novel and elegant observation made in [21]. In [21] it was shown that the non-Abelian Thirring model (NATM) in the large N_f expansion is in the same universality class as QCD at the Wilson-Fisher fixed point in d-dimensions. While the non-Abelian Thirring model is a nonrenormalizable quantum field theory above two dimensions, within the large N_f expansion at its d-dimensional fixed point the d-dimensional critical exponents contain information on the perturbative renormalization group functions of QCD. This has been verified by agreement with the latest set of five loop renormalization group functions [8–15]. The novel feature is the fact that in the non-Abelian Thirring model there are *no* triple and quartic gluon self-interactions as is well known in QCD. These vertices effectively emerge at criticality within large N_f computations via 3- and 4-point quark loops, [21]. More recently this property of critical equivalence has been studied in the simpler $O(N)$ scalar field theories where a similar phenomenon of higher dimensional theory vertices are generated at criticality by triangle and box graphs. In more modern parlance this is known as ultraviolet completion. Indeed in the $O(N)$ nonlinear σ model and $O(N)$ ϕ^4 theory, the Wilson-Fisher fixed point equivalence in $2 < d < 4$ was extended to six dimensional $O(N)$ ϕ^3 theory in [22,23] and then beyond in [24,25].

In light of this the six dimensional extension of the non-Abelian Thirring model and QCD equivalence was provided in [26]. This involved a more intricate Lagrangian but the connection of the two loop renormalization group functions with the universal d-dimensional large N_f critical exponents was verified. Again this reinforced the remarkable connection with the non-Abelian Thirring model in that the six dimensional theory has quintic and sextic gluon self-interactions in addition to cubic and quartic structures which are the only ones present in four dimensions. While formally there are cubic and quartic interactions in both these dimensions, the Feynman rules of the vertices are different in each dimension. So the fact that the large N_f non-Abelian Thirring model exponents encode information on the respective renormalization group functions is remarkable since it is not a gauge theory as such. Given this background it is therefore the purpose of this article to continue the tower of theories to the next link in the chain and construct the eight dimensional non-Abelian theory in what we will now term the non-Abelian Thirring model universality class. This runs parallel to the six and eight dimensional extensions of QED [26,27]. The eight dimensional non-Abelian theory has significantly more structure in its Lagrangian. For instance, there are seven independent quartic field strength operators in general as opposed to two in the QED case [26]. Equally one has a higher power propagator for the gluon and Faddeev-Popov ghost fields which means evaluating Feynman integrals even at one loop becomes a significant task. Therefore in this article we concentrate on a full one loop renormalization of the field anomalous dimensions and all the β-functions. As such one can regard this as proof of concept to launch a two loop computation from. The eight dimensional QED evaluation of [26] was able to probe to two loops partly because of fewer interactions but also as a consequence of the Ward-Takahashi identity.

A parallel reason for examining six and eight dimensional gauge theories rests in the connection to operators in lower dimensions. If one has the viewpoint of an underlying universal theory residing at a fixed point in d-dimensions, then the gauge independent operators corresponding to the interactions of the higher dimensional theory have dimensionless coupling constants in their respective critical dimensions. Below this dimension the coupling constant would become massive. Therefore they would equate to operators in the effective field theory of the lower dimensional gauge theory. In [26] it was noted that in the six dimensional extension of QCD the fully massive gluon propagator in the Landau gauge bore a remarkable qualitative similarity to the infrared behavior of the propagator as computed in the same gauge on the lattice but in *four* dimensions. While there was an observation in [28,29] that the ultraviolet behavior of a higher dimensional theory informs or models the infrared structure of a lower dimensional one, it would seem that an eight dimensional one could only relate to infrared fixed points in its six dimensional partner. However, given that dimension 8 operators are of interest in four dimensional effective field theories of QCD having renormalization group function data in the eight dimensional non-Abelian gauge theory for $SU(N_c)$, where N_c is the number of colors, is an additional motivation for future studies. In four dimensions such dimension 8 operators were studied in [29] for Yang-Mills theories for the $SU(2)$ and $SU(3)$ color groups. Here we extend the set and provide the one loop mixing matrix of dimension 8 operators in four dimensional $SU(N_c)$ Yang-Mills theory. It will turn out that there are qualitative structural similarities between the matrix and the β-functions of the eight dimensional theory.

The article is organized as follows. We discuss the construction of the eight dimensional Lagrangian which will be in the same universality class as the non-Abelian Thirring model and QCD in the next section. The technology used to renormalize the various n-point functions in this Lagrangian is discussed in Sec. III before presenting the main results in Sec. IV. The connection with the large N_f expansion of the critical exponents of the universality class is checked in Sec. V. In Sec. VI we change tack and determine the mixing matrix of anomalous dimensions of dimension 8 operators in four dimensional Yang-Mills theory. Finally, concluding remarks are given in Sec. VII.

II. BACKGROUND

As the first stage to constructing the eight dimensional version of QCD we recall the corresponding Lagrangians of the lower dimensional cases. The four dimensional Lagrangian is

$$L^{(4)} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + i\bar{\psi}^{iI} \not{D} \psi^{iI} - \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 - \bar{c}^a (\partial^\mu D_\mu c)^a \tag{2.1}$$

where we have included the canonical linear covariant gauge fixing term with the associated Faddeev-Popov ghost. In (2.1) and throughout the gluon field will be denoted by A^a_μ, the quark field will be ψ^{iI} and c^a are the Faddeev-Popov ghost fields where $1 \leq i \leq N_f$, $1 \leq I \leq N_F$ and $1 \leq a \leq N_A$. The parameters N_f, N_A and N_F correspond respectively to the number of (massless) quark flavors and the dimensions of the adjoint and fundamental representations of a general color group. We use α as the linear covariant gauge parameter where $\alpha = 0$ will correspond to the Landau gauge. To assist with the process of writing down the Lagrangians which are equivalent to (2.1) in higher dimensions one can regard (2.1) as being comprised of two parts. The first is the set of independent gauge invariant operators of dimension four built from the gluon and quark fields which have canonical dimensions of 1 and $\frac{3}{2}$ in four dimensions. Then in order to be able to carry out explicit computations in perturbation theory, for instance, one has to add in the appropriate gauge fixing term to ensure that a nonsingular propagator can be constructed for the gluon. This is the gauge fixing part of (2.1). From an operator point of view this involves the independent gauge variant dimension four operators. By independent we mean those operators which are not related by linear combinations of total derivative operators. Given this the six dimensional extension of (2.1) was provided in [24] based on similar work given in [30]. With the increase in dimension the canonical dimension of the quark field is now $\frac{5}{2}$ which means that there are no quartic quark interactions. However, there are two independent gauge invariant gluonic operators which are apparent in the Lagrangian [24],

$$L^{(6)} = -\frac{1}{4}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}) + \frac{g_2}{6} f^{abc} G^a_{\mu\nu} G^{b\mu\sigma} G^{c\nu}{}_\sigma - \frac{1}{2\alpha}(\partial_\mu \partial^\nu A^a_\nu)(\partial^\mu \partial^\sigma A^a_\sigma) - \bar{c}^a \Box (\partial^\mu D_\mu c)^a + i\bar{\psi}^{iI} \not{D} \psi^{iI} \tag{2.2}$$

which means that there are two coupling constants. Demonstrating the independence of the gluonic operators lies in part with the use of the Bianchi identity

$$D_\mu G^a_{\nu\sigma} + D_\nu G^a_{\sigma\mu} + D_\sigma G^a_{\mu\nu} = 0. \tag{2.3}$$

The remaining gauge invariant operator is the quark kinetic term wherein lies the quark-gluon interaction which is the core interaction in the tower of theories at the Wilson-Fisher fixed point. Throughout we will always denote the usual gauge coupling constant by g_1 when there are one or more interactions. The remaining part of (2.2) is completed with the dimension six linear covariant gauge fixing term which is the obvious extension of the four dimensional one.

Equipped with this brief review of the construction of the dimension four and six non-Abelian gauge theories, the algorithm is now in place to proceed to eight dimensions. In [31,32] the renormalization of dimension eight operators in four dimensional Yang-Mills theory was considered and those articles serve as the basis for the eight dimensional Lagrangian. As was discussed in [31] there is only one independent dimension eight 2-point gauge invariant operator which therefore serves as the gluon kinetic term. Equally [31,32] there are two independent dimension eight 3-point gluon operators. The new feature in eight dimensions, which derives from the fact that the gluon canonical dimension is unity, is that there will be quartic gluon field strength gauge invariant operators. The same property is present in eight dimensional QED which was introduced in [26] where there were several quartic photon self-interactions. For the non-Abelian case there is the added complication of having to incorporate the color group indices. The upshot is that one has to specify a particular color group as it is not possible to have a finite set of quartic gluon opertors for a general Lie group [31]. Therefore we restrict ourselves to the $SU(N_c)$ Lie group and recall relevant basic properties of this group needed for the Lagrangian. If T^a is the Lie group generator then in $SU(N_c)$ the product of two generators can be written as the linear combination

$$T^a T^b = \frac{1}{2N_c}\delta^{ab} + \frac{1}{2} d^{abc} T^c + \frac{i}{2} f^{abc} T^c \tag{2.4}$$

where d^{abc} is totally symmetric and the structure constants, f^{abc}, are totally antisymmetric. Equally when we have to treat Feynman graphs with quarks, the $SU(N_c)$ relation

$$T^a_{IJ} T^a_{KL} = \frac{1}{2}\left[\delta_{IL}\delta_{KJ} - \frac{1}{N_c}\delta_{IJ}\delta_{KL}\right] \tag{2.5}$$

will be useful. To define gauge independent quartic gluon operators we introduce the rank 4 color tensors

$$f_4^{abcd} \equiv f^{abe} f^{cde}, \qquad d_4^{abcd} \equiv d^{abe} d^{cde} \tag{2.6}$$

and then use the $SU(N_c)$ relation between them [33],

$$f_4^{abcd} = \frac{2}{N_c}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}) + d_4^{acbd} - d_4^{adbc}. \tag{2.7}$$

This in effect [33] is the generalization of the relation between the product of Levi-Civita tensors in $SU(2)$ to the color groups $SU(N_c)$ for $N_c \geq 3$. It means that we use the tensor d_4^{abcd} as the preferred tensor of the gauge invariant operators. One reason for this is that d_4^{abcd} is separately symmetric in the first or last pair of indices from the full symmetry property of d^{abc}. Consequently there are eight gauge independent quartic gluon operators in the eight dimensional extension of the QCD Lagrangian leading to eleven independent coupling constants overall. The full Lagrangian is

$$L^{(8)} = -\frac{1}{4}(D_\mu D_\nu G^a_{\sigma\rho})(D^\mu D^\nu G^{a\sigma\rho}) + \frac{g_2}{4} f^{abc} G^a_{\mu\nu} D^\mu G^{b\sigma\rho} D^\nu G^c_{\sigma\rho} + i\bar{\psi}^{iI} \not{D} \psi^{iI} + \frac{g_3}{2} f^{abc} G^a_{\mu\nu} D_\sigma G^{b\mu\rho} D^\sigma G^{c\nu}{}_\rho$$
$$+ g_4^2 G^a_{\mu\sigma} G^{a\mu\rho} G^{b\sigma\nu} G^b_{\rho\nu} + g_5^2 G^a_{\mu\sigma} G^{b\mu\rho} G^{b\sigma\nu} G^a_{\rho\nu} + g_6^2 G^a_{\mu\sigma} G^a_{\nu\rho} G^{b\sigma\mu} G^{b\rho\nu} + g_7^2 G^a_{\mu\sigma} G^b_{\nu\rho} G^{a\sigma\mu} G^{b\rho\nu} + g_8^2 d_4^{abcd} G^a_{\mu\sigma} G^{b\mu\sigma} G^c_{\nu\rho} G^{d\nu\rho}$$
$$+ g_9^2 d_4^{abcd} G^a_{\mu\sigma} G^{c\mu\rho} G^{b\nu\sigma} G^d_{\nu\rho} + g_{10}^2 d_4^{acbd} G^a_{\mu\sigma} G^{b\mu\sigma} G^c_{\nu\rho} G^{d\nu\rho} + g_{11}^2 d_4^{adbc} G^a_{\mu\sigma} G^{c\mu\rho} G^{b\nu\sigma} G^d_{\nu\rho} - \frac{1}{2\alpha}(\partial_\mu \partial_\nu \partial^\sigma A^a_\sigma)(\partial^\mu \partial^\nu \partial^\rho A^a_\rho)$$
$$- (\Box \bar{c}^a)(\Box \partial^\mu D_\mu c)^a \tag{2.8}$$

where like (2.1) and (2.2) the dimension eight linear covariant gauge fixing term is included. In addition the quark kinetic term is present and is equivalent to those in the lower dimensional Lagrangians which therefore preserves the connection with the Wilson-Fisher fixed point and the underlying universal theory which is accessible from the large N_f expansion. While (2.8) represents the full $SU(N_c)$ Lagrangian those for $N_c = 2$ and 3 are smaller due to properties of the color tensors. For instance, for the $SU(2)$ group $d^{abc} = 0$. So for that group one has $g_8 = g_9 = g_{10} = g_{11} = 0$. For $SU(3)$ $d^{abc} \neq 0$ but d_4^{abcd} satisfies

$$d_4^{adbc} = -d_4^{abcd} - d_4^{acbd} + \frac{1}{3}[\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}]. \tag{2.9}$$

This means that two of the operators involving d_4^{abcd} are absent and within our computations we have set $g_{10} = g_{11} = 0$ for $SU(3)$. Finally we note several useful $SU(N_c)$ group identities, which we used within our graph evaluations, that are [33]

$$d_4^{abcc} = 0, \qquad d_4^{acbc} = \frac{[N_c^2 - 4]}{N_c}\delta^{ab},$$
$$d_4^{apbq} d_4^{cdpq} = \frac{[N_c^2 - 12]}{2N_c} d_4^{abcd}. \tag{2.10}$$

From the quadratic part of (2.8) in momentum space we find that the gluon and ghost propagators are

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle = -\frac{\delta^{ab}}{(p^2)^3}\left[\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right],$$
$$\langle c^a(p) \bar{c}^b(-p)\rangle = -\frac{\delta^{ab}}{(p^2)^3}, \tag{2.11}$$

which are formally the same as those in lower dimensions aside from the cubic power of the overall factor. This is a similar feature to other eight dimensional theories and means that the evaluation of the Feynman graphs we have to compute becomes exceedingly tedious.

While we have constructed the most general non-Abelian gauge theory based on a simple Lie group in (2.8), this is in the case where there are no masses present. The latter would not contribute to the renormalization group functions at the Wilson-Fisher fixed point which is the main reason for not considering them initially. However, one could view the presence of masses as touching the lower dimensional operators which are allowed by power counting renormalizability and which would be a staging point for connecting with the other equivalent Lagrangians for this universality class. Therefore, budgeting for nonzero masses (2.8) generalizes to

$$L_m^{(8)} = L^{(8)} + m_1 \bar{\psi}^{iI}\psi^{iI} - \frac{1}{4} m_2^2 (D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma})$$
$$- \frac{1}{2\alpha} m_3^2 (\partial_\mu \partial^\nu A^a_\nu)(\partial^\mu \partial^\sigma A^a_\sigma) - m_3^2 \bar{c}^a \Box (\partial^\mu D_\mu c)^a$$
$$- \frac{1}{4} m_4^4 G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2\alpha} m_5^4 (\partial^\mu A^a_\mu)^2 - m_5^4 \bar{c}^a (\partial^\mu D_\mu c)^a$$
$$- \frac{1}{2} m_6^6 A^a_\mu A^{a\mu} + m_6^6 \alpha \bar{c}^a c^a + \frac{1}{6} m_7^2 f^{abc} G^a_{\mu\nu} G^{b\mu\sigma} G^{c\nu}{}_\sigma. \tag{2.12}$$

The additional terms fall into two classes which are operators which are gauge invariant or not. In the latter case those operators are Becchi-Rouet-Stora-Tyutin (BRST) invariant. In particular it is evident that the lower dimensional operators are a reflection of the Lagrangians of the lower dimensional massless Lagrangians in the same universality class. In other words in the critical dimension of the lower dimensional Lagrangians the masses would correspond to coupling constants and hence be dimensionless in that spacetime. Implicit in (2.12) is the assumption of locality. If one ignored this and allowed for nonlocal operators then it is possible to construct a completely gauge invariant massive Lagrangian as discussed in [24]. The gluon and ghost propagators of (2.12) have Stingl forms [34], since

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle = -\frac{\delta^{ab} P_{\mu\nu}(p)}{[(p^2)^3 + m_2^2 (p^2)^2 + m_4^4 p^2 + m_6^6]}$$
$$- \frac{\alpha \delta^{ab} L_{\mu\nu}(p)}{[(p^2)^3 + m_3^2 (p^2)^2 + m_5^2 p^2 + \alpha m_6^6]},$$
$$\langle c^a(p) \bar{c}^b(-p)\rangle = -\frac{\delta^{ab}}{[(p^2)^3 + m_3^2 (p^2)^2 + m_5^4 p^2 + \alpha m_6^6]}, \tag{2.13}$$

where

$$P_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad L_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} \tag{2.14}$$

are the respective transverse and longitudinal projection tensors. In this formulation it is apparent that the pole structure of the Faddeev-Popov ghost propagator matches that of the longitudinal part of the gluon. This ensures the cancellation of unphysical degrees of freedom within computations with the massive Lagrangian.

III. TECHNICAL DETAILS

The task of renormalizing (2.8) requires several technical tools some of which were applied to the determination of the two loop renormalization group functions of $L^{(6)}$. However, with the presence of gauge independent 4-point operators built from the field strength, the extraction of the β-functions of the respective coupling constants required a technique not employed in [24]. First, we note that we have constructed an automatic program to renormalize the various 2-, 3- and 4-point functions. The graphs contributing to each Green's function are generated using the FORTRAN based package QGRAF [35]. With the spinor, Lorentz and color group indices added to the electronic representation of the diagrams, each diagram is then passed to the integration routine specific to that particular n-point function. Once the divergences with respect to the regularization are known for each graph, the full set is summed and the renormalization constants determined automatically without the use of the subtraction method but instead using the algorithm provided in [36]. Briefly this is achieved by computing each Green's function as a function of the bare coupling constants and gauge parameter with their respective renormalized versions introduced by multiplicatively rescaling with the constant of proportionality being the renormalization constant. Specifically, at each loop order the renormalization constant associated with the Green's function is fixed by ensuring it is finite which determines the unknown counterterm at that order. Throughout this article we will consider only the $\overline{\text{MS}}$ scheme and regularize the theory using dimensional regularization where the spacetime dimension d is set to $d = 8 - 2\epsilon$ and ϵ is small. It acts as the regularization parameter. To handle the significant amounts of internal algebra of this whole process, use is made of the symbolic manipulation language FORM [37,38]. It is worth noting that the renormalization of (2.8) involves 12 independent parameters as well as color and flavor parameters together with gluon and ghost propagators each of which have an exponent of 3. This means there is a significant amount of integration to be performed, compared to four dimensional QCD, for which FORM is the most efficient and practical tool for the task.

In order to construct the integration routine for each type of n-point function, we follow what is now a well-established procedure which is the application of the integration by parts algorithm devised by Laporta [39]. To evaluate a Feynman graph it is first written as a sum of scalar integrals where scalar products of internal and external momenta are rewritten as combinations of the inverse propagators. For cases where there is no such propagator in an integral, which is termed an irreducible, the basis of propagators is extended or completed. It transpires that for each n-point function at a particular loop order there is a small set of such independent completions which are called integral families. These may or may not correspond to an actual Feynman diagram topology. Irrespective of this it is the mathematical representation of the integral family which is at the center of the Laporta method. One can determine a set of general algebraic relations between integrals in each family by integration by parts and Lorentz identities. The power of the Laporta algorithm is in realizing that these relations can be solved algebraically in terms of a small set of basic or master Feynman integrals [39]. Thus if the ϵ expansion of these master integrals is known then all the Feynman integrals at that loop order can be determined. In particular this includes the specific ones which comprise each of the graphs in the n-point functions of interest. There are various encodings of the Laporta algorithm available but we chose to use both versions of REDUZE [40,41]. While this outlines the general approach we used, there are specific points which required attention. As we are renormalizing an *eight* dimensional Lagrangian we therefore need to have the master integrals in that dimension. Ordinarily the main focus in renormalization computations is four dimensions. However, we have not had to perform the explicit evaluation of master integrals by direct methods which is the normal way to determine their values. Instead we can exploit an elegant technique developed by Tarasov in [42,43]. By considering the graph polynomial representation of a Feynman graph, it is possible to relate a Feynman integral in d-dimensions in terms of a linear combination of the same integrals in $(d+2)$-dimensions. The latter, however, have several propagators with increased powers which is clearly necessary on dimensional grounds. This higher dimensional set of integrals can be reduced to a linear combination of masters in the higher dimension. One of these will be the equivalent topology as the d-dimensional master with the remainder of the combination being masters with a fewer number of propagators [42,43]. As is the case in the Laporta algorithm, some of these lower masters are integrals, such as simple bubble integrals, which are trivial to evaluate without using the Tarasov techniques. Therefore one can connect the more difficult to compute masters in d-dimensions with the unknown ones in $(d+2)$-dimensions. If the lower dimensional ones are available then the higher dimensional ones follow immediately. For our purposes we need to apply this connection twice since the various masters required are known in four dimensions. For instance, the 2-point masters to four loops have been listed in [44] while the 3-point masters for completely off-shell external legs were calculated to two loops in [45,46]. Also the one loop 4-point box integral is known [47]. Although we will not require the higher loop masters here, it is worth noting what has been achieved over several years.

This leads naturally to a brief discussion of the treatment of each set of n-point functions separately. For the 2-point functions and hence wave function renormalization constants, we carried out the renormalization to two loops. The main reason for this is that the double pole in ϵ of the two loop renormalization constant is already predetermined by the one loop computation. Therefore this provides a partial check on the leading order renormalization. For the 2-point function we used the massless Lagrangian and constructed the one and two loop masters by direct evaluation as these are straightforward bubble integrals. By contrast for the 3-point functions, since nullifying an external leg leads to infrared issues, we had to extend the four dimensional off-shell massless master 3-point function of [44,46] to eight dimensions using the Tarasov method [42,43]. For instance, if we define the one loop triangle integral at the completely symmetric point by

$$I(\alpha,\beta,\gamma) = \int_k \frac{1}{(k^2)^\alpha((k-p)^2)^\beta((k+q)^2)^\gamma} \tag{3.1}$$

where p and q are the external momenta satisfying

$$p^2 = q^2 = -\mu^2 \tag{3.2}$$

and $\int_k = d^dk/(2\pi)^d$ then

$$\begin{aligned} I(1,1,1)|_{d=8-2\epsilon} = -\mu^2\Bigg[&-\frac{1}{8\epsilon} - \frac{61}{144} - \frac{2\pi^2}{81} + \frac{1}{27}\psi'\left(\frac{1}{3}\right) \\ &+ \left[\frac{1}{18}\psi'\left(\frac{1}{3}\right) - \frac{895}{864} - \frac{23\pi^2}{864} - \frac{2}{3}s_3\left(\frac{\pi}{6}\right) \right. \\ &\left. + \frac{35}{5832}\pi^3\sqrt{3} + \frac{\pi}{216}\ln^2(3)\sqrt{3}\right]\epsilon + O(\epsilon^2)\Bigg] \end{aligned} \tag{3.3}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ and

$$s_n(z) = \frac{1}{\sqrt{3}}\Im\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right] \tag{3.4}$$

in terms of the polylogarithm function $\mathrm{Li}_n(z)$. While only the simple pole in ϵ is relevant for the renormalization of (2.8), we have included the subsequent terms in the ϵ expansion for comparison with the analogous lower dimensional masters. The finite part for instance is directly correlated with the finite four dimensional master. The simple pole in (3.3) by contrast derives from the one loop bubble integrals which emerge in the Laporta reduction after the construction of the $(d+2)$-dimensional integrals from the d-dimensional master across two iterations. Equipped with (3.3) the three coupling constants associated with the three independent 3-point gluonic operators as well as those of the quark and ghost vertices of (2.8) were renormalized using this strategy. For the latter vertices the quark-gluon vertex renormalization, for instance, determines the renormalization constant for g_1 which can be checked in the ghost-gluon vertex computation. For the remaining two couplings in this set, g_2 and g_3, their renormalization can be determined from the gluon 3-point vertex which provides a third check on the β-function of g_1. From examining the Feynman rule for the 3-gluon vertex it can be seen that there are three independent tensor channels to provide three independent linear relations between the renormalization constants for these couplings.

For the final part of the renormalization we have to extract the renormalization constants for the couplings associated with the purely quartic operators of each eight dimensional Lagrangian. For this we used the vacuum bubble expansion of [48,49] as it was more efficient than constructing a large integration by parts database using REDUZE. This would be time consuming to construct due to the high pole propagators for the gluon and ghost. By contrast, in the vacuum bubble expansion massless propagators are recursively replaced by massive ones in such a way that the new propagators eventually produce Feynman integrals which are ultraviolet finite. Hence by Weinberg's theorem [50], these do not contribute to the overall renormalization of the Green's function and so such terms can be neglected. Subsequently the expansion terminates after a finite number of iterations. The expansion is based on the exact identity [48,49],

$$\frac{1}{(k-p)^2} = \frac{1}{[k^2+m^2]} + \frac{2kp - p^2 + m^2}{(k-p)^2[k^2+m^2]}. \tag{3.5}$$

The contribution to the overall degree of divergence of each of the numerator pieces in the second term is less than that of the original propagator. In addition, the first term does not depend on the external momentum. So when all such terms are collected within a Feynman integral, it becomes a massive vacuum integral. Of course to produce the contributions which are purely vacuum bubbles and contain the ultraviolet divergence of the Feynman graph, the identity has to be repeated sufficient times. Once this has been achieved a simple Laporta reduction of one loop vacuum bubbles is constructed to reduce the only one loop master vacuum bubble which is a simple standard integral in eight dimensions. Another advantage of this approach is that the tensor structure arising from the external momenta together with the scalar products of external momenta derived from (3.5) emerge relatively quickly. In the summation of all the contributions to the gluon 4-point function such terms are central to disentangling the coupling constant renormalization constants for each of the independent quartic operators. A useful check on the procedure is the absence of the parameter of the linear covariant gauge fixing in each of the coupling constant renormalizations in the three separate color group computations we have to perform.

IV. RESULTS

We turn now to the task of recording the results of our renormalization. First, we have followed the conventions of previous analyses [24] and note that the renormalization of the parameter of the linear covariant gauge fixing is not independent of the gluon wave function renormalization in that

$$\gamma_A(g_i) + \gamma_\alpha(g_i) = 0. \tag{4.1}$$

We have checked that this is true for all the $SU(N_c)$ color groups. For $SU(2)$ the anomalous dimensions of the fields are

$$\begin{aligned}
\gamma_A^{SU(2)}(g_i)|_{\alpha=0} = {} & [24N_f g_1^2 + 871g_1^2 - 4158g_1g_2 - 1386g_1g_3 + 567g_2^2 + 378g_2g_3 + 63g_3^2]\frac{1}{1680} \\
& + [-57594816N_f g_1^4 - 2754788105g_1^4 + 37417536N_f g_1^3g_2 + 406217016g_1^3g_2 + 18601152N_f g_1^3g_3 \\
& + 191078016g_1^3g_3 - 4398624N_f g_1^2g_2^2 - 1747949454g_1^2g_2^2 - 3900096N_f g_1^2g_2g_3 - 2040796188g_1^2g_2g_3 \\
& - 1053216N_f g_1^2g_3^2 - 261984978g_1^2g_3^2 + 137535552g_1^2g_4^2 - 275071104g_1^2g_5^2 - 1124500608g_1^2g_6^2 \\
& + 2249001216g_1^2g_7^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 + 500362128g_1g_2g_3^2 + 155288448g_1g_2g_4^2 \\
& + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 310576896g_1g_2g_5^2 + 234033408g_1g_2g_6^2 + 425614392g_1g_2^3 \\
& + 881618976g_1g_2^2g_3 - 468066816g_1g_2g_7^2 + 84640248g_1g_3^3 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 \\
& + 200785536g_1g_3g_4^2 - 401571072g_1g_3g_5^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 21337344g_1g_3g_6^2 \\
& + 42674688g_1g_3g_7^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 26643897g_2^4 - 87736068g_2^3g_3 \\
& + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 89488602g_2^2g_3^2 - 52581312g_2^2g_4^2 + 425614392g_1g_2^3 \\
& + 881618976g_1g_2^2g_3 + 105162624g_2^2g_5^2 + 19813248g_2^2g_6^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 \\
& - 39626496g_2^2g_7^2 - 35913276g_2g_3^3 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 75696768g_2g_3g_4^2 \\
& + 151393536g_2g_3g_5^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 + 40303872g_2g_3g_6^2 - 80607744g_2g_3g_7^2 \\
& + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 - 5230701g_3^4 - 19389888g_3^2g_4^2 + 425614392g_1g_2^3 \\
& + 881618976g_1g_2^2g_3 + 38779776g_3^2g_5^2 + 11233152g_3^2g_6^2 + 425614392g_1g_2^3 + 881618976g_1g_2^2g_3 \\
& - 22466304g_3^2g_7^2]\frac{1}{338688000} + O(g_i^6) \\
\gamma_c^{SU(2)}(g_i)|_{\alpha=0} = {} & -\frac{7}{24}g_1^2 + [12312N_f g_1^2 - 3321487g_1^2 - 628614g_1g_2 - 241878g_1g_3 + 77301g_2^2 \\
& + 192654g_2g_3 + 108549g_3^2]\frac{g_1^2}{2419200} + O(g_i^6) \\
\gamma_\psi^{SU(2)}(g_i)|_{\alpha=0} = {} & \frac{7}{16}g_1^2 + [-17352N_f g_1^2 + 3509752g_1^2 + 1722294g_1g_2 + 973938g_1g_3 - 196371g_2^2 \\
& - 272034g_2g_3 - 121779g_3^2]\frac{g_1^2}{1612800} + O(g_i^6)
\end{aligned} \tag{4.2}$$

in the Landau gauge which is chosen for presentational reasons. The full α dependent results are contained in the Supplemental Material [51]. One of the reasons for proceeding to two loops for this is as a check on the computation. The double pole in ϵ at two loops of the respective renormalization constants is not independent as it depends on the simple pole at one loop. We have verified that this is indeed the case in the explicit renormalization constants for arbitrary α. This checks the one loop coupling constant renormalization as well as the application of the Tarasov method [42,43], to raise the four and six dimension massless two loop 2-point master integrals to eight dimensions. The one loop β-functions are

$$
\begin{aligned}
\beta_1^{SU(2)}(g_i) &= [24N_f g_1^2 - 109g_1^2 - 4158g_1g_2 - 1386g_1g_3 + 567g_2^2 + 378g_2g_3 + 63g_3^2]\frac{g_1}{3360} + O(g_i^5) \\
\beta_2^{SU(2)}(g_i) &= [-272N_f g_1^3 + 32152g_1^3 + 216N_f g_1^2g_2 + 17919g_1^2g_2 - 19908g_1^2g_3 - 32634g_1g_2^2 - 2646g_1g_2g_3 + 3528g_1g_3^2 \\
&\quad + 5103g_2^3 + 2898g_2^2g_3 - 441g_2g_3^2 - 168g_3^3]\frac{1}{10080} + O(g_i^5) \\
\beta_3^{SU(2)}(g_i) &= [-128N_f g_1^3 - 18573g_1^3 + 14889g_1^2g_2 + 36N_f g_1^2g_3 + 8163g_1^2g_3 - 2520g_1g_2^2 - 7539g_1g_2g_3 - 777g_1g_3^2 \\
&\quad + 5544g_1g_4^2 - 11088g_1g_5^2 - 3696g_1g_6^2 + 7392g_1g_7^2 + 819g_2^2g_3 + 378g_2g_3^2 - 1512g_2g_4^2 + 3024g_2g_5^2 \\
&\quad + 1008g_2g_6^2 - 2016g_2g_7^2 - 21g_3^3 - 504g_3g_4^2 + 1008g_3g_5^2 + 336g_3g_6^2 - 672g_3g_7^2]\frac{1}{1680} + O(g_i^5) \\
\beta_4^{SU(2)}(g_i) &= [800N_f g_1^4 + 73999g_1^4 - 82068g_1^3g_2 - 48426g_1^3g_3 + 13734g_1^2g_2^2 + 12852g_1^2g_2g_3 + 3360g_1^2g_3^2 + 1152N_f g_1^2g_4^2 \\
&\quad - 89904g_1^2g_4^2 - 32592g_1^2g_5^2 - 113568g_1^2g_6^2 - 193536g_1^2g_7^2 - 42g_1g_2^2g_3 + 1008g_1g_2g_3^2 - 179424g_1g_2g_4^2 \\
&\quad - 15456g_1g_2g_5^2 - 2688g_1g_2g_6^2 + 8064g_1g_2g_7^2 + 2058g_1g_3^3 - 6720g_1g_3g_4^2 - 7392g_1g_3g_5^2 - 43008g_1g_3g_6^2 \\
&\quad - 45696g_1g_3g_7^2 + 27216g_2^2g_4^2 + 18144g_2g_3g_4^2 - 903g_3^4 - 23184g_3^2g_4^2 - 5712g_3^2g_5^2 - 12768g_3^2g_6^2 - 13440g_3^2g_7^2 \\
&\quad - 169344g_4^4 - 188160g_4^2g_5^2 - 177408g_4^2g_6^2 - 139776g_4^2g_7^2 - 124992g_5^4 - 145152g_5^2g_6^2 - 21504g_5^2g_7^2 \\
&\quad - 37632g_6^4 - 43008g_6^2g_7^2 - 43008g_7^4]\frac{1}{40320} + O(g_i^6) \\
\beta_5^{SU(2)}(g_i) &= [-1192N_f g_1^4 - 101355g_1^4 + 84756g_1^3g_2 + 19194g_1^3g_3 - 14070g_1^2g_2^2 - 16884g_1^2g_2g_3 + 1848g_1^2g_3^2 + 52416g_1^2g_4^2 \\
&\quad + 1152N_f g_1^2g_5^2 + 16608g_1^2g_5^2 + 92064g_1^2g_6^2 + 193536g_1^2g_7^2 + 42g_1g_2^2g_3 + 336g_1g_2g_3^2 - 12096g_1g_2g_4^2 \\
&\quad - 181440g_1g_2g_5^2 + 8064g_1g_2g_6^2 - 8064g_1g_2g_7^2 + 966g_1g_3^3 - 11424g_1g_3g_4^2 - 43008g_1g_3g_5^2 + 75264g_1g_3g_6^2 \\
&\quad + 45696g_1g_3g_7^2 + 27216g_2^2g_5^2 + 18144g_2g_3g_5^2 - 21g_3^4 + 10080g_3^2g_4^2 + 3360g_3^2g_5^2 + 2016g_3^2g_6^2 + 13440g_3^2g_7^2 \\
&\quad - 10752g_4^4 - 107520g_4^2g_5^2 - 37632g_4^2g_6^2 - 10752g_4^2g_7^2 - 12096g_5^4 - 26880g_5^2g_6^2 \\
&\quad - 129024g_5^2g_7^2 - 26880g_6^4 - 43008g_6^2g_7^2]\frac{1}{40320} + O(g_i^6) \\
\beta_6^{SU(2)}(g_i) &= [272N_f g_1^4 - 248207g_1^4 + 14742g_1^3g_2 + 134925g_1^3g_3 + 231g_1^2g_2^2 - 7728g_1^2g_2g_3 - 1323g_1^2g_3^2 - 222432g_1^2g_4^2 \\
&\quad - 343392g_1^2g_5^2 + 2304N_f g_1^2g_6^2 - 1440480g_1^2g_6^2 - 228480g_1^2g_7^2 + 147g_1g_2^2g_3 - 4326g_1g_2g_3^2 + 26880g_1g_2g_4^2 \\
&\quad + 48384g_1g_2g_5^2 - 204288g_1g_2g_6^2 + 2688g_1g_2g_7^2 - 4557g_1g_3^3 + 52416g_1g_3g_4^2 + 118272g_1g_3g_5^2 \\
&\quad + 247296g_1g_3g_6^2 + 34944g_1g_3g_7^2 + 54432g_2^2g_6^2 + 36288g_2g_3g_6^2 - 42g_3^4 + 3360g_3^2g_4^2 + 7392g_3^2g_5^2 + 50400g_3^2g_6^2 \\
&\quad - 21504g_4^4 - 80640g_4^2g_5^2 - 451584g_4^2g_6^2 - 21504g_4^2g_7^2 - 77952g_5^4 - 806400g_5^2g_6^2 - 43008g_5^2g_7^2 - 1666560g_6^4 \\
&\quad - 301056g_6^2g_7^2]\frac{1}{80640} + O(g_i^6) \\
\beta_7^{SU(2)}(g_i) &= [8N_f g_1^4 - 472989g_1^4 + 154266g_1^3g_2 + 155883g_1^3g_3 - 10647g_1^2g_2^2 - 31584g_1^2g_2g_3 + 651g_1^2g_3^2 + 637056g_1^2g_4^2 \\
&\quad + 480480g_1^2g_5^2 + 1704192g_1^2g_6^2 + 2304N_f g_1^2g_7^2 + 3470496g_1^2g_7^2 - 147g_1g_2^2g_3 - 1050g_1g_2g_3^2 - 72576g_1g_2g_4^2 \\
&\quad - 52416g_1g_2g_5^2 - 202944g_1g_2g_6^2 - 751296g_1g_2g_7^2 - 3339g_1g_3^3 - 124992g_1g_3g_4^2 - 81984g_1g_3g_5^2 \\
&\quad - 307776g_1g_3g_6^2 - 651840g_1g_3g_7^2 + 54432g_2^2g_7^2 + 36288g_2g_3g_7^2 - 42g_3^4 - 9408g_3^2g_4^2 - 6048g_3^2g_5^2 \\
&\quad - 41664g_3^2g_6^2 - 74592g_3^2g_7^2 - 91392g_4^4 - 80640g_4^2g_5^2 - 408576g_4^2g_6^2 - 1580544g_4^2g_7^2 - 5376g_5^4 - 64512g_5^2g_6^2 \\
&\quad - 913920g_5^2g_7^2 - 118272g_6^4 - 3440640g_6^2g_7^2 - 4773888g_7^4]\frac{1}{80640} + O(g_i^6).
\end{aligned}
\tag{4.3}
$$

The main perturbative check on these expressions is the absence of the gauge parameter. We computed the various 4-point functions with nonzero α and verified that it canceled in the final Green's function as it ought since we are using the $\overline{\text{MS}}$ scheme.

The results for the case of $SU(3)$ are somewhat similar aside from the additional two couplings. We have

$$\gamma_A^{SU(3)}(g_i)|_{\alpha=0} = [16g_1^2N_f + 871g_1^2 - 4158g_1g_2 - 1386g_1g_3 + 567g_2^2 + 378g_2g_3 + 63g_3^2]\frac{1}{1120}$$
$$+ [-110877632g_1^4N_f - 8264364315g_1^4 + 74835072g_1^3g_2N_f + 1218651048g_1^3g_2 + 37202304g_1^3g_3N_f$$
$$+ 573234048g_1^3g_3 - 8797248g_1^2g_2^2N_f - 5243848362g_1^2g_2^2 - 7800192g_1^2g_2g_3N_f - 6122388564g_1^2g_2g_3$$
$$- 2106432g_1^2g_3^2N_f - 785954934g_1^2g_3^2 + 275071104g_1^2g_4^2 - 550142208g_1^2g_5^2 - 2249001216g_1^2g_6^2$$
$$+ 4498002432g_1^2g_7^2 + 3748335360g_1^2g_8^2 + 229225920g_1^2g_9^2 + 1276843176g_1g_2^3 + 2644856928g_1g_2^2g_3$$
$$+ 1501086384g_1g_2g_3^2 + 310576896g_1g_2g_4^2 - 621153792g_1g_2g_5^2 + 468066816g_1g_2g_6^2 - 936133632g_1g_2g_7^2$$
$$- 780111360g_1g_2g_8^2 + 258814080g_1g_2g_9^2 + 253920744g_1g_3^3 + 401571072g_1g_3g_4^2 - 803142144g_1g_3g_5^2$$
$$- 42674688g_1g_3g_6^2 + 85349376g_1g_3g_7^2 + 71124480g_1g_3g_8^2 + 334642560g_1g_3g_9^2 - 79931691g_2^4$$
$$- 263208204g_2^3g_3 - 268465806g_2^2g_3^2 - 105162624g_2^2g_4^2 + 210325248g_2^2g_5^2 + 39626496g_2^2g_6^2$$
$$- 79252992g_2^2g_7^2 - 66044160g_2^2g_8^2 - 87635520g_2^2g_9^2 - 107739828g_2g_3^3 - 151393536g_2g_3g_4^2$$
$$+ 302787072g_2g_3g_5^2 + 80607744g_2g_3g_6^2 - 161215488g_2g_3g_7^2 - 134346240g_2g_3g_8^2 - 126161280g_2g_3g_9^2$$
$$- 15692103g_3^4 - 38779776g_3^2g_4^2 + 77559552g_3^2g_5^2 + 22466304g_3^2g_6^2 - 44932608g_3^2g_7^2 - 37443840g_3^2g_8^2$$
$$- 32316480g_3^2g_9^2]\frac{1}{451584000} + O(g_i^6)$$

$$\gamma_c^{SU(3)}(g_i)|_{\alpha=0} = -\frac{7}{16}g_1^2 + [8208g_1^2N_f - 3321487g_1^2 - 628614g_1g_2 - 241878g_1g_3 + 77301g_2^2 + 192654g_2g_3$$
$$+ 108549g_3^2]\frac{g_1^2}{1075200} + O(g_i^6)$$

$$\gamma_\psi^{SU(3)}(g_i)|_{\alpha=0} = \frac{7}{9}g_1^2 + [-3856g_1^2N_f + 1147459g_1^2 + 574098g_1g_2 + 324646g_1g_3 - 65457g_2^2$$
$$- 90678g_2g_3 - 40593g_3^2]\frac{g_1^2}{201600} + O(g_i^6)$$

$$\beta_1^{SU(3)}(g_i) = [16g_1^2N_f - 109g_1^2 - 4158g_1g_2 - 1386g_1g_3 + 567g_2^2 + 378g_2g_3 + 63g_3^2]\frac{g_1}{2240} + O(g_i^5)$$

$$\beta_2^{SU(3)}(g_i) = [-544g_1^3N_f + 96456g_1^3 + 432g_1^2g_2N_f + 53757g_1^2g_2 - 59724g_1^2g_3 - 97902g_1g_2^2 - 7938g_1g_2g_3$$
$$+ 10584g_1g_3^2 + 15309g_2^3 + 8694g_2^2g_3 - 1323g_2g_3^2 - 504g_3^3]\frac{1}{20160} + O(g_i^5)$$

$$\beta_3^{SU(3)}(g_i) = [-256g_1^3N_f - 55719g_1^3 + 44667g_1^2g_2 + 72g_1^2g_3N_f + 24489g_1^2g_3 - 7560g_1g_2^2 - 22617g_1g_2g_3$$
$$- 2331g_1g_3^2 + 11088g_1g_4^2 - 22176g_1g_5^2 - 7392g_1g_6^2 + 14784g_1g_7^2 + 12320g_1g_8^2 + 9240g_1g_9^2 + 2457g_2^2g_3$$
$$+ 1134g_2g_3^2 - 3024g_2g_4^2 + 6048g_2g_5^2 + 2016g_2g_6^2 - 4032g_2g_7^2 - 3360g_2g_8^2 - 2520g_2g_9^2 - 63g_3^3$$
$$- 1008g_3g_4^2 + 2016g_3g_5^2 + 672g_3g_6^2 - 1344g_3g_7^2 - 1120g_3g_8^2 - 840g_3g_9^2]\frac{1}{3360} + O(g_i^5)$$

$$\beta_4^{SU(3)}(g_i) = [-784g_1^4N_f - 61551g_1^4 + 6048g_1^3g_2 - 65772g_1^3g_3 - 756g_1^2g_2^2 - 9072g_1^2g_2g_3 + 11718g_1^2g_3^2 + 3456g_1^2g_4^2N_f$$
$$- 168696g_1^2g_4^2 - 208656g_1^2g_5^2 - 417312g_1^2g_6^2 - 245952g_1^2g_8^2 - 85680g_1^2g_9^2 + 3024g_1g_2g_3^2 - 861840g_1g_2g_4^2$$
$$+ 6048g_1g_2g_5^2 + 12096g_1g_2g_6^2 + 6048g_1g_2g_9^2 + 6804g_1g_3^3 - 81648g_1g_3g_4^2 - 30240g_1g_3g_5^2 - 60480g_1g_3g_6^2$$
$$- 88704g_1g_3g_8^2 + 14112g_1g_3g_9^2 + 122472g_2^2g_4^2 + 81648g_2g_3g_4^2 - 2079g_3^4 - 58968g_3^2g_4^2 - 27216g_3^2g_5^2$$
$$- 54432g_3^2g_6^2 - 20160g_3^2g_8^2 - 17136g_3^2g_9^2 - 870912g_4^4 - 806400g_4^2g_5^2 - 1016064g_4^2g_6^2 - 419328g_4^2g_7^2$$
$$- 344064g_4^2g_8^2 - 365568g_4^2g_9^2 - 395136g_5^4 - 516096g_5^2g_6^2 - 64512g_5^2g_7^2 - 290304g_5^2g_8^2 - 303744g_5^2g_9^2$$
$$- 193536g_6^4 - 129024g_6^2g_7^2 - 150528g_6^2g_8^2 - 252672g_6^2g_9^2 - 129024g_7^4 - 21504g_7^2g_9^2 - 50176g_8^4$$
$$- 129024g_8^2g_9^2 - 62720g_9^4]\frac{1}{120960} + O(g_i^6)$$

$$
\begin{aligned}
\beta_5^{SU(3)}(g_i) = {} & [-3576 g_1^4 N_f - 445839 g_1^4 + 380394 g_1^3 g_2 + 97335 g_1^3 g_3 - 63189 g_1^2 g_2^2 - 74466 g_1^2 g_2 g_3 + 6363 g_1^2 g_3^2 \\
& + 157248 g_1^2 g_4^2 + 3456 g_1^2 g_5^2 N_f + 95400 g_1^2 g_5^2 + 383040 g_1^2 g_6^2 + 580608 g_1^2 g_7^2 + 360864 g_1^2 g_8^2 + 88200 g_1^2 g_9^2 \\
& + 189 g_1 g_2^2 g_3 + 1008 g_1 g_2 g_3^2 - 36288 g_1 g_2 g_4^2 - 841680 g_1 g_2 g_5^2 + 28224 g_1 g_2 g_6^2 - 24192 g_1 g_2 g_7^2 - 20160 g_1 g_2 g_8^2 \\
& - 27216 g_1 g_2 g_9^2 + 3213 g_1 g_3^3 - 34272 g_1 g_3 g_4^2 - 194544 g_1 g_3 g_5^2 + 294336 g_1 g_3 g_6^2 + 137088 g_1 g_3 g_7^2 \\
& + 69888 g_1 g_3 g_8^2 - 21504 g_1 g_3 g_9^2 + 122472 g_2^2 g_5^2 + 81648 g_2 g_3 g_5^2 + 252 g_3^4 + 30240 g_3^2 g_4^2 + 15624 g_3^2 g_5^2 \\
& + 8064 g_3^2 g_6^2 + 40320 g_3^2 g_7^2 + 23520 g_3^2 g_8^2 + 16632 g_3^2 g_9^2 - 32256 g_4^4 - 322560 g_4^2 g_5^2 - 112896 g_4^2 g_6^2 - 32256 g_4^2 g_7^2 \\
& - 10752 g_4^2 g_8^2 - 21504 g_4^2 g_9^2 - 56448 g_5^4 - 161280 g_5^2 g_6^2 - 387072 g_5^2 g_7^2 - 172032 g_5^2 g_8^2 - 118272 g_5^2 g_9^2 - 161280 g_6^4 \\
& - 129024 g_6^2 g_7^2 - 129024 g_6^2 g_8^2 - 59136 g_6^2 g_9^2 - 10752 g_7^2 g_9^2 - 7168 g_8^4 + 25088 g_8^2 g_9^2 + 10080 g_9^4] \frac{1}{120960} + O(g_i^6)
\end{aligned}
$$

$$
\begin{aligned}
\beta_6^{SU(3)}(g_i) = {} & [1632 g_1^4 N_f - 1152069 g_1^4 - 120834 g_1^3 g_2 + 778113 g_1^3 g_3 + 17703 g_1^2 g_2^2 - 10584 g_1^2 g_2 g_3 - 10899 g_1^2 g_3^2 \\
& - 1334592 g_1^2 g_4^2 - 1846656 g_1^2 g_5^2 + 13824 g_1^2 g_6^2 N_f - 8246880 g_1^2 g_6^2 - 1370880 g_1^2 g_7^2 - 747264 g_1^2 g_8^2 \\
& - 391440 g_1^2 g_9^2 + 1323 g_1 g_2^2 g_3 - 30870 g_1 g_2 g_3^2 + 161280 g_1 g_2 g_4^2 + 298368 g_1 g_2 g_5^2 - 2407104 g_1 g_2 g_6^2 \\
& + 16128 g_1 g_2 g_7^2 - 26880 g_1 g_2 g_8^2 + 55776 g_1 g_2 g_9^2 - 29169 g_1 g_3^3 + 314496 g_1 g_3 g_4^2 + 846720 g_1 g_3 g_5^2 \\
& + 1358784 g_1 g_3 g_6^2 + 209664 g_1 g_3 g_7^2 + 118272 g_1 g_3 g_8^2 + 139104 g_1 g_3 g_9^2 + 489888 g_2^2 g_6^2 + 326592 g_2 g_3 g_6^2 \\
& - 252 g_3^4 + 20160 g_3^2 g_4^2 + 48384 g_3^2 g_5^2 + 328608 g_3^2 g_6^2 + 7728 g_3^2 g_9^2 - 129024 g_4^4 - 483840 g_4^2 g_5^2 - 2709504 g_4^2 g_6^2 \\
& - 129024 g_4^2 g_7^2 - 43008 g_4^2 g_8^2 - 86016 g_4^2 g_9^2 - 548352 g_5^4 - 5160960 g_5^2 g_6^2 - 258048 g_5^2 g_7^2 - 258048 g_5^2 g_8^2 \\
& - 204288 g_5^2 g_9^2 - 10321920 g_6^4 - 1806336 g_6^2 g_7^2 - 946176 g_6^2 g_8^2 - 989184 g_6^2 g_9^2 - 43008 g_7^2 g_9^2 - 28672 g_8^4 \\
& - 60928 g_8^2 g_9^2 - 20160 g_9^4] \frac{1}{483840} + O(g_i^6)
\end{aligned}
$$

$$
\begin{aligned}
\beta_7^{SU(3)}(g_i) = {} & [608 g_1^4 N_f - 5338695 g_1^4 + 1641906 g_1^3 g_2 + 1839159 g_1^3 g_3 - 111447 g_1^2 g_2^2 - 343224 g_1^2 g_2 g_3 + 4851 g_1^2 g_3^2 \\
& + 8797824 g_1^2 g_4^2 + 3673152 g_1^2 g_5^2 + 11805696 g_1^2 g_6^2 + 27648 g_1^2 g_7^2 N_f + 59727168 g_1^2 g_7^2 + 1537536 g_1^2 g_8^2 \\
& + 1832880 g_1^2 g_9^2 - 1323 g_1 g_2^2 g_3 - 17514 g_1 g_2 g_3^2 - 983808 g_1 g_2 g_4^2 - 395136 g_1 g_2 g_5^2 - 1378944 g_1 g_2 g_6^2 \\
& - 13491072 g_1 g_2 g_7^2 - 53760 g_1 g_2 g_8^2 - 213024 g_1 g_2 g_9^2 - 41895 g_1 g_3^3 - 1620864 g_1 g_3 g_4^2 - 604800 g_1 g_3 g_5^2 \\
& - 2072448 g_1 g_3 g_6^2 - 11313792 g_1 g_3 g_7^2 - 231168 g_1 g_3 g_8^2 - 385056 g_1 g_3 g_9^2 + 979776 g_2^2 g_7^2 + 653184 g_2 g_3 g_7^2 \\
& - 504 g_3^4 - 129024 g_3^2 g_4^2 - 36288 g_3^2 g_5^2 - 249984 g_3^2 g_6^2 - 1342656 g_3^2 g_7^2 - 25872 g_3^2 g_9^2 - 2145024 g_4^4 \\
& - 1128960 g_4^2 g_5^2 - 5225472 g_4^2 g_6^2 - 41545728 g_4^2 g_7^2 - 258048 g_4^2 g_8^2 - 838656 g_4^2 g_9^2 - 64512 g_5^4 - 774144 g_5^2 g_6^2 \\
& - 14192640 g_5^2 g_7^2 - 43008 g_5^2 g_8^2 - 204288 g_5^2 g_9^2 - 1419264 g_6^4 - 47738880 g_6^2 g_7^2 - 516096 g_6^2 g_8^2 - 989184 g_6^2 g_9^2 \\
& - 147603456 g_7^4 - 4300800 g_7^2 g_8^2 - 8644608 g_7^2 g_9^2 - 172032 g_8^4 - 60928 g_8^2 g_9^2 - 82880 g_9^4] \frac{1}{967680} + O(g_i^6)
\end{aligned}
$$

$$
\begin{aligned}
\beta_8^{SU(3)}(g_i) = {} & [-512 g_1^4 N_f - 2093313 g_1^4 + 881370 g_1^3 g_2 + 530523 g_1^3 g_3 - 64575 g_1^2 g_2^2 - 166320 g_1^2 g_2 g_3 + 7875 g_1^2 g_3^2 \\
& + 1334592 g_1^2 g_4^2 + 2487744 g_1^2 g_5^2 + 9434880 g_1^2 g_6^2 + 1370880 g_1^2 g_7^2 + 9216 g_1^2 g_8^2 N_f + 10474176 g_1^2 g_8^2 \\
& + 1021440 g_1^2 g_9^2 - 1323 g_1 g_2^2 g_3 + 6678 g_1 g_2 g_3^2 - 161280 g_1 g_2 g_4^2 - 274176 g_1 g_2 g_5^2 - 1137024 g_1 g_2 g_6^2 \\
& - 16128 g_1 g_2 g_7^2 - 3360000 g_1 g_2 g_8^2 - 110208 g_1 g_2 g_9^2 - 6363 g_1 g_3^3 - 314496 g_1 g_3 g_4^2 - 435456 g_1 g_3 g_5^2 \\
& - 1733760 g_1 g_3 g_6^2 - 209664 g_1 g_3 g_7^2 - 2037504 g_1 g_3 g_8^2 - 165312 g_1 g_3 g_9^2 + 326592 g_2^2 g_8^2 + 217728 g_2 g_3 g_8^2 \\
& - 126 g_3^4 - 20160 g_3^2 g_4^2 - 36288 g_3^2 g_5^2 - 249984 g_3^2 g_6^2 - 197568 g_3^2 g_8^2 - 14784 g_3^2 g_9^2 - 1720320 g_4^2 g_8^2 - 75264 g_4^2 g_9^2 \\
& - 2967552 g_5^2 g_8^2 - 129024 g_5^2 g_9^2 - 11956224 g_6^2 g_8^2 - 666624 g_6^2 g_9^2 - 1204224 g_7^2 g_8^2 - 43008 g_7^2 g_9^2 - 9619456 g_8^4 \\
& - 1627136 g_8^2 g_9^2 - 47488 g_9^4] \frac{1}{322560} + O(g_i^6)
\end{aligned}
$$

$$\beta_9^{SU(3)}(g_i) = [6368g_1^4N_f + 830127g_1^4 - 754740g_1^3g_2 - 260442g_1^3g_3 + 125622g_1^2g_2^2 + 139860g_1^2g_2g_3 - 1008g_1^2g_3^2 - 314496g_1^2g_4^2 + 82656g_1^2g_5^2 - 124992g_1^2g_6^2 - 1161216g_1^2g_7^2 - 262080g_1^2g_8^2 + 2304g_1^2g_9^2N_f - 101376g_1^2g_9^2 - 378g_1g_2^2g_3 + 1008g_1g_2g_3^2 + 72576g_1g_2g_4^2 - 100800g_1g_2g_5^2 - 32256g_1g_2g_6^2 + 48384g_1g_2g_7^2 + 48384g_1g_2g_8^2 - 536256g_1g_2g_9^2 + 378g_1g_3^3 + 68544g_1g_3g_4^2 - 4032g_1g_3g_5^2 - 177408g_1g_3g_6^2 - 274176g_1g_3g_7^2 + 86016g_1g_3g_8^2 - 63840g_1g_3g_9^2 + 81648g_2^2g_9^2 + 54432g_2g_3g_9^2 - 2583g_3^4 - 60480g_3^2g_4^2 + 2016g_3^2g_5^2 - 4032g_3^2g_6^2 - 80640g_3^2g_7^2 - 22848g_3^2g_8^2 - 30240g_3^2g_9^2 - 107520g_4^2g_8^2 - 172032g_4^2g_9^2 + 451584g_5^2g_8^2 + 177408g_5^2g_9^2 + 43008g_6^2g_8^2 + 43008g_6^2g_9^2 - 172032g_7^2g_8^2 - 236544g_7^2g_9^2 - 7168g_8^4 + 28672g_8^2g_9^2 - 32704g_9^4]\frac{1}{80640} + O(g_i^6) \tag{4.4}$$

for the full set or renormalization group functions.

The results for $SU(N_c)$ are more involved partly because of the increase in the number of independent couplings but also because of the explicit N_c dependence. First, the Landau gauge field dimensions for $SU(N_c)$ are

$$\gamma_A(g_i)|_{\alpha=0} = [871N_cg_1^2 + 48N_fg_1^2 - 4158N_cg_1g_2 - 1386N_cg_1g_3 + 567N_cg_2^2 + 378N_cg_2g_3 + 63N_cg_3^2]\frac{1}{3360} + O(g_i^4)$$
$$\gamma_c(g_i)|_{\alpha=0} = -\frac{7}{48}g_1^2N_c + [-3321487N_cg_1^2 + 24624N_fg_1^2 - 628614N_cg_1g_2 - 241878N_cg_1g_3 + 77301N_cg_2^2 + 192654N_cg_2g_3 + 108549N_cg_3^2]\frac{g_1^2N_c}{9676800} + O(g_i^6)$$
$$\gamma_\psi(g_i)|_{\alpha=0} = \frac{7[N_c^2-1]}{24N_c}g_1^2 + [3388477N_c^4g_1^2 - 34704N_c^3N_fg_1^2 - 2903377N_c^2g_1^2 + 34704N_cN_fg_1^2 - 485100g_1^2 + 1722294N_c^4g_1g_2 - 1722294N_c^2g_1g_2 + 973938N_c^4g_1g_3 - 973938N_c^2g_1g_3 - 196371N_c^4g_2^2 + 196371N_c^2g_2^2 - 272034N_c^4g_2g_3 + 272034N_c^2g_2g_3 - 121779N_c^4g_3^2 + 121779N_c^2g_3^2]\frac{g_1^2}{4838400N_c^2} + O(g_i^6) \tag{4.5}$$

where we only present the two loop terms of the ghost and quark for compactness. That for $\gamma_A(g_i)$ is given in the Supplemental Material together with all the other renormalization group functions. For the β-functions we found

$$\beta_1(g_i) = \frac{3}{320}N_cg_1g_3^2 + \frac{9}{160}N_cg_1g_2g_3 + \frac{27}{320}N_cg_1g_2^2 - \frac{33}{160}N_cg_1^2g_3 - \frac{99}{160}N_cg_1^2g_2 - \frac{109}{6720}N_cg_1^3 + \frac{1}{140}N_fg_1^3 + O(g_i^5)$$
$$\beta_2(g_i) = -\frac{1}{120}N_cg_3^3 - \frac{7}{320}N_cg_2g_3^2 + \frac{23}{160}N_cg_2^2g_3 + \frac{81}{320}N_cg_2^3 + \frac{7}{40}N_cg_1g_3^2 - \frac{21}{160}N_cg_1g_2g_3 - \frac{259}{160}N_cg_1g_2^2 - \frac{79}{80}N_cg_1^2g_3 + \frac{1991}{2240}N_cg_1^2g_2 + \frac{4019}{2520}N_cg_1^3 + \frac{3}{140}N_fg_1^2g_2 - \frac{17}{630}N_fg_1^3 + O(g_i^5)$$
$$\beta_3(g_i) = -\frac{6}{5N_c}g_3g_{11}^2 - \frac{2}{5N_c}g_3g_{10}^2 + \frac{3}{5N_c}g_3g_9^2 + \frac{4}{5N_c}g_3g_8^2 - \frac{18}{5N_c}g_2g_{11}^2 - \frac{6}{5N_c}g_2g_{10}^2 + \frac{9}{5N_c}g_2g_9^2 + \frac{12}{5N_c}g_2g_8^2 + \frac{66}{5N_c}g_1g_{11}^2 + \frac{22}{5N_c}g_1g_{10}^2 - \frac{33}{5N_c}g_1g_9^2 - \frac{44}{5N_c}g_1g_8^2 - \frac{2}{5}g_3g_7^2 + \frac{1}{5}g_3g_6^2 + \frac{3}{5}g_3g_5^2 - \frac{3}{10}g_3g_4^2 - \frac{6}{5}g_2g_7^2 + \frac{3}{5}g_2g_6^2 + \frac{9}{5}g_2g_5^2 - \frac{9}{10}g_2g_4^2 + \frac{22}{5}g_1g_7^2 - \frac{11}{5}g_1g_6^2 - \frac{33}{5}g_1g_5^2 + \frac{33}{10}g_1g_4^2 + \frac{3}{10}N_cg_3g_{11}^2 + \frac{1}{10}N_cg_3g_{10}^2 - \frac{3}{20}N_cg_3g_9^2 - \frac{1}{5}N_cg_3g_8^2 - \frac{1}{160}N_cg_3^3 + \frac{9}{10}N_cg_2g_{11}^2 + \frac{3}{10}N_cg_2g_{10}^2 - \frac{9}{20}N_cg_2g_9^2 - \frac{3}{5}N_cg_2g_8^2 + \frac{9}{80}N_cg_2g_3^2 + \frac{39}{160}N_cg_2^2g_3 - \frac{33}{10}N_cg_1g_{11}^2 - \frac{11}{10}N_cg_1g_{10}^2 + \frac{33}{20}N_cg_1g_9^2 + \frac{11}{5}N_cg_1g_8^2 - \frac{37}{160}N_cg_1g_3^2 - \frac{359}{160}N_cg_1g_2g_3 - \frac{3}{4}N_cg_1g_2^2 + \frac{2721}{1120}N_cg_1^2g_3 + \frac{709}{160}N_cg_1^2g_2 - \frac{6191}{1120}N_cg_1^3 + \frac{3}{140}N_fg_1^2g_3 - \frac{8}{105}N_fg_1^3 + O(g_i^5)$$

$$
\begin{aligned}
\beta_4(g_i) = {} & \frac{92}{5N_c^2} g_{11}^4 + \frac{184}{15N_c^2} g_{10}^2 g_{11}^2 + \frac{8}{5N_c^2} g_{10}^4 + \frac{76}{5N_c^2} g_9^2 g_{11}^2 + \frac{24}{5N_c^2} g_9^2 g_{10}^2 + \frac{22}{3N_c^2} g_9^4 + \frac{208}{15N_c^2} g_8^2 g_{11}^2 + \frac{64}{15N_c^2} g_8^2 g_{10}^2 + \frac{224}{15N_c^2} g_8^2 g_9^2 \\
& + \frac{32}{5N_c^2} g_8^4 - \frac{1}{30N_c^2} g_3^2 g_{11}^2 + \frac{1}{15N_c^2} g_3^2 g_{10}^2 + \frac{1}{N_c^2} g_3^2 g_9^2 + \frac{4}{3N_c^2} g_3^2 g_8^2 + \frac{1}{15N_c^2} g_1 g_3 g_{11}^2 + \frac{44}{15N_c^2} g_1 g_3 g_{10}^2 - \frac{17}{15N_c^2} g_1 g_3 g_9^2 \\
& + \frac{68}{15N_c^2} g_1 g_3 g_8^2 + \frac{5}{3N_c^2} g_1 g_2 g_{11}^2 + \frac{8}{15N_c^2} g_1 g_2 g_{10}^2 - \frac{6}{5N_c^2} g_1 g_2 g_9^2 - \frac{4}{5N_c^2} g_1 g_2 g_8^2 - \frac{41}{30N_c^2} g_1^2 g_{11}^2 + \frac{31}{15N_c^2} g_1^2 g_{10}^2 \\
& + \frac{26}{5N_c^2} g_1^2 g_9^2 + \frac{96}{5N_c^2} g_1^2 g_8^2 + \frac{32}{5N_c} g_6^2 g_{11}^2 + \frac{16}{15N_c} g_6^2 g_{10}^2 + \frac{56}{15N_c} g_6^2 g_9^2 + \frac{32}{15N_c} g_6^2 g_8^2 + \frac{16}{5N_c} g_5^2 g_{11}^2 + \frac{8}{15N_c} g_5^2 g_{10}^2 \\
& + \frac{28}{15N_c} g_5^2 g_9^2 + \frac{16}{15N_c} g_5^2 g_8^2 + \frac{48}{5N_c} g_4^2 g_{11}^2 + \frac{16}{5N_c} g_4^2 g_{10}^2 + \frac{32}{5N_c} g_4^2 g_9^2 + \frac{32}{5N_c} g_4^2 g_8^2 - \frac{2}{3N_c} g_3^2 g_7^2 - \frac{1}{30N_c} g_3^2 g_6^2 \\
& + \frac{1}{60N_c} g_3^2 g_5^2 - \frac{1}{2N_c} g_3^2 g_4^2 - \frac{34}{15N_c} g_1 g_3 g_7^2 - \frac{22}{15N_c} g_1 g_3 g_6^2 - \frac{1}{30N_c} g_1 g_3 g_5^2 + \frac{17}{30N_c} g_1 g_3 g_4^2 + \frac{2}{5N_c} g_1 g_2 g_7^2 \\
& - \frac{4}{15N_c} g_1 g_2 g_6^2 - \frac{5}{6N_c} g_1 g_2 g_5^2 + \frac{3}{5N_c} g_1 g_2 g_4^2 - \frac{48}{5N_c} g_1^2 g_7^2 - \frac{31}{30N_c} g_1^2 g_6^2 + \frac{41}{60N_c} g_1^2 g_5^2 - \frac{13}{5N_c} g_1^2 g_4^2 - \frac{23}{5} g_{11}^4 - \frac{46}{15} g_{10}^2 g_{11}^2 \\
& - \frac{2}{5} g_{10}^4 - \frac{19}{5} g_9^2 g_{11}^2 - \frac{6}{5} g_9^2 g_{10}^2 - \frac{11}{6} g_9^4 - \frac{52}{15} g_8^2 g_{11}^2 - \frac{16}{15} g_8^2 g_{10}^2 - \frac{56}{15} g_8^2 g_9^2 - \frac{8}{5} g_8^4 - \frac{16}{15} g_7^4 - \frac{16}{15} g_6^2 g_7^2 - \frac{2}{5} g_6^4 - \frac{8}{15} g_5^2 g_7^2 \\
& - \frac{46}{15} g_5^2 g_6^2 - \frac{89}{30} g_5^4 - \frac{52}{15} g_4^2 g_7^2 - \frac{6}{5} g_4^2 g_6^2 - \frac{46}{15} g_4^2 g_5^2 - \frac{9}{5} g_4^4 + \frac{1}{120} g_3^2 g_{11}^2 - \frac{1}{60} g_3^2 g_{10}^2 - \frac{1}{4} g_3^2 g_9^2 - \frac{1}{3} g_3^2 g_8^2 - \frac{43}{1920} g_3^4 \\
& - \frac{1}{60} g_1 g_3 g_{11}^2 - \frac{11}{15} g_1 g_3 g_{10}^2 + \frac{17}{60} g_1 g_3 g_9^2 - \frac{17}{15} g_1 g_3 g_8^2 + \frac{49}{960} g_1 g_3^3 - \frac{5}{12} g_1 g_2 g_{11}^2 - \frac{2}{15} g_1 g_2 g_{10}^2 + \frac{3}{10} g_1 g_2 g_9^2 + \frac{1}{5} g_1 g_2 g_8^2 \\
& + \frac{1}{40} g_1 g_2 g_3^2 - \frac{1}{960} g_1 g_2^2 g_3 + \frac{41}{120} g_1^2 g_{11}^2 - \frac{31}{60} g_1^2 g_{10}^2 - \frac{13}{10} g_1^2 g_9^2 - \frac{24}{5} g_1^2 g_8^2 + \frac{1}{12} g_1^2 g_3^2 + \frac{51}{160} g_1^2 g_2 g_3 + \frac{109}{320} g_1^2 g_2^2 \\
& - \frac{1153}{960} g_1^3 g_3 - \frac{977}{480} g_1^3 g_2 + \frac{73999}{40320} g_1^4 - \frac{8}{5} N_c g_6^2 g_{11}^2 - \frac{4}{15} N_c g_6^2 g_{10}^2 - \frac{14}{15} N_c g_6^2 g_9^2 - \frac{8}{15} N_c g_6^2 g_8^2 - \frac{4}{5} N_c g_5^2 g_{11}^2 \\
& - \frac{2}{15} N_c g_5^2 g_{10}^2 - \frac{7}{15} N_c g_5^2 g_9^2 - \frac{4}{15} N_c g_5^2 g_8^2 - \frac{12}{5} N_c g_4^2 g_{11}^2 - \frac{4}{5} N_c g_4^2 g_{10}^2 - \frac{8}{5} N_c g_4^2 g_9^2 - \frac{8}{5} N_c g_4^2 g_8^2 - \frac{3}{20} N_c g_3^2 g_6^2 \\
& - \frac{3}{40} N_c g_3^2 g_5^2 - \frac{13}{80} N_c g_3^2 g_4^2 + \frac{9}{40} N_c g_2 g_3 g_4^2 + \frac{27}{80} N_c g_2^2 g_4^2 - \frac{1}{6} N_c g_1 g_3 g_6^2 - \frac{1}{12} N_c g_1 g_3 g_5^2 - \frac{9}{40} N_c g_1 g_3 g_4^2 \\
& + \frac{1}{30} N_c g_1 g_2 g_6^2 + \frac{1}{60} N_c g_1 g_2 g_5^2 - \frac{19}{8} N_c g_1 g_2 g_4^2 - \frac{23}{20} N_c g_1^2 g_6^2 - \frac{23}{40} N_c g_1^2 g_5^2 - \frac{781}{1680} N_c g_1^2 g_4^2 - \frac{2}{15} N_c^2 g_6^4 - \frac{2}{15} N_c^2 g_5^2 g_6^2 \\
& - \frac{1}{30} N_c^2 g_5^4 - \frac{4}{5} N_c^2 g_4^2 g_6^2 - \frac{2}{5} N_c^2 g_4^2 g_5^2 - \frac{3}{5} N_c^2 g_4^4 + \frac{5}{126N_c} N_f g_1^4 + \frac{1}{35} N_f g_1^2 g_4^2 + O(g_i^6) \\
\beta_5(g_i) = {} & \frac{4}{5N_c^2} g_{11}^4 + \frac{8}{15N_c^2} g_{10}^2 g_{11}^2 + \frac{8}{15N_c^2} g_{10}^4 + \frac{52}{5N_c^2} g_9^2 g_{11}^2 + \frac{56}{15N_c^2} g_9^2 g_{10}^2 + \frac{6}{5N_c^2} g_9^4 + \frac{176}{15N_c^2} g_8^2 g_{11}^2 + \frac{64}{15N_c^2} g_8^2 g_{10}^2 \\
& + \frac{32}{15N_c^2} g_8^2 g_9^2 + \frac{32}{15N_c^2} g_8^4 + \frac{1}{30N_c^2} g_3^2 g_{11}^2 - \frac{1}{15N_c^2} g_3^2 g_{10}^2 - \frac{1}{N_c^2} g_3^2 g_9^2 - \frac{4}{3N_c^2} g_3^2 g_8^2 - \frac{1}{15N_c^2} g_1 g_3 g_{11}^2 - \frac{44}{15N_c^2} g_1 g_3 g_{10}^2 \\
& + \frac{17}{15N_c^2} g_1 g_3 g_9^2 - \frac{68}{15N_c^2} g_1 g_3 g_8^2 - \frac{5}{3N_c^2} g_1 g_2 g_{11}^2 - \frac{8}{15N_c^2} g_1 g_2 g_{10}^2 + \frac{6}{5N_c^2} g_1 g_2 g_9^2 + \frac{4}{5N_c^2} g_1 g_2 g_8^2 + \frac{41}{30N_c^2} g_1^2 g_{11}^2 \\
& - \frac{31}{15N_c^2} g_1^2 g_{10}^2 - \frac{26}{5N_c^2} g_1^2 g_9^2 - \frac{96}{5N_c^2} g_1^2 g_8^2 + \frac{16}{15N_c} g_6^2 g_{10}^2 + \frac{8}{15N_c} g_6^2 g_9^2 + \frac{32}{15N_c} g_6^2 g_8^2 + \frac{8}{15N_c} g_5^2 g_{10}^2 + \frac{4}{15N_c} g_5^2 g_9^2 \\
& + \frac{16}{15N_c} g_5^2 g_8^2 + \frac{2}{3N_c} g_3^2 g_7^2 + \frac{1}{30N_c} g_3^2 g_6^2 - \frac{1}{60N_c} g_3^2 g_5^2 + \frac{1}{2N_c} g_3^2 g_4^2 + \frac{34}{15N_c} g_1 g_3 g_7^2 + \frac{22}{15N_c} g_1 g_3 g_6^2 + \frac{1}{30N_c} g_1 g_3 g_5^2 \\
& - \frac{17}{30N_c} g_1 g_3 g_4^2 - \frac{2}{5N_c} g_1 g_2 g_7^2 + \frac{4}{15N_c} g_1 g_2 g_6^2 + \frac{5}{6N_c} g_1 g_2 g_5^2 - \frac{3}{5N_c} g_1 g_2 g_4^2 + \frac{48}{5N_c} g_1^2 g_7^2 + \frac{31}{30N_c} g_1^2 g_6^2 - \frac{41}{60N_c} g_1^2 g_5^2 \\
& + \frac{13}{5N_c} g_1^2 g_4^2 - \frac{1}{5} g_{11}^4 - \frac{2}{15} g_{10}^2 g_{11}^2 - \frac{2}{15} g_{10}^4 - \frac{13}{5} g_9^2 g_{11}^2 - \frac{14}{15} g_9^2 g_{10}^2 - \frac{3}{10} g_9^4 - \frac{44}{15} g_8^2 g_{11}^2 - \frac{16}{15} g_8^2 g_{10}^2 - \frac{8}{15} g_8^2 g_9^2 - \frac{8}{15} g_8^4
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{16}{15}g_6^2g_7^2-\frac{2}{15}g_6^4-\frac{16}{5}g_5^2g_7^2-\frac{2}{15}g_5^2g_6^2-\frac{1}{6}g_5^4-\frac{4}{15}g_4^2g_7^2-\frac{14}{15}g_4^2g_6^2-\frac{8}{3}g_4^2g_5^2-\frac{4}{15}g_4^4-\frac{1}{120}g_3^2g_{11}^2+\frac{1}{60}g_3^2g_{10}^2\\
&+\frac{1}{4}g_3^2g_9^2+\frac{1}{3}g_3^2g_8^2-\frac{1}{1920}g_3^4+\frac{1}{60}g_1g_3g_{11}^2+\frac{11}{15}g_1g_3g_{10}^2-\frac{17}{60}g_1g_3g_9^2+\frac{17}{15}g_1g_3g_8^2+\frac{23}{960}g_1g_3^3+\frac{5}{12}g_1g_2g_{11}^2\\
&+\frac{2}{15}g_1g_2g_{10}^2-\frac{3}{10}g_1g_2g_9^2-\frac{1}{5}g_1g_2g_8^2+\frac{1}{120}g_1g_2g_3^2+\frac{1}{960}g_1g_2^2g_3-\frac{41}{120}g_1^2g_{11}^2+\frac{31}{60}g_1^2g_{10}^2+\frac{13}{10}g_1^2g_9^2+\frac{24}{5}g_1^2g_8^2\\
&+\frac{11}{240}g_1^2g_3^2-\frac{67}{160}g_1^2g_2g_3-\frac{67}{192}g_1^2g_2^2+\frac{457}{960}g_1^3g_3+\frac{1009}{480}g_1^3g_2-\frac{6757}{2688}g_1^4-\frac{4}{15}N_cg_6^2g_{10}^2-\frac{2}{15}N_cg_6^2g_9^2-\frac{8}{15}N_cg_6^2g_8^2\\
&-\frac{2}{15}N_cg_5^2g_{10}^2-\frac{1}{15}N_cg_5^2g_9^2-\frac{4}{15}N_cg_5^2g_8^2+\frac{1}{60}N_cg_3^2g_6^2+\frac{11}{240}N_cg_3^2g_5^2+\frac{9}{40}N_cg_2g_3g_5^2+\frac{27}{80}N_cg_2^2g_5^2+\frac{17}{30}N_cg_1g_3g_6^2\\
&-\frac{13}{24}N_cg_1g_3g_5^2+\frac{1}{30}N_cg_1g_2g_6^2-\frac{59}{24}N_cg_1g_2g_5^2+\frac{53}{60}N_cg_1^2g_6^2+\frac{211}{560}N_cg_1^2g_5^2-\frac{2}{15}N_c^2g_6^4-\frac{2}{15}N_c^2g_5^2g_6^2-\frac{1}{30}N_c^2g_5^4\\
&-\frac{149}{2520N_c}N_fg_1^4+\frac{1}{35}N_fg_1^2g_5^2+O(g_i^6)\\
\beta_6(g_i)={}&\frac{52}{15N_c^2}g_{11}^4+\frac{568}{15N_c^2}g_{10}^2g_{11}^2+\frac{1208}{15N_c^2}g_{10}^4+\frac{56}{15N_c^2}g_9^2g_{11}^2+\frac{112}{5N_c^2}g_9^2g_{10}^2+\frac{6}{5N_c^2}g_9^4+\frac{16}{15N_c^2}g_8^2g_{11}^2+\frac{224}{15N_c^2}g_8^2g_{10}^2\\
&+\frac{32}{15N_c^2}g_8^2g_9^2+\frac{32}{15N_c^2}g_8^4-\frac{3}{10N_c^2}g_3^2g_{11}^2-\frac{31}{15N_c^2}g_3^2g_{10}^2-\frac{1}{6N_c^2}g_3^2g_9^2-\frac{18}{5N_c^2}g_1g_3g_{11}^2-\frac{43}{3N_c^2}g_1g_3g_{10}^2-\frac{13}{5N_c^2}g_1g_3g_9^2\\
&-\frac{26}{15N_c^2}g_1g_3g_8^2-\frac{34}{15N_c^2}g_1g_2g_{11}^2-\frac{47}{5N_c^2}g_1g_2g_{10}^2-\frac{4}{3N_c^2}g_1g_2g_9^2-\frac{2}{15N_c^2}g_1g_2g_8^2+\frac{617}{30N_c^2}g_1^2g_{11}^2+\frac{78}{N_c^2}g_1^2g_{10}^2\\
&+\frac{331}{30N_c^2}g_1^2g_9^2+\frac{34}{3N_c^2}g_1^2g_8^2+\frac{16}{15N_c}g_6^2g_{10}^2+\frac{8}{15N_c}g_6^2g_9^2+\frac{32}{15N_c}g_6^2g_8^2+\frac{8}{15N_c}g_5^2g_{10}^2+\frac{4}{15N_c}g_5^2g_9^2+\frac{16}{15N_c}g_5^2g_8^2\\
&+\frac{31}{30N_c}g_3^2g_6^2+\frac{3}{20N_c}g_3^2g_5^2+\frac{1}{12N_c}g_3^2g_4^2+\frac{13}{15N_c}g_1g_3g_7^2+\frac{43}{6N_c}g_1g_3g_6^2+\frac{9}{5N_c}g_1g_3g_5^2\\
&+\frac{13}{10N_c}g_1g_3g_4^2+\frac{1}{15N_c}g_1g_2g_7^2+\frac{47}{10N_c}g_1g_2g_6^2+\frac{17}{15N_c}g_1g_2g_5^2+\frac{2}{3N_c}g_1g_2g_4^2-\frac{17}{3N_c}g_1^2g_7^2-\frac{39N_c{}^2}{g\;\;{}_1}g_6^2-\frac{617}{60N_c}g_1^2g_5^2\\
&-\frac{331}{60N_c}g_1^2g_4^2-\frac{13}{15}g_{11}^4-\frac{142}{15}g_{10}^2g_{11}^2-\frac{302}{15}g_{10}^4-\frac{14}{15}g_9^2g_{11}^2-\frac{28}{5}g_9^2g_{10}^2-\frac{3}{10}g_9^4-\frac{4}{15}g_8^2g_{11}^2-\frac{56}{15}g_8^2g_{10}^2-\frac{8}{15}g_8^2g_9^2\\
&-\frac{8}{15}g_8^4-\frac{56}{15}g_6^2g_7^2-\frac{302}{15}g_6^4-\frac{8}{15}g_5^2g_7^2-\frac{142}{15}g_5^2g_6^2-\frac{5}{6}g_5^4-\frac{4}{15}g_4^2g_7^2-\frac{28}{5}g_4^2g_6^2-g_4^2g_5^2-\frac{4}{15}g_4^4+\frac{3}{40}g_3^2g_{11}^2\\
&+\frac{31}{60}g_3^2g_{10}^2+\frac{1}{24}g_3^2g_9^2-\frac{1}{1920}g_3^4+\frac{9}{10}g_1g_3g_{11}^2+\frac{43}{12}g_1g_3g_{10}^2+\frac{13}{20}g_1g_3g_9^2+\frac{13}{30}g_1g_3g_8^2-\frac{217}{3840}g_1g_3^3+\frac{17}{30}g_1g_2g_{11}^2\\
&+\frac{47}{20}g_1g_2g_{10}^2+\frac{1}{3}g_1g_2g_9^2+\frac{1}{30}g_1g_2g_8^2-\frac{103}{1920}g_1g_2g_3^2+\frac{7}{3840}g_1g_2^2g_3-\frac{617}{120}g_1^2g_{11}^2-\frac{39}{2}g_1^2g_{10}^2-\frac{331}{120}g_1^2g_9^2-\frac{17}{6}g_1^2g_8^2\\
&-\frac{21}{1280}g_1^2g_3^2-\frac{23}{240}g_1^2g_2g_3+\frac{11}{3840}g_1^2g_2^2+\frac{1285}{768}g_1^3g_3+\frac{117}{640}g_1^3g_2-\frac{248207}{80640}g_1^4-\frac{4}{15}N_cg_6^2g_{10}^2-\frac{2}{15}N_cg_6^2g_9^2\\
&-\frac{8}{15}N_cg_6^2g_8^2-\frac{2}{15}N_cg_5^2g_{10}^2-\frac{1}{15}N_cg_5^2g_9^2-\frac{4}{15}N_cg_5^2g_8^2+\frac{13}{240}N_cg_3^2g_6^2+\frac{1}{120}N_cg_3^2g_5^2+\frac{9}{40}N_cg_2g_3g_6^2+\frac{27}{80}N_cg_2^2g_6^2\\
&-\frac{31}{120}N_cg_1g_3g_6^2+\frac{17}{60}N_cg_1g_3g_5^2-\frac{293}{120}N_cg_1g_2g_6^2+\frac{1}{60}N_cg_1g_2g_5^2+\frac{275}{336}N_cg_1^2g_6^2+\frac{53}{120}N_cg_1^2g_5^2-\frac{2}{15}N_c^2g_6^4\\
&-\frac{2}{15}N_c^2g_5^2g_6^2-\frac{1}{30}N_c^2g_5^4+\frac{17}{2520N_c}N_fg_1^4+\frac{1}{35}N_fg_1^2g_6^2+O(g_i^6)\\
\beta_7(g_i)={}&\frac{52}{15N_c^2}g_{11}^4+\frac{568}{15N_c^2}g_{10}^2g_{11}^2+\frac{1208}{15N_c^2}g_{10}^4+\frac{56}{15N_c^2}g_9^2g_{11}^2+\frac{328}{15N_c^2}g_9^2g_{10}^2+\frac{16}{15N_c^2}g_9^4+\frac{16}{15N_c^2}g_8^2g_{11}^2+\frac{32}{3N_c^2}g_8^2g_{10}^2\\
&+\frac{16}{15N_c^2}g_8^2g_9^2+\frac{32}{15N_c^2}g_8^4+\frac{3}{10N_c^2}g_3^2g_{11}^2+\frac{31}{15N_c^2}g_3^2g_{10}^2+\frac{1}{6N_c^2}g_3^2g_9^2+\frac{18}{5N_c^2}g_1g_3g_{11}^2+\frac{43}{3N_c^2}g_1g_3g_{10}^2+\frac{13}{5N_c^2}g_1g_3g_9^2
\end{aligned}
$$

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\begin{aligned}
&+\frac{26}{15N_c^2}g_1g_3g_8^2+\frac{34}{15N_c^2}g_1g_2g_{11}^2+\frac{47}{5N_c^2}g_1g_2g_{10}^2+\frac{4}{3N_c^2}g_1g_2g_9^2+\frac{2}{15N_c^2}g_1g_2g_8^2-\frac{617}{30N_c^2}g_1^2g_{11}^2-\frac{78}{N_c^2}g_1^2g_{10}^2\\
&-\frac{331}{30N_c^2}g_1^2g_9^2-\frac{34}{3N_c^2}g_1^2g_8^2+\frac{112}{3N_c}g_7^2g_{11}^2+\frac{464}{3N_c}g_7^2g_{10}^2+\frac{64}{3N_c}g_7^2g_9^2+\frac{32}{3N_c}g_7^2g_8^2+\frac{8}{15N_c}g_6^2g_{11}^2+\frac{16}{3N_c}g_6^2g_{10}^2\\
&+\frac{4}{15N_c}g_6^2g_9^2+\frac{4}{15N_c}g_5^2g_{11}^2+\frac{8}{3N_c}g_5^2g_{10}^2+\frac{2}{15N_c}g_5^2g_9^2+\frac{52}{15N_c}g_4^2g_{11}^2+\frac{284}{15N_c}g_4^2g_{10}^2+\frac{28}{15N_c}g_4^2g_9^2+\frac{8}{15N_c}g_4^2g_8^2\\
&-\frac{31}{30N_c}g_3^2g_6^2-\frac{3}{20N_c}g_3^2g_5^2-\frac{1}{12N_c}g_3^2g_4^2-\frac{13}{15N_c}g_1g_3g_7^2-\frac{43}{6N_c}g_1g_3g_6^2-\frac{9}{5N_c}g_1g_3g_5^2-\frac{13}{10N_c}g_1g_3g_4^2-\frac{1}{15N_c}g_1g_2g_7^2\\
&-\frac{47}{10N_c}g_1g_2g_6^2-\frac{17}{15N_c}g_1g_2g_5^2-\frac{2}{3N_c}g_1g_2g_4^2+\frac{17}{3N_c}g_1^2g_7^2+\frac{39}{N_c}g_1^2g_6^2+\frac{617}{60N_c}g_1^2g_5^2+\frac{331}{60N_c}g_1^2g_4^2-\frac{13}{15}g_{11}^4\\
&-\frac{142}{15}g_{10}^2g_{11}^2-\frac{302}{15}g_{10}^4-\frac{14}{15}g_9^2g_{11}^2-\frac{82}{15}g_9^2g_{10}^2-\frac{4}{15}g_9^4-\frac{4}{15}g_8^2g_{11}^2-\frac{8}{3}g_8^2g_{10}^2-\frac{4}{15}g_8^2g_9^2-\frac{8}{15}g_8^4+\frac{232}{15}g_7^4\\
&-\frac{112}{3}g_6^2g_7^2-\frac{22}{15}g_6^4-\frac{26}{3}g_5^2g_7^2-\frac{4}{5}g_5^2g_6^2-\frac{1}{15}g_5^4-\frac{14}{15}g_4^2g_7^2-\frac{24}{5}g_4^2g_6^2-\frac{13}{15}g_4^2g_5^2-\frac{4}{15}g_4^4-\frac{3}{40}g_3^2g_{11}^2-\frac{31}{60}g_3^2g_{10}^2\\
&-\frac{1}{24}g_3^2g_9^2-\frac{1}{1920}g_3^4-\frac{9}{10}g_1g_3g_{11}^2-\frac{43}{12}g_1g_3g_{10}^2-\frac{13}{20}g_1g_3g_9^2-\frac{13}{30}g_1g_3g_8^2-\frac{53}{1280}g_1g_3^3-\frac{17}{30}g_1g_2g_{11}^2-\frac{47}{20}g_1g_2g_{10}^2\\
&-\frac{1}{3}g_1g_2g_9^2-\frac{1}{30}g_1g_2g_8^2-\frac{5}{384}g_1g_2g_3^2-\frac{7}{3840}g_1g_2^2g_3+\frac{617}{120}g_1^2g_{11}^2+\frac{39}{2}g_1^2g_{10}^2+\frac{331}{120}g_1^2g_9^2+\frac{17}{6}g_1^2g_8^2+\frac{31}{3840}g_1^2g_3^2\\
&-\frac{47}{120}g_1^2g_2g_3-\frac{169}{1280}g_1^2g_2^2+\frac{7423}{3840}g_1^3g_3+\frac{3673}{1920}g_1^3g_2-\frac{157663}{26880}g_1^4-\frac{28}{3}N_cg_7^2g_{11}^2-\frac{116}{3}N_cg_7^2g_{10}^2-\frac{16}{3}N_cg_7^2g_9^2\\
&-\frac{8}{3}N_cg_7^2g_8^2-\frac{2}{15}N_cg_6^2g_{11}^2-\frac{4}{3}N_cg_6^2g_{10}^2-\frac{1}{15}N_cg_6^2g_9^2-\frac{1}{15}N_cg_5^2g_{11}^2-\frac{2}{3}N_cg_5^2g_{10}^2-\frac{1}{30}N_cg_5^2g_9^2-\frac{13}{15}N_cg_4^2g_{11}^2\\
&-\frac{71}{15}N_cg_4^2g_{10}^2-\frac{7}{15}N_cg_4^2g_9^2-\frac{2}{15}N_cg_4^2g_8^2-\frac{37}{80}N_cg_3^2g_7^2-\frac{3}{80}N_cg_3^2g_4^2+\frac{9}{40}N_cg_2g_3g_7^2+\frac{27}{80}N_cg_2^2g_7^2-\frac{153}{40}N_cg_1g_3g_7^2\\
&-\frac{7}{60}N_cg_1g_3g_6^2-\frac{7}{120}N_cg_1g_3g_5^2-\frac{9}{20}N_cg_1g_3g_4^2-\frac{557}{120}N_cg_1g_2g_7^2-\frac{1}{12}N_cg_1g_2g_6^2-\frac{1}{24}N_cg_1g_2g_5^2-\frac{17}{60}N_cg_1g_2g_4^2\\
&+\frac{11257}{560}N_cg_1^2g_7^2+\frac{49}{60}N_cg_1^2g_6^2+\frac{49}{120}N_cg_1^2g_5^2+\frac{617}{240}N_cg_1^2g_4^2-\frac{56}{3}N_c^2g_7^4-\frac{4}{3}N_c^2g_6^2g_7^2-\frac{2}{3}N_c^2g_5^2g_7^2-\frac{14}{3}N_c^2g_4^2g_7^2\\
&-\frac{1}{15}N_c^2g_4^2g_6^2-\frac{1}{30}N_c^2g_4^2g_5^2-\frac{13}{60}N_c^2g_4^4+\frac{1}{5040N_c}N_fg_1^4+\frac{1}{35}N_fg_1^2g_7^2+O(g_i^6)
\end{aligned}
$$

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\begin{aligned}
\beta_8(g_i)={}&\frac{58}{15N_c}g_{11}^4+\frac{128}{3N_c}g_{10}^2g_{11}^2+\frac{1328}{15N_c}g_{10}^4+\frac{139}{15N_c}g_9^2g_{11}^2+\frac{764}{15N_c}g_9^2g_{10}^2+\frac{71}{15N_c}g_9^4+\frac{896}{15N_c}g_8^2g_{11}^2+\frac{248N_c{}^2}{g\;{}_8}g_{10}^2\\
&+\frac{788}{15N_c}g_8^2g_9^2+\frac{464}{5N_c}g_8^4+\frac{3}{20N_c}g_3^2g_{11}^2+\frac{31}{30N_c}g_3^2g_{10}^2+\frac{1}{12N_c}g_3^2g_9^2+\frac{9}{5N_c}g_1g_3g_{11}^2+\frac{43}{6N_c}g_1g_3g_{10}^2+\frac{13}{10N_c}g_1g_3g_9^2\\
&+\frac{13}{15N_c}g_1g_3g_8^2+\frac{17}{15N_c}g_1g_2g_{11}^2+\frac{47}{10N_c}g_1g_2g_{10}^2+\frac{2}{3N_c}g_1g_2g_9^2+\frac{1}{15N_c}g_1g_2g_8^2-\frac{617}{60N_c}g_1^2g_{11}^2-\frac{39}{N_c}g_1^2g_{10}^2\\
&-\frac{331}{60N_c}g_1^2g_9^2-\frac{17}{3N_c}g_1^2g_8^2-\frac{4}{15}g_7^2g_9^2-\frac{56}{15}g_7^2g_8^2-\frac{2}{3}g_6^2g_{11}^2-\frac{8}{5}g_6^2g_{10}^2-\frac{24}{5}g_6^2g_9^2-\frac{116}{3}g_6^2g_8^2-\frac{1}{15}g_5^2g_{11}^2-\frac{2}{15}g_5^2g_{10}^2\\
&-\frac{13}{15}g_5^2g_9^2-\frac{28}{3}g_5^2g_8^2-\frac{1}{30}g_4^2g_{11}^2-\frac{2}{15}g_4^2g_{10}^2-\frac{1}{2}g_4^2g_9^2-\frac{82}{15}g_4^2g_8^2-\frac{31}{60}g_3^2g_6^2-\frac{3}{40}g_3^2g_5^2-\frac{1}{24}g_3^2g_4^2-\frac{13}{30}g_1g_3g_7^2\\
&-\frac{43}{12}g_1g_3g_6^2-\frac{9}{10}g_1g_3g_5^2-\frac{13}{20}g_1g_3g_4^2-\frac{1}{30}g_1g_2g_7^2-\frac{47}{20}g_1g_2g_6^2-\frac{17}{30}g_1g_2g_5^2-\frac{1}{3}g_1g_2g_4^2+\frac{17}{6}g_1^2g_7^2+\frac{39}{2}g_1^2g_6^2\\
&+\frac{617}{120}g_1^2g_5^2+\frac{331}{120}g_1^2g_4^2-\frac{1}{4}N_cg_{11}^4-\frac{83}{30}N_cg_{10}^2g_{11}^2-\frac{57}{10}N_cg_{10}^4-\frac{43}{60}N_cg_9^2g_{11}^2-\frac{229}{60}N_cg_9^2g_{10}^2-\frac{41}{80}N_cg_9^4\\
&-\frac{27}{5}N_cg_8^2g_{11}^2-\frac{106}{5}N_cg_8^2g_{10}^2-\frac{223}{30}N_cg_8^2g_9^2-\frac{302}{15}N_cg_8^4-\frac{3}{160}N_cg_3^2g_{11}^2-\frac{31}{240}N_cg_3^2g_{10}^2-\frac{9}{320}N_cg_3^2g_9^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{49}{240}N_c g_3^2 g_8^2-\frac{1}{7680}N_c g_3^4+\frac{9}{40}N_c g_2 g_3 g_8^2+\frac{27}{80}N_c g_2^2 g_8^2-\frac{61}{240}N_c g_1 g_3 g_{11}^2-\frac{223}{240}N_c g_1 g_3 g_{10}^2-\frac{169}{480}N_c g_1 g_3 g_9^2\\
&-\frac{11}{5}N_c g_1 g_3 g_8^2-\frac{13}{1536}N_c g_1 g_3^3-\frac{13}{80}N_c g_1 g_2 g_{11}^2-\frac{31}{48}N_c g_1 g_2 g_{10}^2-\frac{107}{480}N_c g_1 g_2 g_9^2-\frac{211}{60}N_c g_1 g_2 g_8^2\\
&+\frac{7}{3840}N_c g_1 g_2 g_3^2-\frac{7}{7680}N_c g_1 g_2^2 g_3+\frac{143}{96}N_c g_1^2 g_{11}^2+\frac{77}{15}N_c g_1^2 g_{10}^2+\frac{1949}{960}N_c g_1^2 g_9^2+\frac{19267}{1680}N_c g_1^2 g_8^2\\
&+\frac{13}{2560}N_c g_1^2 g_3^2-\frac{259}{1920}N_c g_1^2 g_2 g_3-\frac{383}{7680}N_c g_1^2 g_2^2+\frac{3961}{7680}N_c g_1^3 g_3+\frac{889}{1280}N_c g_1^3 g_2-\frac{29269}{16128}N_c g_1^4+\frac{1}{35}N_f g_1^2 g_8^2\\
&+\frac{1}{10080}N_f g_1^4+O(g_i^6)\\
\beta_9(g_i)=&\,\frac{70}{3N_c}g_{11}^4+\frac{116}{5N_c}g_{10}^2 g_{11}^2+\frac{56}{15N_c}g_{10}^4+\frac{34N_c{}^2}{g_{\,9}}g_{11}^2+\frac{92}{5N_c}g_9^2 g_{10}^2+\frac{58}{3N_c}g_9^4+\frac{232}{15N_c}g_8^2 g_{11}^2+\frac{112}{15N_c}g_8^2 g_{10}^2+\frac{368}{15N_c}g_8^2 g_9^2\\
&+\frac{32}{5N_c}g_8^4-\frac{1}{60N_c}g_3^2 g_{11}^2+\frac{1}{30N_c}g_3^2 g_{10}^2+\frac{1}{2N_c}g_3^2 g_9^2+\frac{2}{3N_c}g_3^2 g_8^2+\frac{1}{30N_c}g_1 g_3 g_{11}^2+\frac{22}{15N_c}g_1 g_3 g_{10}^2-\frac{17}{30N_c}g_1 g_3 g_9^2\\
&+\frac{34}{15N_c}g_1 g_3 g_8^2+\frac{5}{6N_c}g_1 g_2 g_{11}^2+\frac{4}{15N_c}g_1 g_2 g_{10}^2-\frac{3}{5N_c}g_1 g_2 g_9^2-\frac{2}{5N_c}g_1 g_2 g_8^2-\frac{41}{60N_c}g_1^2 g_{11}^2+\frac{31}{30N_c}g_1^2 g_{10}^2\\
&+\frac{13}{5N_c}g_1^2 g_9^2+\frac{48}{5N_c}g_1^2 g_8^2-\frac{8}{15}g_7^2 g_{11}^2-\frac{16}{15}g_7^2 g_{10}^2-\frac{52}{15}g_7^2 g_9^2-\frac{32}{15}g_7^2 g_8^2-\frac{8}{5}g_6^2 g_{11}^2-\frac{4}{5}g_6^2 g_{10}^2-\frac{16}{15}g_6^2 g_9^2-\frac{8}{15}g_6^2 g_8^2\\
&-\frac{26}{5}g_5^2 g_{11}^2-\frac{46}{15}g_5^2 g_{10}^2-3g_5^2 g_9^2-\frac{4}{15}g_5^2 g_8^2-\frac{16}{15}g_4^2 g_{11}^2-\frac{6}{5}g_4^2 g_{10}^2-\frac{16}{5}g_4^2 g_9^2-\frac{28}{15}g_4^2 g_8^2-\frac{1}{3}g_3^2 g_7^2-\frac{1}{60}g_3^2 g_6^2\\
&+\frac{1}{120}g_3^2 g_5^2-\frac{1}{4}g_3^2 g_4^2-\frac{17}{15}g_1 g_3 g_7^2-\frac{11}{15}g_1 g_3 g_6^2-\frac{1}{60}g_1 g_3 g_5^2+\frac{17}{60}g_1 g_3 g_4^2+\frac{1}{5}g_1 g_2 g_7^2-\frac{2}{15}g_1 g_2 g_6^2-\frac{5}{12}g_1 g_2 g_5^2\\
&+\frac{3}{10}g_1 g_2 g_4^2-\frac{24}{5}g_1^2 g_7^2-\frac{31}{60}g_1^2 g_6^2+\frac{41}{120}g_1^2 g_5^2-\frac{13}{10}g_1^2 g_4^2-\frac{23}{15}N_c g_{11}^4-\frac{26}{15}N_c g_{10}^2 g_{11}^2-\frac{1}{3}N_c g_{10}^4-\frac{32}{15}N_c g_9^2 g_{11}^2\\
&-\frac{26}{15}N_c g_9^2 g_{10}^2-\frac{107}{60}N_c g_9^4-\frac{4}{15}N_c g_8^2 g_{11}^2-\frac{4}{15}N_c g_8^2 g_{10}^2-\frac{8}{5}N_c g_8^2 g_9^2-\frac{4}{15}N_c g_8^4-\frac{1}{30}N_c g_3^2 g_{11}^2-\frac{3}{40}N_c g_3^2 g_{10}^2\\
&-\frac{11}{60}N_c g_3^2 g_9^2-\frac{3}{20}N_c g_3^2 g_8^2-\frac{7}{1280}N_c g_3^4+\frac{9}{40}N_c g_2 g_3 g_9^2+\frac{27}{80}N_c g_2^2 g_9^2-\frac{1}{20}N_c g_1 g_3 g_{11}^2-\frac{1}{6}N_c g_1 g_3 g_{10}^2\\
&-\frac{29}{120}N_c g_1 g_3 g_9^2+\frac{13}{1920}N_c g_1 g_3^3-\frac{1}{5}N_c g_1 g_2 g_{11}^2-\frac{1}{30}N_c g_1 g_2 g_{10}^2-\frac{34}{15}N_c g_1 g_2 g_9^2+\frac{2}{15}N_c g_1 g_2 g_8^2+\frac{1}{240}N_c g_1 g_2 g_3^2\\
&-\frac{1}{1920}N_c g_1 g_2^2 g_3-\frac{7}{60}N_c g_1^2 g_{11}^2-\frac{47}{120}N_c g_1^2 g_{10}^2-\frac{583}{840}N_c g_1^2 g_9^2-\frac{91}{60}N_c g_1^2 g_8^2+\frac{3}{320}N_c g_1^2 g_3^2+\frac{59}{320}N_c g_1^2 g_2 g_3\\
&+\frac{331}{1920}N_c g_1^2 g_2^2-\frac{161}{384}N_c g_1^3 g_3-\frac{331}{320}N_c g_1^3 g_2+\frac{87677}{80640}N_c g_1^4+\frac{1}{35}N_f g_1^2 g_9^2+\frac{5}{252}N_f g_1^4+O(g_i^6)\\
\beta_{10}(g_i)=&\,\frac{18}{5N_c}g_{11}^4+\frac{196}{5N_c}g_{10}^2 g_{11}^2+\frac{248}{3N_c}g_{10}^4+\frac{62}{15N_c}g_9^2 g_{11}^2+\frac{116}{5N_c}g_9^2 g_{10}^2+\frac{6}{5N_c}g_9^4+\frac{8}{3N_c}g_8^2 g_{11}^2+\frac{272}{15N_c}g_8^2 g_{10}^2+\frac{32}{15N_c}g_8^2 g_9^2\\
&+\frac{32}{15N_c}g_8^4-\frac{3}{20N_c}g_3^2 g_{11}^2-\frac{31}{30N_c}g_3^2 g_{10}^2-\frac{1}{12N_c}g_3^2 g_9^2-\frac{9}{5N_c}g_1 g_3 g_{11}^2-\frac{43}{6N_c}g_1 g_3 g_{10}^2-\frac{13}{10N_c}g_1 g_3 g_9^2-\frac{13}{15N_c}g_1 g_3 g_8^2\\
&-\frac{17}{15N_c}g_1 g_2 g_{11}^2-\frac{47}{10N_c}g_1 g_2 g_{10}^2-\frac{2}{3N_c}g_1 g_2 g_9^2-\frac{1}{15N_c}g_1 g_2 g_8^2+\frac{617}{60N_c}g_1^2 g_{11}^2+\frac{39}{N_c}g_1^2 g_{10}^2+\frac{331}{60N_c}g_1^2 g_9^2+\frac{17}{3N_c}g_1^2 g_8^2\\
&-\frac{8}{15}g_7^2 g_{11}^2-\frac{56}{15}g_7^2 g_{10}^2-\frac{4}{15}g_7^2 g_9^2-\frac{48}{5}g_6^2 g_{11}^2-\frac{604}{15}g_6^2 g_{10}^2-\frac{82}{15}g_6^2 g_9^2-\frac{16}{5}g_6^2 g_8^2-\frac{26}{15}g_5^2 g_{11}^2-\frac{142}{15}g_5^2 g_{10}^2-\frac{14}{15}g_5^2 g_9^2\\
&-\frac{4}{15}g_5^2 g_8^2-g_4^2 g_{11}^2-\frac{28}{5}g_4^2 g_{10}^2-\frac{8}{15}g_4^2 g_9^2-\frac{4}{15}g_4^2 g_8^2+\frac{31}{60}g_3^2 g_6^2+\frac{3}{40}g_3^2 g_5^2+\frac{1}{24}g_3^2 g_4^2+\frac{13}{30}g_1 g_3 g_7^2+\frac{43}{12}g_1 g_3 g_6^2\\
&+\frac{9}{10}g_1 g_3 g_5^2+\frac{13}{20}g_1 g_3 g_4^2+\frac{1}{30}g_1 g_2 g_7^2+\frac{47}{20}g_1 g_2 g_6^2+\frac{17}{30}g_1 g_2 g_5^2+\frac{1}{3}g_1 g_2 g_4^2-\frac{17}{6}g_1^2 g_7^2-\frac{39}{2}g_1^2 g_6^2-\frac{617}{120}g_1^2 g_5^2\\
&-\frac{331}{120}g_1^2 g_4^2-\frac{1}{30}N_c g_{11}^4-\frac{1}{5}N_c g_{10}^2 g_{11}^2-\frac{4}{15}N_c g_{10}^4-\frac{1}{30}N_c g_9^2 g_{11}^2-\frac{1}{10}N_c g_9^2 g_{10}^2-\frac{1}{120}N_c g_9^4-\frac{2}{15}N_c g_8^2 g_{11}^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8}{15}N_c g_8^2 g_{10}^2 - \frac{1}{15}N_c g_8^2 g_9^2 + \frac{1}{240}N_c g_3^2 g_{11}^2 + \frac{13}{240}N_c g_3^2 g_{10}^2 + \frac{1}{480}N_c g_3^2 g_9^2 + \frac{9}{40}N_c g_2 g_3 g_{10}^2 + \frac{27}{80}N_c g_2^2 g_{10}^2 \\
&+\frac{17}{120}N_c g_1 g_3 g_{11}^2 - \frac{37}{120}N_c g_1 g_3 g_{10}^2 + \frac{17}{240}N_c g_1 g_3 g_9^2 + \frac{1}{10}N_c g_1 g_3 g_8^2 - \frac{29}{7680}N_c g_1 g_3^3 + \frac{1}{120}N_c g_1 g_2 g_{11}^2 \\
&-\frac{289}{120}N_c g_1 g_2 g_{10}^2 + \frac{1}{240}N_c g_1 g_2 g_9^2 - \frac{1}{15}N_c g_1 g_2 g_8^2 - \frac{13}{1280}N_c g_1 g_2 g_3^2 + \frac{7}{7680}N_c g_1 g_2^2 g_3 + \frac{53}{240}N_c g_1^2 g_{11}^2 \\
&+\frac{1879}{1680}N_c g_1^2 g_{10}^2 + \frac{53}{480}N_c g_1^2 g_9^2 - \frac{3}{5}N_c g_1^2 g_8^2 - \frac{47}{7680}N_c g_1^2 g_3^2 + \frac{71}{960}N_c g_1^2 g_2 g_3 + \frac{259}{7680}N_c g_1^2 g_2^2 - \frac{499}{7680}N_c g_1^3 g_3 \\
&-\frac{1661}{3840}N_c g_1^3 g_2 + \frac{112391}{161280}N_c g_1^4 + \frac{1}{35}N_f g_1^2 g_{10}^2 + \frac{17}{5040}N_f g_1^4 + O(g_i^6)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{11}(g_i) = {}& \frac{14}{15N_c}g_{11}^4 + \frac{28}{15N_c}g_{10}^2 g_{11}^2 + \frac{8}{3N_c}g_{10}^4 + \frac{54}{5N_c}g_9^2 g_{11}^2 + \frac{68}{15N_c}g_9^2 g_{10}^2 + \frac{6}{5N_c}g_9^4 + \frac{40}{3N_c}g_8^2 g_{11}^2 + \frac{112}{15N_c}g_8^2 g_{10}^2 \\
&+\frac{32}{15N_c}g_8^2 g_9^2 + \frac{32}{15N_c}g_8^4 + \frac{1}{60N_c}g_3^2 g_{11}^2 - \frac{1}{30N_c}g_3^2 g_{10}^2 - \frac{1}{2N_c}g_3^2 g_9^2 - \frac{2}{3N_c}g_3^2 g_8^2 - \frac{1}{30N_c}g_1 g_3 g_{11}^2 - \frac{22}{15N_c}g_1 g_3 g_{10}^2 \\
&+\frac{17}{30N_c}g_1 g_3 g_9^2 - \frac{34}{15N_c}g_1 g_3 g_8^2 - \frac{5}{6N_c}g_1 g_2 g_{11}^2 - \frac{4}{15N_c}g_1 g_2 g_{10}^2 + \frac{3}{5N_c}g_1 g_2 g_9^2 + \frac{2}{5N_c}g_1 g_2 g_8^2 + \frac{41}{60N_c}g_1^2 g_{11}^2 \\
&-\frac{31}{30N_c}g_1^2 g_{10}^2 - \frac{13}{5N_c}g_1^2 g_9^2 - \frac{48}{5N_c}g_1^2 g_8^2 - \frac{16}{5}g_7^2 g_{11}^2 - \frac{16}{15}g_7^2 g_{10}^2 - \frac{4}{15}g_7^2 g_9^2 - \frac{4}{15}g_6^2 g_{11}^2 - \frac{4}{15}g_6^2 g_{10}^2 - \frac{4}{5}g_6^2 g_9^2 - \frac{8}{15}g_6^2 g_8^2 \\
&-\frac{2}{5}g_5^2 g_{11}^2 - \frac{2}{15}g_5^2 g_{10}^2 - \frac{13}{5}g_5^2 g_9^2 - \frac{44}{15}g_5^2 g_8^2 - \frac{8}{3}g_4^2 g_{11}^2 - \frac{14}{15}g_4^2 g_{10}^2 - \frac{8}{15}g_4^2 g_9^2 - \frac{4}{15}g_4^2 g_8^2 + \frac{1}{3}g_3^2 g_7^2 + \frac{1}{60}g_3^2 g_6^2 - \frac{1}{120}g_3^2 g_5^2 \\
&+\frac{1}{4}g_3^2 g_4^2 + \frac{17}{15}g_1 g_3 g_7^2 + \frac{11}{15}g_1 g_3 g_6^2 + \frac{1}{60}g_1 g_3 g_5^2 - \frac{17}{60}g_1 g_3 g_4^2 - \frac{1}{5}g_1 g_2 g_7^2 + \frac{2}{15}g_1 g_2 g_6^2 + \frac{5}{12}g_1 g_2 g_5^2 - \frac{3}{10}g_1 g_2 g_4^2 \\
&+\frac{24}{5}g_1^2 g_7^2 + \frac{31}{60}g_1^2 g_6^2 - \frac{41}{120}g_1^2 g_5^2 + \frac{13}{10}g_1^2 g_4^2 + \frac{7}{15}N_c g_{11}^4 + \frac{2}{15}N_c g_{10}^2 g_{11}^2 - \frac{4}{15}N_c g_{10}^4 - \frac{8}{15}N_c g_9^2 g_{11}^2 - \frac{4}{15}N_c g_9^2 g_{10}^2 \\
&+\frac{7}{60}N_c g_9^4 - \frac{4}{5}N_c g_8^2 g_{11}^2 - \frac{8}{15}N_c g_8^2 g_{10}^2 + \frac{4}{15}N_c g_8^2 g_9^2 - \frac{1}{16}N_c g_3^2 g_{11}^2 - \frac{1}{40}N_c g_3^2 g_{10}^2 + \frac{13}{240}N_c g_3^2 g_9^2 + \frac{1}{12}N_c g_3^2 g_8^2 \\
&+\frac{1}{384}N_c g_3^4 + \frac{9}{40}N_c g_2 g_3 g_{11}^2 + \frac{27}{80}N_c g_2^2 g_{11}^2 - \frac{3}{8}N_c g_1 g_3 g_{11}^2 + \frac{7}{15}N_c g_1 g_3 g_{10}^2 - \frac{1}{12}N_c g_1 g_3 g_9^2 + \frac{1}{5}N_c g_1 g_3 g_8^2 \\
&+\frac{1}{384}N_c g_1 g_3^3 - \frac{53}{24}N_c g_1 g_2 g_{11}^2 + \frac{1}{12}N_c g_1 g_2 g_{10}^2 - \frac{1}{8}N_c g_1 g_2 g_9^2 - \frac{1}{10}N_c g_1 g_2 g_8^2 + \frac{1}{1920}N_c g_1 g_2^2 g_3 - \frac{361}{1680}N_c g_1^2 g_{11}^2 \\
&+\frac{23}{120}N_c g_1^2 g_{10}^2 + \frac{71}{240}N_c g_1^2 g_9^2 + \frac{83}{60}N_c g_1^2 g_8^2 + \frac{13}{1920}N_c g_1^2 g_3^2 - \frac{63}{320}N_c g_1^2 g_2 g_3 - \frac{111}{640}N_c g_1^2 g_2^2 + \frac{631}{1920}N_c g_1^3 g_3 \\
&+\frac{1001}{960}N_c g_1^3 g_2 - \frac{23629}{20160}N_c g_1^4 + \frac{1}{35}N_f g_1^2 g_{11}^2 - \frac{149}{5040}N_f g_1^4 + O(g_i^6). \qquad (4.6)
\end{aligned}
$$

Again these renormalization group functions, as well as those for $SU(3)$, satisfy the same checks we discussed for the $SU(2)$ case.

V. LARGE N_f CHECK

We devote this section to the final independent check we have on the renormalization group functions in each of the three cases which is the comparison with the large N_f critical exponents which have been computed in the non-Abelian Thirring model universality class. The background to this is the observation that the renormalization group functions depend on the parameter N_f and the various coupling constants for a specific value of N_c. The coefficients of these parameters in each renormalization group function is conventionally determined by perturbative methods as was carried out in the previous section. However one can also determine the coefficients via an ordering of graphs defined by N_f. This is achieved through the known d-dependent critical exponents of the underlying universality class. An alternative view of this is that the exponents already contain information on the perturbative coefficients. The method is to compute the renormalization group functions at the Wilson-Fisher fixed point in $d = 8 - 2\epsilon$, expand in powers of $1/N_f$ and then compare with the ϵ expansion of the corresponding large N_f critical exponents. This constitutes our independent check. The first step in the procedure is to locate the Wilson-Fisher fixed point explicitly order by order in powers of $1/N_f$ and ϵ by finding the solution to

$$\beta_i(g_j) = 0 \tag{5.1}$$

for the d-dimensional β-functions. In four dimensions this is relatively straightforward since there is only one coupling constant in QCD. For eight dimensions we have 11 coupling constants for the case of $SU(N_c)$. So we follow the method introduced in [22,23]. As there are 3- and 4-leg operators in (2.8) we have to be careful in defining the rescaling which is the initial step in the approach of [22,23]. Therefore at the outset we set

$$g_i = \sqrt{\frac{70\epsilon}{N_f}} x_i \qquad i = 1 \text{ to } 3$$
$$g_i^2 = \frac{70\epsilon}{N_f} x_i \qquad i = 4 \text{ to } 11 \tag{5.2}$$

in (5.1) and expand in powers of ϵ and $1/N_f$. First the leading order term in $1/N_f$ of the equations is isolated and then the ϵ expansion of this leading term is found before repeating the exercise for the subsequent term in the large N_f expansion. For the $SU(N_c)$ β-functions the resulting critical couplings are

$$x_1 = 1 + \frac{1933 N_c}{24 N_f} + \frac{3736489 N_c^2}{384 N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_2 = \frac{17}{9} + \frac{287279 N_c}{1944 N_f} + \frac{5066611513 N_c^2}{279936 N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_3 = \frac{16}{3} + \frac{143411 N_c}{324 N_f} + \frac{153781987 N_c^2}{2916 N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_4 = -\frac{25}{9N_c} - \left[\frac{1615081}{46656} + \frac{115591}{432 N_c^2}\right]\frac{1}{N_f} + \left[\frac{4084305085}{11664 N_c^3} - \frac{318375286621}{839808 N_c} + \frac{894758019623 N_c}{3359232}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_5 = \frac{149}{36 N_c} + \left[\frac{39472453}{46656} - \frac{343}{432 N_c^2}\right]\frac{1}{N_f} + \left[-\frac{768922651}{11664 N_c^3} + \frac{6468807373}{839808 N_c} + \frac{144625900963 N_c}{1119744}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_6 = \frac{149}{72} + \left[\frac{18279803 N_c}{46656} - \frac{343}{432 N_c}\right]\frac{1}{N_f} + \left[\frac{161927831371 N_c^2}{2239488} - \frac{28392695975}{1679616} - \frac{1571677793}{46656 N_c^2}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_7 = -\frac{17}{36 N_c} - \left[\frac{997943}{46656} + \frac{343}{432 N_c^2}\right]\frac{1}{N_f} + \left[\frac{285596549}{2916 N_c^3} + \frac{155421633577}{839808 N_c} + \frac{69408246905 N_c}{1119744}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_8 = -\frac{1}{72 N_c} + \left[\frac{343}{216 N_c^2} - \frac{5517727}{46656}\right]\frac{1}{N_f} + \left[\frac{37377424567}{209952 N_c} - \frac{20021939}{23328 N_c^3} - \frac{223918424851 N_c}{3359232}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_9 = -\frac{1}{144} + \left[\frac{343}{216 N_c} - \frac{6535889 N_c}{186624}\right]\frac{1}{N_f} + \left[\frac{388161667565}{3359232} - \frac{5239709503}{93312 N_c^2} - \frac{297237914233 N_c^2}{26873856}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_{10} = -\frac{17}{72} - \left[\frac{343}{432 N_c} + \frac{2137045 N_c}{46656}\right]\frac{1}{N_f} - \left[\frac{11234911345}{104976} + \frac{2318604883}{46656 N_c^2} + \frac{7914271411 N_c^2}{497664}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right)$$
$$x_{11} = -\frac{25}{18} - \left[\frac{115591}{432 N_c} + \frac{8299843 N_c}{93312}\right]\frac{1}{N_f} + \left[\frac{15724650809}{46656 N_c^2} - \frac{429314818345}{1679616} + \frac{261182511995 N_c^2}{6718464}\right]\frac{1}{N_f^2} + O\left(\epsilon; \frac{1}{N_f^3}\right) \tag{5.3}$$

where the double order symbol indicates both the two loop correction and the next order in the large N_f expansion. These values of x_i correspond to the ϵ expansion of all the critical couplings to the order which they are known in the previous section. Next the renormalization group functions for the wave function renormalization are evaluated at the Wilson-Fisher critical point and expanded in powers of both ϵ and $1/N_f$. Subsequently the critical exponents should be in agreement with the coefficients of ϵ in the known large N_f critical exponents of the non-Abelian Thirring universality class when they are expanded around $d = 8 - 2\epsilon$. Substituting the values from (5.3) into (4.5) we find for $SU(N_c)$ that

$$\gamma_A(g_c)|_{\alpha=0} = \epsilon + \frac{245N_c}{12N_f}\epsilon + \frac{473585N_c^2}{144N_f^2}\epsilon + O\left(\epsilon^2;\frac{1}{N_f^3}\right)$$
$$\gamma_c(g_c)|_{\alpha=0} = -\frac{245N_c}{24N_f}\epsilon - \frac{473585N_c^2}{288N_f^2}\epsilon + O\left(\epsilon^2;\frac{1}{N_f^3}\right)$$
$$\gamma_\psi(g_c)|_{\alpha=0} = \left[\frac{245N_c}{12} - \frac{245}{12N_c}\right]\frac{\epsilon}{N_f} + \left[\frac{473585N_c^2}{144} - \frac{473585}{144}\right]\frac{\epsilon}{N_f^2} + O\left(\epsilon^2;\frac{1}{N_f^3}\right) \quad (5.4)$$

where g_c denotes the set of critical couplings defined in (5.2). In order to compare with the large N_f critical exponents of the universal theory founded on the non-Abelian Thirring model at the Wilson-Fisher fixed point, we have to restrict the exponents to the Landau gauge. This is because in effect the gauge parameter α acts as an additional coupling constant and the Landau gauge is the corresponding fixed point in this context. In other words the gauge dependent large N_f critical exponents of the gluon, quark and ghost fields can only be compared with the Landau gauge anomalous dimensions at criticality which has been noted before in [16,18]. We restrict our large N_f comparison to these three anomalous dimensions since they are the only three quantities which are available for eight dimensional QCD. While the large N_f critical exponent of the four dimensional QCD β-function is known at $O(1/N_f)$ [16], that exponent would relate to the renormalization of the operator $\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}$ in (2.12). In four dimensions the gauge coupling constant in four dimensional QCD is dimensionless but in the continuation along the thread of the d-dimensional Wilson-Fisher fixed point the coupling becomes dimensionful and the correction to scaling exponent in four dimensions transcends into a mass parameter in higher dimensions such as the eight dimensional Lagrangian (2.12). Therefore, if we evaluate the leading order d-dimensional large N_f critical exponents for the gluon, quark and ghost fields of [52] near eight dimensions by setting $d = 8 - 2\epsilon$ we find that the coefficients of ϵ match precisely with those of (5.4) in the Landau gauge for $SU(N_c)$. Moreover, since the quark anomalous dimension is also known at $O(1/N_f^2)$ in the Landau gauge [18], it is satisfying to record that the corresponding term of $\gamma_\psi(g_c)|_{\alpha=0}$ is in full agreement. While we have not given explicit details for the $SU(2)$ and $SU(3)$ renormalization group functions, we note that we have carried out the same check as $SU(N_c)$ and found that there is full consistency in these cases too. Consequently the ultraviolet completion of QCD or the non-Abelian Thirring model to eight dimensions via (2.8) has been established at one loop within the large N_f expansion as expected.

VI. DIMENSION 8 OPERATORS IN FOUR DIMENSIONS

In this section we turn to a complementary problem which is the renormalization of dimension 8 operators in four dimensions. Such operators in the case of Yang-Mills theory have been considered in [31,32] where, for instance, the anomalous dimensions for the $SU(2)$ and $SU(3)$ groups were computed at one loop in [31]. The reason for this is that in four dimensions the canonical dimensions of the gluon and ghost fields are such that there is a complicated mixing between gluonic and quark operators. In (2.8) by contrast on dimensional grounds it is not possible to have any other interactions involving quarks aside from the quark-gluon interaction. Therefore in this section we concentrate on the renormalization of four dimensional dimension 8 operators in $SU(N_c)$ Yang-Mills theory for $N_c \geq 4$ as this case has not been considered. In addition we use the *same* operator basis as was used in (2.8), which differs from that of [31,32], in order to ease structural comparisons. First, to set notation the basis for the dimension 8 operators in four dimensions for the color group $SU(N_c)$ we use is

$$\mathcal{O}_{841} = G^a_{\mu\sigma}G^{a\mu\rho}G^{b\sigma\nu}G^b_{\rho\nu}, \qquad \mathcal{O}_{842} = G^a_{\mu\sigma}G^{b\mu\rho}G^{b\sigma\nu}G^a_{\rho\nu}$$
$$\mathcal{O}_{843} = G^a_{\mu\sigma}G^a_{\nu\rho}G^{b\sigma\mu}G^{b\rho\nu}, \qquad \mathcal{O}_{844} = G^a_{\mu\sigma}G^b_{\nu\rho}G^{a\sigma\mu}G^{b\rho\nu}$$
$$\mathcal{O}_{845} = d_4^{abcd}G^a_{\mu\sigma}G^{b\mu\sigma}G^c_{\nu\rho}G^{d\nu\rho},$$
$$\mathcal{O}_{846} = d_4^{abcd}G^a_{\mu\sigma}G^{c\mu\rho}G^{b\nu\sigma}G^d_{\nu\rho}$$
$$\mathcal{O}_{847} = d_4^{acbd}G^a_{\mu\sigma}G^{b\mu\sigma}G^c_{\nu\rho}G^{d\nu\rho},$$
$$\mathcal{O}_{848} = d_4^{adbc}G^a_{\mu\sigma}G^{c\mu\rho}G^{b\nu\sigma}G^d_{\nu\rho}. \quad (6.1)$$

The notation is similar to that used in [31]. However, these operators are *not* the same since we have specified the basis with respect to a specific color group unlike [31]. We have chosen this ordering so that the $SU(2)$ basis corresponds to the first four operators and that for $SU(3)$ involves the first six. Equally the ordering is equivalent to that used in (2.8) for the quartic gluon interactions with coupling constants g_4 to g_{11} respectively.

To renormalize the operators $\mathcal{O}_{84i}$ we use the same technique as that for the 4-point functions of (2.8) but in this case we apply it to the Green's function $\langle A^a_\mu(p_1)A^b_\nu(p_2)A^c_\sigma(p_3)A^d_\rho(p_4)\mathcal{O}_{84i}(p_5)\rangle$ where $p_5 = -\sum_{i=1}^4 p_i$. However, as we are considering an operator renormalization there will be a mixing of the $\mathcal{O}_{84i}$ operators among themselves which will produce a mixing matrix of anomalous dimensions. This is similar to the β-functions for the couplings in (2.8). However for operator renormalization there are aspects to address compared with a Lagrangian renormalization. For instance, for the gauge invariant dimension 8 operators (6.1) there will be mixing into gauge variant and equation of motion operators as well as possibly total derivative operators. The latter can arise when an operator is renormalized in a Green's function where the insertion is at nonzero momentum insertion. Moreover this set includes total derivative operators which are gauge invariant, gauge variant and equation of motion operators. So the mixing matrix in effect is larger than an 8×8 matrix based on (6.1). Not only do the operators of (6.1) mix with all operators of the enlarged set but the gauge variant, equation of motion and total derivative operators can mix with themselves when each is renormalized. However, the overall mixing matrix has a particular structure in that the gauge invariant operators mix with all classes of operators but the gauge variant ones only mix within that class. See, for instance, [53–56]. As we are primarily interested in the gauge invariant operators we restrict the evaluation of the Green's function $\langle A^a_\mu(p_1) A^b_\nu(p_2) A^c_\sigma(p_3) A^d_\rho(p_4) \mathcal{O}_{84i}(p_5)\rangle$ to the case where the external gluon legs are all on-shell. The condition for a gluon $A^a_\mu(p)$ to be on-shell is that its polarization vector and momentum satisfy

$$p_\mu p^\mu = 0, \qquad p^\mu \epsilon_\mu(p) = 0. \tag{6.2}$$

Therefore we multiply the Green's function by $\epsilon^\mu(p_1)\epsilon^\nu(p_2)\epsilon^\sigma(p_3)\epsilon^\rho(p_4)$ and apply (6.2). The terms which remain such as $\epsilon_\mu(p_i) p^\mu_j$ for $i \neq j$ or $p_i p_j$ are resolved by grouping them in terms corresponding to the Feynman rules of the contributing operators such as (6.1) and any gauge invariant total derivative or equation of motion operators. The reason why this list omits gauge variant operators is that the restriction of (6.2) corresponds to taking a physical matrix element. As such no gauge variant operators can be present [53–56].

Necessary to achieve the resolution into this basis of operators is that the operator has to be inserted at nonzero momentum. If it was inserted at zero momentum then certain terms of the Feynman rule of different operators will be similar and hence the extraction of the renormalization constants in the mixing matrix cannot be achieved uniquely and unambiguously. Therefore, formally the set of bare operators, denoted by the subscript o satisfy

$$\mathcal{O}_{i\mathrm{o}} = Z_{ij} \mathcal{O}_j \tag{6.3}$$

where Z_{ij} is the mixing matrix of renormalization constants from which the mixing matrix of anomalous dimensions, $\gamma_{ij}(a)$, can be deduced. In this section $a = g^2/(16\pi^2)$ denotes the coupling constant of four dimensional QCD where g is the coupling present in the covariant derivative. It transpires that for the eight operators (6.1) the matrix needs to be enlarged since there is mixing into an equation of motion operator. In [31] the seven independent equation of motion operators were constructed and are

$$\begin{aligned}
\mathcal{O}_{82e1} &= D^\mu G^a_{\mu\nu} D^\rho D_\sigma D_\rho G^{a\nu\sigma}, \\
\mathcal{O}_{82e2} &= D^\sigma D^\mu G^a_{\mu\nu} D^\rho D^\nu G^a_{\sigma\rho} \\
\mathcal{O}_{82e3} &= D^\sigma D^\mu G^a_{\mu\nu} D_\rho D_\sigma G^{a\nu\rho}, \\
\mathcal{O}_{82e4} &= D_\sigma G^a_{\nu\rho} D^\sigma D^\rho D_\mu G^{a\mu\nu} \\
\mathcal{O}_{82e5} &= G^a_{\nu\sigma} D^\sigma D^\rho D_\rho D_\mu G^{a\mu\nu} \\
\mathcal{O}_{83e1} &= f^{abc} G^a_{\sigma\rho} D^\nu G^{b\sigma\rho} D^\mu G^c_{\mu\nu}, \\
\mathcal{O}_{83e2} &= f^{abc} G^{a\nu}_\sigma G^{b\sigma\rho} D_\rho D^\mu G^c_{\mu\nu}
\end{aligned} \tag{6.4}$$

where the first two labels indicate the operator dimension and gluon leg number respectively and note that each operator is gauge invariant. We recall that in four dimensions the equation of motion of the gluon in Yang-Mills theory is

$$D^\mu G_{\mu\nu} = 0 \tag{6.5}$$

which is relatively simple in contrast to that of (2.8). Unlike (6.1) there is no reduction of the equation of motion set (6.4) depending on which color group we consider. One comment is in order with respect to (2.8) which is that the operators (6.4) are not present in that Lagrangian. The reason why they are considered part of the basis here arises from the different nature of the two types of renormalizations we are carrying out. In (2.8) for the purely gluonic sector we included the set of independent gauge invariant operators involving the field strength. The operators which were dependent, and hence not included, were equivalent to linear combinations of the ones appearing in (2.8) as well as operators which were total derivatives. In a Lagrangian context the latter operators can be integrated out and hence were not included in (2.8). For the renormalization of the dimension 8 operators (6.1) in four dimensions one has to accommodate mixing into the various operator classes noted earlier. As one of these classes involves equation of motion operators we have included these in the set of operators for our mixing. However it is a straightforward exercise to show that the operators $\mathcal{O}_{82ei}$ can each be related to the gluon kinetic operator plus higher leg operators and those with a total derivative. Equally the operators $\mathcal{O}_{83ei}$ in eight dimensions can be mapped to the operators with couplings g_2 and g_3 respectively plus higher leg and total derivative operators in (2.8).

The final stage of the operator renormalization is the evaluation of the divergent part of the on-shell Green's function. Like the renormalization of the 4-point functions of (2.8) we apply the vacuum bubble expansion based on (3.5). The only major difference between its use here and the previous application is that after the expansion and the Laporta reduction the master integral is evaluated in *four* dimensions. Therefore, extracting the renormalization constants we find the elements of the mixing matrix are

$$\gamma_{841,841}(a) = \frac{8}{3N_c}a + O(a^2), \qquad \gamma_{841,842}(a) = -\frac{8}{3N_c}a + O(a^2)$$
$$\gamma_{841,843}(a) = \frac{22}{3N_c}a + O(a^2), \qquad \gamma_{841,844}(a) = -\frac{1}{6N_c}[11N_c^2 + 44]a + O(a^2)$$
$$\gamma_{841,845}(a) = -\frac{11}{3}a + O(a^2), \qquad \gamma_{841,846}(a) = \frac{4}{3}a + O(a^2)$$
$$\gamma_{841,847}(a) = \frac{11}{3}a + O(a^2), \qquad \gamma_{841,848}(a) = -\frac{4}{3}a + O(a^2)$$
$$\gamma_{842,841}(a) = -\frac{1}{3N_c}[14N_c^2 + 4]a + O(a^2), \qquad \gamma_{842,842}(a) = -\frac{1}{3N_c}[10N_c^2 - 4]a + O(a^2)$$
$$\gamma_{842,843}(a) = \frac{1}{3N_c}[12N_c^2 + 22]a + O(a^2), \qquad \gamma_{842,844}(a) = -\frac{1}{6N_c}[-N_c^2 + 44]a + O(a^2)$$
$$\gamma_{842,845}(a) = -\frac{11}{3}a + O(a^2), \qquad \gamma_{842,846}(a) = -\frac{2}{3}a + O(a^2)$$
$$\gamma_{842,847}(a) = \frac{11}{3}a + O(a^2), \qquad \gamma_{842,848}(a) = \frac{2}{3}a + O(a^2)$$
$$\gamma_{843,841}(a) = -\frac{1}{3N_c}[28N_c^2 + 68]a + O(a^2), \qquad \gamma_{843,842}(a) = -\frac{1}{3N_c}[-24N_c^2 - 68]a + O(a^2)$$
$$\gamma_{843,843}(a) = \frac{1}{3N_c}[2N_c^2 + 50]a + O(a^2), \qquad \gamma_{843,844}(a) = -\frac{1}{3N_c}[-N_c^2 + 50]a + O(a^2)$$
$$\gamma_{843,845}(a) = -\frac{25}{3}a + O(a^2), \qquad \gamma_{843,846}(a) = -\frac{34}{3}a + O(a^2)$$
$$\gamma_{843,847}(a) = \frac{25}{3}a + O(a^2), \qquad \gamma_{843,848}(a) = \frac{34}{3}a + O(a^2)$$
$$\gamma_{844,841}(a) = -\frac{56}{N_c}a + O(a^2), \qquad \gamma_{844,842}(a) = \frac{56}{N_c}a + O(a^2)$$
$$\gamma_{844,843}(a) = -\frac{4}{N_c}a + O(a^2), \qquad \gamma_{844,844}(a) = -\frac{1}{3N_c}[22N_c^2 - 12]a + O(a^2)$$
$$\gamma_{844,845}(a) = 2a + O(a^2), \qquad \gamma_{844,846}(a) = -28a + O(a^2)$$
$$\gamma_{844,847}(a) = -2a + O(a^2), \qquad \gamma_{844,848}(a) = 28a + O(a^2)$$
$$\gamma_{845,841}(a) = -\frac{1}{N_c^2}[28N_c^2 - 112]a + O(a^2), \qquad \gamma_{845,842}(a) = \frac{1}{N_c^2}[28N_c^2 - 112]a + O(a^2)$$
$$\gamma_{845,843}(a) = -\frac{1}{N_c^2}[2N_c^2 - 8]a + O(a^2), \qquad \gamma_{845,844}(a) = -\frac{1}{N_c^2}[-2N_c^2 + 8]a + O(a^2)$$
$$\gamma_{845,845}(a) = -\frac{1}{2N_c}[5N_c^2 + 8]a + O(a^2), \qquad \gamma_{845,846}(a) = -\frac{1}{N_c}[6N_c^2 - 56]a + O(a^2)$$
$$\gamma_{845,847}(a) = -\frac{1}{3N_c}[2N_c^2 - 12]a + O(a^2), \qquad \gamma_{845,848}(a) = -\frac{1}{3N_c}[-16N_c^2 + 168]a + O(a^2)$$
$$\gamma_{846,841}(a) = -\frac{1}{3N_c^2}[-4N_c^2 + 16]a + O(a^2), \qquad \gamma_{846,842}(a) = -\frac{1}{3N_c^2}[4N_c^2 - 16]a + O(a^2)$$
$$\gamma_{846,843}(a) = -\frac{1}{3N_c^2}[-11N_c^2 + 44]a + O(a^2), \qquad \gamma_{846,844}(a) = -\frac{1}{3N_c^2}[11N_c^2 - 44]a + O(a^2)$$
$$\gamma_{846,845}(a) = -\frac{1}{3N_c}[4N_c^2 - 22]a + O(a^2), \qquad \gamma_{846,846}(a) = -\frac{1}{3N_c}[3N_c^2 + 8]a + O(a^2)$$

$$\gamma_{846,847}(a) = -\frac{1}{3N_c}[-3N_c^2+22]a+O(a^2), \qquad \gamma_{846,848}(a) = -\frac{1}{3N_c}[3N_c^2-8]a+O(a^2)$$
$$\gamma_{847,841}(a) = -\frac{1}{3N_c^2}[34N_c^2-136]a+O(a^2)$$
$$\gamma_{847,842}(a) = -\frac{1}{3N_c^2}[-34N_c^2+136]a+O(a^2)$$
$$\gamma_{847,843}(a) = -\frac{1}{3N_c^2}[-25N_c^2+100]a+O(a^2)$$
$$\gamma_{847,844}(a) = -\frac{1}{3N_c^2}[25N_c^2-100]a+O(a^2)$$
$$\gamma_{847,845}(a) = -\frac{1}{12N_c}[25N_c^2-200]a+O(a^2), \qquad \gamma_{847,846}(a) = -\frac{1}{3N_c}[19N_c^2-68]a+O(a^2)$$
$$\gamma_{847,847}(a) = -\frac{1}{3N_c}[-3N_c^2+50]a+O(a^2), \qquad \gamma_{847,848}(a) = -\frac{1}{3N_c}[-16N_c^2+68]a+O(a^2)$$
$$\gamma_{848,841}(a) = -\frac{1}{3N_c^2}[2N_c^2-8]a+O(a^2), \qquad \gamma_{848,842}(a) = -\frac{1}{3N_c^2}[-2N_c^2+8]a+O(a^2)$$
$$\gamma_{848,843}(a) = -\frac{1}{3N_c^2}[-11N_c^2+44]a+O(a^2)$$
$$\gamma_{848,844}(a) = -\frac{1}{3N_c^2}[11N_c^2-44]a+O(a^2)$$
$$\gamma_{848,845}(a) = -\frac{1}{6N_c}[5N_c^2-44]a+O(a^2), \qquad \gamma_{848,846}(a) = -\frac{1}{3N_c}[8N_c^2-4]a+O(a^2)$$
$$\gamma_{848,847}(a) = -\frac{1}{3N_c}[-6N_c^2+22]a+O(a^2)$$
$$\gamma_{848,848}(a) = -\frac{1}{3N_c}[4N_c^2+4]a+O(a^2) \tag{6.6}$$

for $SU(N_c)$. For the eight $SU(N_c)$ dimension 8 core operators at one loop there is mixing into only one equation of motion operator which is $\mathcal{O}_{83e2}$. More explicitly we have

$$\gamma_{841,83e2}(a) = -2a+O(a^2), \qquad \gamma_{842,83e2}(a) = 4a+O(a^2),$$
$$\gamma_{843,83e2}(a) = 4a+O(a^2) \qquad \gamma_{844,83e2}(a) = -8a+O(a^2),$$
$$\gamma_{845,83e2}(a) = -\frac{4}{N_c}[N_c^2-4]a+O(a^2)$$
$$\gamma_{846,83e2}(a) = -\frac{1}{N_c}[N_c^2-4]a+O(a^2),$$
$$\gamma_{847,83e2}(a) = \frac{2}{N_c}[N_c^2-4]a+O(a^2)$$
$$\gamma_{848,83e2}(a) = \frac{2}{N_c}[N_c^2-4]a+O(a^2). \tag{6.7}$$

The mixing of the main operators into this specific equation of motion operator is necessary as otherwise divergences would remain in each of the Green's functions. In other words there are not sufficient counterterms and freedom available from the set of operators in (6.1) alone to obtain a finite expression. For $SU(2)$ and $SU(3)$ the respective parts for this sector of the mixing matrix are contained within (6.7). For $SU(2)$ only the first four operators of (6.1) are active and for $SU(3)$ it is the first six. Then for $SU(2)$ the first four entries in (6.7) correspond to the 4-leg operator mixing into the equation of motion operators. Clearly $\gamma_{845,83e2}(a)$ vanishes for $N_c = 2$ as a consistency check. The situation for $SU(3)$ is similar except the first six entries are relevant but $N_c = 3$ has to be set. Finally, the equation of motion operators can mix with themselves and we have determined that sector of the mixing matrix in the same way by inserting each operator in the physical matrix element. The only nonzero entries are

$$\gamma_{83e1,82e4}(a) = -\frac{1}{3N_c}a+O(a^2),$$
$$\gamma_{83e1,82e5}(a) = \frac{1}{2N_c}a+O(a^2) \tag{6.8}$$

which is valid for all the $SU(N_c)$ groups. This completes our dimension 8 operator analysis in four dimensions for the particular $SU(N_c)$ color groups. These results together with the $SU(2)$ and $SU(3)$ cases are all included in the Supplemental Material. While this is a fully separate computation to the renormalization of (2.8) the structural parallels of the respective renormalization group functions are now evident.

VII. DISCUSSION

One of our main goals was to construct the eight dimensional quantum field theory which was in the same universality class as the two dimensional non-Abelian Thirring model and four dimensional QCD at their respective Wilson-Fisher fixed points. We have managed to achieve this by following the guiding principles established for the parallel construction for scalar field theories with an $O(N)$ symmetry. The first of these is to retain the core interaction between the matter and force fields which in the present case were a spin-$\frac{1}{2}$ fermion and spin-1 boson field in the adjoint representation of the color group. This interaction is the only one present in the base theory of the tower of theories lying in the universality class which is the non-Abelian Thirring model [21]. The second aspect is renormalizability. This means that extra interactions have to be included in the critical dimension of each of the subsequent Lagrangians of the tower so that each Lagrangian is renormalizable. These extra independent operators, which are purely gluonic for this universality class, will become irrelevant or relevant away from the critical dimension. So for example including the canonical gluon kinetic operator for QCD in the non-Abelian Thirring model would render it nonrenormalizable in two dimensions. The final main principle is the requirement of gauge fixing. We chose a linear covariant gauge fixing in order to make connections with lower dimensional results and extended the Faddeev-Popov construction to eight dimensions. This last step is necessary as the two dimensional non-Abelian Thirring model has a conserved current, $\bar{\psi}\gamma^\mu T^a \psi$, whose 2-point correlation function is transverse. While there is no gluon as such in the non-Abelian Thirring model, like the four dimensional gauge theory case, the field A^a_μ is an auxiliary in two dimensions and corresponds to this current. In other words the correlation of A^a_μ in two dimensions is in effect akin to a Landau gauge propagator. As the gauge parameter, α, in QCD is effectively a second coupling constant then at criticality one has to effect its critical coupling which corresponds in fact to the Landau gauge. This accords with the establishment of (2.8) as being in the same universality class as the non-Abelian Thirring model and QCD via the large N_f expansion. One can only compare the d-dimensional large N_f critical exponents with the exponents derived from gauge dependent renormalization group functions when the ϵ expansion of the latter have been computed in the Landau gauge. We have checked this off explicitly here for eight dimensional QCD from the one loop renormalization group functions. Put another way the Wilson-Fisher fixed point underlying this particular universality class preserves the transversality of the gluon across the dimensions.

There are several future avenues to pursue in light of our analysis. One is to build the ten dimensional theory of a spin-1 field coupled to a fundamental fermion which lies in the non-Abelian Thirring model universality class. The procedure to do this evidently follows the above outline. It would have no technical obstacles aside from the calculational one of requiring a large amount of integration by parts to determine even just the one loop renormalization group functions. This will be a tedious exercise rather than an insurmountable problem. Another obvious extension is to construct the renormalization group functions of (2.8) at two loops. Indeed this has already been achieved for QED [26,27]. However in eight dimensions the computations were manageable due to there being only four independent interactions and more crucially no quintic or sextic gauge interactions. These were obviously present in the non-Abelian case and also increased the amount of integration needed in order to evaluate the large number of Feynman graphs with high exponent gluon propagators [26]. With the tower of Lagrangians essentially established at the Wilson-Fisher fixed point for the non-Abelian Thirring model universality class, the next focus ought to be on the connection of non-Lagrangian operators in the universal theory. These operators will have massive couplings in the noncritical dimensions but are relevant in constructing effective field Lagrangians in a specific dimension. In other words there should be a drive to study the operator anomalous dimensions at criticality.

We have taken the first step in this direction by renormalizing dimension 8 operators in four dimensions. While laying the foundation to this here by illustrating the structural parallels of the renormalization group functions, the next step is to introduce quark contributions. These are required for the large N_f expansion connection where the underlying operator critical exponents in the universal theory would also need to be found in addition to the mixing matrices in perturbation theory. The perturbative computations to construct such mixing matrices should not be regarded as a straightforward task. One reason for this is due to the canonical dimensions of the quark and gluon fields being different in d-dimensions. Hence quark and gluon operators will have different canonical dimensions except in one particular dimension. Therefore we did not have to consider what would ordinarily be dimension 8 quark operators in the four dimensional sense in the construction of the eight dimensional Lagrangian (2.8). However, in four dimensional QCD there are dimension 8 operators with quark content in addition to the gluon operators of (6.1). This was one of the reasons why our focus was on Yang-Mills operators here as an exploratory exercise in the context of (2.8) and to observe that the structure of the respective four and eight dimensional renormalization group functions were not dissimilar. While (2.8) has a quark operator, it is the kinetic term and it does not have the same canonical dimension as, say, the operators of (6.1) in four dimensions. The first stage in such an investigation will be to set up the large N_f formalism for dimension 6 and 8 gauge invariant operators and compute the mixing matrix of critical exponents at $O(1/N_f)$ in d-dimensions. The former dimension is required for an analysis of (2.2) and we note that the large N_f exponent relating to the QCD β-function in four dimensions [16] was derived from the critical point large N_f renormalization of the dimension four operator $G^a_{\mu\nu}G^{a\mu\nu}$. That in effect was the initial step of the proposal to examine the operator content of the tower of Lagrangians constituting universal non-Abelian Thirring model universality class.

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